WHY IS THE ASYMPTOTIC FORM OF THE AIRY FUNCTION VALID RELATIVELY CLOSE TO TURNING POINT?

$$y \equiv -\left(\frac{2mF_1}{\hbar^2}\right)^{1/3}(x-x_1)$$

THE ASYMPTOTIC FORM OF $Ai(y)$ IS ACCURATE TO 1% AS LONG AS $y \gtrsim 5$

CONSIDER, AS AN EXAMPLE, THE NUCLEAR CASE:

$mc^2 \simeq 931$ MeV

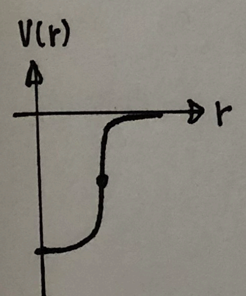
$\hbar c = 197$ MeV$\cdot$fm

SUPPOSE: $F_1 \simeq 10$ MeV/fm

[DIFFUSENESS : $a \simeq 0.65$ fm]

$$|x-x_1| = \frac{\overset{y\simeq 5}{y}}{\left(\frac{2mc^2F_1}{\hbar^2c^2}\right)^{1/3}} \simeq 6 \text{ fm}$$

[MOTION OF ONE NUCLEON IN NUCLEUS]



BEYOND THIS DISTANCE OF 6 fm, $y \gtrsim 5$, AND $Ai(y)$ IS ACCURATE TO 1%

QUESTION: IS THE LINEAR APPROXIMATION OF THE POTENTIAL STILL VALID FOR $|x-x_1| \gtrsim 6$ fm?

[SEE PROB. 8.8 IN GRIFFITHS]

DETERMINATION OF THE MISSING COEFFICIENT C

THE LAST COEFFICIENT, C, IS USUALLY OBTAINED BY REMEMBERING THAT $\psi_{II}^{<}$ AND $\psi_{II}^{>}$ ARE SOLUTIONS IN EXACTLY THE SAME REGIONS AND, THEREFORE, THEY MUST BE IDENTICAL:

$$\underbrace{\frac{\overset{\overset{2}{\parallel}}{A'}}{\sqrt{p(x)}} \sin\left[\frac{1}{\hbar}\int_{x_1}^{x} p(x')\,dx' + \overset{\overset{\pi/4}{\parallel}}{\delta}\right]}_{\psi_{II}^{<}} = \underbrace{\frac{\overset{\overset{2C}{\parallel}}{A''}}{\sqrt{p(x)}} \sin\left[\frac{1}{\hbar}\int_{x}^{x_2} p(x')\,dx' + \overset{\overset{\pi/4}{\parallel}}{\delta}\right]}_{\psi_{II}^{>}}$$

$$\sin\left[\frac{1}{\hbar}\int_{x_1}^{x} p(x')\,dx' + \frac{\pi}{4}\right] = C \sin\left[\frac{1}{\hbar}\int_{x}^{x_2} p(x')\,dx' + \frac{\pi}{4}\right]$$

DEFINE VARIABLES:

$$\eta \equiv \underbrace{\frac{1}{\hbar}\int_{x_1}^{x_2} p(x')\,dx'}_{\text{CONSTANT} > 0} \quad , \quad a \equiv \frac{1}{\hbar}\int_{x}^{x_2} p(x')\,dx' + \frac{\pi}{4}$$

WE FIND:

$$\sin\left(\eta + \frac{\pi}{2} - a\right) = C \sin a$$

VARIES THROUGH x

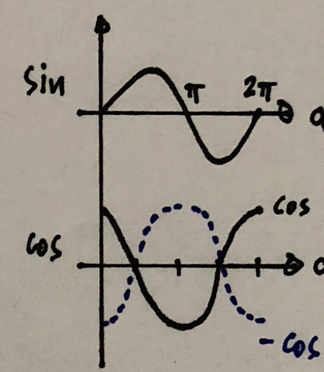
CONSTANT, INDEPENDENT OF a

OR :

$$\underbrace{\sin\left(\eta + \frac{\pi}{2}\right)}\cos a - \cos\left(\eta + \frac{\pi}{2}\right)\sin a = \zeta \sin a$$

MUST BE ZERO, OTHERWISE EQUATION CANNOT BE FULFILLED

[SEE SKETCHES OF SINE AND COSINE]



THEREFORE :

$$\eta + \frac{\pi}{2} = (n+1)\pi \quad , \quad n = 0, 1, 2, \ldots$$

$$\zeta = (-1)^n$$

[NOTE: THE FIRST CONDITION SHOULD NOT BE WRITTEN AS $\eta + \frac{\pi}{2} = n\pi$, $n = 0, 1, 2, \ldots$, BECAUSE THEN $n = 0$ LEADS TO $\eta + \frac{\pi}{2} = 0$, WHICHS CONTRADICTS THE FACT THAT $\eta > 0$; SEE DEFINITION ABOVE]

WE CAN REWRITE THE FIRST CONDITION :

$$\eta + \frac{\pi}{2} = n\pi + \pi$$

$$\frac{1}{\hbar}\int_{x_1}^{x_2} p(x')\,dx' + \frac{\pi}{2} = n\pi + \pi$$

FINALLY, WE FIND FOR THE WKB ENERGIES AND WAVEFUNCTIONS:

GRIFFITHS: $n=1,2,\ldots$

$$\boxed{\int_{x_1}^{x_2} p(x)\,dx \equiv \int_{x_1}^{x_2} \sqrt{2m\left[E_n - V(x)\right]}\,dx = \hbar\pi\left(n+\tfrac{1}{2}\right) \qquad n=0,1,2,\ldots}$$

EXPRESSION DETERMINES QUANTIZED WKB ENERGY LEVELS, E_n, OF BOUND STATES [$V(x)$ HAS NO RIGID WALLS]

$$\boxed{\Psi_{WKB} = D \begin{cases} \dfrac{1}{\sqrt{|p(x)|}}\, e^{+\frac{1}{\hbar}\int_{x_1}^{x} |p(x')|\,dx'} , & x < x_1 \\[2ex] \dfrac{2}{\sqrt{p(x)}} \sin\left[\dfrac{1}{\hbar}\displaystyle\int_{x_1}^{x} p(x')\,dx' + \dfrac{\pi}{4}\right], & x_1 < x < x_2 \\[2ex] \dfrac{(-1)^n}{\sqrt{|p(x)|}}\, e^{-\frac{1}{\hbar}\int_{x_2}^{x} |p(x')|\,dx'} , & x > x_2 \end{cases}}$$

REMARKS:

(i) n IS EQUAL TO THE NUMBER OF NODES BETWEEN TURNING POINTS

(ii) EXPRESSIONS ARE EXPECTED TO BE MORE ACCURATE FOR LARGE E_n, i.e., LARGE n

(iii) CAREFUL: E_n ENTERS EXPLICITLY IN $p(x)$, BUT ALSO IN INTEGRATION LIMITS, SINCE THE TURNING POINTS ARE FOUND FROM $E = V(x_1) = V(x_2)$

$\boxed{\text{EXAMPLE : HARMONIC OSCILLATOR}}$ [GRIFFITHS, PROB. 8.7]

$$H = \underbrace{-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}}_{T} + \underbrace{\frac{1}{2}m\omega^2x^2}_{V}$$

PROVIDE INTEGRAL

(i) CALCULATE [CLASSICAL] MOMENTUM AND FIND TURNING POINTS x_{MIN} AND x_{MAX} [EXPRESS THEM IN TERMS OF E]

(ii) CALCULATE ENERGY LEVELS, E_n

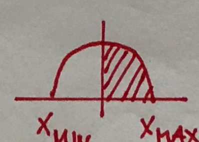
(i) $$p = \sqrt{2m\,(E - V(x))} = \sqrt{2m\,(E - \tfrac{1}{2}m\omega^2x^2)} = \sqrt{2mE - m^2\omega^2x^2}$$

DEFINITION OF TURNING POINTS: $\boxed{E = V(x)}$

$$E = V(x) = \frac{1}{2}m\omega^2x^2 \Rightarrow x_{MIN} = -\sqrt{\frac{2E}{m\omega^2}},\quad x_{MAX} = +\sqrt{\frac{2E}{m\omega^2}}$$

(ii) $$\int_{x_1}^{x_2}\sqrt{2m\,(E - V(x))}\,dx = \boxed{(n+\tfrac{1}{2})\pi\hbar} = \int_{x_{MIN}}^{x_{MAX}}\sqrt{2mE - m^2\omega^2x^2}\,dx$$

$$= m\omega\int_{x_{MIN}}^{x_{MAX}}\sqrt{\frac{2mE}{m^2\omega^2} - \frac{m^2\omega^2x^2}{m^2\omega^2}}\,dx = m\omega\int_{x_{MIN}}^{x_{MAX}}\sqrt{x_{MAX}^2 - x^2}\,dx$$



CONTINUED →

$$\underset{\substack{\text{SYMMETRY}\\ \text{AROUND } x=0}}{=} 2m\omega \int_0^{x_{MAX}} \sqrt{x_{MAX}^2 - x^2}\, dx = \boxed{2m\omega \frac{1}{2}\left[x\sqrt{x_{MAX}^2 - x^2} + x_{MAX}^2 \arcsin\left(\frac{x}{x_{MAX}}\right)\right]_0^{x_{MAX}}}$$

PROVIDE INTEGRAL

$$= m\omega\left[x_{MAX}^2 \underbrace{\arcsin(1)}_{\pi/2} - 0\right] = m\omega x_{MAX}^2 \frac{\pi}{2} = \frac{\pi}{2} m\omega \frac{2E}{m\omega^2} = \boxed{\frac{\pi E}{\omega}}$$

$$\frac{\pi E}{\omega} = \left(n+\frac{1}{2}\right)\pi\hbar \Rightarrow \boxed{E_n = \left(n+\frac{1}{2}\right)\hbar\omega} \quad n = 0, 1, 2, \ldots$$

IN THIS EXAMPLE, THE WKB APPROXIMATION GIVES THE EXACT RESULT!

[IS IT AN ACCIDENT? THE EXACT ENERGIES DEPEND ON $\hbar^1$, AND SINCE WKB APPROXIMATION CONSIDERS IN THE EXPANSION UP TO AND INCLUDING $\hbar^1$, THE EXACT ENERGIES ARE RECOVERED (BUT <u>NOT</u> THE EXACT WAVEFUNCTIONS, SEE BELOW)]

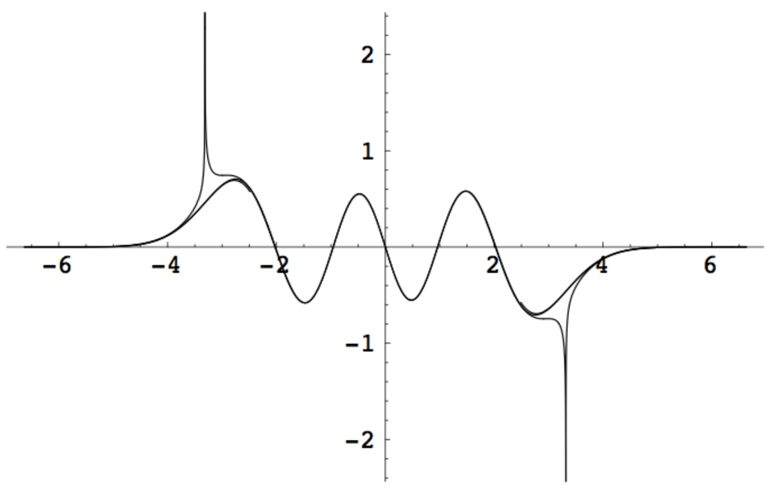
NOTICE THAT WE FOUND THE QUANTIZED ENERGY LEVELS WITHOUT EVER EXPLICITLY SOLVING THE SCHRÖDINGER EQUATION!

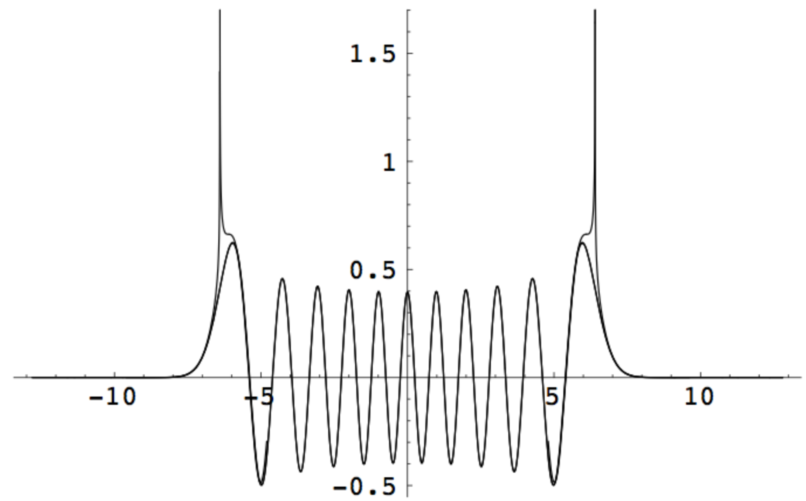
THE QUANTIZED WKB ENERGIES, E_n, CAN BE SUBSTITUTED INTO $p(x)$ OF WKB WAVEFUNCTIONS...

SLIDE ON COMPARISON OF WKB APPROXIMATIONS TO EXACT RESULTS FOR $n=5$ AND $n=20$

FOR AN INTERESTING PAPER ON THE WKB METHOD THAT AVOIDS INFINITIES AT THE TURNING POINTS, SEE MILLER AND GOOD, PHYS. REV. 91, 174 (1953)

More harmonic oscillator: comparison of WKB approximation with exact wave function



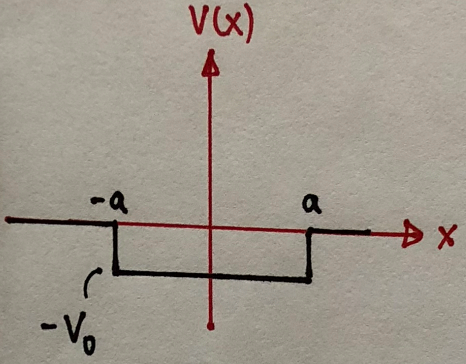


Remember two things:

1. WKB approximation breaks down near turning points
2. WKB approximation improves with increasing n

$\boxed{\text{EXAMPLE}}$ CALCULATE THE ENERGIES FOR THE FINITE SQUARE WELL:

$$V(x) = \begin{cases} -V_0 & , \quad -a \leq x \leq a \\ 0 & , \quad \text{ELSEWHERE} \end{cases}$$



[TRANSCENDENTAL EQUATION]

TWO SOFT WALLS: $\displaystyle\int_{x_1}^{x_2} \sqrt{2m[E_n - V(x)]}\, dx = \hbar\pi\left(n+\frac{1}{2}\right)$, $n = 0, 1, 2, \ldots$

$$\hbar\pi\left(n+\frac{1}{2}\right) = \int_{-a}^{a} \sqrt{2m[E_n+V_0]}\, dx = \sqrt{2m[E_n+V_0]}\,(a-(-a))$$

$$\hbar^2\pi^2\left(n+\frac{1}{2}\right)^2 = 2m(E_n+V_0)\,2^2a^2$$

$$\boxed{E_n^{WKB} = \frac{\hbar^2\pi^2}{8ma^2}\left(n+\frac{1}{2}\right)^2 - V_0 \quad , \; n = 0, 1, 2, \ldots}$$

LET'S CHOSE SOME NUMBERS:

- TYPICAL NUCLEON MASS [$m_p c^2 = 938$ MeV, $\hbar c = 197$ MeV·fm]
- $V_0 = 50$ MeV
- $a = 6$ fm

AND COMPARE EXACT ENERGIES, FOUND BY NUMERICAL INTEGRATION OF ANALYTIC EXPRESSIONS FOR FINITE SQUARE WELL, WITH WKB RESULT:

		E_n [MeV]			
	n	WKB	EXACT		
ø	0	-49.6	-48.8	ø	
2.8	1	-46.8	-45.4	3.4	
8.5	2	-41.1	-39.6	9.2	
17.0	3	-32.6	-31.7	17.1	
28.4	4	-21.2	-21.7	27.1	
42.6	5	-7.0	-10.1	38.7	10%

[NO OTHER LEVELS EXIST]

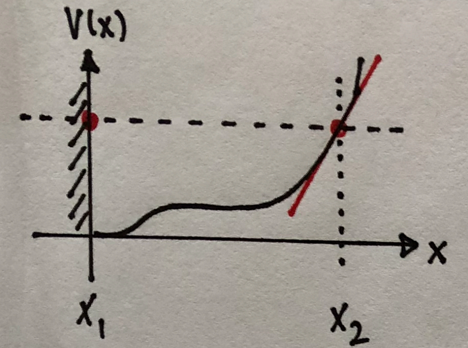
THE RELATIVE AGREEMENT IS { POOR, SINCE: / SURPRISING, CONSIDERING THAT:

- THE POTENTIAL DOES NOT VARY SLOWLY AT TURNING POINTS
- $dV(x)/dx$ IS NOT FINITE AT TURNING POINTS

[- THE TURNING POINTS ARE NOT RIGID WALLS]

SO FAR, WE CONSIDERED THE MOST GENERAL CASE, i.e., TWO "SOFT" TURNING POINTS; WHEN THE PARTICLE ENCOUNTERS RIGID WALLS ($V \to \infty$) THE ABOVE EXPRESSIONS NEED TO BE MODIFIED

A. BOUND STATES FOR POTENTIAL WITH ONE RIGID WALL



$x_1 \leq x \leq x_2$:

$$\psi_{II}^{<}(x) = \frac{A'}{\sqrt{p(x)}} \sin\left[\frac{1}{\hbar}\int_{x_1}^{x} p(x')\,dx' + \delta\right]$$

WAVE FUNCTION MUST VANISH AT $x = x_1$: $\delta = 0$

AT $x = x_2$, WE PROCEED AS BEFORE: LINEAR APPROXIMATION OF $V(x)$ AND MATCHING OF WKB SOLUTIONS

$$\psi_{II}^{>}(x) = \frac{A''}{\sqrt{p(x)}} \sin\left[\frac{1}{\hbar}\int_{x}^{x_2} p(x')\,dx' + \delta\right] \quad \frac{\pi}{4}$$

[SEE PREVIOUSLY]

SINCE $\psi_{II}^{<}$ AND $\psi_{II}^{>}$ ARE IDENTICAL:

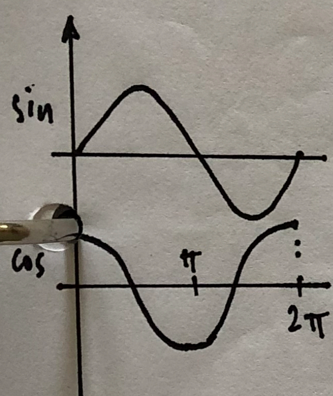
$$\frac{A'}{\sqrt{p(x)}} \sin\left[\frac{1}{\hbar}\int_{x_1}^{x} p(x')\,dx'\right] = \frac{A''}{\sqrt{p(x)}} \sin\left[\frac{1}{\hbar}\int_{x}^{x_2} p(x')\,dx' + \frac{\pi}{4}\right]$$

INTRODUCE: $\underbrace{\eta \equiv \frac{1}{\hbar}\int_{x_1}^{x_2} p(x')\,dx'}_{\text{POSITIVE CONSTANT}}$ AND $\underbrace{a \equiv \frac{1}{\hbar}\int_{x}^{x_2} p(x')\,dx' + \frac{\pi}{4}}_{\text{VARIABLE THROUGH } x}$

EQUATION BECOMES:

$$A' \sin\left(\eta + \frac{\pi}{4} - a\right) = A'' \sin a$$

$$A' \Big[\underbrace{\sin\left(\eta + \frac{\pi}{4}\right)}_{\text{MUST BE ZERO}} \cos a - \underbrace{\cos\left(\eta + \frac{\pi}{4}\right)}_{\Rightarrow \pm 1} \sin a \Big] = A'' \sin a$$



$$\eta + \frac{\pi}{4} = (n+1)\pi \quad , \quad n = 0, 1, 2, \ldots$$

[WE CANNOT CHOSE $\ldots = n\pi$, BECAUSE FOR $n = 0$, $\eta < 0$!]

$$A' = (-1)^n A''$$

THUS: $\eta \equiv \frac{1}{\hbar}\int_{x_1}^{x_2} p(x')\,dx' = n\pi + \pi - \frac{\pi}{4} = \pi\left(n + \frac{3}{4}\right)$ OR

$$\int_{x_1}^{x_2} p(x)\,dx = \hbar\pi\left(n + \frac{3}{4}\right); \quad n = 0, 1, 2, \ldots$$

B. BOUND STATES FOR POTENTIAL WITH TWO RIGID WALLS

IN COMPLETE ANALOGY:

$$\int_{x_1}^{x_2} p(x)\,dx = \hbar\pi\,(n+1)\ ; \quad n = 0, 1, 2, \ldots$$

SUMMARY:

$$\boxed{\int_{x_1}^{x_2} p(x)\,dx = \pi\hbar\left(n + \frac{1}{2} + \frac{1}{4}m\right) \qquad n = 0, 1, 2, \ldots \qquad m: \text{NUMBER OF RIGID } (\infty) \text{ WALLS}}$$

NOTES:

- QUANTIZATION RULES GIVE SAME RESULTS FOR $n \to$ LARGE
- MORE COMPLICATED EXPRESSIONS ARISE FOR SITUATIONS WITH MORE THAN TWO TURNING POINTS [SEE PROB. 8.15 GRIFFITHS]
- FOR BOUND STATES IN CENTRAL POTENTIAL, WE HAVE:

$$E = \frac{p_r^2}{2m} + \underbrace{V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}}_{V_{eff}(r)}$$

[SEC. 4.1.3 GRIFFITHS]

THE TURNING POINTS ARE DEFINED BY $E = V_{eff}(r_1) = V_{eff}(r_2)$, AND THE BOUND ENERGY LEVELS ARE OBTAINED FROM:

$$\int_{r_1}^{r_2} p_r(E,r)\,dr = \int_{r_1}^{r_2} dr \sqrt{2m\left(E - V(r) - \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right)} = \left(n + \frac{1}{2} + \frac{1}{4}m\right)\hbar\pi$$

$$n = 0, 1, 2, \ldots$$

WE CAN APPLY THE WKB METHOD TO SPECIAL 3D SITUATIONS;

FOR EXAMPLE, FOR A CENTRAL POTENTIAL :

$$\psi(r,\theta,\phi) = R(r)\, Y(\theta,\phi)$$

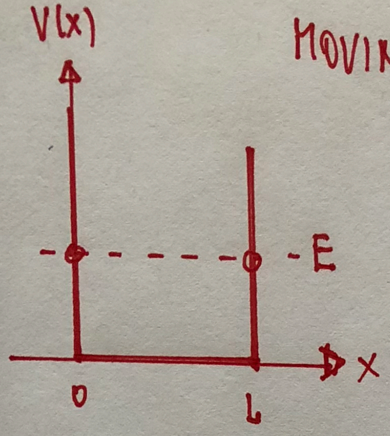
WITH SUBSTITUTION $u(r) = r R(r)$ THE RADIAL EQUATION BECOMES :

$$-\frac{\hbar^2}{2m} \frac{d^2 u(r)}{dr^2} + V(r)\, u(r) = E\, u(r)$$

WHICH IS IDENTICAL IN FORM TO 1D SCHRÖDINGER EQUATION

EXAMPLE

USE THE WKB APPROXIMATION TO CALCULATE ENERGY LEVELS OF A SPINLESS PARTICLE OF MASS m MOVING IN AN INFINITE SQUARE WELL POTENTIAL



TWO RIGID WALLS:

$$\int_{x_1}^{x_2} p(x)\,dx = (n+1)\pi\hbar \quad ; \quad n = 0, 1, 2, \dots$$

$$p(x) = \sqrt{2m(E - V(x))} = \sqrt{2mE} \quad \text{INSIDE WELL}$$

$$\int_0^L p(x)\,dx = \int_0^L \sqrt{2mE}\,dx = \sqrt{2mE}\,[x]_0^L = L\sqrt{2mE}$$

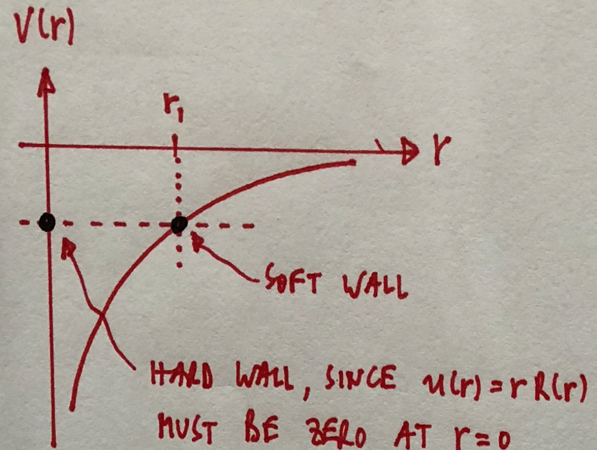
$$\Rightarrow \quad L^2\, 2mE = (n+1)^2 \pi^2 \hbar^2$$

$$E_n = \frac{\pi^2\hbar^2}{2mL^2}(n+1)^2 \quad , \quad n = 0, 1, 2, \dots$$

WHICH IS THE EXACT ENERGY OF A PARTICLE IN AN INFINITE WELL

[IN PREVIOUS EXPRESSIONS: $E_n = \frac{\pi^2\hbar^2}{2mL^2} n^2$; $n = 1, 2, \dots$]

EXAMPLE USE WKB APPROXIMATION TO ESTIMATE BOUND STATE ENERGIES OF ELECTRON IN THE S-STATES OF HYDROGEN ($\ell = 0$)



$$V_c = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}$$

$$\boxed{\text{PROVIDE : } \int_0^{r_1} \frac{\sqrt{r_1 r - r^2}}{r}\,dr = \frac{\pi}{2} r_1}$$

$$= \int_0^{r_1} \sqrt{\frac{r_1}{r} - 1}\,dr$$

ONE RIGID WALL : $\left(n + \frac{3}{4}\right)\pi\hbar = \int_0^{r_1} \sqrt{2m\left(E + \underbrace{\frac{e^2}{4\pi\epsilon_0}}_{-r_1 E}\frac{1}{r}\right)}\,dr$

ELECTRON MOVES BETWEEN ZERO AND : $E = V(r_1) = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r_1} \Rightarrow r_1 = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{E}$

$$\boxed{\left(n + \frac{3}{4}\right)\pi\hbar} = \sqrt{-2m\underbrace{E}_{\text{NEGATIVE}}} \int_0^{r_1} \sqrt{-1 + \frac{r_1}{r}}\,dr = \sqrt{-2mE}\underbrace{\int_0^{r_1} \frac{\sqrt{r_1 r - r^2}}{r}\,dr}_{\frac{\pi}{2} r_1}$$

$$= \sqrt{2m}\,\frac{\pi}{2}\,r_1\sqrt{-E} = \sqrt{2m}\,\frac{\pi}{2}\left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{\sqrt{-E}}{(-E)}$$

$$= \boxed{\frac{e^2}{4\pi\epsilon_0}\sqrt{2m}\,\frac{\pi}{2}\,\frac{1}{\sqrt{-E}}}$$

$$\left(n+\frac{3}{4}\right)^2 \pi^2 \hbar^2 = \left(\frac{e^2}{4\pi\epsilon_0}\right)^2 2m \frac{\pi^2}{4}\frac{1}{(-E)}$$

$$E_n = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{\left(n+\frac{3}{4}\right)^2}\quad,\quad n = \overset{3/4}{0},\ 1,\ 2,\ \ldots$$

OR:

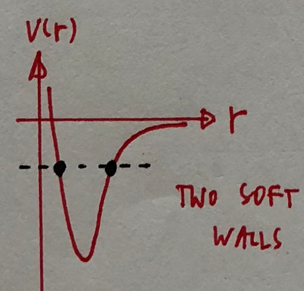
$$\boxed{E_n^{WKB} = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{\left(n-\frac{1}{4}\right)^2}\quad ;\quad n = \overset{3/4}{1},\ 2,\ 3,\ \ldots}$$

COMPARE TO:

$$E_n^{EXACT} = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2}\quad ;\quad n = 1, 2, 3, \ldots$$

COMMENT ON PROBLEM 8.14 GRIFFITHS:

PROBLEM INCLUDES CENTRIPETAL POTENTIAL



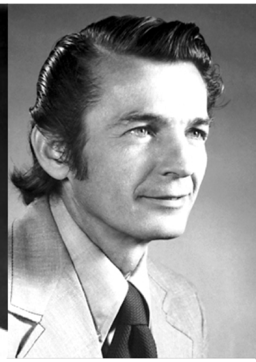
IF WE SET $\ell = 0$ IN GRIFFITH'S EXPRESSION, ONE OBTAINS

$$E_n = - \ldots \frac{1}{\left(n-\frac{1}{2}\right)^2}\quad ;\ n = 1, 2, \ldots$$

n	INCLASS WKB $\frac{1}{(n-\frac{1}{4})^2}$	EXACT $\frac{1}{n^2}$	GRIFFITHS WKB $\frac{1}{(n-\frac{1}{2})^2}$
1	1.77	1	4
2	0.32	0.25	0.44
3	0.13	0.11	0.16
4	0.071 13%	0.0625	0.0816

THE DIFFERENT SOLUTIONS CONVERGE FOR LARGE n

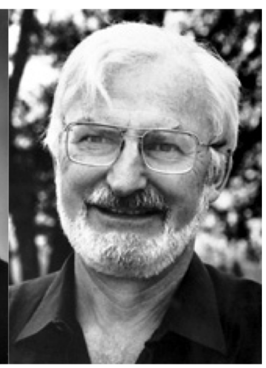
Quantum Tunneling


Brian Josephson (1940-) | Ivar Giaever (1929-) | Leo Esaki (1925-)

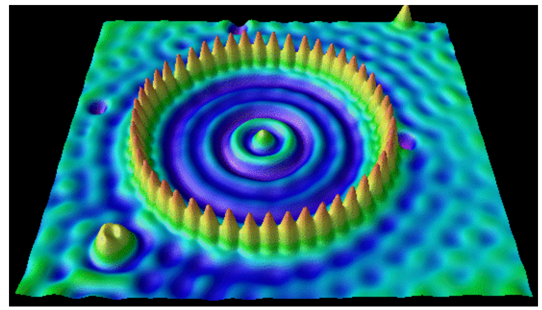
Nobel Prize 1973: Quantum tunneling in superconductors



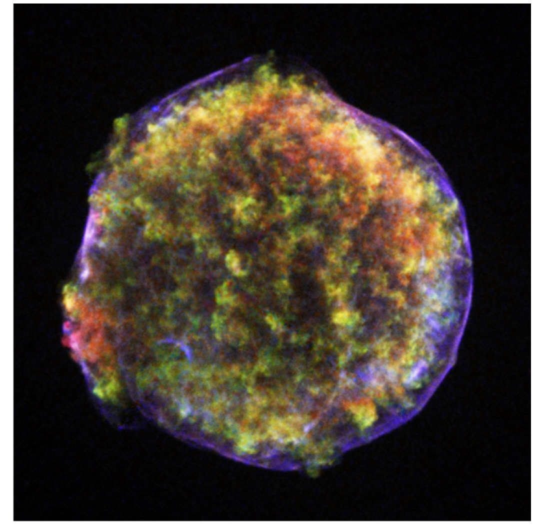


Ernst Ruska (1906-1988) | Gerd Binnig (1947-) | Heinrich Rohrer (1933-2013)

Nobel Prize 1986: Scanning Tunneling Microscope



"Quantum Corral" STM image [credit: IBM]



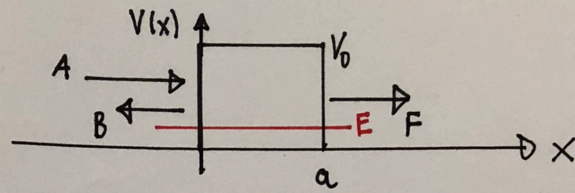
Tycho's Supernova [SN 1572]

Mike Shot [1952]

My cell phone [2015]

TRANSMISSION THROUGH POTENTIAL BARRIERS

RECALL THE SIMPLEST CASE, A RECTANGULAR BARRIER :



EXACT SOLUTION YIELDS : [PROB. 2.33, GRIFFITHS]

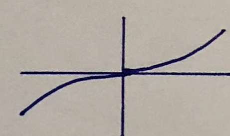
TRANSMISSION COEFFICIENT

$$T \equiv \frac{|F|^2}{|A|^2} = \frac{1}{1 + \frac{V_0^2 \sinh^2(\kappa a)}{4E(V_0 - E)}}, \quad \text{WHERE } \kappa = \frac{\sqrt{2m(V_0-E)}}{\hbar}$$

$|F|^2$: INTENSITY OF TRANSMITTED WAVE

$|A|^2$: INTENSITY OF INCIDENT WAVE

$\sinh z \equiv \frac{1}{2}(e^{z} - e^{-z})$



FOR $\kappa a \gg 1$ [LIMIT OF THICK BARRIER / LOW ENERGY] :

$$T \approx \underbrace{\frac{16E(V_0-E)}{V_0^2}}_{\text{PREFACTOR, VARIES SLOWLY WITH ENERGY}} \underbrace{e^{-\frac{2}{\hbar} a \sqrt{2m(V_0-E)}}}_{\text{DOMINATES ENERGY DEPENDENCE}}$$

WHAT IS THE BASIC REASON FOR QUANTUM TUNNELING?

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

[IF ψ SUDDENLY FALLS TO ZERO AT THE BOUNDARY, $\frac{d\psi}{dx} \to \infty$]