Second Task: Implementation and Analysis of 3-Point Schemes for Linear Advection

Problem statement: Implementation and Analysis of 3-Point Schemes for Linear Advection

Consider the linear advection equation:

$$u_t + au_x = 0, \quad \text{for } (t,x) \in (0,T) \times (0,1),$$

with periodic boundary conditions $u(t,0) = u(t,1)$ and initial condition $u(x,0) = \sin(2\pi x)$. Let $a = 1$ and $T = 1$.

We investigate the behavior and accuracy of several 3-point finite-difference schemes.

a) Consider the following schemes:

(1) **Central Difference in Space:**

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a \cdot \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0$$

(2) **Upwind Scheme:**

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a \cdot \frac{u_{j+1}^n - u_j^n}{\Delta x} = 0$$

(3) **Lax–Friedrichs Scheme:**

$$\frac{1}{\Delta t}\left(u_j^{n+1} - \frac{u_{j+1}^n + u_{j-1}^n}{2}\right) + a \cdot \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0$$

(4) **Lax–Wendroff Scheme:**

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a \cdot \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} - \frac{a^2 \Delta t}{2} \cdot \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} = 0$$

b) Numerical Implementation and Convergence Study

Implement these schemes in a programming language of your choice (e.g., Python, Matlab, or C++). Use periodic boundary conditions.

- Simulate the solution for $t \in \{0, 0.5, 1\}$.
- Plot the numerical solutions alongside the exact solution.
- For each scheme, compute the numerical error at $T = 1$ and determine the observed order of convergence using grid resolutions $N = 50, 100, 200, 400$.