P358 Project: Fokker-Planck Equation

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1 Background

Before the emergence of quantum mechanics and chaos theory, physics up until the nineteenth century mostly consisted of the modelling of natural phenomena by deterministic solutions of differential equations. During that time period it was commonly believed that if one knew all of the initial conditions of a system, one would be able to predict the future with certainty. Quantum mechanics and chaos theory prompted a statistical interpretation of physics due to the probabilistic measurement process in quantum mechanics, the unpredictable behaviors of simple differential equations in chaos theory and Brownian motion.

In 1827, a botanist named Robert Brown investigated the motion of small pollen grains suspended in water. The pollen grains seemed to have an extremely random and irregular state of motion and was called *Brownian motion*. In 1905, Einstein published an article titled *Concerning the motion, as required by the molecular-kinetic theory of heat, of particles suspended in liquids at rest*. Simultaneously an identical explanation was developed by Smoluchowski in 1906. Smoluchowski was responsible for the later systematic development of the theory which is why the Fokker-Planck equation is sometimes called the Smoluchowski equation, especially when applied to particle position distributions. Additionally, the equation is referred to as the Kolmogorov equations in probability theory. In 1931, Andrei Kolmogorov sought to derive a theory of continuous Markov processes by extending discrete Markov processes.

The Fokker-Planck equation is a partial differential equation that describes the time evolution of the probability density function under the influence of arbitrary potentials and stochastic forces. This allows for the equation to be used in various fields outside of physics. For example, in mathematical finance the Fokker-Planck equation is used to inversely find the local volatility of an option. The local volatility is the equivalent of the diffusion coefficient D, see below. The Fokker-Planck equation can also be applied to evolutionary dynamics, or the study of the mathematical principles according to which biological organisms as well as cultural ideas evolve.

2 Motivation

We first consider the Fokker-Planck equation in one dimension:

$$\frac{\partial W(x,t)}{\partial t} = \frac{1}{kT}\frac{\partial}{\partial x}(-F(x)W(x,t)) + D\frac{\partial^2 W}{\partial x^2}. \tag{1}$$

Here, W is a time dependent probability distribution, k is the Boltzmann constant, T is the temperature, $F(x)$ is the force acting on the particle distribution, and D is the diffusion coefficient. For our purposes D is viewed as constant, but D can also depend on both position and time. The first term relates the change in the probability distribution, W, to an external force, F. The second term incorporates the diffusion due to surrounding particles.

The canonical example of this equation is Brownian motion. Brownian motion is the random motion of a particle suspended within a fluid. The motion is due to the impulses caused by collision with fast moving particles within the medium. It is not feasible to model the motion exactly due to the large number

of collisions. Instead, the motion can be modeled using stochastic variables. Classical Brownian motion is modeled according to:

$$\frac{\partial W(x,t)}{\partial t} = D\frac{\partial^2 W}{\partial x^2}. \tag{2}$$

This is simply (1) without external forces, and with a constant diffusion coefficient D. The constant diffusion is due to the homogeneity of the surrounding medium. Einstein's derivation of the above equation will be discussed in the Theory section and was important in the development of the Fokker-Planck equation and other stochastic methods. However, this simple example shows the power of the Fokker-Planck equation to allow one to create models for processes that involve a large number of particles.

Overall, our motivation to study the Fokker-Planck equation comes from the desire to model distributions of particles that are subject to both external forces and diffusion. This can be parsed into many fields outside of physics when it is difficult to model objects individually but easier to model discrete objects subject to both diffusive and non-diffusive forces as a distribution.

3 Theory

We will motivate the Fokker-Planck equation by echoing the derivation by Albert Einstein. We will start with a physical example, specifically the motion of small particles on the surface of water. An important assumption made in Einstein's approach is that the observed motion of the particles is caused by the interaction between the water molecules and the particles resting on the surface. The collisions between the observed particles and the water molecules are assumed to be very frequent and alters the trajectory of the particle based on some probability distribution ϕ. Moreover, the trajectory of one particle is independent of the trajectories of other particles. Also, the trajectories of the same particle over different time intervals are independent of each other.

For simplicity we will consider the particles in one dimension. Assume the system has a total of N particles in the system and a small time step τ. The position of the of the particles will change by a small step size of Δ according to our model. Note that the step size Δ can have different values for each particle and can be either positive or negative. We will assume the number of particles that experience these spatial shifts over the time interval τ is governed by the probability distribution ϕ. Note that over the time interval τ the spatial shift goes from Δ to $\Delta + d\Delta$. Thus, $N\phi(\Delta)d\Delta$ is the number of particles that experience the spatial shift. We assume that ϕ is normalized $\int_{-\infty}^{\infty}\phi(\Delta)d\Delta = 1$. We also assume that there is an equal likelihood of a particles taking a positive Δ displacement and a negative Δ displacement, implying that ϕ is even, $\phi(-\Delta) = \phi(\Delta)$.

We will define the function $W(x,t)$ to be the number of particles per unit volume. We can determine how the number of particles per unit volume change over the time interval τ according to our physical assumptions. For simplicity, we will consider the change in volume in a single spatial dimension x,

$$W(x,t+\tau) = \int_{-\infty}^{\infty}\phi(\Delta)W(x+\Delta,t)d\Delta. \tag{3}$$

We can expand the left hand side of the above equation about the variable τ since it is a small value we have;

$$W(x,t+\tau) = W(x,t) + \tau\frac{\partial W}{\partial t} + \mathcal{O}(\tau^2). \tag{4}$$

We can also taylor expand the right hand side of the equation about the variable of Δ since it is also assumed to be a small value;

$$W(x+\Delta,t) = W(x,t) + \Delta\frac{\partial W}{\partial x} + \frac{\Delta^2}{2}\frac{\partial^2 W}{\partial x^2} + \mathcal{O}(\Delta^3). \tag{5}$$

Using the results from (3), (4), and (5), we arrive at the following result;

$$\begin{aligned} W(x,t)+\tau\frac{\partial W}{\partial t} &= \int_{-\infty}^{\infty} W(x,t)+\Delta\frac{\partial W}{\partial x}+\frac{\Delta^2}{2}\frac{\partial^2 W}{\partial x^2}\phi(\Delta)d\Delta \\ &= W\int_{-\infty}^{\infty}\phi(\Delta)d\Delta+\frac{\partial W}{\partial x}\int_{-\infty}^{\infty}\Delta\phi(\Delta)d\Delta+\frac{\partial^2 W}{\partial x^2}\int_{-\infty}^{\infty}\frac{\Delta^2}{2}\phi(\Delta)d\Delta. \end{aligned} \tag{6}$$

Note that we move the partial derivatives outside of the integral since they are not dependent on the small displacement Δ. Recall that we have constrained the distribution ϕ, such that it is an even function and is normalized. Based on these constraints, it follows that the second term of (6) will be zero, since it is an odd integrand over symmetric bounds. The result from this cancellation is the following expression;

$$\frac{\partial W}{\partial t} = \frac{\partial^2 W}{\partial x^2}\frac{1}{\tau}\int_{-\infty}^{\infty}\frac{\Delta^2}{2}\phi(\Delta)d\Delta = D\frac{\partial^2 W}{\partial x^2}. \tag{7}$$

We have arrived at the Diffusion equation, where $D=\frac{1}{\tau}\int_{-\infty}^{\infty}\frac{\Delta^2}{2}\phi(\Delta)d\Delta$. This is the Fokker-Planck equation (see section 1) in the simplest ensemble, i.e. with no drift present in the particles.

Now we will physically motivate the so-called drift term of the Fokker-Planck equation. Consider a constant time-independent potential well U acting on the ensemble of particles. Clearly we can assume that there will also be a force F present on each of the particles due to this potential well. Now the small displacements Δ for each particle will be driven by both this potential well and the probabilistic interactions. We can describe the drift on the particles via the force at each position $F(x)W(x,t)$. The change in the distribution of particles at each time interval τ will be a balancing match between the statistical diffusion process and the presence of the potential well. Note the drift term used is divided by kT since this is a measure of the average kinetic energy of the particles, where k is the Boltzmann constant and T is the temperature of the system. In short, this argument is saying that the potential well of the system will have more influence on the trajectory of the particle if the system is at a low temperature and less influence when the thermal bath is at a high temperature. Note that the temperature of the system is assumed to be constant in this case. Again recall that $W(x,t)$ is the number of particles per unit volume and at a small time step we want to see how this distribution is changing due to the drift in only the x-dimension over a small change in time. This physical argument gives us the following drift correction to the equation;

$$\text{Drift} = -\frac{1}{kT}\frac{\partial}{\partial x}\Big(F(x)W(x,t)\Big). \tag{8}$$

Note that this term is negative since these interactions from the potential well will 'resist' or 'counteract' the statistical diffusion term. Adding this term to the Diffusion equation we arrive at the one dimensional Fokker-Planck equation;

$$\frac{\partial W(x,t)}{\partial t} = -\frac{1}{kT}\frac{\partial}{\partial x}\Big(F(x)W(x,t)\Big)+D\frac{\partial^2 W}{\partial x^2}. \tag{9}$$

4 Methods

4.1 Explicit method for Fokker-Planck

We initially consider an explicit numerical method for (1) with the force given by

$$F(x) = -\frac{\partial U}{\partial x}, \tag{10}$$

and potential given by

$$U(x) = a_0 kT(a_4x^4+a_3x^3+a_2x^2a_1x), \tag{11}$$

with known constants a_n, and with boundary conditions

$$W(x=0,t=0) = W(x=1,t=0) = 0. \tag{12}$$

We use finite-differencing with symmetric first spacial derivatives and forward time derivatives to discretize Equation 1. We apply the chain rule to Equation 1 and obtain

$$\frac{\partial W(x,t)}{\partial t} = \frac{1}{kT}\left[-\frac{\partial F(x)}{\partial x}W(x,t) - \frac{\partial W(x,t)}{\partial x}F(x)\right] + D\frac{\partial^2 W}{\partial x^2}. \tag{13}$$

Using a central finite difference approximation, (13) is simplified to be

$$\frac{W_i^{n+1} - W_i^n}{\Delta t} = \frac{1}{kT}\left[-\frac{F_{i+1}^n - F_{i-1}^n}{2\Delta x}W_i^n - \frac{W_{i+1}^n - W_{i-1}^n}{2\Delta x}F_i^n\right] + D\frac{W_{i+1}^n - 2W_i^n + W_{i-1}^n}{(\Delta x)^2}, \tag{14}$$

where there are time steps Δt and position steps Δx, with W_i^n for $W(x_i, t_n)$ and with F_i^n for $F(x_i, t_n)$ (given the potential supplied in (11), we can calculate the partial derivative directly rather than use the finite differencing shown and decrease the truncation error).

From (14), we obtain an explicit recurrence relation for the probability density function of a particle in a potential well,

$$W_i^{n+1} = W_i^n + \frac{\Delta t}{kT}\left[-\frac{F_{i+1}^n - F_{i-1}^n}{2\Delta x}W_i^n - \frac{W_{i+1}^n - W_{i-1}^n}{2\Delta x}F_i^n\right] + \Delta t D\frac{W_{i+1}^n - 2W_i^n + W_{i-1}^n}{(\Delta x)^2}. \tag{15}$$

Using the boundary conditions (12), and starting with a Gaussian distribution as an initial condition

$$W(x, t=0) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

(or some other initial distribution) we obtain a system of linear equations from (15) that we step through and solve directly. It should be noted that our truncation error is on the order $O(h^2)$ for the central difference approximation for the local position steps, but $O(h)$ for the forward difference approximation in the time steps.

In order to enforce the boundary conditions (12) we add ghost points on either side of our set of support points. For node centered support points we set $W_{-1}^n = 0$ and $W_{J+1}^n = 0$. For cell centered support points we set $W_{-1/2}^n = -W_{1/2}^n$ and $W_{J+1/2}^n = -W_{J-1/2}^n$.

We can determine the stability of the explicit method using Fourier analysis. We begin by replacing all W_j^n with $\xi^n(k)e^{ijk\Delta x}$. After some simplification we arrive at:

$$\xi(k) = 1 - \Delta t\frac{\partial F}{\partial x} - \frac{2D\Delta t}{(\Delta x)^2} + i\sin k\Delta x\left[\frac{\Delta t}{\Delta x}F_j^n + \frac{2D\Delta t}{(\Delta x)^2}\right], \tag{16}$$

we have set $kT = 1$. The method is stable when $|\xi(k)|^2 \leq 1$. In the limit of $D \to 0$ this reduces to the following condition:

$$\Delta t \leq \frac{2\frac{\partial F}{\partial x}}{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{F_n}{\Delta x}\right)^2}. \tag{17}$$

Additionally, in the limit of U being a constant we instead get the following:

$$\Delta t \leq \frac{(\Delta x)^2}{2D}. \tag{18}$$

Recall that this is the same condition as just the diffusion equation.

4.2 Implicit method for Fokker-Planck

We next consider an implicit scheme in order to guarantee stability. For more than one dimension, we might consider an Alternating-Direction Implicit (ADI) method, however for a Fokker-Planck equation in

one dimension we simply apply finite differences fully implicitly,

$$\begin{aligned}\frac{W_i^{n+1}-W_i^n}{\Delta t} =& \frac{1}{kT}\left[-\frac{F_{i+1}^n-F_{i-1}^n}{2\Delta x}W_i^{n+1} - \frac{W_{i+1}^{n+1}-W_{i-1}^{n+1}}{2\Delta x}F_i^n\right] \\ &+ D\frac{W_{i+1}^{n+1}-2W_i^{n+1}+W_{i-1}^{n+1}}{(\Delta x)^2}.\end{aligned} \tag{19}$$

Although the implicit is more expensive to compute, it guarantees stability.

We determine the stability of the implicit method using Fourier analysis. We replace all W_j^n with $\xi^n(k)e^{ijk\Delta x}$. After some simplification we obtain

$$\xi(k) = \frac{1}{1+\Delta t\frac{\partial F}{\partial x} + \frac{2D\Delta t}{(\Delta x)^2}\Big(1-\cos k\Delta x\Big) + \frac{F_i\Delta t}{\Delta x}\Big(i\sin k\Delta x\Big)}, \tag{20}$$

where we have allowed $kT = 1$. Moreover, we find that $|\xi(k)| \leq 1$, and the scheme is unconditionally stable.

It is important to note the error accumulation in this technique as the distribution approaches the equilibrium solution. Chang and Cooper[1] claim that the error will increase monotonically once the distribution gets close to the equilibrium solution since the distribution changes very slowly. These authors recommend to terminate a calculation when the distribution begins to change slowly to avoid results being tampered from this error accumulation.

4.3 Semi-Implicit method for Fokker-Planck

We also considered a semi-implicit scheme to solve the Fokker-Planck equation. Our implementation of the semi-implicit method takes advantage of operator splitting to use methods with forgiving stability criteria for both the drift portion and diffusion portion of the Fokker-Planck equation.

For the drift portion we have the change in W_j^n is:

$$\Delta W = -\frac{F_{i+1}^n-F_{i-1}^n}{2\Delta x}W_i^n - \frac{W_{i+1}^n-W_{i-1}^n}{2\Delta x}F_i^n. \tag{21}$$

After doing a step of drift we then feed the updated W^n vector into an implicit step to solve the diffusion portion of the Fokker-Planck equation. We treat these two steps together as a single time step. Our implicit step solves the equation $\boldsymbol{A}W^{n+1} = W^n$. Where,

$$A = \begin{bmatrix} 1+2*\alpha & -\alpha & 0 & ... & ... & 0 \\ -\alpha & 1+2*\alpha & -\alpha & ... & ... & ... \\ 0 & -\alpha & ... & ... & ... & ... \\ ... & ... & ... & ... & -\alpha & 0 \\ ... & ... & ... & -\alpha & 1+2*\alpha & -\alpha \\ 0 & ... & ... & 0 & -\alpha & 1+2*\alpha \end{bmatrix} \tag{22}$$

The boundary conditions in the implicit step are dealt with within the right-hand side of the matrix equation.

The stability condition for this method is identical to that of the explicit method with $D = 0$. This is because the implicit method for solving the diffusion equation is unconditionally stable.

5 Test Cases and Results

Before we discuss our test cases we will talk about initial conditions and boundary conditions. Unless mentioned otherwise all of our following results will be for an initial distribution that is normally distributed with mean 0.5 and variance 0.1. These parameters were chosen such that the initial distribution is contained mostly within the x-boundaries. Additionally, the initial distribution rarely has impact on the distribution after a few hundred steps.

5.1 Explicit Method

We used the explicit method as described in section 4.1 to solve the Fokker-Planck equation for a potential well defined by the following fourth order polynomial.

$$U(x) = a_0 kT \left(a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x\right) \tag{23}$$

We used the following numerical values for the constants; $a_0 = 300, a_1 = -0.38, a_2 = 1.37, a_3 = -2$, and $a_4 = 1$, which corresponds to the above graph.

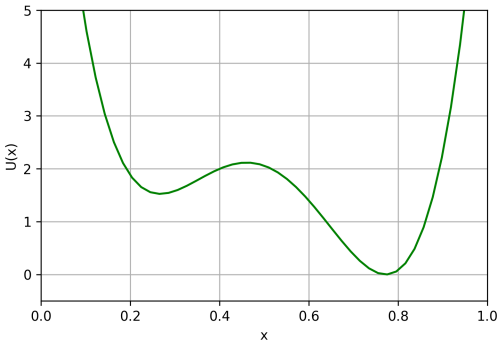
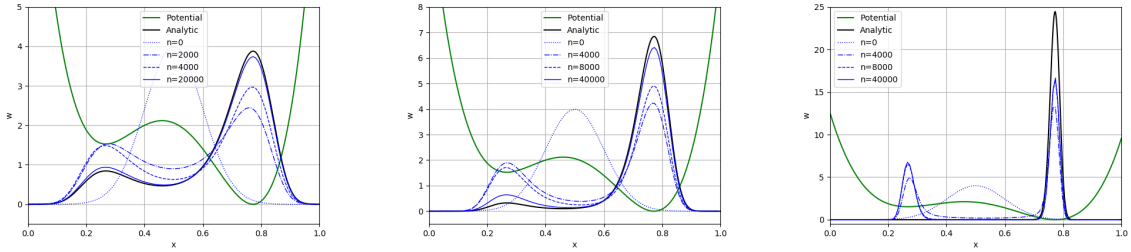


Figure 1: Potential Well

If we have no diffusion, it follows that the equilibrium positions of this potential are at the minimums, i.e. where the force is zero ($-\frac{\partial U}{\partial x} = 0$). The distribution also cannot travel "uphill" away from a minimum for zero diffusion, so once it is in one minimum it will stay there permanently. The global minimum of the potential is at ≈ 0.78. There is, however, a local minimum at the position ≈ 0.26, note that this will correspond to the system being meta-stable. Prior to using our explicit integration solver we know that the solution as $t \to \infty$ is the following expression,

$$W(t \to \infty) = Ae^{-U(x)/D}, \tag{24}$$

where $U(x)$ is our potential as a function of only space, D is a constant diffusion, and A is a normalization constant.



(a) D = 1.0 (b) D = 0.5 (c) D = 0.05

Figure 2: Explicit method of given potential

Figure 2 shows the outputs of our explicit scheme for various diffusion values. From the analytic solutions we can see that as $D \to 0$ that the final solution becomes more restricted to the deeper potential well. We can also see from our numeric solutions that it takes longer for the distribution to reach the final state at

lower values of D. Our explicit method begins to become unstable around $D = 0.05$. This is not the case for our semi-implicit. From now on we will limit our discussion to the semi-implicit results due to its improved stability conditions. This can best be illustrated for a value $D = 0.03$.

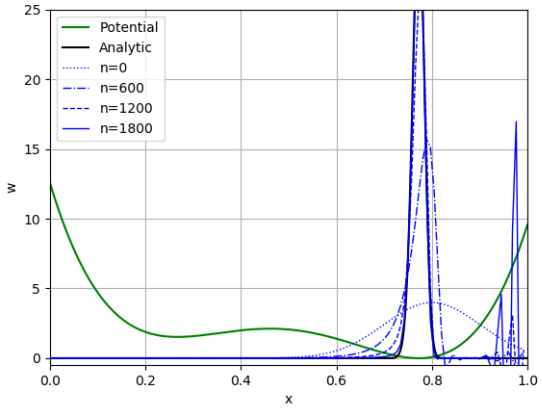


Figure 3: Plot of explicit scheme with the given potential where D = 0.03, Dirichlet Boundaries, and average for the initial condition of the Gaussian distribution of 0.8.

5.2 Semi-Implicit

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Figure 4: Plot of semi-implicit scheme with the given potential where D = 0.03, Dirichlet boundaries, and average for the initial condition of the Gaussian distribution of 0.8.

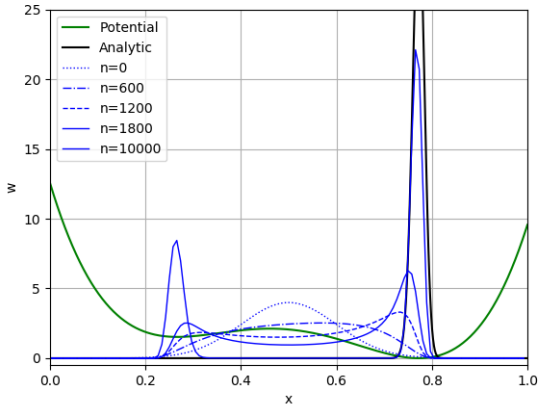

Figure 5: Plot of semi-implicit scheme with the given potential where D = 0.03, Dirichlet Boundaries, and average for the initial condition of the Gaussian distribution of 0.5.

The explicit and matching semi-implicit plots (Figures 3 and 4) display a case where only one of our two integrators functions correctly (the semi-implicit). Additionally, it gives us an opportunity to discuss the effects of initial conditions. We see that for low diffusion at the same time step the results are vastly different (Figures 4 and 5). For the central initial condition (Figure 5)we see that a large portion of the distribution is stuck in the shallower of the two minimums. The low diffusion makes it extremely unlikely for our distribution to move out of the well even when a deeper well is present. For a initial mean of 0.8 we hardly interact with the shallower well at all. The entire distribution quickly fits itself into the lower well and has a negligible chance of leaving the well.

In Figure 6 we have an initial condition which is not our regular Gaussian, but instead a uniform distribution from 0 to 1. When we compare to Figure 2a we can see that the results converge to the same value and are only significantly different for a few initial time steps. The plots quickly match up due to the high diffusion constant.

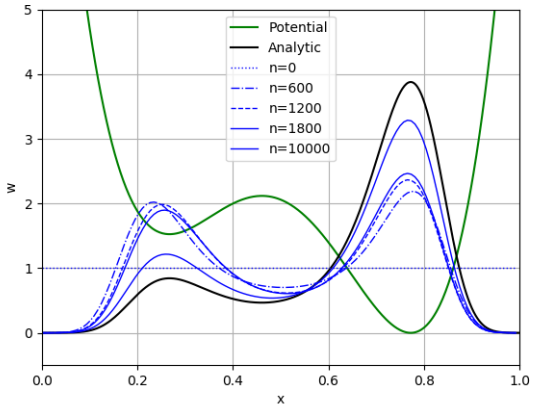

Figure 6: Semi Implicit scheme with uniform initial condition

We initialized a piece wise potential well such it was linear at the bottom of the well and was quadratic on the edges. The physical motivation for such a well is that we would expect the probability distribution to peak along one of the walls of the potential well. Also, the rate at which this process occurs is of physical interest since this will give a notion of whether the drift or diffusion term more strongly governs the dynamics of the probability distribution. The analytical realization of this well is as follows,

$$U(x) = \begin{cases} \frac{412}{3}(x^4 + x^3 + x^2) - 100x + \frac{927}{64} & x < 0.25 \\ \frac{1552}{37}x^4 + \frac{-2508}{37}x + \frac{23571}{592} & x > 0.75 \\ 3x \qquad otherwise. \end{cases} \tag{25}$$

Note that the first derivative is continuous for this potential well. Our plot for the solution of the Fokker-Planck equation of this linear well is provided below. We observed from the solution (at equilibrium) that there is a peak in the distribution near the one of the walls of the well, as we would expect. We also, note that to right of this peak that the probability distribution decays until it reaches a cusp. What is interesting is that this cusp in the distribution coincides with the the point 0.75 defined for our potential, where the second derivative is not continuous. Also we can observe that there is a cusp corresponding to the peak of our distribution at the other cusp at 0.25 where the second derivative is not continuous. Therefore, we observe for this potential well that the discontinuities in the derivative of the probability distribution, echo the discontinuities in the second derivative of the potential well.

Now we will explore two final potentials that require us to develop new boundary conditions. The first of these potentials is just a constant diffusion. That is: $U(x) = C$. We can now no longer enforce (12) as our boundary conditions because the potential function no longer has a minimum between 0 and 1. We must

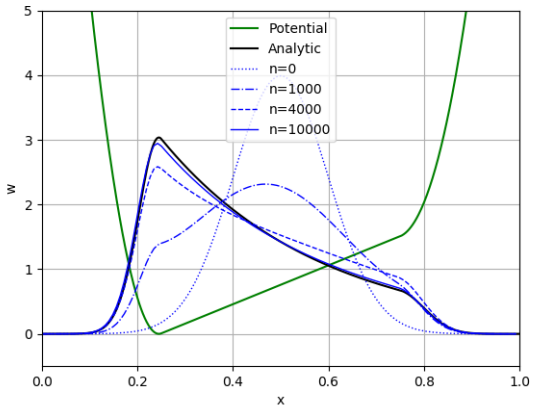

Figure 7: Semi-Implicit solver of linear well

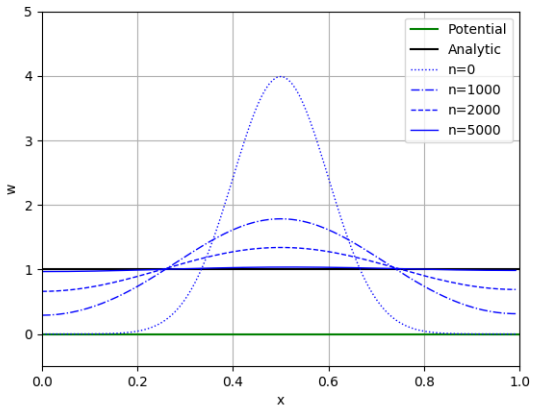

Figure 8: Semi-Implicit solver of constant potential

allow for the distributions to be non-zero at our boundaries. We now choose to enforce a boundary condition that keeps the slope along our boundary constant. Written as a finite difference that is:

$$\frac{W^n_{1/2} - W^n_{-1/2}}{\Delta x} = \frac{W^n_{3/2} - W^n_{1/2}}{\Delta x}. \tag{26}$$

With our constant potential we see that this solution matches exactly with a pure diffusion equation (Figure 8). Our last potential is a square well made with two hyperbolic tangents:

$$U(x) = 10\Big[(1 + \tanh(1000(x - 0.9))) + (1 - \tanh(1000(x - 0.1)))\Big]. \tag{27}$$

In Figure 9, we notice that for the region from 0.1 to 0.9 our integrator does a good job reaching the analytic solution. Outside of this region our integrator over compensates for the steep walls of the well and gives us a negative probability (which the Chang and Cooper method discussed in Section 6.2 would not allow). This error is reminiscent of the error near equilibrium solutions we encountered in purely implicit methods.

In summary, we first analyzed the effect of changing the diffusion constant and saw that for lower diffusion it takes much longer to reach the analytical solution (as expected, with smaller diffusion). Next, we looked at

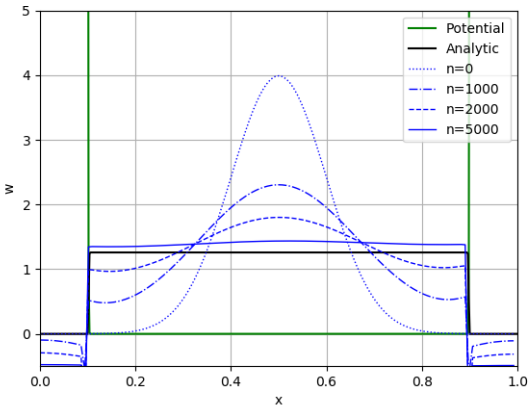

Figure 9: Semi-Implicit solver of box well

the effects of changing initial conditions. This had a much more noticeable impact for low diffusion. Finally, we moved to looking at new potential functions. Using our new potential functions we could look at the relative importance of the drift and diffusion portions of the Fokker-Planck equation by making the drift term try to force the whole distribution into the wall of our well. Additionally, the new potentials required us to reanalyze our boundary conditions to make physical sense for potentials that do not have a steep well in the region from 0 to 1.

6 Other Methods

As mentioned in the above section, we know that after a long time the equilibrium condition of the probability distribution of our particles will correspond to the expression $Ae^{-U(x)/D}$. We can access this equilibrium state through other means besides the methods described above. We give two examples below.

6.1 Monte Carlo Sampling

The Fokker-Planck equation has many applications in fields such as quantum electrodynamics that utilize the path integral formulation of quantum mechanics. The Fokker-Planck equation can similarly be formulated using path integrals. A common way to numerically solve such path integrals is Monte Carlo sampling. We use Metropolis-Hastings sampling of a path of from zero to one in space, using the change in action of our path to determine whether or not we accept a new path. This scheme will give a distribution corresponding to our equilibrium distribution. We have attached an example of such a scheme for the mid-peak potential well. We observe that the resulting histogram mimics the structure of the equilibrium solution. In implementing the Monte Carlo scheme we have to pick a step size that will sample the entirety of the potential well. However, if our diffusion constant in the Fokker-Planck equation was a low value then the distribution may not reach the entirety of the well. This limitation of diffusion is analogous to the sampling step size in the Monte Carlo scheme. Therefore, we would expect the above distribution to correspond to a highly diffusive system.

6.2 Chang and Cooper

Chang and Cooper[1] namely proposes a flux-conservative, implicit scheme that uses a centered differencing for the diffusion term and a weighted differencing method for the advection term. This weight is determined

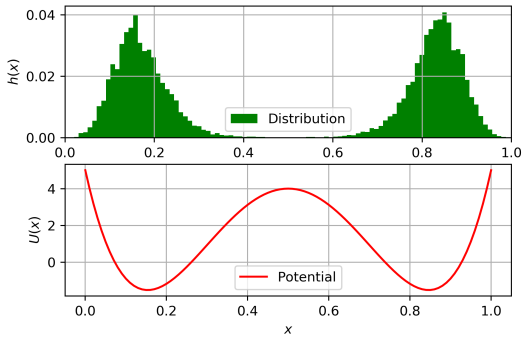

Figure 10: Monte Carlo Sampling

in order for all W's for each Δx to be positive, finding that we require

$$\frac{W_{j+1}^{n+1}}{W_j^{n+1}} \approx \exp\left[\frac{F(x)}{D}\Delta x\right],$$

with the calculation of the weights shown in Chang and Cooper. For our chosen time-independent distributions, the Chang-Cooper method was found to give results (with non-changing weights) comparable to that of the unweighted implicit method.

7 Conclusion

We have shown through this paper that the Fokker-Planck equation can be applied effectively to an assortment of time-independent physical potentials, corresponding to various physical systems. Similarly, we deal with a few of the techniques that for the evolution of the equation with time. Specifically, we used a semi-implicit operator splitting scheme and an explicit scheme. We found one example where the semi-implicit scheme is stable when the explicit scheme is not, and we expect the semi-implicit to be more stable in general. Thus, for other cases (other potential wells) we considered only the semi-implicit scheme. We also explored different methods to solve the Fokker-Planck equation such as the fully implicit method, Monte Carlo sampling by representing the Fokker-Planck equation as a path integral, and a flux-conservative fully implicit Chang-Cooper method.

The Fokker-Planck equation is a partial differential equation that describes the time evolution of the probability density function under the influence of arbitrary potentials and stochastic forces. This allows it to be extremely versatile in its applications, used extensively in physics, such as for plasma simulation, in finance, for modeling the volatility of options, or even molecular biology to describe diffusion processes. Fokker-Planck can be useful where the time evolution of stochastic processes need to be modelled, and accordingly, numerical methods can be used to effectively accomplish this task.

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