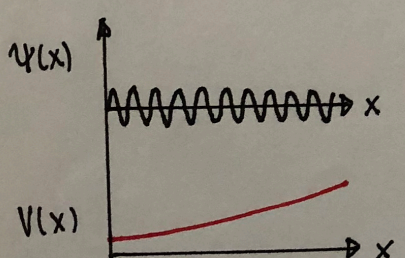
WENTZEL - KRAMERS - BRILLOUIN (WKB) METHOD

[JEFFREYS]

- FIRST TIME APPLIED TO QUANTUM MECHANICS IN 1926
- APPROXIMATE METHOD FOR CALCULATING BOUND STATE ENERGIES AND BARRIER TRANSMISSIONS FOR <u>SLOWLY VARYING POTENTIALS</u> IN 1D [INCLUDING CENTRAL POTENTIALS IN 3D]

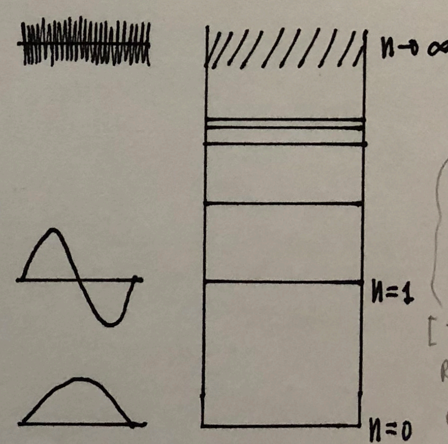
THAT IS, FOR A POTENTIAL THAT REMAINS <u>NEARLY CONSTANT</u> OVER DE BROGLIE WAVELENGTH λ

SINCE THIS CONDITION IS ALWAYS SATISFIED IN THE CLASSICAL LIMIT, THIS METHOD IS FREQUENTLY CALLED A SEMI-CLASSICAL APPROXIMATION:



$\boxed{\lambda \to 0}$

$\lambda = \frac{h}{p}$



$\boxed{\hbar \to 0}$

$[\Delta x \cdot \Delta p_x \approx \hbar]$

[THE LIMIT THAT REDUCES THE GAUSSIAN WAVEPACKET TO THE CLASSICAL PARTICLE]

METHOD IS NOT RESTRICTED TO SMALLNESS OF PERTURBING POTENTIAL!

[SINCE $\hbar \to 0$: $\lambda = \frac{h}{p} \to 0$]

CONSIDER 1D MOTION OF PARTICLE IN TIME-INDEPENDENT POTENTIAL $V(x)$:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\,\psi(x) = E\,\psi(x)\,, \quad E > V$$

$$\frac{d^2}{dx^2}\psi(x) + \frac{1}{\hbar^2}\,p(x)^2\,\psi(x) = 0$$

- i.e., THE MOMENTUM WHICH THE PARTICLE WOULD POSSESS CLASSICALLY
- FOR $\hbar = 0$ [CLASSICAL LIMIT], THIS WOULD BE THE LINEAR MOMENTUM OF PARTICLE

WHERE $p(x) \equiv \sqrt{2m\,(E - V(x))}$ IS THE CLASSICAL FORMULA FOR MOMENTUM OF PARTICLE WITH TOTAL ENERGY E AND POTENTIAL ENERGY $V(x)$

$$\left(\pm\frac{i}{\hbar}\right)\left(\pm\frac{i}{\hbar}\right)p_0^2\,e^{\pm\frac{i}{\hbar}p_0x} + \frac{1}{\hbar^2}p_0^2\,e^{\pm\frac{i}{\hbar}p_0x} = 0$$

$\boxed{\text{IF } V(x) = V_0 = \text{CONST}}$

SOLUTION IS A LINEAR COMBINATION OF

$$\psi(x) = A\,e^{\pm\frac{i}{\hbar}p_0x}$$

A, p_0 : CONSTANTS!

MULTIPLY BY $e^{-i\omega t}$:

+ : PLANE WAVE TRAVELING $\rightarrow$

− : PLANE WAVE TRAVELING $\leftarrow$

$e^{kx} \rightarrow e^{kx-c}$

$e^{-kx} \rightarrow e^{-[kx+c]}$

[UNITS IN EXPONENTIAL : SINCE $p = \hbar k \rightarrow$ "kx"]

$$\lambda \equiv \frac{2\pi}{k}$$

$$\boxed{\text{IF } V(x) \text{ VARIES SLOWLY OVER DE BROGLIE WAVELENGTH}}$$

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2m(E - V(x))}}$$

TRY SOLUTION OF FORM :
$$\boxed{\psi(x) = A\, e^{\frac{i}{\hbar} S(x)}}$$

AMPLITUDE PHASE

ARE FUNCTIONS YET TO BE DETERMINED

SUBSTITUTION INTO SCHRÖDINGER EQUATION :

$$-\frac{\hbar^2}{2m} \underbrace{\frac{d^2}{dx^2}\left[A e^{\frac{i}{\hbar}S(x)}\right]} + V(x)\left[A e^{\frac{i}{\hbar}S(x)}\right] - E\left[A e^{\frac{i}{\hbar}S(x)}\right] = 0$$

$$A \frac{d}{dx} e^{\frac{i}{\hbar}S(x)} \left[\frac{i}{\hbar}\frac{d}{dx}S(x)\right]$$

$$= A\left[e^{\frac{i}{\hbar}S(x)} \frac{i}{\hbar}\frac{d}{dx}S(x) \frac{i}{\hbar}\frac{d}{dx}S(x) + e^{\frac{i}{\hbar}S(x)} \frac{i}{\hbar}\frac{d^2}{dx^2}S(x)\right]$$

THIS GIVES:

$$-\frac{\hbar^2}{2m}\left(\frac{i}{\hbar}\right)^2\left[\frac{d}{dx}S(x)\right]^2 - \frac{\hbar^2}{2m}\frac{i}{\hbar}\left[\frac{d^2}{dx^2}S(x)\right] + V(x) - E = 0$$

$$\boxed{-\frac{i\hbar}{2m}\frac{d^2 S(x)}{dx^2}} + \frac{1}{2m}\left[\frac{dS(x)}{dx}\right]^2 + V(x) - E = 0 \quad *$$

- NO APPROXIMATION HAS BEEN MADE SO FAR !
- EQUIVALENT TO ORIGINAL SCHRÖDINGER EQUATION

IN FACT, NON-LINEAR EQUATION LOOKS MORE COMPLICATED THAN BEFORE!

SOLVE APPROXIMATELY:

- IF $V(x) = \text{CONST}$ $\Rightarrow$ $S(x) = \pm p_0 x$ AND FIRST TERM VANISHES

- IF $V(x) \neq \text{CONST}$, BUT VARIES SLOWLY $\Rightarrow$ FIRST TERM REMAINS SMALL

- FIRST TERM IS PROPORTIONAL TO $\hbar$ AND HENCE VANISHES IN THE CLASSICAL LIMIT ($\hbar \to 0$)

THEREFORE, EXPAND $S(x)$ INTO POWERS OF $\hbar$:

EXAMINE SOLUTIONS OF ABOVE EQUATION IN LIMIT $\hbar \to \emptyset$

$$S(x) = S_0(x) + \hbar \underbrace{S_1(x)}_{\left[\frac{dS}{d\hbar}\right]_{\hbar=0}} + \hbar^2 \underbrace{\frac{1}{2} S_2(x)}_{\frac{1}{2!}\left[\frac{d^2S}{d\hbar^2}\right]_{\hbar=0}} + \dots$$

[$\hbar$, OF COURSE, IS A NATURAL CONSTANT; SITUATION IS SIMILAR TO PERTURBATION THEORY, WITH λ NOT BEING A PHYSICAL VARIABLE; THESE ARE JUST MATHEMATICAL TOOLS]

SUBSTITUTION INTO LAST EQUATION :

$$-\frac{i\hbar}{2m}\frac{d^2}{dx^2}\left[S_0+\hbar S_1+\frac{\hbar^2}{2}S_2\right]+\frac{1}{2m}\left[\frac{dS_0}{dx}+\hbar\frac{dS_1}{dx}+\frac{\hbar^2}{2}\frac{dS_2}{dx}\right]^2+V(x)-E=0$$

$$\boxed{-\frac{i\hbar}{2m}\frac{d^2S_0}{dx^2}}-\frac{i\hbar}{2m}\hbar\frac{d^2S_1}{dx^2}-\frac{i\hbar}{2m}\frac{\hbar^2}{2}\frac{d^2S_2}{dx^2}$$
$$+\frac{1}{2m}\left[\frac{dS_0}{dx}\right]^2+\boxed{\frac{1}{2m}\hbar\frac{dS_0}{dx}\frac{dS_1}{dx}}+\frac{1}{2m}\frac{\hbar^2}{2}\frac{dS_0}{dx}\frac{dS_2}{dx}$$
$$+\boxed{\frac{1}{2m}\hbar\frac{dS_0}{dx}\frac{dS_1}{dx}}+\frac{1}{2m}\hbar^2\left[\frac{dS_1}{dx}\right]^2+\frac{1}{2m}\hbar\frac{\hbar^2}{2}\frac{dS_1}{dx}\frac{dS_2}{dx}$$
$$+\frac{1}{2m}\frac{\hbar^2}{2}\frac{dS_0}{dx}\frac{dS_2}{dx}+\frac{1}{2m}\hbar\frac{\hbar^2}{2}\frac{dS_1}{dx}\frac{dS_2}{dx}+\frac{1}{2m}\frac{\hbar^4}{4}\left[\frac{dS_2}{dx}\right]^2+V(x)-E=0$$

[EQUATION MUST BE SATISFIED FOR SMALL BUT ARBITRARY VALUES OF $\hbar$: COEFFICIENT OF EACH POWER OF $\hbar$ MUST VANISH SEPARATELY

$a+b\hbar+c\hbar^2=0$]

YIELDS EQUATION FOR EACH POWER OF $\hbar$ SEPARATELY :

[SIMILAR TO PERTURBATION THEORY]

$$\hbar^0 \;:\; \frac{1}{2m}\left[\frac{dS_0(x)}{dx}\right]^2+V(x)-E=0$$

$$\hbar^1 \;:\; \frac{dS_0(x)}{dx}\frac{dS_1(x)}{dx}-\frac{i}{2}\frac{d^2S_0(x)}{dx^2}=0$$

$$\hbar^2 \;:\; \frac{dS_0(x)}{dx}\frac{dS_2(x)}{dx}+\left[\frac{dS_1(x)}{dx}\right]^2-i\frac{d^2S_1(x)}{dx^2}=0$$

$\vdots$

SOLVE COUPLED EQUATIONS SUCCESSIVELY TO FIND $S_0(x)$, $S_1(x)$, $S_2(x)$:

① $$\frac{dS_0(x)}{dx} = \pm \underbrace{\sqrt{2m(E - V(x))}}_{\text{CLASSICAL MOMENTUM}} = \pm\, p(x)$$

$$\int_{x_0}^{x} \frac{dS_0(x')}{dx'}\, dx' = \boxed{\pm \int_{x_0}^{x} p(x')\, dx' = S_0(x) - S_0(x_0)}$$

VARIABLE

ABSORB INTEGRATION CONSTANT INTO NORMALIZATION CONSTANT A

FUNDAMENTAL THEOREM OF CALCULUS:

$$\int_a^b \frac{df(x')}{dx'}\, dx' = f(b) - f(a)$$

② SUBSTITUTE INTO SECOND EQUATION AND INTEGRATE :

$$p(x)\frac{dS_1(x)}{dx} - \frac{i}{2}\frac{d}{dx}p(x) = 0$$

$$\int_{x_0}^{x} \frac{dS_1(x')}{dx'}\, dx' = \frac{i}{2}\int_{x_0}^{x} \frac{\frac{dp(x')}{dx'}}{p(x')}\, dx'$$

$$\int \frac{f'(x')}{f(x')}\, dx' = \ln|f(x)| + C$$

$$\boxed{S_1(x) = \frac{i}{2}\ln|p(x)|}$$

INTEGRATION CONSTANT AGAIN ABSORBED INTO NORMALIZATION

[OR EXPLICITLY, IN TERMS OF INTEGRATION CONSTANT :

$$S_1(x) = \frac{i}{2}\ln\frac{p(x)}{p(x_0)}$$

SO THAT $S_1(x)$ IS UNITLESS]

$p(x) = \sqrt{2m(E - V(x))}$

③ SUBSTITUTE BOTH SOLUTIONS AND INTEGRATE:
[WITHOUT PROOF]

$$S_2(x) = \frac{1}{2} \frac{m}{[p(x)]^3} \frac{dV(x)}{dx} - \frac{1}{4} m^2 \int^x \frac{1}{[p(x')]^5} \left[\frac{dV(x')}{dx'}\right]^2 dx'$$

THIS TERM CAN BE DISREGARDED AS LONG AS:

- $dV(x)/dx$ IS SMALL [i.e., A SLOWLY VARYING POTENTIAL]
- $E - V$ NOT TOO CLOSE TO ZERO

SIMILARLY, IF ALL HIGHER DERIVATIVES, $\frac{d^n V(x)}{dx^n}$, ARE SMALL, HIGHER ORDER TERMS CAN BE DISREGARDED

RETAINING THE FIRST TWO TERMS ONLY, WE OBTAIN THE <u>WKB APPROXIMATION</u> OF THE WAVE FUNCTION:

$$\psi(x) = A\, e^{\frac{i}{\hbar} S(x)} = A\, e^{\frac{i}{\hbar}\left[\pm \int^{x} p(x')\,dx' + \hbar \frac{i}{2} \ln p(x)\right]}$$

$$= A\, \underbrace{e^{-\frac{1}{2}\ln p(x)}}_{e^{\ln p(x)^{-1/2}} \,=\, \frac{1}{\sqrt{p(x)}}}\; e^{\pm \frac{i}{\hbar}\int^{x} p(x')\,dx'}$$

THE GENERAL WKB SOLUTION IS GIVEN BY COMBINATIONS:

$$\boxed{\psi(x) = \underbrace{\frac{A}{\sqrt{p(x)}}\, e^{+\frac{i}{\hbar}\int^{x} p(x')\,dx'}}_{\text{WAVE MOVING TO RIGHT}} + \underbrace{\frac{B}{\sqrt{p(x)}}\, e^{-\frac{i}{\hbar}\int^{x} p(x')\,dx'}}_{\text{WAVE MOVING TO LEFT}} \,,\; E > V}$$

COMPLEX EXPONENTIALS → OSCILLATORY FUNCTIONS

A, B: CONSTANTS

IF $V(x) = V_0 = \text{CONST}$, WAVES BECOME PLANE WAVES WITH $p(x) = p_0$ [INTEGRAL BECOMES $p_0 \cdot x$]

NOTE: AMPLITUDES ARE PROPORTIONAL TO $1/\sqrt{p(x)}$, OR, PROBABILITY OF FINDING PARTICLE BETWEEN <u>x AND $x+dx$ IS PROPORTIONAL TO $1/p(x)$</u>; THIS IS EXACTLY WHAT YOU EXPECT FOR A CLASSICAL PARTICLE [TIME TO TRAVEL DISTANCE dx PROPORTIONAL TO INVERSE OF ITS SPEED]

"SEMI-CLASSICAL APPROXIMATION"

$p(x)$: CLASSICAL MOMENTUM

CRITERION FOR VALIDITY OF WKB APPROXIMATION:

$$\Psi(x) = A e^{\frac{i}{\hbar} S(x)}$$

THIRD TERM IN S(X) EXPANSION SHOULD BE SMALL

$$\overbrace{\frac{1}{\hbar}\left[\ldots + \frac{\hbar^2}{2} S_2(x)\right]}^{\text{EXPONENTIAL}} \rightarrow \left|\frac{\hbar}{2} S_2(x)\right| \ll 1$$

$$S_2(x) = \underbrace{\frac{1}{2}\frac{m}{[p(x)]^3}\frac{dV(x)}{dx}} - \underbrace{\frac{1}{4} m^2 \int^x \ldots}$$

SUPPOSE BOTH TERMS ARE OF SAME ORDER OF MAGNITUDE

$$\approx 2\,\frac{1}{2}\frac{m}{[p(x)]^3}\frac{dV(x)}{dx}$$

$$\Rightarrow \left|\frac{\hbar}{2} S_2(x)\right| = \left|\frac{\hbar}{2}\frac{m}{[p(x)]^3}\frac{dV(x)}{dx}\right| = \boxed{\left|\frac{\hbar}{2}\frac{m\frac{dV(x)}{dx}}{[2m(E-V(x))]^{3/2}}\right| \ll 1}$$

- V(x) MUST VARY SLOWLY
- KINETIC ENERGY [E - V(x)] NOT BE TOO SMALL

WHAT ABOUT CLASSICALLY FORBIDDEN REGION, $E < V(x)$?

$$p(x) = \sqrt{2m\underbrace{(E - V(x))}_{\text{NEGATIVE}}} = \sqrt{-i^2 2m(E - V(x))} = i\sqrt{\underbrace{-2m(E - V(x))}_{\text{POSITIVE, REAL}}}$$

$\Rightarrow$ $p(x)$ BECOMES PURELY IMAGINARY

$|z| = \sqrt{z^* \cdot z}$

REPEAT OF ABOVE DERIVATION, BUT FOR $E < V(x)$, GIVES:

$$\psi(x) = A \frac{1}{|p(x)|^{1/2}} e^{\pm \frac{1}{\hbar}\int^x |p(x')| dx'}$$

GENERAL WKB APPROXIMATION:

$$\boxed{\psi(x) = \frac{C}{|p(x)|^{1/2}} e^{-\frac{1}{\hbar}\int^x |p(x')| dx'} + \frac{D}{|p(x)|^{1/2}} e^{+\frac{1}{\hbar}\int^x |p(x')| dx'}, \quad E < V}$$

REAL EXPONENTIALS

C, D: CONSTANTS

AGAIN, APPROXIMATION IS ACCURATE AS LONG AS:

- $V(x)$ VARIES SLOWLY
- $[V-E]$ IS NOT TOO SMALL

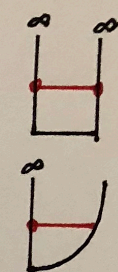
$$\lambda = \frac{h}{\sqrt{2m(E - V(x))}}$$

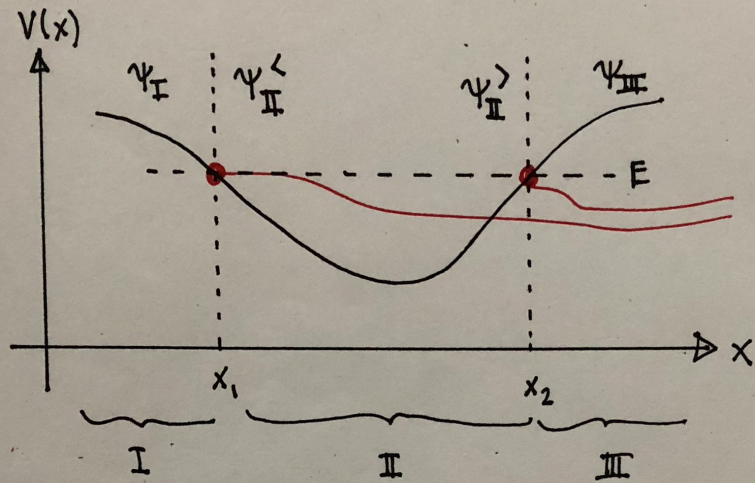
WKB APPROXIMATION BREAKS DOWN NEAR TURNING POINTS:

- λ BECOMES LARGE $\rightarrow$ $V(x)$ DOES NOT VARY SLOWLY ANYMORE OVER λ
- AT TURNING POINTS: $E = V(x)$, $p(x) = 0$, AND EXPRESSIONS DIVERGE

MATCHING AT THE TURNING POINTS TO FIND THE COMPLETE SOLUTION : CONNECTION FORMULAS

CONSIDER FIRST BOUND STATES, WITH NO RIGID WALLS





"CLASSICAL TURNING POINTS"

[CLASSICAL PARTICLE MUST RETURN]

SOLUTION IN REGION I [AWAY FROM TURNING POINT x_1]:

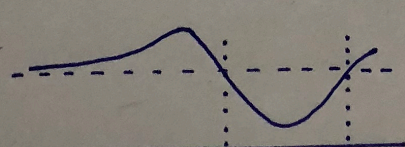
$$\psi_I(x) = \frac{C}{\sqrt{|p(x)|}}\, e^{\overbrace{-\frac{1}{\hbar}\underbrace{\int_{x_1}^{x}|p(x')|dx'}_{\text{NEGATIVE}}}^{}} + \frac{D}{\sqrt{|p(x)|}}\, e^{+\frac{1}{\hbar}\underbrace{\int_{x_1}^{x}|p(x')|dx'}_{\text{NEGATIVE}}}$$

$$e^{+\frac{1}{\hbar}\int_{x}^{x_1}|p(x')|dx'} \quad \text{DIVERGES FOR } x \to -\infty$$

$$= \frac{D}{\sqrt{|p(x)|}}\, e^{+\frac{1}{\hbar}\int_{x_1}^{x}|p(x')|dx'}$$

$$|p(x)| = \sqrt{-2m(E - V(x))}$$

WHAT IF $E \approx V(x)$ FOR $x \to -\infty$:



THEN $\psi \sim \frac{1}{\sqrt{|p(x)|}}$ WILL DIVERGE AND IS NOT NORMALIZABLE

SOLUTION IN REGION III [AWAY FROM TURNING POINT x_2]:

$$\psi_{III}(x) = \frac{C}{\sqrt{|p(x)|}} e^{-\frac{1}{\hbar}\int_{x_2}^{x} |p(x')| dx'} + \underbrace{\frac{D}{\sqrt{|p(x)|}} e^{+\frac{1}{\hbar}\int_{x_2}^{x} |p(x')| dx'}}_{\text{DIVERGES FOR } x \to \infty}$$

$$= \frac{C}{\sqrt{|p(x)|}} e^{-\frac{1}{\hbar}\int_{x_2}^{x} |p(x')| dx'}$$

SOLUTION IN REGION II [AWAY FROM TURNING POINTS]:

$$\psi_{II}(x) = \frac{A}{\sqrt{p(x)}} e^{\frac{i}{\hbar}\int^{x} p(x') dx'} + \frac{B}{\sqrt{p(x)}} e^{-\frac{i}{\hbar}\int^{x} p(x') dx'}$$

$$= \frac{A'}{\sqrt{p(x)}} \sin\left[\frac{1}{\hbar}\int^{x} p(x') dx' + \delta\right]$$

PHASE

$\int_x \ldots$

$$A e^{i\beta z} + B e^{-i\beta z}$$

$$= C_1 \cos(\beta z) + C_2 \sin(\beta z)$$

$$= C' \sin(\beta z + \delta)$$

$\int^x$ CAN BE WRITTEN AS $\int_{x_1}^{x}$ OR $\int_{x_2}^{x}$; IT IS OF ADVANTAGE TO EXPRESS THE SOLUTIONS AS FOLLOWS, DEPENDING ON WHICH TURNING POINT IS APPROACHED:

$$\psi_{II}^{<}(x) = \frac{A'}{\sqrt{p(x)}} \sin\left[\frac{1}{\hbar}\int_{x_1}^{x} p(x') dx' + \delta\right]$$

$$\psi_{II}^{>}(x) = \frac{A''}{\sqrt{p(x)}} \sin\left[\frac{1}{\hbar}\int_{x}^{x_2} p(x') dx' + \delta\right]$$

THE CHANGE $\int_{x_2}^{x} \to \int_{x}^{x_2}$ SIMPLY CORRESPONDS TO A FLIP IN SIGN OF COMPLEX EXPONENTIALS IN EULER'S EQUATIONS

- REGIONS OF VALIDITY FOR THESE FORMS OF THE WAVE FUNCTION ARE SEPARATED BY CLASSICAL TURNING POINTS, NEAR WHICH APPROXIMATION FAILS

- ψ_I, $\psi_{II}^{<}$, $\psi_{II}^{>}$, ψ_{III} ARE APPROXIMATIONS TO THE SAME FUNCTION ψ

  $\Rightarrow$ COEFFICIENTS D, C, A', A'', δ CANNOT BE ARBITRARY

- WE ARE FACING A SITUATION, WHERE WE NEED TO MATCH THE VARIOUS SOLUTIONS AT THE BOUNDARIES [TURNING POINTS], ALTHOUGH THE SOLUTIONS DIVERGE AT THESE LOCATIONS!

  [THIS IS DIFFERENT FROM PREVIOUS BOUNDARY MATCHING AND REPRESENTS THE MOST DIFFICULT ASPECT OF THE WKB METHOD]

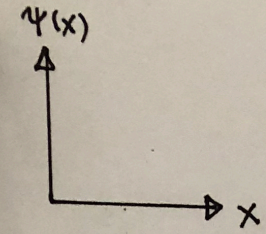
STRATEGY:

(i) APPROXIMATE $V(x)$ BY LINEAR FUNCTION AT TURNING POINTS
  $\rightarrow$ SCHRÖDINGER EQUATION CAN THEN BE SOLVED EXACTLY AT AND NEAR THESE POINTS

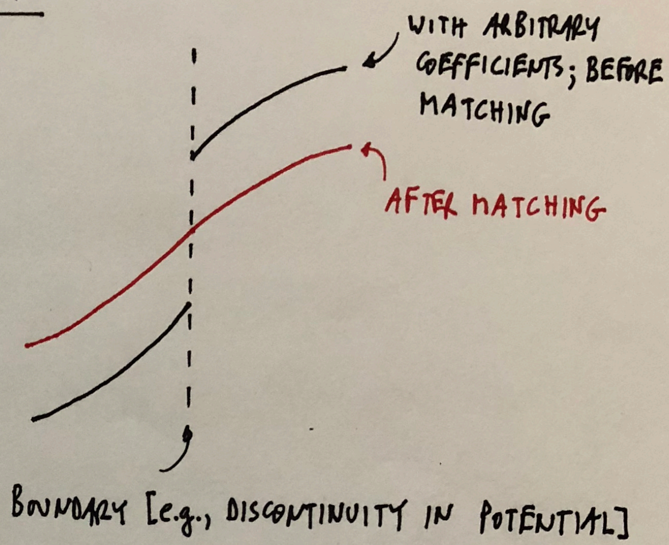
(ii) USE THIS EXACT SOLUTION NEAR TURNING POINTS AS A GUIDE TO FIND UNKNOWN COEFFICIENTS OF WKB WAVE FUNCTIONS

  [INSTEAD OF APPLYING CONTINUITY CONDITION - MATCHING OF WAVE FUNCTIONS AT A SINGLE POINT - WE MATCH THE WKB SOLUTIONS TO A WAVEFUNCTION THAT BEHAVES CORRECTLY ACROSS TURNING POINTS]

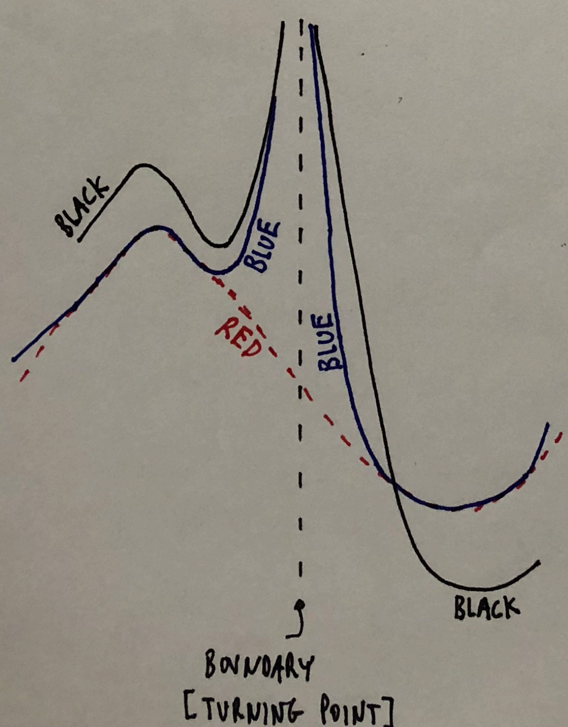
SKETCH:



PREVIOUSLY:



WKB METHOD:



BLACK —— APPROXIMATE [WKB] SOLUTIONS FOR EXACT POTENTIAL [ARB. COEFFICIENTS]

RED - - - - EXACT SOLUTION FOR LINEAR POTENTIAL APPROXIMATION

BLUE —— AFTER MATCHING

MATCHING MUST BE PERFORMED AWAY FROM TURNING POINT [WHERE WKB SOLUTION DIVERGES], BUT NOT TOO FAR AWAY FROM IT [BECAUSE LINEAR POTENTIAL APPROXIMATION BECOMES INACCURATE]

MATCHING OF WKB SOLUTIONS AT TURNING POINTS : CONNECTION FORMULAS

HANDOUT DISCUSSES THE FOLLOWING:

(i) EXACT SOLUTIONS FOR APPROXIMATE LINEAR POTENTIAL

(ii) WKB SOLUTIONS FOR APPROXIMATE LINEAR POTENTIAL

(iii) MATCHING OF SOLUTIONS

WE START WITH EXACT SOLUTIONS FOR APPROXIMATE LINEAR POTENTIAL

APPROXIMATE $V(x)$ AT TURNING POINTS BY A STRAIGHT LINE WITH A SLOPE EQUAL TO THE SLOPE OF $V(x)$ AT x_1 OR x_2 [i.e., FIRST-ORDER EXPANSION]

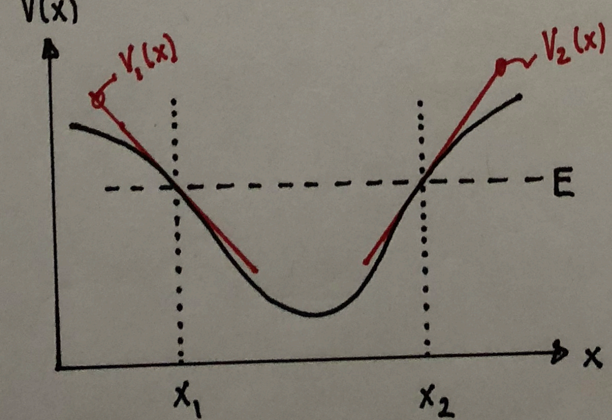
NEG. SLOPE

$$V(x) \approx \underbrace{V(x_1)}_{E} + (x-x_1)\underbrace{\left[\frac{dV(x)}{dx}\right]_{x_1}}_{\equiv -F_1} = E - (x-x_1)F_1$$

POSITIVE NUMBERS

POS. SLOPE

$$V(x) \approx V(x_2) + (x-x_2)\underbrace{\left[\frac{dV(x)}{dx}\right]_{x_2}}_{\equiv F_2} = E + (x-x_2)F_2$$



EXACT SOLUTION $\phi(x)$ OF SCHRÖDINGER EQUATION:

$$-\frac{\hbar^2}{2m}\frac{d^2\phi(x)}{dx^2} + V(x)\,\phi(x) - E\,\phi(x) = 0$$

$$\frac{d^2\phi(x)}{dx^2} - \frac{2m}{\hbar^2}\left[V(x) - E\right]\phi(x) = 0$$

$$\begin{cases} \dfrac{d^2\phi(x)}{dx^2} + \dfrac{2mF_1}{\hbar^2}(x - x_1)\,\phi(x) = 0 \,, & x \approx x_1 \\[2ex] \dfrac{d^2\phi(x)}{dx^2} - \dfrac{2mF_2}{\hbar^2}(x - x_2)\,\phi(x) = 0 \,, & x \approx x_2 \end{cases}$$

SIMPLIFY BY CHANGING VARIABLE:

$$y \equiv -\left(\frac{2mF_1}{\hbar^2}\right)^{1/3}(x - x_1) \qquad \text{AND} \qquad y \equiv +\left(\frac{2mF_2}{\hbar^2}\right)^{1/3}(x - x_2)$$

NEAR TURNING POINT x_1, WE FIND:

$$\frac{dy}{dx} = -\left(\frac{2mF_1}{\hbar^2}\right)^{1/3} \quad \rightarrow$$

$$\frac{d^2\phi(y)}{dy^2}\left(\frac{2mF_1}{\hbar^2}\right)^{2/3} \underbrace{-\left(\frac{2mF_1}{\hbar^2}\right)^{1/3}}_{} (-1)\left(\frac{2mF_1}{\hbar^2}\right)^{2/3} \underbrace{(x - x_1)}_{}\,\phi(y) = 0$$

y

$$\left(\frac{2mF_1}{\hbar^2}\right)^{2/3}\left[\frac{d^2\phi(y)}{dy^2} - y\,\phi(y)\right] = 0$$

SIMILAR EXPRESSION FOR TURNING POINT x_2

FOR BOTH TURNING POINTS, SCHRÖDINGER EQUATION REDUCES TO

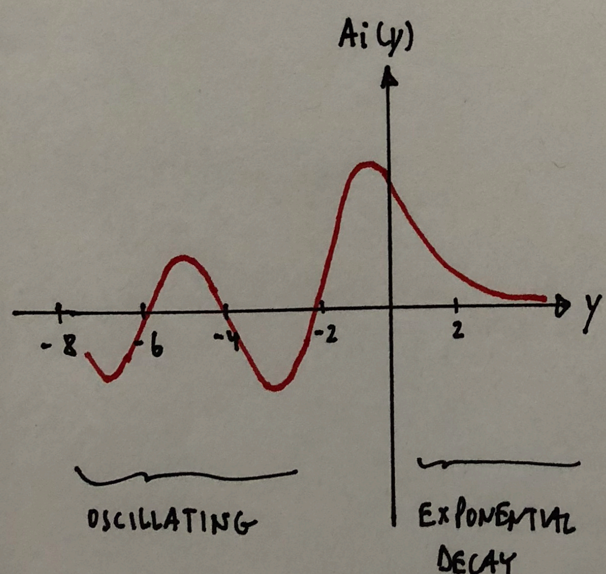
$$\boxed{\frac{d^2\phi}{dy^2} - y\phi = 0 \qquad \text{AIRY'S EQUATION}}$$

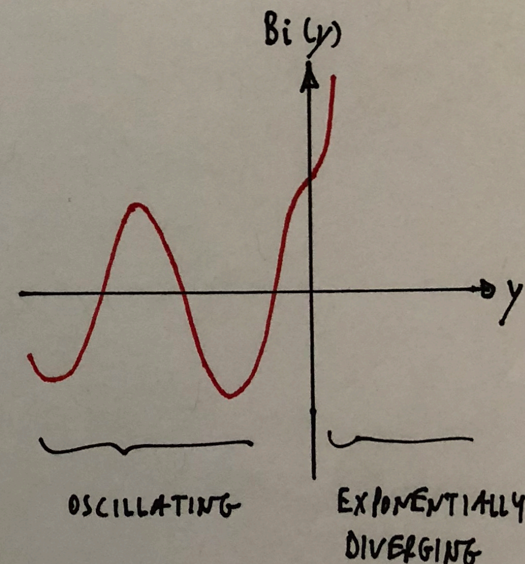
PROPERTIES AND SOLUTIONS OF THIS EQUATION ARE WELL KNOWN

"AIRY FUNCTIONS" :

$$Ai(y) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{s^3}{3} + sy\right) ds$$

$$Bi(y) \equiv \frac{1}{\pi}\int_0^\infty \left[e^{sy - \frac{1}{3}s^3} + \sin\left(\frac{s^3}{3} + sy\right)\right] ds$$





[SOME PROPERTIES OF THE AIRY FUNCTIONS ARE LISTED ON PAGE 257 IN GRIFFITHS]

YOU CAN ALREADY SEE THAT THE SOLUTION $Ai(y)$ IS APPROPRIATE FOR THE PROBLEM AT HAND:

- IT OSCILLATES ON ONE SIDE AND DECAYS EXPONENTIALLY ON THE OTHER SIDE [UNLIKE $Bi(y)$]

- THE ASYMPTOTIC FORMS OF $Ai(y)$ ARE GIVEN BY:

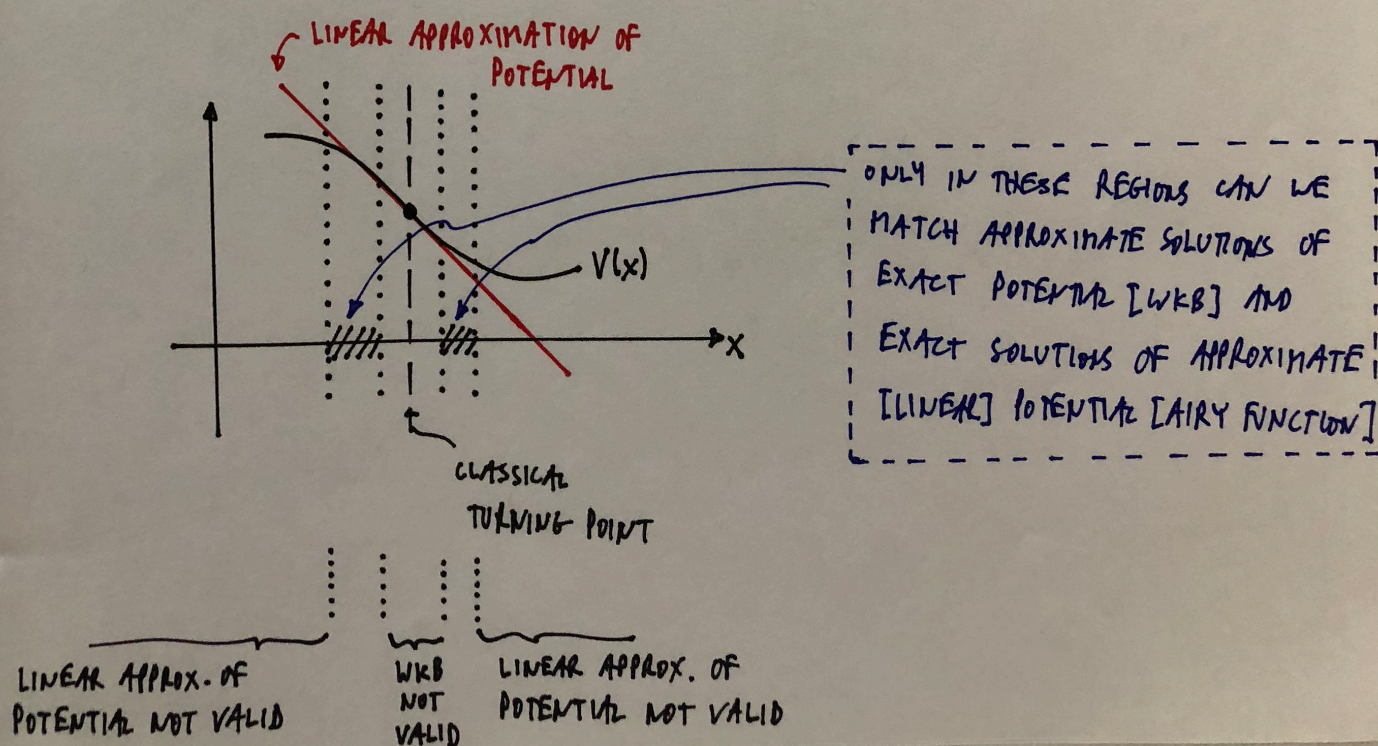
$$Ai(y) \longrightarrow \frac{1}{2\sqrt{\pi}\, y^{1/4}} e^{-\frac{2}{3}y^{3/2}} \quad , \quad y \gg 0$$

EXPONENTIAL DECAY IN CLASSICALLY FORBIDDEN REGION

$$Ai(y) \longrightarrow \frac{1}{\sqrt{\pi}\,(-y)^{1/4}} \sin\left[\frac{2}{3}(-y)^{3/2} + \frac{\pi}{4}\right] , \quad y \ll 0$$

OSCILLATING IN CLASSICALLY ALLOWED REGION

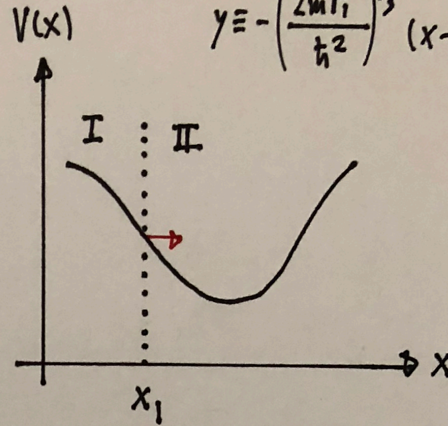
USEFUL MENTAL IMAGE:



CONSIDER NOW WKB SOLUTIONS FOR APPROXIMATE [LINEAR] POTENTIAL

FIRST, RIGHT SIDE OF x_1 [CLASSICALLY ALLOWED]

$$y \equiv -\left(\frac{2mF_1}{\hbar^2}\right)^{1/3}(x-x_1)$$



$$\frac{dy}{dx} = -\left(\frac{2mF_1}{\hbar^2}\right)^{1/3}$$

$$p^2 = 2m[E - V(x)] \approx 2m[E - E + \overset{\text{POS. NUMBER}}{F_1}(x-x_1)]$$

$$= 2mF_1(x-x_1) = -\hbar^2\left(-\frac{2mF_1}{\hbar^2}\right)(x-x_1)$$

$$= -\hbar^2(-1)\left(\frac{2mF_1}{\hbar^2}\right)^{2/3}\underbrace{\left(\frac{2mF_1}{\hbar^2}\right)^{1/3}(x-x_1)}_{\equiv -y}$$

$$= -\hbar^2\left(\frac{2mF_1}{\hbar^2}\right)^{2/3} y = -(2mF_1\hbar)^{2/3} y$$

WE FIND FOR ARGUMENT OF SIN(...) IN $\psi^<_{II}(x)$:

$$\frac{1}{\hbar}\int_{x_1}^{x} p(x')\,dx' = \frac{1}{\hbar}\int_0^{y \leftarrow \text{NEG.}} (2mF_1\hbar)^{1/3}(-y')^{1/2}(-1)\left(\frac{\hbar^2}{2mF_1}\right)^{1/3} dy' = -\frac{1}{\hbar}\int_0^y \hbar^{1/3}\,\hbar^{2/3}(-y')^{1/2}\,dy'$$

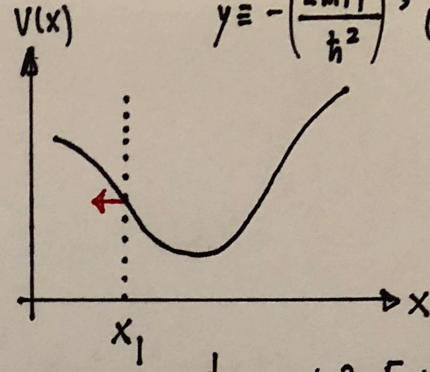
$$= -\int_0^y (-y')^{1/2}\,dy' = +\frac{2}{3}\left[(-y')^{3/2}\right]_0^y = +\frac{2}{3}\left[(-y)^{3/2} - 0\right]$$

$$\tfrac{2}{3}\,\tfrac{3}{2}\,(-y')^{1/2}(-1)$$

$$= +\frac{2}{3}(-y)^{3/2}, \quad y < 0$$

$y \equiv -\left(\frac{2mF_1}{\hbar^2}\right)^{1/3}(x-x_1)$

SECOND, LEFT SIDE OF x_1 [CLASSICALLY FORBIDDEN]



$$p(x) = i\underbrace{\sqrt{-2m[E-V(x)]}}_{\text{POSITIVE, REAL}}$$ [SEE EARLIER]

$$\frac{dy}{dx} = -\left(\frac{2mF_1}{\hbar^2}\right)^{1/3}$$

$$|p(x)|^2 = \left[i\sqrt{-2m[E-V(x)]}\right]\left[-i\sqrt{-2m[E-V(x)]}\right]$$

$$= -2m[E-V(x)] \approx -2m[E-E+F_1(x-x_1)]$$

$$= -2mF_1(x-x_1) = (2mF_1\hbar)^{2/3}\,y$$

SEE LAST PAGE

WE FIND FOR ARGUMENT IN EXPONENTIAL OF $\psi_I(x)$:

$$\frac{1}{\hbar}\int_{x_1}^{x}|p(x')|\,dx' = \frac{1}{\hbar}\int_0^{y}(2mF_1\hbar)^{1/3}(y')^{1/2}(-1)\left(\frac{\hbar^2}{2mF_1}\right)^{1/3}dy'$$

$\leftarrow$ POS.

$$= -\frac{1}{\hbar}\int_0^{y}\hbar^{1/3}\hbar^{2/3}(y')^{1/2}\,dy' = -\int_0^{y}(y')^{1/2}\,dy'$$

$$= -\frac{2}{3}\left[(y')^{3/2}\right]_0^{y} = -\frac{2}{3}y^{3/2}\ ,\quad y>0$$

LET US COLLECT RESULTS SO FAR FOR MATCHING NEAR TURNING POINT x_1

$$\phi(y) = G \; Ai(y) \longrightarrow \begin{cases} \dfrac{G}{2\sqrt{\pi}} \boxed{\dfrac{1}{y^{1/4}} e^{-\frac{2}{3}y^{3/2}}}, & y \gg 0 \\ \dfrac{G}{\sqrt{\pi}} \boxed{\dfrac{1}{(-y)^{1/4}} \sin\left[\frac{2}{3}(-y)^{3/2} + \frac{\pi}{4}\right]}, & y \ll 0 \end{cases}$$

NORMALIZATION

$$\psi_I(x) = \frac{D}{\sqrt{|p|}} e^{+\frac{1}{\hbar}\int_{x_1}^{x} |p(x')| dx'} = \frac{D}{(2mF_1\hbar)^{1/6}} \boxed{\frac{1}{y^{1/4}} e^{-\frac{2}{3}y^{3/2}}}, \; y > 0$$

$$|p(x)| = (2mF_1\hbar)^{1/3}\sqrt{y}$$

$$\psi_{II}^{<}(x) = \frac{A'}{\sqrt{p}} \sin\left[\frac{1}{\hbar}\int_{x_1}^{x} p(x')\,dx' + \delta\right] = \frac{A'}{(2mF_1\hbar)^{1/6}} \boxed{\frac{1}{(-y)^{1/4}} \sin\left[\frac{2}{3}(-y)^{3/2} + \delta\right]},$$

$y < 0$

$$p(x) = (2mF_1\hbar)^{1/3}\sqrt{-y}$$

MATCHING OF WKB SOLUTIONS ON BOTH SIDES OF x_1 TO ASYMPTOTIC FORM OF AIRY FUNCTION YIELDS:

$$\delta = \frac{\pi}{4}$$

$$D = \frac{A'}{2} \Rightarrow A' = 2D$$

IF WE ABSORB D INTO OVERALL NORMALIZATION OF COMPLETE WAVE FUNCTION: $D = 1$ AND $A' = 2$

REPEATING THE LENGTHY PROCEDURE FOR TURNING POINT x_2 AND MATCHING $\psi_{II}^{>}$ AND ψ_{III} TO ASYMPTOTIC FORM OF AIRY FUNCTION YIELDS :

$$A'' = 2G$$

SUMMARY SO FAR FOR COEFFICIENTS :

$$\underset{\substack{\downarrow \\ \text{OVERALL NORMALIZATION}}}{D}\ ,\ C\ ,\ \underset{\substack{\| \\ 2}}{A'}\ ,\ \underset{\substack{\| \\ 2G}}{A''}\ ,\ \underset{\substack{\| \\ \frac{\pi}{4}}}{\delta}$$

WHY IS THE ASYMPTOTIC FORM OF THE AIRY FUNCTION VALID RELATIVELY CLOSE TO TURNING POINT ?

$$y \equiv -\left(\frac{2mF_1}{\hbar^2}\right)^{1/3}(x-x_1)$$

THE ASYMPTOTIC FORM OF $Ai(y)$ IS ACCURATE TO 1% AS LONG AS $y \gtrsim 5$

CONSIDER, AS AN EXAMPLE, THE NUCLEAR CASE :

$mc^2 \simeq 931$ MeV

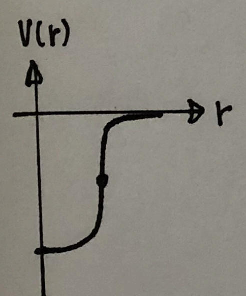
$\hbar c = 197$ MeV·fm

SUPPOSE : $F_1 \simeq 10$ MeV/fm

[DIFFUSENESS : $a \simeq 0.65$ fm]

$$|x-x_1| = \frac{\overset{y \simeq 5}{y}}{\left(\frac{2mc^2F_1}{\hbar^2c^2}\right)^{1/3}} \simeq 6 \text{ fm}$$

[MOTION OF ONE NUCLEON IN NUCLEUS]



BEYOND THIS DISTANCE OF 6 fm, $y \gtrsim 5$, AND $Ai(y)$ IS ACCURATE TO 1%

QUESTION : IS THE LINEAR APPROXIMATION OF THE POTENTIAL STILL VALID FOR $|x-x_1| \gtrsim 6$ fm ?

[SEE PROB. 8.8 IN GRIFFITHS]