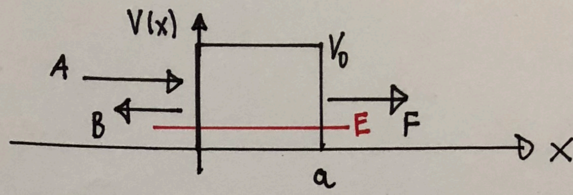
# TRANSMISSION THROUGH POTENTIAL BARRIERS

RECALL THE SIMPLEST CASE, A RECTANGULAR BARRIER :




EXACT SOLUTION YIELDS : [PROB. 2.33, GRIFFITHS]

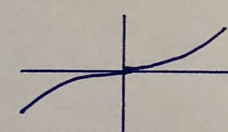
TRANSMISSION COEFFICIENT

$$T \equiv \frac{|F|^2}{|A|^2} = \frac{1}{1 + \frac{V_0^2 \sinh^2(\kappa a)}{4E(V_0 - E)}} \quad , \text{ WHERE } \kappa = \frac{\sqrt{2m(V_0-E)}}{\hbar}$$

$|F|^2$ — INTENSITY OF TRANSMITTED WAVE

$|A|^2$ — INTENSITY OF INCIDENT WAVE

$\sinh z \equiv \frac{1}{2}(e^{z} - e^{-z})$



FOR $\kappa a \gg 1$ [LIMIT OF THICK BARRIER / LOW ENERGY] :

$$T \approx \underbrace{\frac{16E(V_0-E)}{V_0^2}}_{\text{PREFACTOR, VARIES SLOWLY WITH ENERGY}} \underbrace{e^{-\frac{2}{\hbar} a \sqrt{2m(V_0-E)}}}_{\text{DOMINATES ENERGY DEPENDENCE}}$$

WHAT IS THE BASIC REASON FOR QUANTUM TUNNELING ?

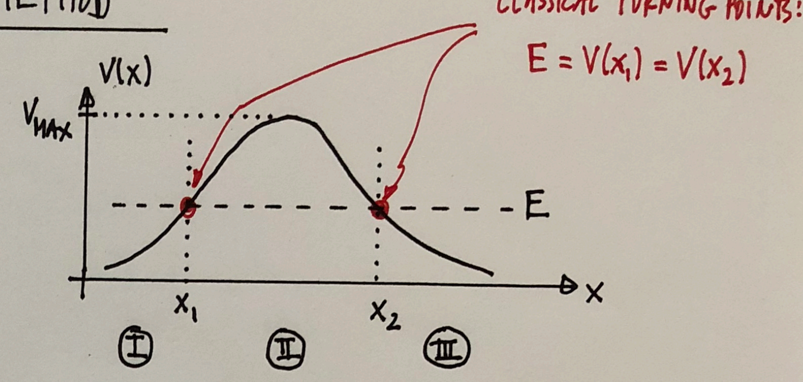
$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

[IF $\psi$ SUDDENLY FALLS TO ZERO AT THE BOUNDARY, $\frac{d\psi}{dx} \to \infty$ ]

IF BARRIER IS NOT SQUARE: APPROXIMATE METHODS MUST BE USED TO CALCULATE TRANSMISSION

## WKB IS THE MOST POPULAR METHOD

GENERAL STATEMENT OF PROBLEM




E.G., BEAM OF INCIDENT PARTICLES FROM LEFT WITH $E < V_{MAX}$:
SOME PARTICLES ARE TRANSMITTED INTO REGION III, SOME ARE REFLECTED IN REGION I

PROCEEDING IN THE SAME MANNER AS BEFORE [WKB WAVEFUNCTIONS, CONNECTION FORMULAS (LINEAR POTENTIAL APPROX.), ETC.], WE FIND FOR THE TRANSMISSION COEFFICIENT:

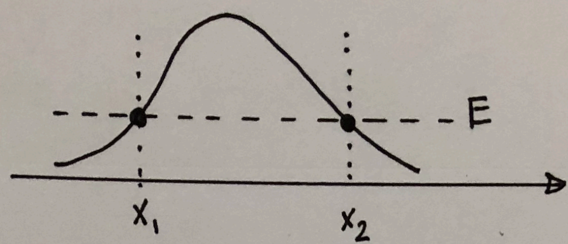
$$\boxed{T^{WKB} = e^{-\frac{2}{\hbar}\int_{x_1}^{x_2} \underbrace{\sqrt{2m[V(x)-E]}}_{p(x)}\, dx}}$$

FOR A DERIVATION, SEE, E.G., MERZBACHER, QUANTUM MECHANICS [GRADUATE-LEVEL TEXT]

COMPARISON TO EXACT EXPRESSION SHOWS THAT WKB EXPRESSION IS THE NATURAL EXTENSION TO BARRIER WITH VARIABLE HEIGHT

[p. 126 IN MERZBACHER: DEFINITION OF TRANSMISSION COEFFICIENT FOR WKB WAVE FUNCTIONS YIELDS INDEED $T = |F|^2/|A|^2$ IF POTENTIAL ON LEFT AND RIGHT OF BARRIER ARE NOT AT THE SAME LEVEL; $1/\sqrt{p(x)}$ IN WKB EXPRESSIONS CANCELS WAVE NUMBER RATIOS]

# SITUATIONS WHERE WKB APPROXIMATION IS SURELY APPLICABLE




- POTENTIAL IS "SLOWLY VARYING" [SEE EARLIER]

- TURNING POINTS MUST BE SUFFICIENTLY WELL SEPARATED [THICK BARRIER; SEE BELOW]

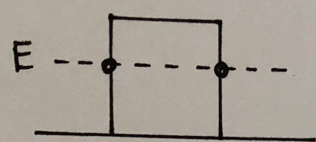
IN THIS CASE, THE ABSOLUTE VALUE OF THE EXPONENT IN $T^{WKB}$ BECOMES LARGE

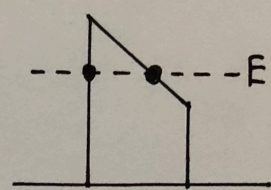
$\Rightarrow$ $T^{WKB}$ BECOMES VERY SMALL

YOU SHOULD BE SUSPICIOUS IF, E.G., $T^{WKB} \gtrsim 0.5$ !

# SITUATIONS WHERE YOU SHOULD BE CAREFUL WHEN APPLYING THE WKB METHOD

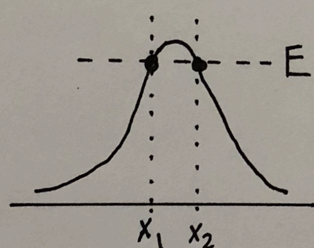
## (i) RECTANGULAR OR TRAPEZOIDAL BARRIERS




SLOPE OF POTENTIAL IS INFINITE AT ONE OR TWO TURNING POINTS AND THUS CANNOT BE APPROXIMATED BY LINEAR POTENTIAL

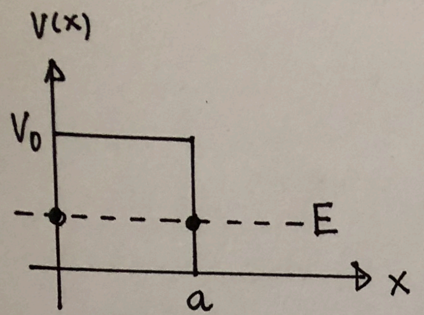
## (ii) THIN BARRIERS OR TURNING POINTS NEAR TOP OF BARRIER




IF TURNING POINTS ARE NOT SUFFICIENTLY WELL SEPARATED: WKB WAVEFUNCTIONS ARE NOT VALID NEAR TURNING POINTS, AND THERE IS NO GOOD OVERLAP REGION FOR MATCHING WKB SOLUTIONS TO ASYMPTOTIC FORMS OF AIRY FUNCTIONS

$\Rightarrow$ $\boxed{\text{THE WKB APPROXIMATION IS POOR OVER A TOO LARGE REGION}}$

NEVERTHELESS, LET'S CONSIDER AN EXTREME CASE:




RECTANGULAR BARRIER [$V(x)$ = CONST]

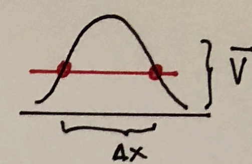
$$T^{WKB} = e^{-\frac{2}{\hbar}\int_{x_1}^{x_2}\sqrt{2m[V(x)-E]}\,dx} = \boxed{e^{-\frac{2}{\hbar}a\sqrt{2m(V_0-E)}}}$$

$$T^{EXACT}_{E\ll V_0} \cong \underbrace{\frac{16E(V_0-E)}{V_0^2}}_{\text{PREFACTOR}}\ \boxed{e^{-\frac{2}{\hbar}a\sqrt{2m(V_0-E)}}}$$

- ALTHOUGH THE WKB APPROXIMATION BREAKS DOWN AT THE TURNING POINTS, THE FUNCTIONAL FORM [EXPONENTIAL] OF EXACT TUNNELING PROBABILITY IS PRESERVED [AS LONG AS NEGATIVE EXPONENT IS LARGE, i.e., $>4$]

- EXPRESSIONS ONLY DIFFER BY A "PREFACTOR", WHICH ACCOUNTS FOR THE CORRECT CONTRIBUTIONS NEAR TURNING POINTS TO THE TOTAL TRANSMISSION COEFFICIENT

## WHAT ABOUT AN ARBITRARY-SHAPE BARRIER ?




(i) THE FUNCTIONAL FORM OF THE EXACT TUNNELING PROBABILITY THROUGH AN ARBITRARY BARRIER IS PRESERVED IN THE WKB EXPRESSION FOR CERTAIN CONDITIONS;

E.G., FOR ELECTRONS IN SOLIDS :

$$\Delta x \sqrt{\bar{V}} > 4$$

GUNDLACH AND SIMMONS, THIN SOLID FILMS 4, 61 (1969)

DISTANCE BETWEEN TURNING POINTS IN Å

MEAN BARRIER HEIGHT IN eV

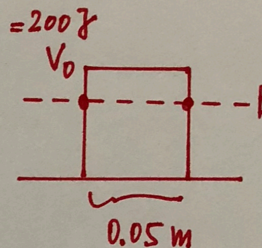
(ii) UNDER SUCH CONDITIONS, EXACT AND WKB EXPRESSIONS ONLY DIFFER BY A PREFACTOR, WHICH IS RELATED TO THE SLOPE OF THE POTENTIAL AT TURNING POINTS [THE LARGER THE SLOPE, THE LARGER THE PREFACTOR]

- IN PRACTICAL CASES, PREFACTOR IS IN THE RANGE 1 - 10
- FOR RELATIVELY SLOWLY VARYING POTENTIALS, PREFACTOR $\approx 1$

## EXAMPLE: QUANTUM TUNNELING IN XMEN

SUPPOSE THE STUDENT $(m = 60\,kg)$ WALKS TOWARDS DOOR WITH A SPEED OF $1\,m/s$. THE DOOR HAS A THICKNESS OF 5 cm. IT TAKES 200 J OF ENERGY TO BREAK THROUGH THE DOOR. CALCULATE THE TUNNELING PROBABILITY USING THE WKB APPROXIMATION

$[\hbar = 1.054 \cdot 10^{-34}\,m^2 kg/s]$




$$T^{WKB} = e^{-\frac{2}{\hbar} a \sqrt{2m}\sqrt{V_0 - E}}$$

$$V_0 - E = 200\,J - \underbrace{\frac{1}{2}\underset{\overset{\parallel}{60\,kg}}{m}\,\overset{\overset{1\,m/s}{\parallel}}{v}{}^2}_{E = 30\,J} = 170\,J$$

$$\sqrt{2m\,(V_0 - E)} = \sqrt{2 \cdot (60\,kg)(170\,kg\,m^2/s^2)} = 142.8\,kg\,m/s$$

$$T^{WKB} = e^{-\frac{2}{1.054 \cdot 10^{-34}\,m^2 kg/s}(0.05\,m)(142.8\,kg\,m/s)} = e^{-1.35 \cdot 10^{35}}$$

$$\approx 10^{-6 \cdot 10^{34}} \approx 10^{-10^{35}}$$

[ABOUT $5 \cdot 10^{17}$ s HAVE PASSED SINCE THE UNIVERSE WAS BORN]

Alexander Litvinenko [former detective and critic of the Russian government]

Fell ill on November 1, 2006, and died on November 23.

Radiation poisoned with $^{210}Po$

$^{210}Po$ facts:

1 g of $^{210}Po$ sufficient to kill 20 million people
10 mg $^{210}Po$ sufficient to kill 200,000 people [grain of sand]

natural occurence:
100 $\mu g$ per ton of Uranium ore
extraction of $^{210}Po$ from U ore is impractical

artificial production:
neutron irradiation of Bismuth in nuclear reactors
every month, 8 g of $^{210}Po$ are shipped from Russia to the U.S.
$^{210}Po/Be$ is a neutron source for nuclear weapons
used in batteries of spacecraft

# ENERGETICS

<u>EXPERIMENT</u>

"MASS DEFECT"

$$m_{nuc} = Z m_p + N m_n - \Delta m$$

[MANIFESTATION OF EQUIVALENCE OF MASS AND ENERGY]

MULTIPLY BY $c^2$ AND REARRANGE :

$$B(Z,N) \equiv \Delta m c^2 = [Z m_p + N m_n - m_{nuc}] c^2$$

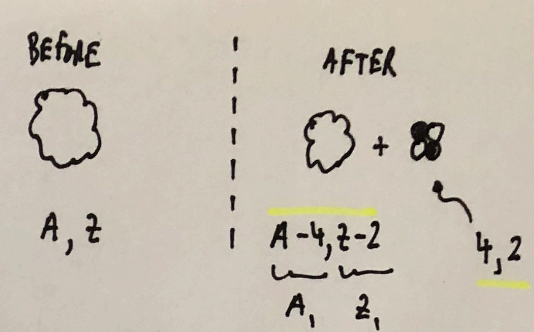
"BINDING ENERGY" : - ENERGY RELEASED IN ASSEMBLING NUCLEUS FROM FREE NUCLEONS [NUCLEAR FORCE IS ATTRACTIVE]

- ENERGY REQUIRED TO SEPARATE NUCLEUS INTO FREE NUCLEONS

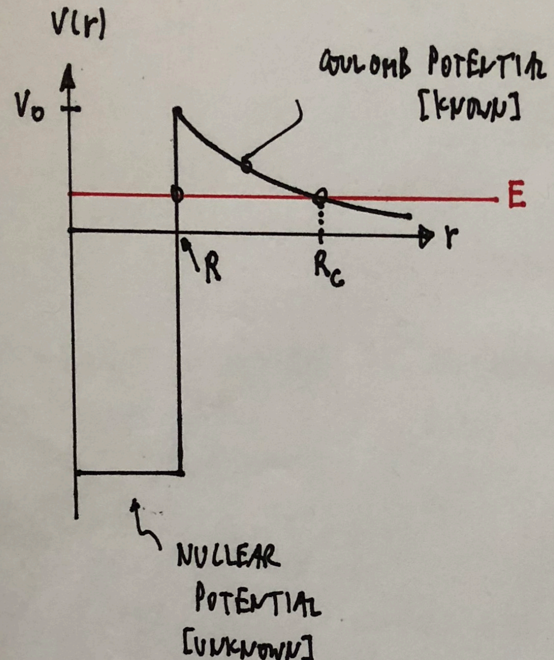
<u>THEORY</u>

$B(Z,N)$ CAN ALSO BE CALCULATED USING NUCLEAR MODELS; MASS FORMULAS EXPRESS B IN TERMS OF DIFFERENT POTENTIAL CONTRIBUTIONS [VOLUME TERM, SURFACE TERM, COULOMB TERM, ...]

# α- PARTICLE DECAY




[CM AND LAB SYSTEM ARE THE SAME]




$$V(r) = \frac{\overset{=2}{Z_\alpha} Z_1 e^2}{4\pi\epsilon_0 r} \quad , \; r > R$$

$\frac{e^2}{4\pi\epsilon_0} = 1.440$ MeV·fm

$$= 1.440 \frac{Z_\alpha Z_1}{r\,(\text{fm})} \quad (\text{MeV})$$

CLASSICAL TURNING POINT: $\quad E = V(R_c) = \frac{2 Z_1 e^2}{4\pi\epsilon_0 R_c}$

$$\Rightarrow \quad R_c = \frac{2 Z_1 e^2}{4\pi\epsilon_0} \frac{1}{E}$$

HEIGHT OF COULOMB BARRIER: $\quad V_0 = \frac{2 Z_1 e^2}{4\pi\epsilon_0} \frac{1}{R}$

$$M c^2 = m_\alpha c^2 + m_1 c^2 + K_\alpha + K_1$$

$$\Rightarrow Q \equiv K_\alpha + K_1 = [M - m_\alpha - m_1] c^2 = E$$

[BLATT/WEISSKOPF, p. 567]

TOTAL E AVAILABLE FOR DECAY IS EQUAL TO SUM OF $K_\alpha + K_1$

**EXAMPLE:** $\quad {}^{210}_{84}Po \rightarrow {}^{206}_{82}Pb + \alpha$

$\qquad\qquad\quad 0^+ \qquad\quad 0^+ \qquad\quad 0^+$

$Z = 84$

$Z_1 = 82$

$A = 210$

$A_1 = 206$

TOTAL DECAY ENERGY [SUM OF KINETIC ENERGIES OF α-PARTICLE AND $^{206}Pb$ DAUGHTER]:

$$Q = E = 5.4075 \text{ MeV}$$

$$K_\alpha = \frac{A_1}{A_1 + 4} Q$$

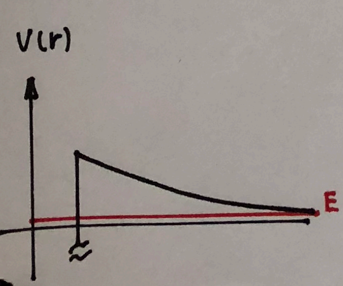
- NUCLEAR RADII ARE BETTER DEFINED THAN ATOMIC RADII, BECAUSE THE NUCLEAR FORCE IS OF SHORT RANGE:

$$R = R_0 A^{1/3} \approx (1.2 \text{ fm})(210)^{1/3} = 7.1 \text{ fm}$$

[SINCE CENTRAL CHARGE DENSITY IS CONSTANT FOR ALL NUCLEI:

$$\frac{A}{\frac{4}{3}\pi r^3} = \text{CONST} \,]$$

- CLASSICAL TURNING POINT: $\quad R_c = (1.440 \text{ MeV fm}) \dfrac{2 \cdot 82}{5.4075 \text{ MeV}} = 43.7 \text{ fm}$




- α-PARTICLE ENERGY IS WAY BELOW COULOMB BARRIER HEIGHT
- α-PARTICLE MUST TUNNEL THROUGH A THICK BARRIER

[IN THE FOLLOWING, WE ASSUME ZERO ANGULAR MOMENTUM; OTHERWISE MORE COMPLICATED EXPRESSIONS ARISE]

# TRANSMISSION COEFFICIENT FOR $\alpha$-PARTICLE DECAY USING WKB:

$$T^{WKB} = e^{-\frac{2}{\hbar}\int_{r_1}^{r_2}\sqrt{2m[V(r)-E]}\,dr} = e^{-\frac{2}{\hbar}\sqrt{2m}\int_R^{R_c}\sqrt{\frac{2Z_1e^2}{4\pi\epsilon_0}\frac{1}{r}-E}\,dr}$$

($m$: REDUCED MASS)

WITH $E = V(R_c) = \frac{2Z_1e^2}{4\pi\epsilon_0 R_c}$ WE FIND

$$T^{WKB} = \exp\left[-\frac{\sqrt{8m}}{\hbar}\int_R^{R_c}\sqrt{\frac{2Z_1e^2}{4\pi\epsilon_0}\frac{1}{r} - \frac{2Z_1e^2}{4\pi\epsilon_0}\frac{1}{R_c}}\,dr\right] = \exp\left[-\frac{\sqrt{8m}}{\hbar}\sqrt{\frac{2Z_1e^2}{4\pi\epsilon_0}}\int_R^{R_c}\sqrt{\frac{1}{r}-\frac{1}{R_c}}\,dr\right]$$

SUBSTITUTION:

$z \equiv \frac{r}{R_c}$

$\frac{dz}{dr} = \frac{1}{R_c}$

$$\int_R^{R_c}\sqrt{\frac{1}{r}-\frac{1}{R_c}}\,dr = \int_{R/R_c}^{1}\sqrt{\frac{1}{zR_c}-\frac{1}{R_c}}\,R_c\,dz = \sqrt{R_c}\int_{R/R_c}^{1}\sqrt{\frac{1}{z}-1}\,dz$$

$$= \sqrt{R_c}\left[\arccos\sqrt{\frac{R}{R_c}} - \sqrt{\frac{R}{R_c}\left(1-\frac{R}{R_c}\right)}\right]$$

SINCE $\frac{R}{R_c} \ll 1$ : $\arccos\sqrt{x} - \sqrt{x(1-x)} \approx \frac{\pi}{2} - 2\sqrt{x} + \dots$

$$T^{WKB} \cong \exp\left[-\frac{\sqrt{8m}}{\hbar}\sqrt{\frac{2Z_1e^2}{4\pi\epsilon_0}}\sqrt{R_c}\left(\frac{\pi}{2}-2\sqrt{\frac{R}{R_c}}\right)\right] = \exp\Bigg[\underbrace{-\frac{\sqrt{8m}}{\hbar}\frac{\pi}{2}\sqrt{\frac{2Z_1e^2R_c}{4\pi\epsilon_0}}}_{-\frac{2\pi}{\hbar}\sqrt{\frac{Z_1e^2}{4\pi\epsilon_0}m\frac{2Z_1e^2}{4\pi\epsilon_0}\frac{1}{E}}} + \underbrace{\frac{2\sqrt{8m}}{\hbar}\sqrt{\frac{2Z_1e^2R}{4\pi\epsilon_0}}}_{+\frac{8}{\hbar}\sqrt{\frac{Z_1e^2}{4\pi\epsilon_0}mR}}\Bigg]$$

$$= -\frac{2\pi}{\hbar}\frac{Z_1e^2}{4\pi\epsilon_0}\sqrt{\frac{2m}{E}}$$

EQ. (8.230)
BRANSDEN

$$\boxed{T^{WKB} = e^{-\frac{2\pi}{\hbar}\frac{Z_1 e^2}{4\pi\epsilon_0}\sqrt{\frac{2m}{E}}} \cdot e^{+\frac{8}{\hbar}\sqrt{\frac{Z_1 e^2}{4\pi\epsilon_0} mR}}}$$

"GAMOW FACTOR"
MAIN FACTOR DETERMINING TRANSMISSION THROUGH COULOMB BARRIER

CORRECTION FOR FINITE NUCLEAR SIZE; NOTE "+" SIGN

IF WE WOULD HAVE TAKEN INTO ACCOUNT 3. TERM IN EXPANSION, WE FIND NUMERICALLY:

$$T^{WKB} = e^{-3.95804\, Z_1\sqrt{\frac{M_1}{M_1+4}}\frac{1}{\sqrt{E}}\left[1+0.043418\left(\frac{E\cdot R}{Z_1}\right)^{3/2}\right]} \cdot e^{+2.969646\sqrt{Z_1}\sqrt{\frac{M_1}{M_1+4}}\sqrt{R}}$$

CORRECTION IF E BECOMES SIGNIFICANT FRACTION OF COULOMB BARRIER HEIGHT:

$$V_0 = \frac{2e^2}{4\pi\epsilon_0}\frac{Z_1}{R} \Rightarrow \frac{ER}{Z_1} \sim \frac{E}{V_0}$$

IN THIS EXPRESSION:

E IN MeV [TOTAL DECAY $E = Q_\alpha$]

$M_1$ IN amu [DAUGHTER]

$Z_1$ [DAUGHTER]

R IN fm

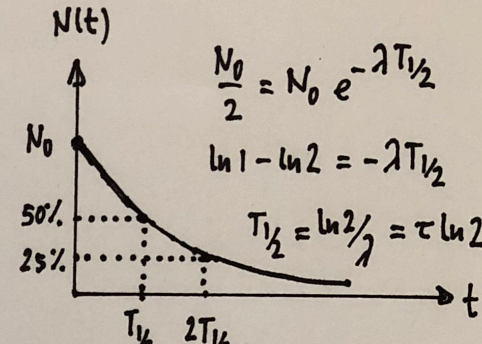
[THAT "m" IN ABOVE EXPRESSION IS THE REDUCED MASS IS EXPLAINED IN EVANS, p. 874/875]

# DECAY CONSTANTS, LIFETIMES, HALF LIVES

DECAY CONSTANT: PROBABILITY OF DECAY PER SECOND, $\lambda$

$$\frac{1}{\lambda} \equiv \tau = \frac{T_{1/2}}{\ln 2} = 1.4427\, T_{1/2}$$

"MEAN LIFETIME"

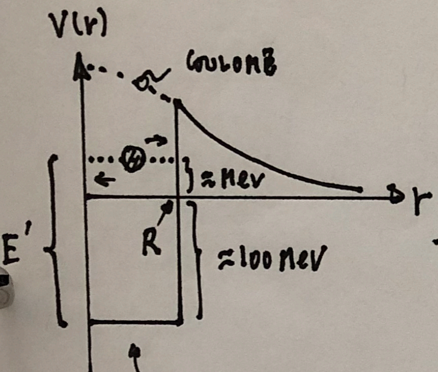



$$\frac{N_0}{2} = N_0\, e^{-\lambda T_{1/2}}$$

$$\ln 1 - \ln 2 = -\lambda T_{1/2}$$

$$T_{1/2} = \ln 2/\lambda = \tau \ln 2$$

$$N(t) = N_0\, e^{-\lambda t}$$




IN THE INTERIOR, KINETIC ENERGY APPROXIMATELY EQUAL TO POTENTIAL DEPTH; FOR $\alpha$-DECAY: ≈ 100 MeV

$$\nu = \frac{1}{\Delta t} = \frac{v}{2R}$$

$\Delta t$: TIME IT TAKES FOR KNOCKING AT BARRIER

$v$: VELOCITY OF $\alpha$-PARTICLE IN INTERIOR

$$\boxed{\lambda_{\alpha\text{-DECAY}} = \nu\, S_\alpha\, T^{WKB}}$$

$\nu$: KNOCKING FREQUENCY

$S_\alpha$: SPECTROSCOPIC FACTOR [PREFORMATION PROBABILITY]

$T^{WKB}$: TRANSMISSION COEFFICIENT

NUMERICALLY:

$E'$: ENERGY IN INTERIOR

$$\nu = \frac{\sqrt{\frac{2E'}{m}}}{2R} = \frac{\sqrt{E'}\,c}{\sqrt{2}\sqrt{mc^2}\,R} = \frac{\sqrt{E'}\,c}{\sqrt{2}\,R\sqrt{\frac{M_1\cdot 4}{M_1+4}}\sqrt{m_u c^2}} = 3.475\cdot 10^{21}\,\frac{\sqrt{E'}}{R}\,\frac{1}{\sqrt{\frac{M_1}{M_1+4}}}\ \left(\frac{1}{s}\right)$$

($E'$ in MeV, $R$ in fm)

$m_u c^2 = 931.4940$ MeV; $\hbar c = 197.327$ MeV$\cdot$fm

$\boxed{\text{EXAMPLE}}$ CALCULATE THE HALFLIFE OF $^{210}Po$ IN UNITS OF DAYS

USE THE FOLLOWING INFORMATION:

DECAY ENERGY $Q_\alpha = E = 5.4075$ MeV [SUM OF $\alpha$-PARTICLE AND DAUGHTER KINETIC ENERGIES]

KINETIC ENERGY IN NUCLEAR INTERIOR $E' \approx 100$ MeV UNCERTAIN

RADIUS $R = R_0 (A_1^{1/3} + 4^{1/3}) = 8.62$ fm

$R_0 = 1.15$ fm UNCERTAIN

SPECTROSCOPIC FACTOR: $S_\alpha \cong 0.02$ [FROM FIG. 2 OF QIAN et al., SCIENCE CHINA - PHYSICS MECH. ASTRON. 56, 1520 (2013)]

$\Rightarrow$ $\nu = 4.07 \cdot 10^{21}$ 1/s ; $T^{WKB} = e^{-138.2355 [1.0186]} \cdot e^{+78.1968} = e^{-62.6098}$

$= 6.44 \cdot 10^{-28}$

$\Rightarrow$ $\lambda = \nu S_\alpha T^{WKB} = 5.242 \cdot 10^{-8}$ /s AND $\tau = 1.90 \cdot 10^{7}$ s

$T_{1/2} = \ln 2 \, \tau = 1.322 \cdot 10^{7}$ s $= \underline{\underline{153 \text{ DAYS}}}$

[EXPERIMENTAL VALUE: 138.4 DAYS]

WE COMPLETELY DISREGARDED THE "PRE-FACTOR", PRECISELY BECAUSE OF THE UNCERTAINTIES INVOLVED IN THE DERIVATION OF $T_{1/2}$:

(i) GAMOW FACTOR DOMINATES ENERGY DEPENDENCE; SLIGHTLY CHANGING E CHANGES THE HALF-LIFE DRAMATICALLY

[THAT'S WHY α-DECAY HALF-LIVES VARY FROM SECONDS TO BILLIONS OF YEARS]

[$Q_\alpha$ IS USUALLY WELL KNOWN; CALCULATED FROM MASSES]

(ii) LESS DRAMATIC, BUT STILL IMPORTANT, IS THE ASSUMPTION OF THE NUCLEAR RADIUS; VARYING R A LITTLE AFFECTS $T^{WKB}$ BY ORDERS-OF-MAGNITUDE

IT IS DANGEROUS WHEN $S_\alpha$ IS DETERMINED FROM:

$$T_{1/2}^{EXP} \cdot \underbrace{\nu}_{\text{DEPENDS ON } R \text{ AND } E'} \cdot \underbrace{T^{WKB}}_{\text{DEPENDS ON } R} \cdot 1.4427 = \frac{1}{S_\alpha}$$

[MANY PAPERS IN THE LITERATURE]

[R-DEPENDENCE OF PRODUCT $\nu \cdot T^{WKB}$ LESSENS!]

(iii) ALTHOUGH AT THE INNER TURNING POINT [NUCLEAR RADIUS] THE SLOPE OF THE POTENTIAL IS INFINITE, THE WKB METHOD GIVES REASONABLE RESULTS [BUT, AGAIN, UNCERTAINTIES ARE LARGE]

Phenomenological formulas for $\alpha$-particle decay:

Important for:

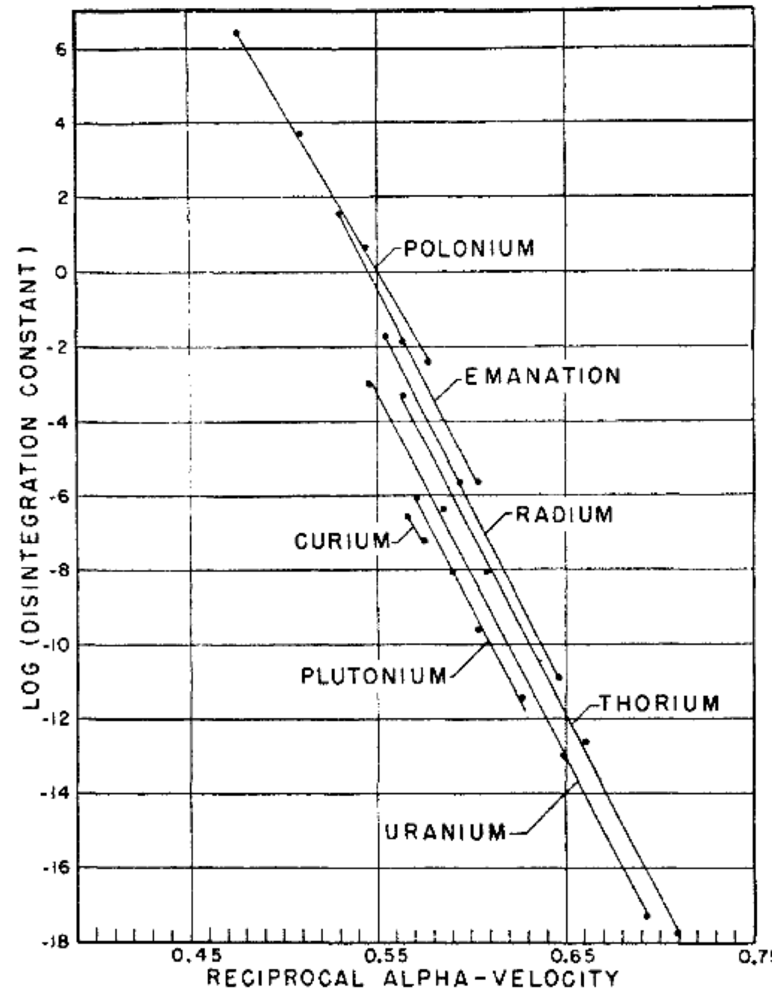
- predicting $T_{1/2}$ for nuclides not yet measured
- searching for stability of super-heavy elements

From results obtained in class we expect:

$$\log \lambda = - a Z_1 E^{-1/2} + b$$

we conclude that a factor of 2-3 change in the decay energy, E, corresponds to a Factor of $10^{24}$ in $T_{1/2}$ or $\lambda$

Kaplan, Phys. Rev. 81, 962 (1951)




"Geiger-Nuttall law"