HW #6

6a

1. $T^{n+1} = AT^n$, $T_j^{n+1} = T_j^n + \frac{k\Delta t}{(\Delta x)^2}\left(T_{j+1}^n - 2T_j^n + T_{j-1}^n\right)$

In lecture we set $\alpha = \frac{k\Delta t}{(\Delta x)^2} \Rightarrow T_j^{n+1} = T_j + \alpha\left(T_{j+1}^n - 2T_j^n + T_{j-1}^n\right)$

$$A = \begin{pmatrix} 0 \;\; \alpha \;\; 1-2\alpha \;\; \alpha \;\; 0 \cdots \\ 0 \cdots 0 \;\; \alpha \;\; 1-2\alpha \;\; \alpha \;\; 0 \cdots 0 \\ \cdots 0 \;\; \alpha \;\; 1-2\alpha \;\; \alpha \;\; 0 \cdots \end{pmatrix} \begin{pmatrix} T_0^n \\ T_1^n \\ \vdots \\ T_j^n \\ \vdots \\ T_{J-1}^n \\ T_J \end{pmatrix}$$

First/Last rows & cols can change due to boundary conditions. Ignoring the boundary conditions, A is a tridiagonal matrix of $[\alpha, 1-2\alpha, \alpha]$

2. $T_j^{n+1} = T_j^n + \frac{k\Delta t}{(\Delta x)^2}\left(T_{j+1}^N - 2T_j^N + T_{j-1}^N\right)$, $T_j^N = \xi^N(k)e^{ikj\Delta x}$

$$\xi^{N+1}(k)e^{ikj\Delta x} = \xi^N(k)e^{ikj\Delta x} + \frac{k\Delta t}{(\Delta x)^2}\left(\xi^N(k)e^{ik(j+1)\Delta x} - 2\xi^N(k)e^{ikj\Delta x} + \xi^N(k)e^{ik(j-1)\Delta x}\right)$$

$\Rightarrow \xi(k) = 1 + \frac{k\Delta t}{(\Delta x)^2}\left(e^{ik\Delta t} - 2 + e^{-ik\Delta t}\right)$, $\left(\frac{e^{ix}+e^{-ix}}{2i}\right)^2 = \sin^2(x)$

$$\Rightarrow \boxed{\xi(k) = 1 - 4\frac{k\Delta t}{(\Delta x)^2}\sin^2\left(\frac{k\Delta t}{2}\right)}$$

3. We need $\sin^2\left(\frac{k\Delta t}{2}\right) = 1$

$\Rightarrow 1 \geq |\xi(k)| = 1 - 4\frac{k\Delta t}{(\Delta x)^2}\sin^2\left(\frac{k\Delta x}{2}\right) \Rightarrow 1 \geq \left|1 - 4\frac{k\Delta t}{(\Delta x)^2}\right|$

$\Rightarrow 2 \geq 4\frac{k\Delta t}{(\Delta x)^2} \Rightarrow \boxed{\Delta t = \frac{1}{2}\frac{(\Delta x)^2}{k\Delta t}}$

4. Since Δt correlates with $(\Delta x)^2$, you would need to reduce Δt by a factor of 4 in order to double the resolution.

6b

1. Implicit $\Rightarrow$ $T^{n+1} = A^{-1} T^{N}$

$$\Rightarrow \boxed{T_j^{N} = -\alpha T_{j+1}^{N+1} + (1+2\alpha) T_j^{N+1} - \alpha T_{j-1}^{N+1}}, \text{ where } \alpha = \frac{\kappa \Delta t}{(\Delta x)^2}$$

$$A = \begin{pmatrix} 1+\alpha & -\alpha & 0 & \cdots & & & \\ -\alpha & 1+\alpha & \alpha & 0 & \cdots & & \\ 0 & -\alpha & 1+\alpha & 0 & \cdots & & \\ 0 & 0 & -\alpha & 1+\alpha & 0 & \cdots & \\ \vdots & & & & & & \\ & \cdots & 0 & 0 & -\infty & 1+\infty & -\infty \\ & & \cdots & 0 & 0 & -\infty & 1+\infty \end{pmatrix}$$

Tridiagonal matrix of $[-\alpha, 1+\alpha, \alpha]$,

2.

$$\xi^{N+1}(k) e^{ikj\Delta x} = \xi^{N}(k) e^{ikj\Delta x} + \frac{\kappa \Delta t}{(\Delta x)^2}\left(\xi^{N}(k) e^{ik(j+1)\Delta x} - 2\xi^{N}(k) e^{ikj\Delta x} + \xi^{N}(k) e^{ik(j-1)\Delta x}\right)$$

$$\Rightarrow 1 = \xi^{-1}(k) + \frac{\kappa \Delta t}{(\Delta x)^2}\left(e^{ik\Delta x} - 2 + e^{ik\Delta x}\right) = \xi^{-1}(k) + \frac{\kappa \Delta t}{(\Delta x)^2}\left(e^{\frac{ik\Delta x}{2}} - e^{\frac{-ik\Delta x}{2}}\right)$$

$$\Rightarrow \xi^{-1}(k) = 1 + 4\frac{\kappa \Delta t}{(\Delta x)^2}\sin^2\left(\frac{k\Delta t}{2}\right) \Rightarrow \boxed{\xi(k) = \left(1 + 4\frac{\kappa \Delta t}{(\Delta x)^2}\sin^2\left(\frac{k\Delta t}{2}\right)\right)^{-1}}$$

3. Since $|\xi(k)| = \left|\dfrac{1}{1 + 4\frac{\kappa \Delta t}{(\Delta x)^2}\sin^2\left(\frac{k\Delta t}{2}\right)}\right| \le 1$,

$\boxed{\text{The implicit method is unconditionally stable for any } \Delta t.}$

4. $T_j^{N} = \xi^{N}(k) e^{ikj\Delta x}$, $\xi(k) = \dfrac{1}{1 + 4\frac{\kappa \Delta t}{(\Delta x)^2}\sin^2\left(\frac{k\Delta t}{2}\right)}$

As $\Delta t \to \infty$, $\xi(k) \to 0$ $\Rightarrow \boxed{T_j^{N} \to 0}$

HW#6

6c. 1. $T_j^{N+1} = T_j^N + \frac{1}{2}\left(\frac{K\Delta t}{(\Delta x)^2}\left(\underset{\text{explicit}}{T_j^N} + \underset{\text{implicit}}{T_j^{N+1}}\right)\right)$, (We take a half step explicitly and implicity)

$$\Rightarrow T_j^{N+1} = T_j^N + \frac{1}{2}\frac{K\Delta t}{(\Delta x)^2}\left(T_{j+1}^N - 2T_j^N + T_{j-1}^N + T_{j+1}^{N+1} - 2T_j^{N+1} + T_{j-1}^{N+1}\right)$$

$$\Rightarrow \xi^{N+1}(k)e^{ikj\Delta x} = \xi^N(k)e^{ikj\Delta x} + \frac{1}{2}\frac{K\Delta t}{(\Delta x)^2}\left(\xi^N(k)e^{ik(j+1)\Delta x} - 2\xi^N(k)e^{ikj\Delta x} + \xi^N(k)e^{ik(j-1)\Delta x} + \xi^{N+1}(k)e^{ik(j+1)\Delta x} - 2\xi^{N+1}(k)e^{ikj\Delta x} + \xi^{N+1}(k)e^{ik(j-1)\Delta x}\right)$$

$$\Rightarrow 1 = \xi^{-1}(k) + \frac{1}{2}\frac{K\Delta t}{(\Delta x)^2}\left[\xi^{-1}(k)\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right) + \left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)\right]$$

$$\Rightarrow \xi^{-1}(k)\left(1 - 2\frac{K\Delta t}{(\Delta x)^2}\sin^2\left(\frac{K\Delta t}{2}\right)\right) = 1 + 2\frac{K\Delta t}{(\Delta x)^2}\sin^2\left(\frac{K\Delta t}{2}\right)$$

$$\Rightarrow \boxed{\xi(k) = \frac{1 - 2\frac{K\Delta t}{(\Delta x)^2}\sin^2\left(\frac{K\Delta t}{2}\right)}{1 + 2\frac{K\Delta t}{(\Delta x)^2}\sin^2\left(\frac{K\Delta t}{2}\right)}}$$

2. $$\xi(k) = \frac{1 - 2\frac{K\Delta t}{(\Delta x)^2}\sin^2\left(\frac{K\Delta t}{2}\right)}{1 + 2\frac{K\Delta t}{(\Delta x)^2}\sin^2\left(\frac{K\Delta t}{2}\right)} \le 1,$$

so similarly to the implicit method, the crank-nicolson is stable for any Δt.

6d. All in the code.

6e. 1. $$u(x,t) = \frac{4u_0}{\pi}\sum_{n=1}^{\infty}\frac{\sin((2n-1)\pi x)}{2n-1}\exp\left(-(2n-1)^2\pi^2 t\right)$$

$$\Rightarrow \boxed{u(x,t) = \frac{4u_0}{\pi}\sum_{n=0}^{\infty}\frac{\sin((2n+1)\pi x)}{2n+1}\exp\left(-(2n+1)^2\pi^2 t\right)}$$

2. Seems to be provided.

3. Also seems to be provided

4. $\Delta t \le \frac{(\Delta x)^2}{2K}$, where $\Delta x = \frac{1}{20}$, $K = 1$

$\Rightarrow \boxed{\Delta t \le 1.25 \cdot 10^{-3}}$, so it will take $\frac{1}{1.25\cdot 10^{-3}} = \boxed{800 \text{ iterations}}$

$\Delta t \le \frac{(\Delta x)^2}{2K}$, where $\Delta x = \frac{1}{40}$, $K = 1$ $\times 4$

$\boxed{\Delta t \le 3.125 \cdot 10^{-4}}$ $\boxed{3200 \text{ iterations}}$

Discussion seperately.

$J = 20$
$t = 1.25 \times 10^{-3}$

ftcs vs implicit vs. CN

For ftcs, the rms difference increases linearly with time to about -5.5. Results are very similar to the analytical results.

For implicit, the rms difference increases slower than ftcs to about -4.4, with similar results to the analytical solution.

For CN, the rms differerce increases at a rate between ftcs and implicit to about 4.6, with similar results to the analytical solution.

$J = 40$
$t = 1.25 \times 10^{-3}$

For ftcs, the method does not produce useful results since it is unstable, since we doubled J, we would need to divide Δt by a factor of 4 to stay stable.

Both the implicit and CN similarly ran with the rms difference being about -4.5, but with twice the spatial points.

$J = 40$
$t = 3.125 \cdot 10^{-4}$

For ftcs, the rms error increases linearly over time to about -6. Results are very similar to the analytical solution.

The implicit and CN methods ran very similarly with rms differences about -5.

All three methods produced results very similar to analytical results.