PHYS358: Session 18

Partial Differential Equations: Laplace Equation

Equation and Solution: Our goal is to solve the boundary value problem

$$\nabla^2 u = 0 \tag{1}$$

in one dimension, with periodic boundary conditions. The initial shape of u shall be

$$u(x) = \sum_{k=1}^{5} \sin\left(2\pi 2^k x\right). \tag{2}$$

Q1: If you did a Fourier analysis of $u(x)$, which wave numbers would have non-zero power?

Q2: What is the solution to eq. 1?

Jacobi Method: Using the finite-difference expression for eq. 1, write down the update rule for the Jacobi method, i.e. $\boldsymbol{u}^{n+1} = f(\boldsymbol{u}^n)$, with the iteration number n.

Q3: Describe in words the value of u_j^{n+1}.

Q4: If you had some perturbations on large scales (small wave numbers, long wave lengths) and some on small scales (large wave numbers, short wave length), which do you expect to be changing faster, given your insight from Q3?

Gauss-Seidel Method: Write down the update rule for the Gauss-Seidel method in the same way as above.

Q5: Do you expect the Gauss-Seidel method to converge faster or slower than the Jacobi method? Why?

Resolution-dependent convergence: Assume $u = \sin(2\pi x)$ on $0 \leq x \leq 1$ on a grid with J support points.

Q6: How small can you choose J to be to still get an accurate solution? (*Hint:* Think in terms of Fourier modes)