3 1)

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Homework 8

a) $q_j^{N+1} = q_j^N - \frac{\alpha}{2}(q_j^N - q_{j-1}^N)$

$q_j^N = \xi^N(k)e^{ijk\Delta x}$

$\Rightarrow \xi^{N+1} e^{ijk\Delta x} = \xi^N e^{ijk\Delta x} - \frac{\alpha}{2}\left(\xi^N e^{i(j+1)k\Delta x} - \xi^N e^{i(j-1)k\Delta x}\right)$

$\xi e^{ijk\Delta x} = e^{ijk\Delta x} - \frac{\alpha}{2}\left(\xi^N e^{i(j+1)k\Delta x} - e^{i(j-1)k\Delta x}\right)$

$\xi = 1 - \frac{\alpha}{2}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right) = 1 - \alpha i \sin(k\Delta x)$ ✓

This method converges when $|\xi| \le 1$

$\Rightarrow \sqrt{|1 - i\alpha\sin(k\Delta x)|^2} = \sqrt{1+\alpha^2\sin^2(k\Delta x)} \le 1 \Rightarrow 1+\alpha^2\sin^2(k\Delta x) \le 1$

$\Rightarrow$ since $\sin^2$ ranges between 0 and 1, and $k \neq 0$, this could only be stable when $\Delta x = 0$, which would accomplish nothing. Since $\Delta x \neq 0$, this method would be unconditionally unstable.

b) $q_j^{N+1} = \frac{1}{2}(q_j^N + q_{j+1}^N) - \frac{\alpha}{2}(q_{j+1}^N - q_{j-1}^N)$

$\xi^{N+1} e^{ijk\Delta x} = \frac{1}{2}\left(\xi^N e^{i(j-1)k\Delta x} + e^{i(j+1)k\Delta x}\right) - \frac{\alpha}{2}\left(\xi^N e^{i(j+1)k\Delta x} - \xi^N e^{i(j-1)k\Delta x}\right)$

$\xi e^{ijk\Delta x} = \frac{1}{2}\left(e^{i(j-1)k\Delta x} + e^{i(j+1)k\Delta x}\right) - \frac{\alpha}{2}\left(e^{i(j+1)k\Delta x} - e^{i(j-1)k\Delta x}\right)$

$\xi = \frac{1}{2}\left(e^{-ik\Delta x} + e^{ik\Delta x}\right) - \frac{\alpha}{2}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)$

$\Rightarrow \xi = \cos(k\Delta x) - i\alpha\sin(k\Delta x)$ ✓, This method converges when $|\xi| \le 1$

$\Rightarrow \cos^2(k\Delta x) + \alpha^2\sin^2(k\Delta x) \le 1$, $1 - \sin^2(k\Delta x) + \alpha^2\sin^2(k\Delta x) \le 1$

$\Rightarrow 1 + \sin^2(k\Delta x)(\alpha^2 - 1) \le 1$, we can replace $\sin^2$ term with 1 since its largest value is 1. $\Rightarrow 1 + \alpha^2 - 1 \le 1 \Rightarrow \alpha^2 \le 1$. This method is stable when $\alpha \le 1$.

2)

c) $$\varrho_j^{N+1} = \frac{1}{2}\left(\varrho_{j-1}^N + \varrho_{j+1}^N\right) - \frac{c\Delta t}{\Delta x}\left(\varrho_{j+1}^N - \varrho_{j-1}^N\right)$$

$$\frac{\varrho_j^{N+1} - \varrho_j^N}{\Delta t} = \frac{1}{2\Delta t}\left(\varrho_{j-1}^N + \varrho_{j+1}^N\right) - \frac{c}{2\Delta x}\left(\varrho_{j+1}^N - \varrho_{j-1}^N\right) - \frac{\varrho_j^N}{\Delta t}$$

$$\frac{\varrho_j^{N+1} - \varrho_j^N}{\Delta t} = \frac{1}{2\Delta t}\left(\varrho_{j-1}^N - 2\varrho_j^N + \varrho_{j+1}^N\right) - c\left(\frac{\varrho_{j+1}^N - \varrho_{j-1}^N}{2\Delta x}\right)$$

$$\Rightarrow \frac{\varrho_j^{N+1} - \varrho_j^N}{\Delta t} = \frac{\Delta x^2}{2\Delta t}\left(\frac{\varrho_{j-1}^N - 2\varrho_j^N + \varrho_{j+1}^N}{\Delta x^2}\right) - c\left(\frac{\varrho_{j+1}^N - \varrho_{j-1}^N}{2\Delta x}\right)$$

$$\Rightarrow \frac{\partial \varrho}{\partial t} = \frac{(\Delta x)^2}{2\Delta t}\frac{\partial^2 \varrho}{\partial x^2} - c\frac{\partial \varrho}{\partial x} \checkmark$$

When this term approaches 0, the diffusion term will not have an effect on the solution. When Δt is very large or Δx is very small then this will occur. When the derivative of ϱ is not changing much ($\frac{\partial^2 \varrho}{\partial x^2} \approx 0$) this may also occur.

d) Done in Code.

e) 1. I can confirm, very unstable. For $\alpha = 0.1$, the grid changes from mostly constant to very oscillatory with an increasing amplitude.

2. The lax method works as expected for $\alpha = 1$. The numerical results agree with the analytical results. This works since we are using periodic boundary conditions, $\alpha = 1$, and we are simply shifting every index over by 1.

3. For $\alpha = 0.5$, the numerical results have a gaussian-like shape. for $\alpha = 0.1$, it seems to have a gaussian like shape where ϱ ranges between ≈ 0.4 and 0.6. As $\alpha \approx 0$, $\varrho \to \frac{1}{2}$ because $\alpha \to 0$ $\varrho_j^{N+1} = \frac{1}{2}\left(q_{j-1}^N + q_{j+1}^N\right) \Rightarrow \frac{1}{2}$, for this scenario.

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f) 1. $q_j^{N+1} = q_j^{N-1} - \alpha\left(q_{j+1}^N - q_{j-1}^N\right)$

$$q_j^{N+1} = q_j^{N-1} - \frac{c\Delta t}{\Delta x}\left(q_{j+1}^N - q_{j-1}^N\right), \quad \frac{\partial q}{\partial t} = \frac{q_j^{N+1} - q_j^{N-1}}{2\Delta t}.$$

We know that $\frac{q_j^{N+1} - q_j^{N-1}}{2\Delta t}$ is something that is $O(n^2)$.

2. leapfrog agrees with the analytic solution for $\alpha = 1$ on the tophat scenario. When $\alpha = 0.5$, the numerical results roughly follow the analytical solution, however there is a lot of oscillatory noise. When $\alpha = 0.1$, results look extremely similar to $\alpha = 0.5$.

Since the gaussian distribution is continuous and differentiable it works better than tophat. It can not handle discontinuity well.

g) The Lax-Wendroff handles discontinuities better than the leapfrog method. This means it is better for the tophat scenario. They both handle the gaussian scenario similarly.

h) 1. implemented in code

2. The upwind scheme has no oscillation, but is more diffusive than the Lax-Wendroff method. It is better in some aspects, but worse in others. In certain situations it may be beneficial to choose either.

3. The upwind loses some information due to its diffusive nature, but does not add oscillations like the leapfrog and Lax-Wendroff methods do. Thus, the upwind performed the task of advecting the tophat profile for $\alpha = 0.5$ best since it did not create undesirable oscillations.