PHYS358: Session 07

Monte Carlo Integration (1): LLN and CLT

Both the Law of Large Numbers and the Central Limit Theorem are at the basis of why Monte Carlo integration works in the first place (as long as the function to be integrated has a mean...). Here, we explore both laws with a simple numerical experiment.

We construct a random variable by taking the mean of N draws $u_i \in U(0,1)$, $i \in 1 \ldots N$, i.e.

$$X = \frac{1}{N}\sum_{i=1}^{N} u_i, \tag{1}$$

with variance

$$\sigma_X^2 = \frac{1}{N}\sum_{i=1}^{N}(u_i - X)^2. \tag{2}$$

The key point to understand here is that X should be seen as a "measurement" with an error σ_X.

Step 1: Determine the expectation value of X and its variance σ_X^2.

Step 2: Run `mci_llnclt.py 1 N` with $N = 10, 100, 1000, 10000$. You should see the running mean and the variance. Do the results agree with your estimate in Step 1? What does the code actually do?

Now we construct a new random variable $\bar{X}$ from R realizations of our "measurement" X,

$$\bar{X} = \frac{1}{R}\sum_{r=1}^{R} X_r. \tag{3}$$

Since the X_r all have the same expectation value and the same variance, the CLT informs us that the expectation value for $\bar{X}$ is

$$E[\bar{X}] = E[X], \tag{4}$$

with the variance

$$Var[\bar{X}] = \frac{\sigma_X^2}{R}, \tag{5}$$

i.e. if I quadruple the number of measurements of my quantity X, the standard deviation of the **sample mean** reduces by a factor of 2.

Step 3: Run `mci_llnclt.py 2 N -R r` with $N = 100$ and $r = 100, 300, 1000, 3000$. You can test larger r, but it will take a while. Again, what does the code do? Describe the results in terms of the LLN and CLT.