1B) Lunar Lander

$\ddot{z} = -g + T_{max}/M_{tot}$

$M_{tot} = M_S + M_s$

$z(0) = z_0$

$V_z(0) = 0$

(1) $\dot{z} = V_z$

(2) $\dot{V}_z = -g + T_{max} k / M_{tot}$

(3) $\dot{M}_f = -T_{max} \cdot k / V_{ex}$

1C) Kepler Problem

$\ddot{r}_i = \vec{F}_i = -G m_i \sum_{j \neq i} m_j \frac{r_i - r_j}{|r_i - r_j|^3}$

$\ddot{x}_i = -G \sum_{j \neq i} m_j \frac{x_i - x_j}{|r_i - r_j|^3}$

$\ddot{y}_i = -G \sum_{j \neq i} m_j \frac{y_i - y_j}{|r_i - r_j|^3}$

The low ordered steppers were less accurate in creating a circular orbit.
RK4 allows us to get a closed orbit.

1d) H_2 Formation

k_1 (1)	$A + A + G \rightarrow B + G$
k_2 (2)	$B + \gamma \rightarrow 2A$
k_3 (3)	$A + CR \rightarrow C + e^-$
k_4 (4)	$C + e^- \rightarrow A + \gamma$

$$\dot{A} = -2k_1A^2 + 2k_2B - k_3A + k_4C^2$$
$$\dot{B} = k_1A^2 - k_2B$$
$$\dot{C} = k_3A - k_4C^2$$

$$\underset{3\times 4}{\begin{bmatrix} -2 & 2 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}} \underset{4\times 1}{\begin{bmatrix} k_1A^2 \\ k_2B \\ k_3A \\ k_4C \end{bmatrix}} =$$

Part B.

1e

$y'(x) = \exp(y(x) - 2x) + 2 \quad , \quad y(0) = -\ln(2)$

$z(x) = -2x + y(x) \quad \Rightarrow \quad \frac{dz}{dx} = \frac{dy}{dx} - 2$

$\frac{dz}{dx} + 2 = e^{z(x)} + 2 \quad \Rightarrow \quad dz/dx = e^{z(x)}$

$e^{-z(x)} \frac{dz}{dx} = 1 = \int e^{-z(x)} \frac{dz}{dx}\, dx = \int 1\, dx$

$\Rightarrow \quad -e^{-z(x)} = x + c \quad \Rightarrow \quad z(x) = -\ln(-x - c_1)$

$\Rightarrow$ subsitute into $z(x) = -2x + y(x)$

$\Rightarrow \quad -\ln(-x + c_1) = -2x + y(x)$

$\Rightarrow \quad \boxed{y(x) = 2x - \ln(-x + c_1)}$

$y(0) = -\ln(2)$

$\Rightarrow \quad \boxed{y(x) = 2x - \ln(-x + c_1)}$

$y(0) = -\ln(2)$

$y(0) = 0 - \ln(0 + c_1) = -\ln(2) \quad \Rightarrow \quad c_1 = 2$

$\Rightarrow \quad \boxed{y(x) = 2x - \ln(2 - x)}$

$y(1) = 2(1) - \ln(2-1) = 2$

$\boxed{y(1) = 2}$

(iii) **Slopes Expected:**

Euler $\Rightarrow -1$ since it is $O(h)$ and $\log(h) \propto 1$

RK2 $\Rightarrow -2$ since it is $O(h^2)$ and $\log(h^2) \propto 2$

RK4 $\Rightarrow -4$ since it is $O(h^4)$ and $\log(h^4) \propto 4$

However, RK4 does not seem to have a slope of -4, instead it hits a minimum error at $N = 10^3$, with $N > 10^3$ it's error increases. $N = 10^3$ was also the only error that is > 0. This may be due to machine error.

Part C

Coupled ODE

$u' = 998u + 1998v$ $\quad u(0) = 1$

$v' = -999u - 1999v$ $\quad v(0) = 1$

$u = 2e^{-x} - e^{-1000x}$ $\quad v = -e^{-x} + e^{-1000x}$

(i) $(d/dx)u(x) = \frac{d}{dx}(2e^{-x} - e^{-1000x}) = d/dx(2e^{-x}) - d/dx\, e^{-1000x}$

$= -2e^{-x} - (-1000e^{-1000x}) = -2e^{-x} + 1000e^{-1000x}$

$= 998(2e^{-x} - e^{-1000x}) + 1998(-e^{-x} + e^{-1000x})$

$= 998u + 1998v \Rightarrow \underline{u' = 998u + 1998v}$

$(d/dx)v(x) = d/dx(-e^{-x} + e^{-1000x}) = d/dx(-e^{-x}) + d/dx(e^{-1000x})$

$= e^{-x} + (-1000e^{-x}) = -999(2e^{-x} - e^{-1000x}) - 1999(-e^{-x} + e^{-1000x})$

$= -999u - 1999v \Rightarrow \underline{v' = -999u - 1999v}$

$\boxed{h < 2/1000}$

$\boxed{h < 2/1000}$

$y' = -C \cdot y$, where C is a positive definite matrix

$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} 998 & 1998 \\ -999 & -1999 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \Rightarrow C = \begin{bmatrix} -998 & -1998 \\ 999 & 1999 \end{bmatrix}$

$y' = -\begin{bmatrix} -998 & -1998 \\ 999 & 1999 \end{bmatrix} y \Rightarrow h < 2/\lambda_{max}$

$[A - \lambda I] \Rightarrow \begin{bmatrix} 1 & 1 \\ 999 & 1999 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1000 \end{bmatrix}$

$[A - \lambda I] \Rightarrow \lambda = 1, 1000 \Rightarrow h > 2/\lambda_{max} > 2/1000$

$\Rightarrow \boxed{h < 2/1000}$

(ii) $998u + 1998v$ $\quad -C = $ Jacobian

$-999u - 199v$

Back Euler Num Iterations: [1 ... 1]

RK45 Num Iterations: [17, 5, 2, 3 ... 3]

Implicit has a constant, predictable number of iterations, can be quicker.