

## 1B) Lunar Lander

$$\ddot{z} = -g + T_{\max}/M_{\text{tot}}$$

$$M_{\text{tot}} = M_S + M_s$$

$$z(0) = z_0$$

$$\dot{z}(0) = 0$$

$$(1) \quad \dot{z} = v_z$$

$$(2) \quad \dot{v}_z = -g + T_{\max}^k / M_{\text{tot}}$$

$$(3) \quad \dot{M}_f = -T_{\max} \cdot \delta_z / v_{\text{ex}}$$

## 1C) Kepler Problem

$$\ddot{\vec{r}}_i = \vec{F}_i = -G m_i \sum_{j \neq i} m_j \frac{\vec{r}_i - \vec{r}_j}{|\vec{r}_i - \vec{r}_j|^3}$$

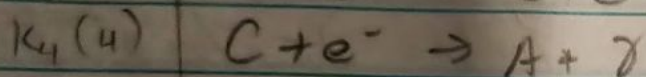
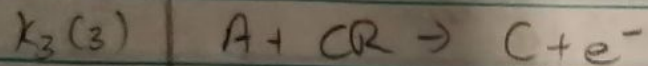
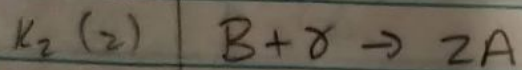
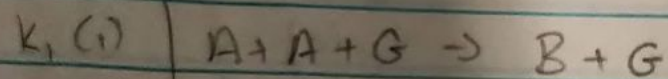
$$\ddot{x}_i = -G \sum_{j \neq i} m_j \frac{x_i - x_j}{|\vec{r}_i - \vec{r}_j|^3}$$

$$\ddot{y}_i = -G \sum_{j \neq i} m_j \frac{y_i - y_j}{|\vec{r}_i - \vec{r}_j|^3}$$

The low ordered steppers were less accurate in creating a circular orbit.

RK4 allows us to get a closed orbit.

# 1d) $H_2$ Formation



$$\dot{A} = -2k_1 A^2 + 2k_2 B - k_3 A + k_4 C^2$$

$$\dot{B} = k_1 A^2 - k_2 B$$

$$\dot{C} = k_3 A - k_4 C^2$$

$$\begin{matrix} 3 \times 4 & & 4 \times 1 \\ \begin{bmatrix} -2 & 2 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} & \begin{bmatrix} k_1 A^2 \\ k_2 B \\ k_3 A \\ k_4 C \end{bmatrix} & = \end{matrix}$$



## Part B.

1e

$$y'(x) = \exp(y(x) - 2x) + 2, \quad y(0) = -\ln(2)$$

$$z(x) = -2x + y(x) \Rightarrow \frac{dz}{dx} = \frac{dy}{dx} - 2$$
$$\frac{dz}{dx} + 2 = e^{z(x)} + 2 \Rightarrow \frac{dz}{dx} = e^{z(x)}$$

$$e^{-z(x)} \frac{dz}{dx} = 1 = \int e^{-z(x)} \frac{dz}{dx} dx = \int 1 dx$$

$$\Rightarrow -e^{-z(x)} = x + c \Rightarrow z(x) = -\ln(-x - c_1)$$

$$\Rightarrow \text{substitute into } z(x) = -2x + y(x)$$

$$\Rightarrow -\ln(-x + c_1) = -2x + y(x)$$

$$\Rightarrow \boxed{y(x) = 2x - \ln(-x + c_1)}$$

$$y(0) = -\ln(2)$$

$$\Rightarrow \boxed{y(x) = 2x - \ln(-x + c_1)}$$

$$y(0) = -\ln(2)$$

$$y(0) = 0 - \ln(0 + c_1) = -\ln(2) \Rightarrow c_1 = 2$$

$$\Rightarrow \boxed{y(x) = 2x - \ln(2 - x)}$$

$$y(1) = 2(1) - \ln(2-1) = 2$$

$$\boxed{y(1) = 2}$$

Slopes Expected:

(iii)

Euler  $\Rightarrow -1$  since it is  $O(h)$  and  $\log(h) \propto 1$

RK2  $\Rightarrow -2$  since it is  $O(h^2)$  and  $\log(h^2) \propto 2$

RK4  $\Rightarrow -4$  since it is  $O(h^4)$  and  $\log(h^4) \propto 4$

However, RK4 does not seem to have a slope of -4, instead it hits a minimum error  $\propto N^{-10^3}$ , with  $N > 10^3$  its error increases.  $10^{-3}$  was also the only error that is  $> 0$ . This may be due to machine error.



# Part C

Coupled  
ODE

$$u' = 998u + 1998v$$

$$u(0) = 1$$

$$v' = -999u - 1999v$$

$$v(0) = 1$$

$$u = 2e^{-x} - e^{-1000x}$$

$$v = -e^{-x} + e^{-1000x}$$

$$\begin{aligned} (i) \quad \frac{d}{dx}u(x) &= \frac{d}{dx}(2e^{-x} - e^{-1000x}) = \frac{d}{dx}(2e^{-x}) - \frac{d}{dx}(e^{-1000x}) \\ &= -2e^{-x} - (-1000e^{-1000x}) = -2e^{-x} + 1000e^{-1000x} \\ &= 998(2e^{-x} - e^{-1000x}) + 1998(-e^{-x} + e^{-1000x}) \\ &= 998u + 1998v \Rightarrow u' = 998u + 1998v \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}v(x) &= \frac{d}{dx}(-e^{-x} + e^{-1000x}) = \frac{d}{dx}(e^{-x}) + \frac{d}{dx}(e^{-1000x}) \\ &= e^{-x} + (-1000e^{-x}) = -999(2e^{-x} - e^{-1000x}) - 1999(-e^{-x} + e^{-1000x}) \\ &= -999u - 1999v \Rightarrow v' = -999u - 1999v \end{aligned}$$

$$h < 2/1000$$

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$$y' = -C \cdot y, \text{ where } C \text{ is a positive definite matrix}$$

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} 998 & 1998 \\ -999 & -1999 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \Rightarrow C = \begin{bmatrix} -998 & -1998 \\ 999 & 1999 \end{bmatrix}$$

$$y' = - \begin{bmatrix} -998 & -1998 \\ 999 & 1999 \end{bmatrix} y \Rightarrow h < \frac{2}{\lambda_{\max}}$$

$$[A - \lambda I] \Rightarrow \begin{bmatrix} -1 & 1 \\ 999 & 1999 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1000 \end{bmatrix}$$

$$[A - \lambda I] \Rightarrow \lambda = 1, 1000 \Rightarrow h > \frac{2}{\lambda_{\max}} > \frac{2}{1000} \Rightarrow h < \frac{2}{1000}$$

(ii)

$$\begin{aligned} 998u + 1998v \\ -999u - 1999v \end{aligned} \quad -C = \text{Jacobian}$$

Back Euler Num Iterations: [1 ... 1]

RK45 Num Iterations: [17, 5, 2, 3 ... 3]

Implicit has a constant, predictable number of iterations, can be quicker.