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## Second Task: Implementation and Analysis of 3-Point Schemes for Linear Advection

### Problem statement: Implementation and Analysis of 3-Point Schemes for Linear Advection

Consider the linear advection equation:

$$u_t + au_x = 0, \quad \text{for } (t, x) \in (0, T) \times (0, 1),$$

with periodic boundary conditions  $u(t, 0) = u(t, 1)$  and initial condition  $u(x, 0) = \sin(2\pi x)$ . Let  $a = 1$  and  $T = 1$ .

We investigate the behavior and accuracy of several 3-point finite-difference schemes.

#### a) Consider the following schemes:

##### (1) Central Difference in Space:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a \cdot \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0$$

##### (2) Upwind Scheme:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a \cdot \frac{u_{j+1}^n - u_j^n}{\Delta x} = 0$$

##### (3) Lax–Friedrichs Scheme:

$$\frac{1}{\Delta t} \left( u_j^{n+1} - \frac{u_{j+1}^n + u_{j-1}^n}{2} \right) + a \cdot \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0$$

##### (4) Lax–Wendroff Scheme:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a \cdot \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} - \frac{a^2 \Delta t}{2} \cdot \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} = 0$$

#### b) Numerical Implementation and Convergence Study

Implement these schemes in a programming language of your choice (e.g., Python, Matlab, or C++). Use periodic boundary conditions.

- Simulate the solution for  $t \in \{0, 0.5, 1\}$ .
- Plot the numerical solutions alongside the exact solution.
- For each scheme, compute the numerical error at  $T = 1$  and determine the observed order of convergence using grid resolutions  $N = 200, 300, 400$ .