

TIME - DEPENDENT PERTURBATION THEORY

UP TO NOW, WE CONSIDERED TIME-INDEPENDENT POTENTIALS. IN THAT CASE:

$$V(\vec{r}, t) = V(\vec{r})$$

$$\hat{H} \psi(\vec{r}, t) = i\hbar \frac{\partial \psi(\vec{r}, t)}{\partial t}$$

SEPARATION OF VARIABLES, $\psi(\vec{r}, t) = \psi(\vec{r}) \phi(t)$, YIELDS:

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{r}) + V(\vec{r}) \psi(\vec{r}) = E \psi(\vec{r}) \quad \textcircled{*}$$

$$\psi(\vec{r}, t) = \psi(\vec{r}) e^{-\frac{i}{\hbar} Et}$$

- EQUATION $\textcircled{*}$ DOES NOT CONTAIN IMAGINARY NUMBER; IF $V(\vec{r})$ IS REAL, WE EXPECT $\psi(\vec{r})$ TO BE REAL

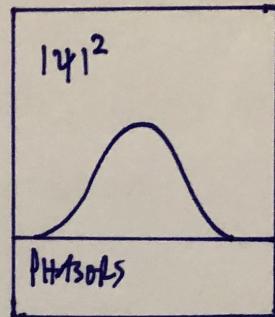
- BUT $\psi(\vec{r}, t)$ WILL BE COMPLEX

- IF PARTICLE IS IN SINGLE EIGENSTATE [PURE STATE], $|\psi|^2$ IS CONSTANT WITH TIME [$e^{-\frac{i}{\hbar} Et}$ CANCELS]; "STATIONARY STATES"

- FOR A SUPERPOSITION OF EIGENSTATES, $\psi(\vec{r}, t) = \sum c_n \psi_n(\vec{r}) e^{-\frac{i}{\hbar} E_n t}$, $|\psi|^2$ CHANGES WITH TIME [$e^{-\frac{i}{\hbar} Et}$ DOES NOT CANCEL, CROSS TERMS REMAIN]

- EVEN IF WE FORM LINEAR COMBINATIONS OF STATES, EXPECTATION VALUE OF ENERGY REMAINS CONSTANT [ENERGY CONSERVATION IN QM]

PICK "INFINITE SQUARE WELL"



COMPUTER ANIMATIONS:

WWW.FALSTAD.COM/QM1D

- CLICK ON EIGENSTATE [START WITH G.S.]:

$$\psi(x,t) = \psi(x) e^{-i\omega t} ; \text{ Re } \psi \text{ AND } \text{Im } \psi \text{ BOTH EVOLVE WITH TIME}$$

REAL
↓

[NOTICE THAT NODES ARE STATIONARY \Rightarrow STANDING WAVE PATTERN]

$$\psi(x,t) = \psi(x) [\cos(\omega t) - i \sin(\omega t)]$$

()

GIVE RISE TO OSCILLATIONS IN TIME; IF Re ψ IS MAX, Im ψ IS ZERO, AND VICE VERSA

- NODES ARE STATIONARY

$$|\psi(x,t)|^2 = |\psi(x)|^2 \quad \text{"STATIONARY STATES"}$$

- SUPERPOSITION OF TWO EIGENSTATES:

$$\psi(x,t) = c_1 \psi_1(x) e^{-i\omega_1 t} + c_2 \psi_2(x) e^{-i\omega_2 t} ; c_i : \text{OCCUPATION PROBABILITIES}$$

$$|\psi(x,t)|^2 = |c_1|^2 |\psi_1|^2 + |c_2|^2 |\psi_2|^2 + 2 |c_1 c_2| |\psi_1 \psi_2| e^{-i(\omega_1 + \omega_2)t}$$

- $|\psi|^2$ DEPENDS ON TIME

- NODES ARE NOT STATIONARY ANYMORE

- IN APPLET, YOU CAN PICK THE OCCUPATION PROBABILITIES BY THE LENGTH OF THE PHASORS

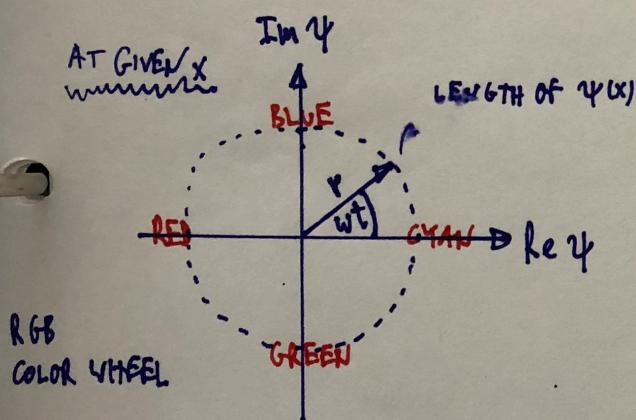
- THE LENGTH OF THE PHASORS REMAINS CONSTANT [$c_i = \text{CONST}$] [E EXPECTATION = CONST]

HOW CAN WE BEST VISUALIZE ψ , NOT $|\psi|^2$?

PLOTTING $\text{Re } \psi$ AND $\text{Im } \psi$ IS NOT VERY MEANINGFUL, BECAUSE MULTIPLYING ψ BY AN ARBITRARY PHASE $e^{i\phi}$ LEAVES $|\psi|^2$ UNCHANGED, BUT COMPLETELY CHANGES APPEARANCE OF $\text{Re } \psi$ AND $\text{Im } \psi$

PURE STATE: $e^{i\phi} \cdot \psi(x, t) = \psi(x) [\cos(\omega t) - i \sin(\omega t)] [\cos \phi + i \sin \phi]$

A BETTER WAY OF VISUALIZATION IS TO SHOW $|\psi|^2$ AND PHASE [FULL INFO]



$$\zeta = r \cdot e^{i\theta}$$

$$\psi(x, t) = \psi(x) \cdot e^{-i\omega t}$$

[SINGLE EIGENSTATE]

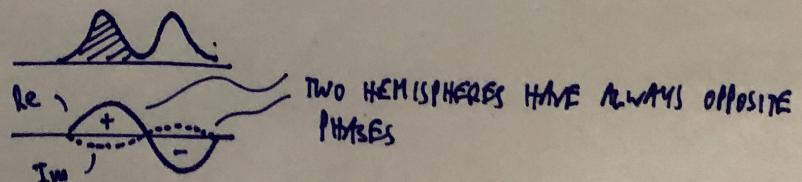
PHASE CHANGES ORIENTATION
IF ψ VECTOR WITH TIME
[ROTATES WITH TIME]

THE PHASE TELLS US THE RELATIVE MAGNITUDE
OF $\text{Re } \psi$ AND $\text{Im } \psi$

- LENGTH $|\psi|^2$ CONSTANT IN TIME [STATIONARY STATE]
- PHASE DETERMINES TIME DEPENDENCE OF STATIONARY STATE $[\omega_n t]$

- SHOW G.S. STATE FIRST IN $|\psi|^2 + \text{PHASE}$

- 1. EXCITED STATE:



- Show SUPERPOSITION OF TWO STATIONARY STATES ["PROBABILITY AND PHASE"]:
AT ANY GIVEN TIME, PHASES ARE NOT SIMPLY COLOR COMPLEMENTS
[NODES ARE NOT CONSTANT ANYMORE]

SUMMARY:

- | PHASE OF ψ [i.e., RELATIVE MAGNITUDE OF $\text{Re}\psi$ AND $\text{Im}\psi$] IMPORTANT FOR:
|
| (i) TIME-DEVELOPMENT OF EIGEN [STATIONARY] STATES [$|\psi|^2$ constant]
|
| (ii) TIME - DEVELOPMENT OF ψ FOR SUPERPOSITION OF EIGENSTATES
| [PHASE $e^{-i(\omega_1 + \omega_2)t}$ DETERMINES EVOLUTION] [$|\psi|^2$ NOT constant]
|

USE OTHER APPLET : PREPARE SYSTEM IN ONE EIGENSTATE, TURN ON EXTERNAL RADIATION

TIME EVOLUTION :

- PHASORS OF BOTH STATES CHANGE LENGTHS WITH TIME
- $c_n(t)$ ARE NOT CONSTANT ANYMORE
- RELATIVE LENGTHS OF PHASORS INDICATE WHICH STATE THE PARTICLE OCCUPIES: IT MAKES TRANSITIONS FROM ONE STATE TO THE OTHER STATE
[NO OTHER PHASORS CHANGE LENGTHS]

WHEN HAMILTONIAN DEPENDS ON TIME:

- ENERGY STATES ARE NOT STATIONARY ANYMORE
- EXPECTATION VALUE OF ENERGY NOT CONSTANT
[ENERGY NOT CONSERVED]
- TRANSITIONS BETWEEN STATES CAN OCCUR

VERY FEW PROBLEMS EXIST FOR TIME-DEPENDENT POTENTIALS
THAT ARE EXACTLY SOLVABLE

IF TIME-DEPENDENT PART OF HAMILTONIAN IS SMALL COMPARED
TO TIME-INDEPENDENT PART

⇒ APPLY PERTURBATION THEORY

USE UNPERTURBED EIGENSTATES AS BASIS FOR EXPANDING
GENERAL STATES OF PERTURBED SYSTEM, EVEN THOUGH THESE
UNPERTURBED STATES ARE IN GENERAL NOT EIGENSTATES OF
THE PERTURBED SYSTEM

$$H\psi = i\hbar \frac{\partial \psi}{\partial t}$$

BEFORE PERTURBATION IS TURNED ON

$$H^{(0)} \psi_n = E_n \psi_n$$

UNPERTURBED ENERGIES AND WAVE FUNCTIONS
 [SUPPRESS (0) SUPERSCRIPT ON ψ_n AND E_n ;
 NOT NEEDED; SEE BELOW]

$$\psi_{H=0}(t=0) = \sum_n c_n \psi_n$$

$|c_n|^2$: PROBABILITY THAT PARTICLE IS IN STATE ψ_n
 [THAT A MEASUREMENT OF ITS ENERGY WILL
 YIELD E_n]

$$\psi_{H=0}(t) = \sum_n c_n e^{-\frac{i}{\hbar} E_n t} \psi_n$$

CONSTANT WITH
TIME

EACH TERM ACQUIRES A TIME-DEPENDENT PHASE
 EVOLUTION FACTOR THAT DEPENDS ON E_n

AFTER PERTURBATION IS TURNED ON [$t > 0$]

$$H = H^{(0)}(\vec{r}) + \lambda H'(\vec{r}, t)$$

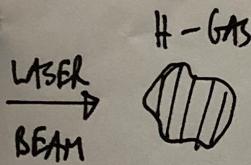
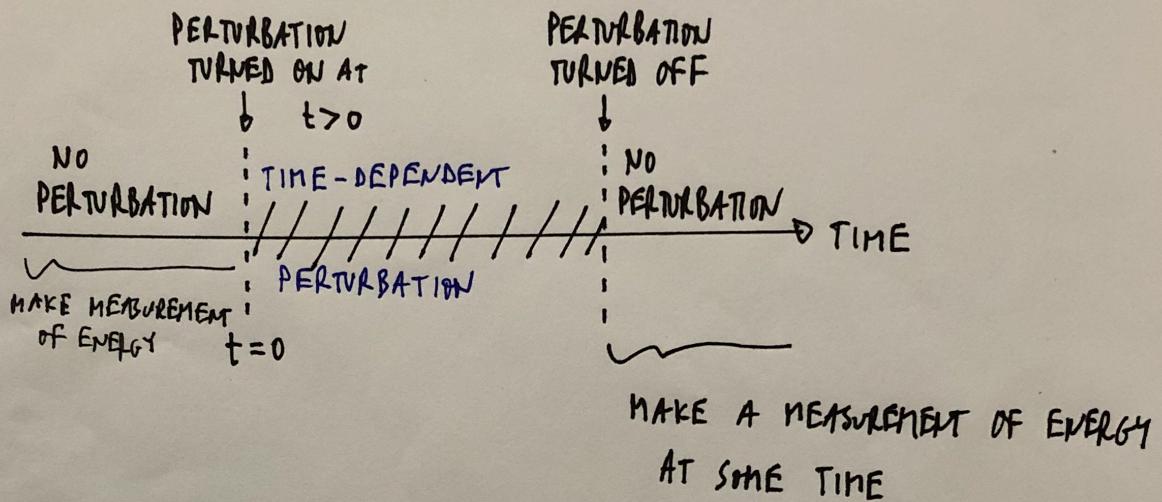
$$\psi_{H+0}(t) = \sum_n c_n(t) e^{-\frac{i}{\hbar} E_n t} \psi_n$$

WE ASSUME THAT ADDITIONAL TIME DEPENDENCE CAUSED BY PERTURBATION
 CAN BE INCLUDED BY LETTING EXPANSION COEFFICIENTS BECOME
 TIME-DEPENDENT

A TIME - DEPENDENT PERTURBATION WILL SHIFT ENERGY LEVELS, $E(t)$,
 BUT WE ARE NOT INTERESTED IN SUCH SMALL, TIME - DEPENDENT
 ENERGY SHIFTS

OUR MAIN INTEREST IS THE FOLLOWING GENERAL SITUATION:

IT DOESN'T MATTER
 IF ENERGIES SHIFT
 A LITTLE DURING
 //////////////// INTERVAL



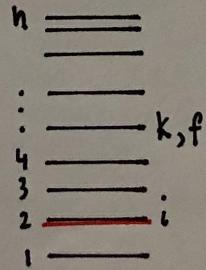
SYSTEM IS INITIALLY IN
 THIS EIGENSTATE...

$|1^3p\rangle$

$|1^1s\rangle$

INDICES FOR FOLLOWING

DISCUSSION:



$|1^3p\rangle$

$|1^1s\rangle$

WHAT IS THE PROBABILITY THAT A
 TRANSITION TO ANOTHER LEVEL HAS
 OCCURRED?

OUR TASK IS TO DETERMINE THE COEFFICIENTS $c_n(t)$

$$H \psi = i\hbar \frac{d}{dt} \psi$$

$$H = H^{(0)}(\vec{r}) + \lambda H'(\vec{r}, t)$$

PERTURBATION
ON/OFF
[KEEKS TRACK OF
PERTURBATION ORDER]

SOME TIME
DEPENDENT
FUNCTION

$$\psi_{H \neq 0}(t) = \sum_n c_n(t) e^{-\frac{i}{\hbar} E_n t} \psi_n$$

$$[H^{(0)} + \lambda H'(t)] \sum_n c_n(t) e^{-\frac{i}{\hbar} E_n t} \psi_n = i\hbar \frac{d}{dt} \sum_n c_n(t) e^{-\frac{i}{\hbar} E_n t} \psi_n$$

ITES NOT DEPEND
 ON TIME
 (CANCEL)

NO \vec{r} DEPENDENCE

$$\underbrace{\sum_n H^{(0)} c_n(t) e^{-\frac{i}{\hbar} E_n t} \psi_n}_{+ \lambda \sum_n H'(t) c_n(t) e^{-\frac{i}{\hbar} E_n t} \psi_n}$$

$$= \sum_n E_n c_n(t) e^{-\frac{i}{\hbar} E_n t} \psi_n = i\hbar \sum_n \frac{dc_n(t)}{dt} e^{-\frac{i}{\hbar} E_n t} \psi_n$$

SINCE $H^{(0)} \psi_n = E_n \psi_n$

$$+ \underbrace{-i\hbar \frac{i}{\hbar} \sum_n E_n c_n(t) e^{-\frac{i}{\hbar} E_n t} \psi_n}_{\text{CANCEL}}$$

$$\Rightarrow \lambda \sum_n H'(t) c_n(t) e^{-\frac{i}{\hbar} E_n t} \psi_n = i\hbar \sum_n \frac{dc_n(t)}{dt} e^{-\frac{i}{\hbar} E_n t} \psi_n$$

PROJECT EQUATION ONTO PARTICULAR UNPERTURBED STATE, ψ_k , AND

USE ORTHOGONALITY $\langle \psi_k | \psi_n \rangle = \delta_{kn}$:

$$\lambda \langle \psi_k | \sum_n H'(t) (c_n(t) e^{-\frac{i}{\hbar} E_n t}) | \psi_n \rangle = \langle \psi_k | i\hbar \sum_n \frac{dc_n(t)}{dt} e^{-\frac{i}{\hbar} E_n t} | \psi_n \rangle$$

HAS \vec{P}
 DEPENDENCE
 [HAS
 NO $\frac{\partial}{\partial t}$]
 NO \vec{P} DEPENDENCE;
 CAN BE TAKEN OUT OF INTEGRAL

$$\lambda \sum_n c_n(t) e^{-\frac{i}{\hbar} E_n t} \langle \psi_k | H'(t) | \psi_n \rangle = i\hbar \frac{dc_k(t)}{dt} e^{-\frac{i}{\hbar} E_k t}$$

REARRANGING, WE OBTAIN FOR EACH COEFFICIENT IN EXPANSION:

$$\boxed{i\hbar \frac{dc_k(t)}{dt} = \lambda \sum_n c_n(t) e^{\frac{i}{\hbar} (E_k - E_n)t} \langle \psi_k | H'(t) | \psi_n \rangle}$$

MEANING: RATE OF CHANGE OF $c_k(t)$ DEPENDS ON AMPLITUDES OF ALL STATES, ψ_n , AND ALSO ON MATRIX ELEMENTS THAT CONNECT STATE k VIA H' WITH ALL OTHER STATES ψ_n

THE ABOVE RESULT IS EXACT! BUT THE SYSTEM OF COUPLED DIFFERENTIAL EQUATIONS IS DIFFICULT TO SOLVE ... SIMPLIFY BY USING EXPANSION:

$$c_n(t) = c_n^{(0)}(t) + \lambda c_n^{(1)}(t) + \lambda^2 c_n^{(2)}(t) + \dots$$

PERTURBATION ORDER
 WITHOUT PERTURBATION

$$c_n^{(1)}(t) = \left[\frac{dc_n(t)}{d\lambda} \right]_{\lambda=0}$$

SUBSTITUTING EXPANSION INTO EXPRESSION ABOVE AND EQUATING COEFFICIENTS OF EQUAL POWERS IN λ :

$$\begin{aligned} i\hbar \left[\frac{dc_k^{(0)}}{dt} + \lambda \frac{dc_k^{(1)}}{dt} + \lambda^2 \frac{dc_k^{(2)}}{dt} + \dots \right] \\ = \lambda \sum_n [c_n^{(0)} + \lambda c_n^{(1)} + \lambda^2 c_n^{(2)} + \dots] e^{\frac{i}{\hbar}(E_k - E_n)t} \langle \psi_k | H'(t) | \psi_n \rangle \end{aligned}$$

GIVES FOLLOWING SYSTEM OF EQUATIONS:

$$\lambda = 0 : i\hbar \frac{dc_k^{(0)}(t)}{dt} = 0$$

$$\lambda = 1 : i\hbar \frac{dc_k^{(1)}(t)}{dt} = \sum_n c_n^{(0)}(t) e^{\frac{i}{\hbar}(E_k - E_n)t} \langle \psi_k | H'(t) | \psi_n \rangle$$

$$\lambda = m : i\hbar \frac{dc_k^{(m)}(t)}{dt} = \sum_n c_n^{(m-1)}(t) e^{\frac{i}{\hbar}(E_k - E_n)t} \langle \psi_k | H'(t) | \psi_n \rangle$$

THESE EQUATIONS CAN, IN PRINCIPLE, BE INTEGRATED SUCCESSIVELY TO ANY ORDER IN THE PERTURBATION, BUT WE WILL GO ONLY UP TO FIRST ORDER [OF MAIN INTEREST IN ATOMIC AND NUCLEAR PHYSICS]

EQUATION FOR $\lambda = 0$ MEANS:

$c_n^{(0)}(t) = \text{CONST}$ \Rightarrow THERE IS NO TIME DEPENDENCE WITHOUT PERTURBATION.

EQUATION FOR $\lambda = 1$:

ASSUMPTION OF MAIN INTEREST HERE: IF WE ASSUME THAT SYSTEM BEFORE PERTURBATION IS IN PARTICULAR EIGENSTATE, ψ_i OF ENERGY E_i , THEN

TO BE MORE
SPECIFIC

$$c_n^{(0)}(t) = \delta_{ni}$$

[ALL $c_n^{(0)}$ ARE ZERO, EXCEPT FOR $n=i$; THIS DOES NOT CHANGE WITH TIME,
INITIALLY: $\underbrace{|c_i|^2 + |c_1|^2 + \dots + |c_n|^2 + \dots}_{=1} = 1$ i.e., WITHOUT PERTURBATION]

SUBSTITUTE INTO $\lambda = 1$ EQUATION:

$$\text{i}\hbar \frac{dc_k^{(1)}(t)}{dt} = \sum_n \delta_{ni} e^{\frac{i}{\hbar}(E_k - E_n)t} \langle \psi_k | H'(t) | \psi_n \rangle$$

$$= e^{\frac{i}{\hbar}(E_k - E_i)t} \langle \psi_k | H'(t) | \psi_i \rangle$$

-143

DIRECT INTEGRATION BETWEEN $t=0$ AND TIME t [WHEN PERTURBATION IS ON] GIVES:

$$\underbrace{\int_0^t \frac{dc_k^{(1)}(t')}{dt'} dt'}_{\text{Left side}} = \frac{1}{i\hbar} \int_0^t e^{\frac{i}{\hbar}(E_k - E_i)t'} \langle \psi_k | H'(t') | \psi_i \rangle dt'$$

$$[c_k^{(1)}(t')]_0^t = c_k^{(1)}(t) - \underbrace{c_k^{(1)}(0)}$$

$= 0$ SINCE INITIAL STATE [$t \leq 0$]
HAS NO PERTURBATION

$$c_k^{(1)}(t) = \frac{1}{i\hbar} \int_0^t e^{\frac{i}{\hbar}(E_k - E_i)t'} \langle \psi_k | H'(t') | \psi_i \rangle dt'$$

DESCRIBES TIME EVOLUTION OF EXPANSION COEFFICIENTS $c_k(t)$
FOR EIGENSTATE ψ_k , SUBJECT TO PERTURBATION $H'(t)$, GIVEN
THAT SYSTEM STARTED OUT IN EIGENSTATE ψ_i

OR

UNDER THE INFLUENCE OF TIME-DEPENDENT PERTURBATION, THE
SYSTEM MAKES TRANSITIONS TO OTHER EIGENSTATES

TRANSITION PROBABILITY CORRESPONDING TO TRANSITION $i \rightarrow k$
 [FROM INITIAL EIGENSTATE ψ_i AT $t=0$ TO FINAL EIGENSTATE ψ_k
 AT TIME t]:

$$P_{i \rightarrow k}(t) = \left| \underbrace{\langle \psi_k | \psi(t) \rangle}_{\text{FINAL EIGENSTATE}} \right|^2 = \left| \langle \psi_k | \sum_n c_n(t) e^{-\frac{i}{\hbar} E_n t} | \psi_n \rangle \right|^2$$

NO \vec{p} DEPENDENCE

$$= \underbrace{|c_k(t)|^2}_{\text{FINAL EIGENSTATE}}$$

TO FIRST-ORDER PERTURBATION, WE HAVE:

$$c_k(t) = \underbrace{c_k^{(0)}(t)}_{\substack{\text{U} \\ \text{= 0}}} + \lambda \underbrace{c_k^{(1)}(t)}_{\substack{\text{U} \\ \text{1}}} = c_k^{(1)}(t)$$

SEE EARLIER: $c_k^{(0)}(t) = \delta_{ki}$

INITIALLY: $|c_1|^2 + |c_2|^2 + \dots + |c_k|^2 + \dots = 1$; DOES NOT CHANGE WITH TIME [SEE EARLIER]

BUT HERE WE ARE EXPLICITLY INTERESTED IN TRANSITIONS BETWEEN DIFFERENT EIGENSTATES:

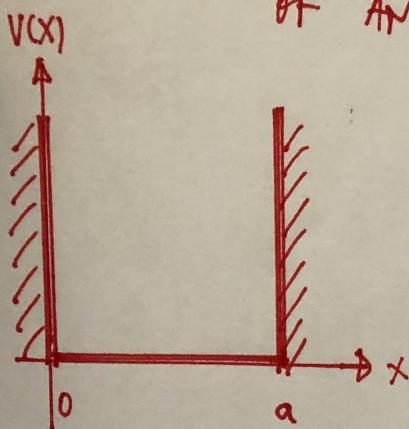
$$P_{i \rightarrow f}(t) = \frac{1}{\hbar^2} \left| \int_0^t e^{\frac{i}{\hbar} (E_f - E_i)t'} \langle \psi_f | H'(t') | \psi_i \rangle dt' \right|^2$$

[REPLACED "k" BY "f"]

CONVENTION IN TIME-DEPENDENT PERTURBATION THEORY:

$\langle \text{FINAL STATE} | \text{PERTURBING INTERACTION} | \text{INITIAL STATE} \rangle$

EXAMPLE: A PARTICLE IS INITIALLY ($t=0$) IN THE GROUND STATE OF AN INFINITE 1-D POTENTIAL BOX.



FOR $0 < t \leq \infty$ IT IS SUBJECT TO A PERTURBATION OF

$$H'(t) = V_0 x^2 e^{-t/\tau}$$

WHERE τ IS THE CHARACTERISTIC TIME CONSTANT OF THE PERTURBATION. CALCULATE THE FIRST-ORDER PROBABILITY OF FINDING THE PARTICLE IN THE FIRST EXCITED STATE AT VERY LONG TIMES [i.e., IN THE LIMIT $t \rightarrow \infty$]

UNPERTURBED STATES:

$$E_n = \frac{\pi^2 \hbar^2}{2ma^2} n^2$$

$n=1$: GROUND STATE

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

$n=2$: FIRST-EXCITED STATE

$$\text{DEFINE: } \omega_{21} = \frac{1}{\hbar} (E_2 - E_1) = \frac{3\pi^2 \hbar}{2ma^2}$$

$$\text{INTEGRAL NEEDED: } \int_0^a x^2 \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{\pi x}{a}\right) dx = -\frac{8a^3}{9\pi^2}$$

$$P_{n=1 \rightarrow n=2}(t) = \frac{1}{\hbar^2} \left| \int_0^t \langle \psi_{n=2} | H'(t') | \psi_{n=1} \rangle e^{\frac{i}{\hbar}(E_2 - E_1)t'} dt' \right|^2$$

$$\omega_{21} = \frac{1}{\hbar} (E_2 - E_1)$$

$$= \frac{V_0^2}{\hbar^2} \left| \underbrace{\langle \psi_{n=2} | x^2 | \psi_{n=1} \rangle}_I \right|^2 \left| \underbrace{\int_0^t e^{-t'[\frac{1}{\hbar} - i\omega_{21}]} dt'}_I \right|^2$$

$$\textcircled{I} \quad \langle \psi_2 | x^2 | \psi_1 \rangle = \frac{2}{a} \underbrace{\int_0^a x^2 \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{\pi x}{a}\right) dx}_{= -\frac{8a^3}{9\pi^2}} = \frac{2}{a} \left(-\frac{8a^3}{9\pi^2} \right) = -\frac{16a^2}{9\pi^2}$$

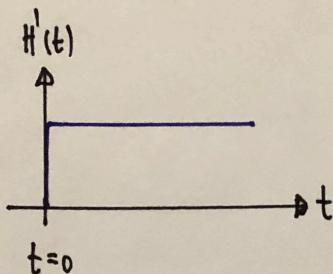
$$\textcircled{II} \quad \int_0^t e^{-t'[\frac{1}{\hbar} - i\omega_{21}]} dt' = \left[-\frac{e^{-t'[\frac{1}{\hbar} - i\omega_{21}]}}{\frac{1}{\hbar} - i\omega_{21}} \right]_0^t = \frac{-e^{-t[\frac{1}{\hbar} - i\omega_{21}]}}{\frac{1}{\hbar} - i\omega_{21}} + 1$$

→ 0 for $t \rightarrow \infty$
 ↗ Just an oscillating function

PUTTING EVERYTHING TOGETHER:

$$P_{n=1 \rightarrow n=2}(t \rightarrow \infty) = \frac{V_0^2}{\hbar^2} \left(\frac{16a^2}{9\pi^2} \right)^2 \frac{1}{\omega_{21}^2 + \frac{1}{\hbar^2}} = \frac{\frac{V_0^2}{\hbar^2} \left(\frac{16a^2}{9\pi^2} \right)^2}{\left(\frac{3\pi^2 \hbar}{2ma^2} \right)^2 + \frac{1}{\hbar^2}}$$

SPECIAL

CASE #1: CONSTANT [i.e., TIME-INDEPENDENT] PERTURBATION

ONLY TIME-DEPENDENCE:
TURNING POTENTIAL ON

$$c_f^{(1)}(t) = \frac{1}{i\hbar} \langle \psi_f | H' | \psi_i \rangle \int_0^t e^{\frac{i}{\hbar}(E_f - E_i)t'} dt'$$

$$\text{DEFINE : } \omega_{fi} \equiv \frac{E_f - E_i}{\hbar}$$

$$\begin{aligned} \int_0^t e^{i\omega_{fi}t'} dt' &= \left[\frac{e^{i\omega_{fi}t'}}{i\omega_{fi}} \right]_0^t = \frac{e^{i\omega_{fi}t} - 1}{i\omega_{fi}} = e^{i\omega_{fi}\frac{t}{2}} \frac{e^{i\omega_{fi}\frac{t}{2}} - e^{-i\omega_{fi}\frac{t}{2}}}{i\omega_{fi}} \\ &= 2 e^{i\omega_{fi}\frac{t}{2}} \cdot \frac{\sin(\omega_{fi}\frac{t}{2})}{\omega_{fi}} \end{aligned}$$

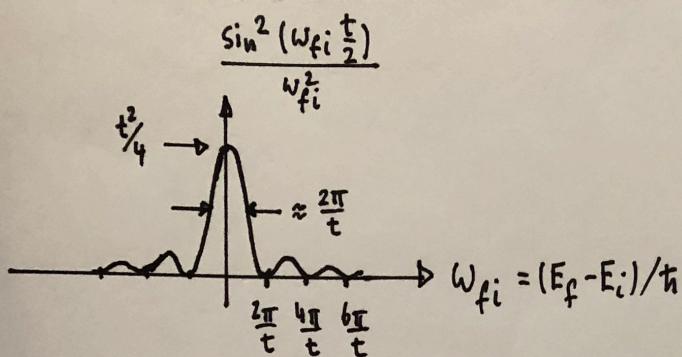
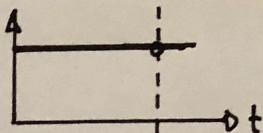
PROBABILITY THAT SYSTEM IS MEASURED TO BE IN FINAL EIGENSTATE ψ_f
AT TIME t :

$$P_{i \rightarrow f}(t) = |c_f^{(1)}(t)|^2 = \frac{4 |\langle \psi_f | H' | \psi_i \rangle|^2}{\hbar^2} \frac{\sin^2(\omega_{fi}\frac{t}{2})}{\omega_{fi}^2}$$

- TO MAKE TRANSITION $\psi_i \rightarrow \psi_f$: MATRIX ELEMENT $\langle \psi_f | H' | \psi_i \rangle$ MUST BE
NON-ZERO

- LET'S HAVE A CLOSER LOOK AT FACTOR $\frac{\sin^2(\omega_{fi}\frac{t}{2})}{\omega_{fi}^2}$

AS A FUNCTION OF ω_{fi} , AT A FIXED TIME:

 $H'(t)$ 

- INTERFERENCE PATTERN

- WE ASSUMED HERE THAT ω_{fi} IS CONTINUOUS [TO DRAW CURVE], i.e., LARGE FINAL STATE DENSITY

TRANSITION PROBABILITY GREATEST FOR THOSE STATES ψ_f FOR WHICH $\omega_{fi} \approx 0$ OR $E_f \approx E_i$, i.e., IF ENERGY IS APPROXIMATELY CONSERVED IN PROCESS

BY FAR, MOST OF STRENGTH GOES INTO ENERGY BAND, AROUND E_i , OF WIDTH $\delta\omega_{fi} = 2\pi/t$ → UNPERTURBED ENERGY E_i IS CONSERVED WITHIN $\Delta E = \hbar \cdot \delta\omega = \hbar \frac{2\pi}{t}$

WHICH IS CONSISTENT WITH TIME-ENERGY RELATION $\Delta E \cdot \Delta t \geq \hbar$

OR: THE ONLY FINAL STATES POPULATED FOR AN APPRECIABLE AMOUNT OF TIME ARE THOSE THAT CONSERVE THE ENERGY OF THE SYSTEM

[INTERESTINGLY, ENERGY CONSERVATION, SUITABLY MODIFIED BY UNCERTAINTY PRINCIPLE, IS AN AUTOMATIC CONSEQUENCE OF THE CALCULATION AND DOES NOT HAVE TO BE INSERTED AS A SEPARATE ASSUMPTION]

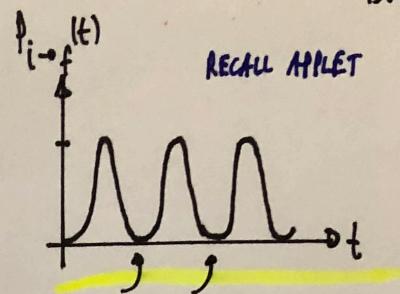
AS A FUNCTION OF TIME THE PERTURBATION IS TURNED ON:

FOR FIXED ω_{fi} : $P_{i \rightarrow f}(t)$ OSCILLATES WITH PERIOD

$$\frac{2\pi}{\omega_{fi}}$$

- BETWEEN SOME MAXIMUM ($\ll 1$!) AND ZERO (!)

- TYPICAL IN ATOMIC / CM PHYSICS : $E_f - E_i \approx \text{eV}$

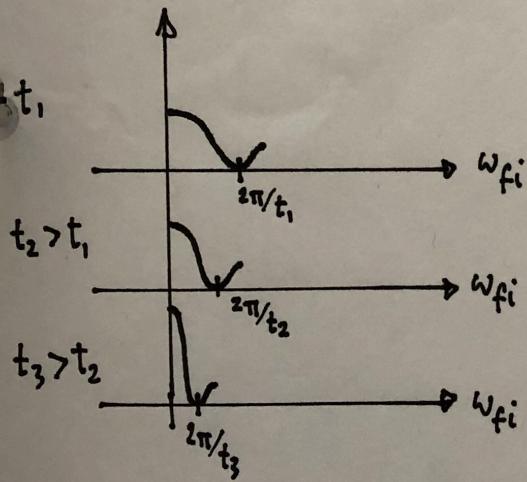


PARTICLE MOVES COMPLETELY OUT OF FINAL STATE

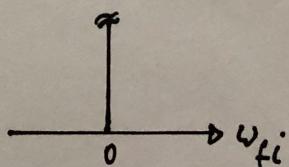
$$\Rightarrow \frac{2\pi}{\omega_{fi}} = \frac{2\pi}{\text{eV}/\hbar} = 2\pi\hbar/\text{eV} = 2\pi \cdot 6.58 \cdot 10^{-16} \text{ eV} \cdot \text{s/eV} \approx 4.1 \cdot 10^{-15} \text{ s}$$

[DIFFICULT TO OBSERVE IN PRACTICE]

- AT DIFFERENT TIMES, PATTERN EVOLVES :



- CENTRAL PEAK BECOMES NARROWER AND HIGHER
- BECOMES A δ-FUNCTION IN LIMIT $t \rightarrow \infty$



FOR A CONSTANT PERTURBATION, IN THE LIMIT $t \rightarrow \infty$, THE ENERGY OF THE FINAL STATE IS THE SAME AS THE ENERGY OF THE INITIAL STATE

IN MANY PRACTICAL APPLICATIONS, A DISTRIBUTION OF FINITE WIDTH IMPLIES THAT E_f [NEAR E_i] IS NOT AN ISOLATED LEVEL, BUT NEAR E_i A LARGE DENSITY OF LEVELS EXIST:

- ATOMIC OR NUCLEAR SCATTERING
- IONIZATION OF ATOMS [TRANSITIONS INTO CONTINUUM]
- BAND STRUCTURE OF SOLIDS

GIVING RISE TO A CONTINUOUS SPECTRUM

$$\text{TRANSITION RATE : } W_{i \rightarrow f} \equiv \frac{P_{i \rightarrow f}}{t}$$

DENSITY OF STATES PER UNIT ENERGY INTERVAL NEAR $E_i \approx E_f$: $\rho(E_f) = \frac{dn_f}{dE_f}$

$$w_{fi} \equiv \frac{E_f - E_i}{\hbar}$$

$$\frac{dw_{fi}}{dE_f} = \frac{1}{\hbar}$$

- WE ASSUME A CONSTANT DENSITY OF STATES NEAR E_f
- WE ALSO ASSUME THAT MATRIX ELEMENT $\langle \psi_f | H' | \psi_i \rangle$ IS CONSTANT WITH ψ_f [FREQUENTLY THE CASE, ESPECIALLY OVER NARROW CENTRAL MAXIMUM]

TOTAL INTEGRATED TRANSITION RATE:

$$W_{i \rightarrow f} = \int \frac{P_{i \rightarrow f}(t)}{t} \underbrace{\rho(E_f) dE_f}_{\frac{dn_f}{dE_f}} = \int dE_f \rho(E_f) \frac{1}{t} \frac{4}{\hbar^2} |\langle \psi_f | H' | \psi_i \rangle|^2 \frac{\sin^2(\omega_{fi} t/2)}{w_{fi}^2}$$

$$= \frac{4}{\hbar^2} \frac{t}{4} \rho(E_f) \frac{1}{t} |\langle \psi_f | H' | \psi_i \rangle|^2 \int d\omega_{fi} \frac{t^2}{4} \frac{\sin^2(\omega_{fi} t/2)}{(\omega_{fi} t/2)^2}$$

$$= \frac{4}{\hbar} \frac{t}{4} \rho(E_f) |\langle \psi_f | H' | \psi_i \rangle|^2 \frac{2}{t} \int_{-\infty}^{+\infty} dx \frac{\sin^2(x)}{x^2}$$

$= \pi$

$$x = \omega_{fi} t/2$$

$$dx = \frac{t}{2} d\omega_{fi}$$

BY INTEGRATING OVER ω_{fi} ,
THE OSCILLATORY BEHAVIOR WITH t
DISAPPEARS!
[PERIOD OF $P_{i \rightarrow f}$ IS $2\pi/\omega_{fi}$]

MOST IMPORTANT RESULT OF FIRST-ORDER TIME-DEPENDENT PERTURBATION THEORY

$$\lambda = W_{i \rightarrow [f]} = \frac{2\pi}{\hbar} |\langle \psi_f | H' | \psi_i \rangle|^2 \rho(E_f)$$

"GOLDEN RULE"

[SOMETIMES ALSO
CALLED:

"FERMI'S GOLDEN RULE
#2"]

[f] DENOTES A TRANSITION TO A GROUP OF FINAL STATES

- FIRST OBTAINED BY DIRAC; HAS WIDE APPLICATIONS IN QUANTUM MECHANICS
- MEANS: IF PERTURBATION IS CONSTANT, TOTAL TRANSITION RATE BECOMES CONSTANT WITH TIME
- DERIVED HERE FOR A CONSTANT PERTURBATION, BUT CAN BE GENERALIZED TO HARMONIC PERTURBATION; IN THAT CASE, H' DENOTES ONLY PART OF PERTURBATION THAT DEPENDS ON \vec{r} [NOT t]; ALSO, $E_f = E_i + \hbar\omega$
 $\rho(E_f) = \rho(E_i + \hbar\omega)$
- A TOTAL TRANSITION RATE, $W_{i \rightarrow [f]}$, IS CALLED "DECAY CONSTANT" IN NUCLEAR AND PARTICLE PHYSICS, WHERE THE MEAN LIFETIME IS DEFINED AS $\tau \equiv \frac{1}{\lambda}$

LETTERS TO THE EDITOR

Letters are selected for their expected interest for our readers. Some letters are sent to reviewers for advice; some are accepted or declined by the editor without review. Letters must be brief and may be edited, subject to the author's approval of significant changes. Although some comments on published articles and notes may be appropriate as letters, most such comments are reviewed according to a special procedure and appear, if accepted, in the Notes and Discussions section. (See the "Statement of Editorial Policy" at <http://www.kzoo.edu/ajp/docs/edpolicy.html>.) Running controversies among letter writers will not be published.

WHOSE GOLDEN RULE IS IT ANYWAY?

In an enlightening article,¹ Jackson writes that scientific discoveries more often than not are named after the wrong person. He provides ample evidence for this principle, dubbed "the zeroth theorem of the history of science," by giving a detailed description of several examples from physics.

To this list I would like to add another misattribution whose persistence has always struck me: in almost any textbook on quantum mechanics the treatment of time-dependent perturbations culminates in an expression for the transition probability per unit time between two states, which is then called "Fermi's Golden Rule."² In fact, in his classic text *Nuclear Physics*,³ Fermi gives two results, one for first-order transitions and one for second-order transitions. He coins the names "Golden Rule #1" and "Golden Rule #2," but does not give a derivation. For this he refers to Schiff's textbook.⁴ Clearly the names suggested by Fermi were hugely successful. So successful even that Schiff adopted this terminology in a later edition of his book,⁵ in which he writes "Eq. (35.14)... is so useful that it was called 'Golden Rule No. 2' by E. Fermi." Nevertheless, Schiff was well aware that this formalism originated elsewhere. In another footnote he mentions its discoverer: Paul Dirac. More than 20 years before Fermi's book appeared, Dirac published a beautiful and comprehensive treatment of quantum mechanical perturbation theory⁶ in which the first-order result is presented and applied to absorption and emission of radiation. In older texts, for example the book by Kramers⁷ or that by Condon and Shortley,⁸ Dirac is given full credit for his work. After Fermi published his

book, that habit seems to have gone out of style. But Fermi is in no need of extra accolades; the key formula of perturbation theory is really *Dirac's Golden Rule*.

¹J. D. Jackson, "Examples of the zeroth theorem of the history of science," Am. J. Phys. **76**(8), 704–719 (2008).

²One example out of at least a dozen such books is *Quantum Mechanics* by C. Cohen-Tannoudji, B. Diu, and F. Laloë (Wiley, New York, 1977).

³E. Fermi, *Nuclear Physics*, revised ed. (U. of Chicago, Chicago, 1950).

⁴L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1949).

⁵L. I. Schiff, *Quantum Mechanics*, 3rd ed., International Student Edition (McGraw-Hill Kogakusha, Tokyo, 1968). See footnotes on pp. 280 and 285.

⁶P. A. M. Dirac, "The quantum theory of the emission and absorption of radiation," Proc. R. Soc. London, Ser. A **114**, 243–265 (1927).

⁷H. A. Kramers, *Die Grundlagen der Quantentheorie* (Akademische Verlagsgesellschaft, Leipzig, 1938).

⁸E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge U. P., Cambridge, 1951).

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BOHREN'S EDITORIAL

Craig Bohren's guest editorial on errors in textbooks is characteristically incisive and correct.¹ What authors ought to do is clear, but many authors are too rushed to perform due diligence. Then how can we reduce the number of textbook errors in physics *per se* and, perhaps more numerously, in the history of physics? The key lies in one observation: textbook authors prefer to get it right.

Let me illustrate. One of my pet peeves is the treatment that introductory textbooks accord to Einstein's relation $E=mc^2$ and its implications. A

few books get it right, but others present misleading or, worse yet, indefensible implications. Having helped a retired chemist to set his community on the road to a better conception of entropy, I decided to try his strategy. First, I published a substantial expository and historical article on $E=mc^2$. Then I read meticulously the relevant pages in the prominent current introductory textbooks and wrote each author a personal letter (with a reprint enclosed). Where I could praise, I did; where I could not, I offered suggestions. The response from authors was consistently positive.

Thus, if you want to see improvements and you know something about a topic, *write to the authors*.

The AJP may even find it productive to establish a clearinghouse for corrections, suggestions, and authors' addresses.

¹C. F. Bohren, "Physics textbook writing: Medieval, monastic mimicry," Am. J. Phys., **77**(2), 101–103 (2009).

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REPEATED PROBLEM SOLVING REVISITED

The motivation of this Letter is to advocate the teaching of physics via strategies, which simultaneously incorporate the teaching of conceptual physics and quantitative mathematical reasoning. The pertinence of this concern is based on the observation that most of the recent published articles in *Physics Education Research* favors an emphasis on conceptual learning over quantitative reasoning. Thus, one

AVERAGE DISTANCE TRAVELED BY NEUTRINOS:

$$1 \text{ ly} = 9.46 \cdot 10^{17} \text{ cm}^{-159-}$$

MEAN FREE PATH

$$L = \frac{1}{n \cdot \sigma} = \frac{1}{(6.7 \cdot 10^{22} \text{ PROTONS/cm}^3)(4.9 \cdot 10^{-43} \text{ cm}^2)}$$

↓
OF NUCLEI
PER cm^3

$$= 3.0 \cdot 10^{-19} \text{ cm} \approx \underline{\underline{32 \text{ ly}}}$$

$$\rho_{\text{WATER}} = 1 \text{ g/cm}^3$$

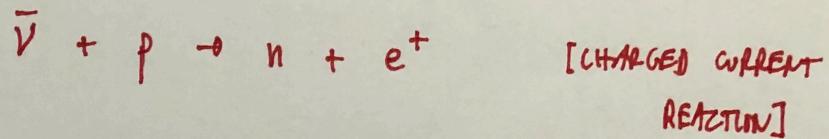
$$18 \text{ g OF H}_2\text{O} \approx 6.022 \cdot 10^{23} \text{ MOLECULES}$$

$$\rightarrow 2 \times 6.022 \cdot 10^{23} \text{ PROTONS}$$

$$1 \text{ g OF H}_2\text{O} \approx 6.7 \cdot 10^{22} \text{ PROTONS}$$

EXAMPLE

CROSS SECTION FOR "INVERSE β -DECAY":



SLIDE ON COWAN/REINES

ASSUME AN INCIDENT $\bar{\nu}$ ENERGY OF 3.6 MeV [i.e., SOMEWHERE NEAR AVERAGE ENERGY OF DETECTABLE $\bar{\nu}$ IN COWAN/REINES EXPERIMENT]

- ASSUME INFINITELY HEAVY NUCLEONS:

$$E_{\bar{\nu}} + m_p c^2 = m_n c^2 + E_e \Rightarrow \underbrace{E_e}_{[\text{NEUTRON KINETIC ENERGIES: } \propto 10 \text{ keV}]} = E_{\bar{\nu}} - [m_n - m_p] c^2 = 1.293 \text{ MeV}$$

[NEUTRON-PROTON MASS DIFF.]

- CROSS SECTIONS:

$$\Gamma = \frac{\# \text{ INTERACTIONS PER TARGET NUCLEUS PER TIME}}{\# \text{ PROJECTILES PER AREA PER TIME}} = \frac{\# \text{ OF INTERACTIONS PER TARGET PER TIME}}{\text{INCIDENT FLUX OF ONE PROJECTILE}}$$



$$V = A \cdot d = A \cdot c t \rightarrow \frac{1}{A \cdot t} = \frac{c}{V} \quad \begin{matrix} \leftarrow \text{SPEED OF LIGHT} \\ \text{VOLUME [ARBITRARILY, SEE LATER]} \end{matrix}$$

$$c = \frac{d}{t}$$

$$\text{THUS: } \Gamma = \frac{1}{c/V}$$

PROCESS IS GOVERNED BY THE WEAK NUCLEAR FORCE, WHICH IS A SMALL PERTURBATION TO THE NUCLEAR HAMILTONIAN [FACTOR 10^{13} WEAKER]

$$\lambda = \frac{2\pi}{\hbar} |\langle \psi_f | H' | \psi_i \rangle|^2 \rho(E_f)$$

$$= \Phi_n \Phi_e$$

$$= \Phi_p \Phi_\nu$$

$$\frac{dne}{dE_f}$$

RANGE OF WEAK FORCE:
 $\approx 10^{-18} \text{ m}$

LEPTONS [ANTI-NEUTRINO AND POSITRON] ARE NEARLY FREE PARTICLES [INTERACTION WITH p/n NEGIGIBLE]:

$$\phi_\nu(\vec{r}) = \frac{1}{V} e^{-i\vec{p} \cdot \vec{r}/\hbar} \times \frac{1}{V} \left(1 + \underbrace{\frac{i\vec{p} \cdot \vec{r}}{\hbar}}_{\text{SMALL}} + \dots \right) \approx \frac{1}{V}$$

PLANE WAVE,
NORMALIZED WITHIN
SOME VOLUME

WE RETAIN ONLY THE LEADING TERM SINCE ν/e WAVEFUNCTIONS ARE APPROXIMATELY CONSTANT OVER NUCLEAR VOLUME

CHECK:

$$E_\nu \approx p_\nu \cdot c$$

$$\frac{p_\nu \cdot r}{\hbar} = \frac{E_\nu \cdot r}{(\hbar c)} = 0.051$$

$\sim 1 \text{ MeV}$
 $\sim 10 \text{ fm}$
 197.3 MeV fm

FINAL STATE DENSITY

[SEC. 5.3 GRIFFITHS]

$$dn_e = \frac{p_e^2 dE_e V}{2\pi^2 \hbar^3}$$

SEE INTERESTING FOOTNOTE IN MAYER-KUCKUK, p. 309,
ON MULTIPOLY FROM SPINS; THESE ARE IMPLICITLY
INCLUDED IN MATRIX ELEMENTS!

$$\frac{dn_e}{dE_f} = \frac{p_e^2 V}{2\pi^2 \hbar^3} \frac{dE_e}{dE_f}$$

POSITION: $p_e = \frac{1}{c} \sqrt{E_e^2 - (m_e c^2)^2}$ $\Rightarrow \frac{dE_e}{dE_f} = \frac{1}{c} \frac{1}{2} \frac{2 E_e}{\sqrt{E_e^2 - (m_e c^2)^2}} = \frac{1}{c} \frac{E_e}{p_e c}$

MATRIX ELEMENT

$$|\langle \Phi_n | H' | \Phi_p \rangle|^2 = |g \langle \Phi_n | \Sigma | \Phi_p \rangle|^2 = g^2 M^2$$

STRENGTH OF INTERACTION

A MORE RIGOROUS TREATMENT SHOWS THAT THERE ARE TWO DIFFERENT MATRIX ELEMENTS, DEPENDING ON NUCLEAR STRUCTURE OF FINAL AND INITIAL NUCLEI:

$$|\langle \Phi_n | H' | \Phi_p \rangle|^2 = G_V^2 M_F^2 + G_A^2 M_{GT}^2$$

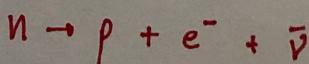
EFFECTIVE VECTOR AND AXIAL VECTOR COUPLING CONSTANTS

FERMI AND GAMOW-TELLER MATRIX ELEMENTS

$$G_V = 8.9618 \cdot 10^{-5} \text{ MeV fm}^3$$

$$G_A = 1.1202 \cdot 10^{-4} \text{ MeV fm}^3$$

FOR (FREE)
NEUTRON DECAY:



$$M_F^2 = 1$$

$$M_{GT}^2 = 3$$

$\frac{G_A}{G_V} = -1.270 \pm 0.003$

PUTTING EVERYTHING TOGETHER:

- 158 -

$$\lambda = \frac{2\pi}{\hbar} \frac{1}{V^2} (G_V^2 M_F^2 + G_A^2 M_{GT}^2) \frac{p_e^2 V E_e}{2\pi^2 \hbar^3 p_e c^2}$$

$$\Gamma = \frac{\lambda}{c/V} = \frac{1}{\pi} \frac{1}{(\hbar c)^4} (p_e c) E_e (G_V^2 M_F^2 + G_A^2 M_{GT}^2)$$

$\bar{\nu} + p \rightarrow n + e^+$

$\hbar c = 197.327 \text{ Mev fm}$

$\sqrt{E_e^2 - (m_e c^2)^2}$

$$\Gamma (E_\nu = 3.6 \text{ MeV}) = \frac{1}{\pi} \frac{(2.24 \text{ Mev})(2.3 \text{ Mev})}{(197.327 \text{ Mev fm})^4} \underbrace{[(8.9618 \cdot 10^{-5})^2 \cdot 1 + (1.1202 \cdot 10^{-4})^2 \cdot 3]}_{4.587 \cdot 10^{-8}}$$

$$= 4.9 \cdot 10^{-17} \text{ fm}^2$$

$$= \underline{4.9 \cdot 10^{-43} \text{ cm}^2}$$

AGREES WITH σ GIVEN IN ZACEK et al.,
PRD 34, 2621 (1986)

- MEASURED BY COWAN/REIMERS : $\bar{\Gamma} \approx 1.1 \cdot 10^{-43} \text{ cm}^2$ [ENERGY AND ANGLE INTEGRATED]

[CAN'T COMPARE VALUES DIRECTLY BECAUSE
WE NEED TO KNOW ENERGY SPECTRUM OF
EMITTED ANTI-NEUTRINOS]

- CALCULATED VALUE IS CLOSE TO MORE RIGOROUS CALCULATION; CORRECTIONS
FOR (i) RECOIL, (ii) WEAK MAGNETISM ARE SMALL; THERE IS NO
COULOMB FIELD TO DISTORT PLANE WAVES OF LEPTONS!

- REMARKABLE : WE USED MATRIX ELEMENTS FROM FREE NEUTRON DECAY,
 $n \rightarrow p + e^- + \bar{\nu}$, TO PREDICT NEUTRINO CROSS SECTION

