26

HW #7

7a. Prove $F[f]F[g] = F[f*g]$

$$F[f] = \int f(x) e^{-2\pi i kx} dx, \quad F[g] = \int g(x) e^{-2\pi i kx} dx$$

$$[f*g](s) = \int f(x) g(s-x) dx \Rightarrow F[f*g] = \iint f(x) g(s-x) dx\, e^{-2\pi i sk} ds$$
$$= \int f(x) \int g(s-x) e^{-2\pi i sk} ds\, dx$$

Let $z = s-x \Rightarrow dz = -dx$

$$F[f*g] = \int f(x) \int g(z) e^{-2\pi i (z+x)k} dz\, dx = \int f(x) e^{-2\pi i kx} dx \int g(z) e^{-2\pi i kz} dz$$

$$\Rightarrow F[f*g] = F[f]F[g]$$

7b. $F[f](k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx, \quad f_j = f(x_j), \quad x_j = jL/J$

$$F(k) = \sum_{j=0}^{\infty} f_j e^{-ikxj} = \sum_{j=0}^{J-1} f_j e^{-2\pi ijk/J}$$

7c. $F_k = \sum_{j=0}^{J-1} f_j e^{-2\pi ijk/J}, \quad f_j = \sum_{k=0}^{J-1} F_k e^{2\pi ijk/J}$

$$F^{-1}[F[f_j]] = \sum_{k=0}^{J-1} \sum_{j=0}^{J-1} f_j e^{-2\pi ijk/J} \cdot e^{2\pi ijk/J} = f_\ell$$

$J=4, \ell=0$

$$f_0 = f_j \left(\sum_{k=0}^{3} \sum_{j=0}^{3} e^{2\pi ijk/J(\ell-j)} \right) = f_j \sum_{k=0}^{3} \left(e^0 + e^{-2\pi ijk/J} + \dots \right)$$
$$= f_j (4) = 4f_j$$

7e. 1) $\int_{-\infty}^{\infty} \sin(2\pi n x)\, e^{-2\pi i k x}\, dx = \int_{-\infty}^{\infty} \sin(2\pi n x)\left[\cos(-2\pi i k x) + i\sin(-2\pi i k x)\right] dx$

$= i\int_{-\infty}^{\infty} \sin(2\pi n x)\sin(2\pi k x)\, dx$, this is 0 when $n \neq k \Rightarrow n = k$

$= i\int_{0}^{1/N} \sin^2(2n\pi x)\, dx = i\left[\frac{x}{2} - \frac{\sin(4\pi n x)}{8\pi n}\right]\Big|_0^{1/n}$

$= i\,\frac{1}{2N}$, since $n = k \Rightarrow -\frac{i}{2}\,\delta(n-k)$

2) The results work as expected for the values $J = 32$, $n = 1, 4, 8, 16, 17, 24, 32, 33$. However when $n \geq J/2$ the results are mirrored (they appear as if $n = J - n$.)

3) Computation time increases by the number of points squared. $\rightarrow O(J^2)$

f) $\Phi_{j-1} - 2\Phi_j + \Phi_{j+1} = 4\pi G\rho_j(\Delta x)^2$, $\nabla^2\Phi(x) = 4\pi G\rho(x)$

$\frac{1}{J}\sum_{k=0}^{J-1} \Phi_k\left(-2e^{-2\pi i j k/J} + e^{-2\pi i (j+1)k/J} + e^{-2\pi i (j-1)k/J}\right)$

$= \frac{1}{J}\sum_{k=0}^{J-1} \hat{\rho}_k e^{-2\pi i j k/J}(\Delta x)^2 \Rightarrow \Phi_k = \hat{\rho}_x(\Delta x)^2 \frac{e^{-2\pi i j k/J}}{e^{-2\pi i j k/J}} \frac{1}{e^{2\pi i j k l} - 2 + e}$

$= \hat{\rho}_k(\Delta x)^2 \cdot \frac{1}{\left[e^{\pi i k/J} - e^{-(\pi i k/J)}\right]^2} = \frac{\rho_k(\Delta x)^2}{4\sin^2\left(\frac{\pi x}{J}\right)}$

$\Rightarrow \boxed{\hat{\Phi}_k = \frac{\rho_k(\Delta x)^2}{2\left(\cos\left(\frac{2\pi k}{J}\right) - 1\right)}}$

D and G all in the code.