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Math 2106 Fall '20
Foundations of Mathematical Proofs
Portfolio

1. Direct Proof - Homework 11 Question 5

   Theorem Prove that all groups of order strictly less than 6 are abelian.

   Proof. As given by Lagrange's theorem, we know that for H to be a subgroup of G, where both H and G are finite groups, the order of H must divide the order of G. Let $g \in G, g \neq e$ where e is the identity element. Assume $|G| = p$ where p is a prime number $p \neq 1$. Now let us denote the cyclic group generated by g as $\langle g \rangle$. Since g is within the group G, $\langle g \rangle$ must be a subgroup of G in order for G to be closed under a binary operation. We therefore know that $|\langle g \rangle|$ must divide $|G|$ by Lagrange's theorem. Since the order of G is a prime number we know that the only divisors of p are p and 1. Since we said that $g \neq e$, then $|\langle g \rangle|$ is not equal to one. It must also be true that $|\langle g \rangle| = p$. We can further see that $\langle g \rangle \ = G$ implying that G is cyclic. In summary, if the order of G is prime and not equal to one, then G is cyclic and therefore is abelian. This shows that groups of order 2,3,5 are all abelian. If G has order 1, then by the definition of a group, it only contains the identity element e and is therefore abelian.

   Now when we look at the case where $|G| = 4$, there are several possible groups we could create of order 4. The first scenario is when G contains an element of order 4, $\{e, x, x^2, x^3\}$ meaning that G is cyclic and therefore abelian. The next scenario arises if G contains an element of order 3, $\{e, x, x^2, y\}$, however, since the elements xy, yx, x^2y, yx^2 are not within the set, the group is not closed and therefore is not valid. Lastly, G can be comprised of 3 elements of order 2, $\{e, x, y, z\}$. Let us define xy to be equal to z so we may rewrite the set as $\{e, x, y, xy\}$. If every element has order 2, then we can see that $x^2 = y^2 = (xy)^2 = e$. In order to show that this particular group is abelian, then we must show that $xy = yx$ and we can do so as follows

   $$(xy)^2 = e = x^2y^2$$
   $$xyxy = xxyy$$
   $$xxyxy = xxxyy$$
   $$eyxy = exyy$$
   $$yxy = xyy$$
   $$yxyy = xyyy$$
   $$yxe = xye$$
   $$yx = xy$$

   We have then shown that all possible groups of order 4 are either abelian or cannot exist so we can conclude that all groups of order 4 are abelian.

   Since we have shown that sets of order 1,2,3,4, and 5 are all abelian we have thus shown that every group of order strictly less than 6 is abelian as desired. □

2. Contrapositive Proof - Homework 12 Question 1

   Lemma 2.1 If $x \in G$, $x \notin H$ and $H \leq G$ then $xh \notin H$ for all $h \in H$

   For some group G and subgroup H, if $x \in G$ and $x \notin H$ then $xh \notin H$ for all $h \in H$. If we know that $h \in H$, then by the definition of a group, then $h^{-1} \in H$ as well. Assume that $x \notin H$ and $xh \in H$. Since H is a group, it must be closed under some operation. If we take xh and multiply by h^{-1}, then the result should be in H as well. Therefore we get that $xhh^{-1} \in H$. We see that $xhh^{-1} = xe = x$, so we get that $x \in H$ which is a contradiction as we stated that $x \notin H$ so we can conclude that $xh \notin H$ for all $h \in H$.

   Theorem Suppose that $H \leq G$ with $[G : H] = 2$. If a and b are not in H, then prove that $ab \in H$.

   Proof. We look to show the fact that "If a and b are not in H, then $ab \in H$". We start by taking the contrapositive of this statement to get: If $ab \notin H$, then either a or b is in H. Since we know that the index of $[G : H] = 2$, there are only two possible cosets. By definition, we know that H contains the identity element so $eh \in H$ for all $h \in H$ and $eH = H$. Additionally, if we take x where $x \in G$ and $x \notin H$ then from Lemma 2.1 we can see that $xh \notin H$ for all $h \in H$ and therefore $xH \neq H$. This shows us that one of these cosets is H itself, while the other is every element not in H. Since $ab \notin H$, then we know that $abH \neq H$. Next we must show that either a or b is in H. Assume $a \in H$, then our contrapostive statement is true, meaning that our original statement is true.

   Next, assume $a \notin H$, then we know $aH \neq H$. Since there is only one coset not equal to H, we can deduce that since $abH \neq H$ and $aH \neq H$, then we can see that $abH = aH$. By multiplying a^{-1} on both sides, we can get $a^{-1}abH = a^{-1}aH$, this then simplifies to $ebH = eH \rightarrow bH = H$, which we can use to conclude that b is in H. Therefore, we have shown that if ab is not in H, then either a or b is in H, and can therefore conclude that the contrapositive and therefore our original statement is true as well. □

3. Proof by Contradiction - Homework 12 Question 4

Lemma 3.1 If $x \in G$, $x \notin H$ and $H \leq G$ then $xh \notin H$ for all $h \in H$

For some group G and subgroup H, if $x \in G$ and $x \notin H$ then $xh \notin H$ for all $h \in H$. If we know that $h \in H$, then by the definition of a group, then $h^{-1} \in H$ as well. Assume that $x \notin H$ and $xh \in H$. Since H is a group, it must be closed under some operation. If we take xh and multiply by h^{-1}, then the result should be in H as well. Therefore we get that $xhh^{-1} \in H$. We see that $xhh^{-1} = xe = x$, so we get that $x \in H$ which is a contradiction as we stated that $x \notin H$ so we can conclude that $xh \notin H$ for all $h \in H$.

Theorem Let $A \leq S_4$ be defined by

$$A = \{(1), (12)(34), (13)(24), (14)(23), (123), (132), (124), (142), (134), (143), (234), (243)\}.$$

Prove that A has no subgroup of order 6.

Proof. Let H be a subgroup of A_4 such that $|H| = 6$. A_4 contains the identity element, 3 cycles of order 2, and 8 cycles of order 3. By the pigeonhole principle, at least 2 cycles of order 3 cannot be within H. Let us denote these two cycles as x and y. By Lagrange's theorem, we define the index of a subgroup to be the number of left/right cosets and can calculate the index by diving the order of the groups. We can then derive that the index of H is $[A_4 : H] = 2$, so we know that there are 2 distinct cosets. By definition, we know that H contains the identity element so $eh \in H$ for all $h \in H$ and $eH = H$. Additionally, if we take x where $x \in A_4$ and $x \notin H$ then from Lemma 3.1 we can see that $xh \notin H$ for all $h \in H$ and $xH \neq H$. From this we see that of the 2 cosets, 1 coset is H and the other coset is not equal to H. Now let us proceed and look at the cosets generated by x and y.

When looking at possible cycles for x and y, we again arrive at two distinct possibilities, either that ($x^2 = y$ and $x = y^2$) or that ($x^2 \neq y$ and $x \neq y^2$). Let us look at the first possibility of where ($x^2 = y$ and $x = y^2$). Since $x, y \notin H$, then $xH \neq H$ and $yH \neq H$. Since we stated that there is only 1 coset not equal to H, it follows that $xH = yH = x^2H$. However, if we multiply both sides by x, we get $x^2H = xyH = x^3H$. By definition, x is a 3-cycle meaning that $x^3 = e$ so we can reduce our statement to be that $x^2H = eH = H$ so $x^2H = yH = H$ which is a contradiction as we said that x and y are not in H.

Let us now look at the other case where $x^2 \neq y$ and $x \neq y^2$. Similarly, we see that $xH \neq H$ and $yH \neq H$. Since we know that $x \notin H$ and $xH \neq H$, x^2H must be equal to either H or xH. If $x^2H = H$, then since x is a 3-cycle $x^2 = x^{-1}$ and $x^{-1} \in H$ but in order to fufil the properties of a group, if $x^{-1} \in H$ then $x \in H$ as well which is a contradiction since we stated that $x \notin H$. If $x^2H \neq H$ then $x^2H = xH$, and by multiplying both sides by x, we get that $x^3H = x^2H \rightarrow eH = x^2H \rightarrow H = x^2H$ which is a contradiction since we stated that $x^2H \neq H$.

Therefore, we can see that it is impossible for H to be a subgroup of A_4 meaning that it is impossible for A_4 to have a subgroup of order 6. □

4. If and only if - Homework 10 Question 2

Theorem Let G be a group. Prove that G is abelian if and only if $(ab)^2 = a^2b^2$ for all $a, b \in G$. As a corollary show that if $a = a^{-1}$ for all $a \in G$ then G is abelian.

Proof. First we must show that if G is abelian, then $(ab)^2 = a^2b^2$ for all $a, b \in G$

Assume that G is an abelian group. Let $a, b, c \in G$ and let $c = ab$. Then $cc = c^2$. Now we can substitute in ab for c.

$$(ab)(c) = (ab)^2$$

Since we know G is abelian, then we can rewrite c as ba giving us

$$(ab)(ba) = (ab)^2$$

$$ab^2a = (ab)^2$$

$$aab^2 = (ab)^2$$

$$a^2b^2 = (ab)^2$$

Therefore showing that G is abelian implies $(ab)^2 = a^2b^2$ for all $a, b \in G$.

Next suppose that $(ab)^2 = a^2b^2$ is true for all $a, b \in G$. Lets show that G is abelian.

$$(ab)^2 = a^2b^2$$

$$(ab)(ab) = aabb$$

$$abab = aabb$$

Multiply both sides by a^{-1} and b^{-1}

$$a^{-1}ababb^{-1} = a^{-1}aabbb^{-1}$$

$$ba = ab$$

Since we have shown that $ab = ba$, then we know G must be abelian by the definition of commutative so we have thus show that G is abelian if and only if $(ab)^2 = a^2b^2$ for all $a, b \in G$. □

Proof. For the corollary, if $a = a^{-1}$, then we can see

$$aa^{-1} = e$$

based on the identity. And by substitution, we can see

$$aa = e$$

$$a^{-1}a = e$$

$$aa^{-1} = a^{-1}a$$

meaning that G is an abelian group. □

5. Existence and Uniqueness - Modified Homework 13 Question 3

   Theorem Let $s_1 = 1$ and $s_{n+1} = \sqrt{s_n + 1}$. Prove that the limit of this sequence exists and is unique.

   Proof. Before we can find the limit of the sequence, we must show that the sequence converges. In order to show that s_n converges, we must show that the sequence is both monotonic and bounded. In order to show that the sequence is monotonic, we must show that it either increases or decreases. In our scenario, the sequence increases and we will show this through induction. We know that $s_0 = 1$ so therefore $s_1 = \sqrt{1+1} = \sqrt{2} > s_0$, which shows that our base case holds. Now we assume that $s_n < s_{n+1}$ is true and wish to show that $s_{n+1} < s_{n+2}$

$$\begin{aligned} s_n &< s_{n+1} \\ s_n + 1 &< s_{n+1} + 1 \\ \sqrt{s_n + 1} &< \sqrt{s_{n+1} + 1} \\ s_{n+1} = \sqrt{s_n + 1} &< \sqrt{s_{n+1} + 1} = s_{n+2} \\ s_{n+1} &< s_{n+2} \end{aligned}$$

   So we know that the sequence s_n is increasing. We can see that the sequence s_n is bounded by the fact that the sequence is bounded by 3. We can prove this by letting $s_x = 3$, then $s_{x+1} = \sqrt{3+1} = 2$ so $s_{x+1} < s_x$. So we know that s_n is bounded and has a supremum less than 3. Now, we have shown that s_n is bounded and monotonic. We must find and prove the limit of s_n. Since s_n is increasing, we know the limit must be the supremeum must also be the limit. By the definition of the supremum we know that

$$s_x = s_{x+1}$$

   We can use this and solve for the supremum as shown:

$$\begin{aligned} s_x &= s_{x+1} \\ x &= \sqrt{x+1} \\ x^2 &= x + 1 \\ x^2 - x - 1 &= 0 \\ x &= \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

   Since the function s_n is increasing we know the supremum must be $\frac{1+\sqrt{5}}{2}$. Now we must prove that this is the limit of s_n. Let x be the supremum of s_n. Let $\epsilon > 0$ and some N such that $n > N$. Let $s_N = x - \epsilon$. We then get

$$s_N < s_n \leq x$$

$$x - \epsilon < s_n \leq x$$

$$x - \epsilon - x < s_n - x \leq x - x$$

$$-\epsilon < s_n - x \leq 0$$
$$-\epsilon < s_n - x$$
$$|s_n - x| < \epsilon$$

Which by the definition of a convergent sequence, means that s_n converges to $x = \frac{1+\sqrt{5}}{2}$, therefore proving that s_n is convergent and has a limit as desired. □

Proof. Now let us show that the limit of this sequence is unique. Assume that s_n has two limits defined as a and b, where $a \neq b$. Let us define ϵ as $\epsilon = |a - b|$ and since the absolute value will always be positive so we can say that $\epsilon > 0$ and $|a - b| > 0$. By definition, since a is a limit of s_n, then there exists some $N_1 \in \mathbb{N}$ such that for all $n > N_1$ $|a - s_n| < \frac{\epsilon}{2}$. Since b is a limit of s_n, we can additionally say that there exists some $N_2 \in \mathbb{N}$ such that for all $n > N_2$ $|b - s_n| < \frac{\epsilon}{2}$. Now let M be defined as $M = max\{N_1, N_2\}$. Then there exists some $n > M$ such that $|a - b| = |a - s_n + s_n - b|$.

$$\begin{aligned}
|a - b| &= |a - s_n + s_n - b| \\
&\leq |a - s_n| + |b - s_n| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&< \epsilon = |a - b| \\
|a - b| &< |a - b|
\end{aligned}$$

This leads us to the statement that $|a - b| < |a - b|$ which is impossible so we have arrived at a contradiction and showing that a must be equal to b. Therefore, s_n may only have one limit and the limit is unique. □

6. Proof using Sets - Homework 5 Question 5

Theorem Prove that

$$\bigcap_{x\in\mathbb{R}}[3-x^2,5+x^2]=[3,5].$$

Proof. In order to show that both of these sets are equivalent, we need to show that they are both subsets of each other. The first thing we can look at is $[3,5]\subseteq\bigcap_{x\in\mathbb{R}}[3-x^2,5+x^2]$. We see that for $\bigcap_{x\in\mathbb{R}}[3-x^2,5+x^2]$ we can look at both sides and determine inequalities. Since we always know x^2 will be positive, we can determine the left side to be $3-x^2\leq 3$ and the right side to be $5+x^2\geq 5$. Since the intervals will always include 3 and 5, we can see that $[3,5]\subseteq\bigcap_{x\in\mathbb{R}}[3-x^2,5+x^2]$.

Next we must take a look at $\bigcap_{x\in\mathbb{R}}[3-x^2,5+x^2]\subseteq[3,5]$. If we allow $a\in\bigcap_{x\in\mathbb{R}}[3-x^2,5+x^2], a\in\mathbb{R}$, then we can write this as the inequality $3-x^2\leq a\leq 5+x^2, x\in\mathbb{R}$. Now we have to look at the overlap for the sets in $\bigcap_{x\in\mathbb{R}}[3-x^2,5+x^2]$. This means we need to maximize $3-x^2$ and minimize $5+x^2$ and since we know $x^2\geq 0$, both of these occur at $x=0$. This will give us that $a\in[3,5]$ meaning that $\bigcap_{x\in\mathbb{R}}[3-x^2,5+x^2]\subseteq[3,5]$. Since we have shown both sides of this statement are subsets of each other, we can now say that $\bigcap_{x\in\mathbb{R}}[3-x^2,5+x^2]=[3,5]$. □

7. Induction Proof - Question Given by Dr. Bloomquist

Lemma 7.1 $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

Let us start with the right-hand side of our equation $\binom{n}{k-1} + \binom{n}{k}$, expanding these expressions out gives us the following

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!}$$

$$= \frac{n!(k) + n!(n-k+1)}{k(k-1)!(n-k+1)!} = \frac{n!(k-k+n+1)}{k!(n+1-k)!} = \frac{n!(n+1)}{k!(n+1-k)!} = \frac{(n+1)!}{k!(n+1-k)!}$$

$$= \binom{n+1}{k}$$

We have thus shown that $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

Theorem Prove by induction that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof. In order to prove by induction, we must first look at our base cases of $n = 0$ and $n = 1$. If we let $n = 0$, then we can see that $\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{0} \binom{0}{0} = \frac{0!}{0!} = 1 = 2^0$ so the case where $n = 0$ is true. Now let us look at the case where $n = 1$. We see that $\sum_{k=0}^{1} \binom{1}{0} + \binom{1}{1} = \frac{1!}{0!} + \frac{1!}{1!} = 1 + 1 = 2 = 2^1$ so our case where $n = 1$ holds true as well. Now let us assume that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ for some constant $n \in \mathbb{N}$. Now we look to show that $\sum_{k=0}^{n+1} \binom{n+1}{k} = 2^{n+1}$.

Looking at $\sum_{k=0}^{n+1} \binom{n+1}{k}$, we can expand this expression out as follows:

$$\sum_{k=0}^{n+1} \binom{n+1}{k} = \binom{n+1}{0} + \binom{n+1}{1} + \binom{n+1}{2} + ... + \binom{n+1}{n} + \binom{n+1}{n+1}$$

We know that $\binom{n+1}{0}$ is equivalent to $\frac{(n+1)!}{(n+1)!} = 1$, we can also see that $\binom{n+1}{n+1}$ is equivalent to $\frac{(n+1)!}{(n+1)!} = 1$ as well. We can take our expansion above and rewrite it as follows.

$$1 + \binom{n+1}{1} + \binom{n+1}{2} + ... + \binom{n+1}{n} + 1$$

which is equivlant to

$$2 + \sum_{k=1}^{n} \binom{n+1}{k}$$

Using Lemma 7.1, we are able to break $\sum_{k=1}^{n} \binom{n+1}{k}$ up into 2 smaller summations

$$2 + \sum_{k=1}^{n} \binom{n+1}{k} = 2 + \sum_{k=1}^{n} \binom{n}{k-1} + \sum_{k=1}^{n} \binom{n}{k}$$

Looking at the summation $\sum_{k=1}^{n}\binom{n}{k-1}$, we can see the following

$$\sum_{k=1}^{n}\binom{n}{k-1} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n-1} =$$

$$\sum_{k=0}^{n}\binom{n}{k} - \binom{n}{n} = \sum_{k=0}^{n}\binom{n}{k} - 1$$

Next let us look at the summation $\sum_{k=1}^{n}\binom{n}{k}$. If we expand this summation, we are able to see that

$$\sum_{k=1}^{n}\binom{n}{k} = \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n-1} + \binom{n}{n}$$

$$= \sum_{k=0}^{n}\binom{n}{k} - \binom{n}{0} = \sum_{k=0}^{n}\binom{n}{k} - 1$$

If we substitute both of these statements back into our original equation we get that:

$$2 + \sum_{k=1}^{n}\binom{n}{k-1} + \sum_{k=1}^{n}\binom{n}{k} = 2 + \sum_{k=0}^{n}\binom{n}{k} - 1 + \sum_{k=0}^{n}\binom{n}{k} - 1$$

$$= 2\sum_{k=0}^{n}\binom{n}{k}$$

Now using our induction hypothesis above, we can substitute $\sum_{k=0}^{n}\binom{n}{k}$ with 2^n. This now gives us:

$$= 2\sum_{k=0}^{n}\binom{n}{k} = 2 \cdot 2^n = 2^{n+1}$$

Thus showing that $\sum_{k=0}^{n+1}\binom{n+1}{k} = 2^{n+1}$ as desired. We have thus shown that by mathematical induction, $\sum_{k=0}^{n}\binom{n}{k} = 2^n$ holds true for all $n, k \in \mathbb{N}$

□

8. An equivalence relation proof - Homework 7 Question 5

Theorem Prove that the intersection of equivalence relations on the same set is an equivalence relation

Proof. In order to show that the intersection of equivalence relations is an equivalence relation, we must show that the intersection is reflexive, symmetric, and transitive. Let us define the two equivalence relations R_1 and R_2. Then we can represent the intersection as $R_1 \cap R_2$

Assume both R_1 and R_2 are equivalence relations on the set A, then we know that if we let $x \in A$, then we know that xR_1x and xR_2x exist since R_1 and R_2 are reflexive so we can say that $(x, x) \in R_1 \cap R_2$ therefore showing that $R_1 \cap R_2$ is reflexive

Now we can show that $R_1 \cap R_2$ is symmetric. Assume $x(R_1 \cap R_2)y$ where $x, y \in A$, then we know that xR_1y and xR_2y exist and since both R_1 and R_2 are symmetric, then yR_1x and yR_2x exist as well and therefore $y(R_1 \cap R_2)x$ is true as well.

Lastly we must show that $R_1 \cap R_2$ is transitive as well. Assume $x(R_1 \cap R_2)y$ and $y(R_1 \cap R_2)z$ both are true, then we know that xR_1y, xR_2y ,yR_1z, yR_2z all exist and are true. Then, since R_1 and R_2 are equivalence relations and therefore are transitive so xR_1z and xR_2z exist so therefore $x(R_1 \cap R_2)z$ exists and shows that $R_1 \cap R_2$ is transitive as well

Since we are able to show that $R_1 \cap R_2$ is reflexive, symmetric, and transitive, then we know that $R_1 \cap R_2$ is an equivalence relation proving that the intersection of two equivalence relations is an equivalence relation as well. □

9. A proof involving injectivity and surjectivity of a function - Midterm Question 4

Theorem Let $f : A \to B$ be a function. For any $Y \subseteq B$, define

$$f^{-1}(Y) := \{a \in A : f(a) \in Y\}$$

Note that this doesn't require that f^{-1} is a function Then f is said to be **midterm injective** if for all $b \in B$

$$|f^{-1}(\{b\})| \leq 1$$

Prove that f is midterm injective if and only if f is injective We also have that f is said to be **midterm surjective** if for all $b \in B$

$$|f^{-1}(\{b\})| > 0$$

Prove that f is midterm surjective if and only if f is surjective.

Use the above definitions to give a description of a bijection as both midterm injective and midterm surjective.

Proof. First let us begin by showing that f is midterm injective if and only if f is injective. Let f be injective. So let $a, b \in A$, then let $f(a), f(b) \in f(A)$, so therefore $f(a), f(b) \in Y$. Then by the definition of injective, each value in A has at most 1 corresponding value in B. Therefore $b \in B$ and $|f^{-1}(b)|$ is either equal to 0 or 1, both of which are less than equal to 1, meaning that f is midterm injective. Now assume f is midterm injective. We then know that $\forall b \in B |f^{-1}(b)| \leq 1$. Let $f^{-1}(b) = a$. By the definition of midterm injective, a must be either $a \in A$ or nonexistent which means that it is unique, proving that f is injective. Therefore we have shown that f is midterm injective if and only if f is injective.

Next we will show that f is midterm surjective if and only if f is surjective. Assume f is midterm surjective, then we know that $|f^{-1}(b)| > 0$, meaning that for all $b \in B$, there exists some $f(a) = b, a \in A$. Sincethere exists some $f(a) = b$ for all b, then by definition f is surjective. Now assume f is surjective, then we know $\forall b \in B, \exists a, f(a) = b$. Since there is always some a that can transform into any b, $|f^{-1}(b)|$ is always at least one meaning that $|f^{-1}(b)| > 0$ showing that f is midterm surjective. Therefore we have shown that f is midterm surjective if and only if f is surjective.

By the definition of bijective, we know that a fucntion must be both injective and surjective to be bijective. Using the definitions of midterm injective and midterm surjective, we can see that for a function to be bijective:

$$0 < |f^{-1}(\{b\})| \leq 1$$

$$|f^{-1}(\{b\})| = 1$$

This means that for every $b \in B$, there is a unique $a \in A$ and the unique value a is guaranteed to exist. □

10. Proving that something is a group - Mastery Quiz 3

Theorem Let $(G,)$ be a group and X a nonempty set. Define $M(G, X)$ to be the set of functions from X to G. Define $\star$ by $(f \star g)(x) = f(x) \cdot g(x)$. Prove that $(M(G, X), \star)$ is a group. Additionally show that if $(G,)$ is abelian then $(M(G, X), \star)$ is abelian.

Proof. In order to show that $(M(G, X), \star)$ is a group, we must show that it satisfies 4 major properties. We must show that the binary operation is closed, the binary operation is associative, the identity element exists within the group, and that every element has an inverse element that exists within the group. First let us show that the binary operation is closed. For the operation to be closed, we must show that the output of the operation is another element within our set. If we take two functions h, k where $h, k \in M(G, X)$, then perform the operation $\star$, we get $(h \star k)(x)$. By definition of $\star$ we know that $(h \star k)(x) = h(x) \cdot k(x)$. Looking at the right hand side, we know that for any $x \in X$, then $h(x), k(x) \in G$ so we can say that the binary operation is closed. Now let us show that the group is associative. If we take functions $f, h, k \in M$, then we can perform $(f \star (h \star k))(x)$. We can expand this out to be

$$(f \star (h \star k))(x) = f(x) \cdot (h(x) \cdot k(x))$$

Since we know that $f(x), h(x), k(x) \in G$ and that G is a group, this means that elements of G are associative so we are able to shift the parenthesis to get

$$(f(x) \cdot h(x)) \cdot k(x) = ((f \star h) \star k)(x)$$

And therefore showing that M is associative.

Now we look to show that the identity element exists within the group. Let e_G be the identity element of G. Now let us define e_M to be the function in M that maps every element in X to e_G. $e_M(x) = e_G \forall x \in X$. Now let us show that e_M is the identity element. If we take an element $f \in M$, $(e_M \star f)(x) = (e_M(x) \cdot f(x)) = (e_G \cdot f(x)) = f(x)$. Additionally $(f \star e_M)(x) = (f(x) \cdot e_M(x)) = (f(x) \cdot e_G) = f(x)$. Therefore we see that e_M is the identity element of M.

Lastly we must show that every element has an inverse within the group such that $(f \star f^{-1})(x) = e_M(x)$. If every $f \in M$ maps to some value $f(x) \in G$, then for every $f \in M$, let us define the inverse f^{-1} to map to $(f(x))^{-1}$. Then we can see that $f \in M$, $(f^{-1} \star f)(x) = ((f(x))^{-1} \cdot f(x)) = e_G = e_M(x)$. Additionally $(f \star f^{-1})(x) = (f(x) \cdot (f(x))^{-1}) = e_G = e_M(x)$. Therefore we can see that every element has an inverse.

Since we have shown that the binary operation is associative, the identity element exists within the group, and that every element has an inverse element that exists within the group, we can conclude that $(M(G, X), \star)$ is a group.

Lastly we must show that if G is abelian, then $(M(G, X), \star)$ is abelian as well. We know $(M(G, X), \star)$ is abelian if for $f, g \in M$, $(f \star g)(x) = (g \star f)(x)$. $(f \star g)(x) = f(x) \cdot g(x)$ and since it is given that G is abelian, we can rearrange these values to get $g(x) \cdot f(x)$, which we can see is equivalent to $(g \star f)(x)$. Therefore showing that $(f \star g)(x) = (g \star f)(x)$ and $(M(G, X), \star)$ is abelian. □

11. Limit Proof - Mastery Quiz 4

Theorem Let $a_n = \frac{100n+2}{100n+1}$. Show that $\lim_{n\to\infty} a_n = 1$

Proof. Let $\epsilon > 0$. Define $N = \frac{1}{100\epsilon} - \frac{1}{100}$. Then for all $n > N$, this implies that $n > \frac{1}{100\epsilon} - \frac{1}{100}$. Through algebra we can see that $100n > \frac{1}{\epsilon} - 1$ and $100n + 1 > \frac{1}{\epsilon}$. Thus we can see that $\epsilon > \frac{1}{100n+1}$.

Expanding the fraction $\frac{1}{100n+1}$ gives us that

$$\frac{1}{100n+1} = \frac{100n+1-100n}{100n+1} = \frac{100n+2-100n-1}{100n+1}$$

$$\frac{100n+2-100n-1}{100n+1} = \frac{100n+2-(100n+1)}{100n} = \frac{100n+2}{100n+1} - 1$$

We now have that $\frac{100n+2}{100n+1} - 1 < \epsilon$ and by taking the absolute value of the left hand side we can get

$$|\frac{100n+2}{100n+1} - 1| < \epsilon$$

and by the definition of a convergent sequence, we see that $a_n = \frac{100n+2}{100n+1}$ converges to 1, meaning that we have shown that $\lim_{n\to\infty} a_n = 1$ as desired. □