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The gap between secondary school and university mathematics

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It seems self-evident that there is a significant gap between secondary school and university mathematics, even though the gap may take different forms, which vary with different education systems in different places and at different times. This paper attempts to capture some common forms of this gap, and in particular, it discusses some core factors that are directly related to the nature of mathematics. Further, it also attempts to find ways to bridge the gap more easily and surely for students.

1. Introduction

The transition from secondary school to university is itself an exciting and confusing experience for students. After tough matriculation examinations and competitive admission applications, the successful entrants have yet to adapt to new learning environments, new modes of study, new peers and professors, and above all, higher expectations. The old and the new, in fact the different natures of secondary school and university education, inevitably create a gap. How the gap manifests itself, how difficult it is for the students and how it is bridged or worsened are probably as varied as the individual universities and the individual students.

The gap seems more pronounced in mathematics. In this paper, we will mainly be concerned with mathematics majors. Thus we shall be considering students who are more motivated to the study of mathematics and who are mathematically better prepared. However, even with this restriction, in view of the different education systems in different places and at different times, the gap between secondary school and university mathematics is not quite well-defined. So, different places may have different transition points for the same mathematical gap. A simple example is the notorious epsilon-delta gap between calculus and analysis, which usually marks the distinct levels of rigour between school and university mathematics. To ease the gap for students, it may have been postponed to a later time in university. Or, to speed up the ways for better students, it may have been introduced earlier in school.

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We would therefore naively separate factors and aspects of the gap between school and university mathematics into two groups. The first group consists of more circumstantial factors and the second group consists of more mathematical factors. The former factors are more susceptible to local differences and continual changes. They include the actual mathematics curricula and examination syllabuses, the systems of assessing students and university admission policies, student qualities and pedagogical adjustments, expectations on students and evaluations of courses, as well as the peers and teachers whom the students meet. It is relative to these factors that the gap manifests itself more apparently. Students obviously feel the changes in curriculum and pedagogy. More significantly, it may well happen that the beginning mathematics major finds himself or herself among much more outstanding peers, some of whom may have been super-trained for well-known mathematics competitions. This could be a good challenge; it could also be a stressful challenge. Teachers, too, may have more diverse views and experiences relative to this group of factors.

The more mathematical factors in the second group refer to factors that are directly related to the nature of mathematics. These factors are more intrinsic and invariant. Roughly speaking, at the school–university juncture, there is a change from 'elementary' to 'advanced' viewpoints, resulting in specific gaps in areas of algebra, geometry and calculus, such as the epsilon-delta gap mentioned above and the gap between pictures of vectors and the axioms of vector spaces. There would be little disagreement that these gaps must be crossed by mathematics majors, in fact, by any student who wishes to really understand mathematics. As explained above, these specific gaps need not occur exactly in the first days in university. However, they are the real issues which underlie the more circumstantial factors, and could be our common discussion points. Yet we would like to go deeper than specific gaps and focus our attention on two fundamental factors: first, the rigorous and abstract nature of mathematics gaining dominance over the more heuristic and concrete approaches in secondary school, and second, the formidable formalism aggravating specific mathematical gaps at this transition.

This simple classification is meant, first of all, for convenience. In view of our different situations and possibly different philosophies on mathematics education, the mathematical nature of the gap could serve as a unifying theme for our discussion. Sharing our special circumstances and innovations would reveal interesting features and phenomena, giving us a global and updated picture. However, local difficulties may be intricate. Indeed, some of us may occasionally find ourselves trapped in peculiar bureaucratic situations which do not quite agree with our own philosophies. In such cases, then, it is hoped that a distinction between circumstantial and intrinsic factors can help us sort out the issues, resolve the mathematical gap and strengthen our own philosophies in our various situations.

Through this distinction between circumstantial and intrinsic factors, we hope also to bring out two types of psychological factors for discussion. The underlying premise is that although the content of mathematics is logical, the conduct of mathematics is psychological as well. In the case of the gap between school and university mathematics, psychological factors clearly play a crucial role, for the logical steps have yet to be learnt. A deep understanding of the psychological factors involved is therefore essential for seeking ways to ease the transition for students.

First then, we would like to focus on those psychological factors that arise from the nature of mathematical thinking. In particular, as teachers we need to probe into the mental processes by which students can understand the abstract concepts, master the formal language, follow rigorous reasoning, get a good feeling for the mathematical objects and acquire so-called mathematical maturity. A main message of this paper is that mathematics is not only a subject but a way of thinking. Insistence on perfect conduct of the mathematical way of thinking right at the start, in terms of axiomatic formulation and formalized proofs, naturally leads to a gap at the school–university juncture. As the conduct of mathematical thinking is psychological, understanding the psychology of the mathematical thoughts will be a necessary first step in bridging the gap for students.

Suppose we regard the psychology of the mathematical way of thinking as the intrinsic psychological factor. We still have to consider it in context, in terms of curriculum, pedagogy, expectations, examinations, course evaluations and so on. We are then led to more circumstantial psychological factors such as teacher–student interaction, peer competition, prerequisites, progress and assessments. Thus, parallel to the earlier classification of circumstantial and mathematical factors, we emphasize the initiation into the mathematical way of thinking as the more intrinsic psychological aspect of the school–university gap among more situational aspects.

It is easy for mathematicians to take the mathematical way of thinking for granted, unaware that they may be talking in a foreign language to students. They may demonstrate certainty in their reasoning sometimes too meticulously, sometimes by unaccountable hand-waving. Without getting into this way of thinking, students cannot really cross the gap. They may be diligently following each logical step, swallowing every word, but not getting a good feeling of the arguments. Have we ourselves not gone through such a stage? How is it that things gradually or suddenly become clear to us? Are there still things we are afraid to touch? The psychology underlying all these phenomena, whether they are successes or failures, is most fascinating and important.

As mathematics teachers, we can also contribute insights on the psychology of mathematical thinking. We may not have the authority of Poincaré [1] on intuition and logic or of Hadamard [2] on the psychology of invention in mathematics. We may not be able to delve into the impenetrable foundation of mathematics like Hilbert or Gödel. We may not be creative mathematicians ourselves. We may not know much about artificial intelligence, nor neurological investigations of the human mind. But we are working with many aspiring students in their most impressionable years. And the school–university gap is a critical point. Looking deep into the psychology of mathematical thinking at this level better prepares us to help students overcome the gap and also provides us with interesting facts about rational thinking and perhaps the working of the human mind, at least at the formative stage. As a highest model of human reasoning, mathematics should instruct us further on the human thinking process and the nature of human mental phenomena. It would seem ironic if we, as mathematics teachers, stiffen ourselves in the logic of our subject matter and do not convey to our students the art and science of mathematical thinking. Of course, we have to be clearer than Riemann when he observed, concerning mental processes, 'a continual activity of our soul' and drew out the 'Fact: thus something permanent underlies every act of our soul, which enters our soul together with this act but disappears completely from the world of phenomena at that very moment' [3, p. 282]. Perhaps we have to be more rigorous than the same

Riemann when he left a gap in his celebrated mapping theorem. But who among us can provide more hints than this great master when he suggested that it is ‘very likely’ that all the complex zeros of the zeta function lie on the 1/2-line [4]? Here I am jokingly saying that although we must teach our students mathematical rigour, the mathematical way of thinking seems more complex, intuitive and adventurous. Our students, and ourselves as well, would miss out a lot if, owing to the logic and rigour of our subject, we turn our eyes away from the human psychological elements in mathematics education and mathematical thoughts. In fact, understanding the psychological factors would help us provide natural motivation for certain mathematical formulations and find easy bridges across certain difficult gaps, even if we never get to the level of Riemann.

Indeed, at the level under consideration, the mathematical way of thinking is already powerful in its clarity and accuracy. For that very reason, a gap is inevitable. It is probably the first time a student learns to use words exactly as defined or undefined, to mean ‘for all’ or ‘for some’ exactly as stated, to write clearly the conditions for a conclusion and to make statements that stand ‘all’ tests. Imagine if we could reason as clearly and unambiguously in everyday life! But probably we cannot, because the mathematical way of thinking, as it is, applies to mathematical objects and structures which may not include everything. However, if more students can cross the gap to substantial mathematics and get into the way of mathematical thinking with clear understanding of its powers and limits, we would have rendered a worthwhile service. To do that, we would be dealing with the psychology of contemporary young people who probably have wider exposure, more choices but perhaps less patience or commitment. When they are quick and smart, we have to strike them with the simplicity and elegance of good mathematics. When they seem to know nothing and cannot stride out one step, we need to know where they are and understand how we ourselves would react in such circumstances. And what intolerable mistakes they will make! Then we know where the gap lies and how tolerant we are, while the students have a valuable experience of realizing and reflecting on their mistakes.

To recapitulate, we shall emphasize the psychological difficulties in the mathematical way of thinking as we investigate the major mathematical gaps that occur at the school–university transition. In contrast to more circumstantial gaps, the mathematical gaps have more to do with the nature as well as the historical development of mathematics, namely the discovery of new mathematical phenomena, objects or structures, the creation of new methods or concepts, and the solution of deep problems. These mathematical gaps then correspond to significant breakthroughs in the subject. Though they are difficult, they may well be the places where mathematics captures the imagination and the aspirations of students.

All these sound self-contained. But the more circumstantial factors do haunt us, particularly concerning assessments and evaluations. On the one hand, we may feel uncertain how to measure students’ efforts and progress. On the other hand, we may be pressed to show and advertise results, as education becomes more and more business-like. From the beginning to the end of this paper, I can only attempt a feeble message: there are some things more important than ranking students.

Section 2 discusses a case in Hong Kong of the transition from secondary school mathematics to university mathematics, detailing some bridging courses. Section 3 tries to identify and discuss some basic issues, following the framework described above. It is hoped that on the one hand, we can find better ways to help students through the transition and on the other hand, we may understand better our own perspectives, practices and philosophies amidst shifting circumstances that define our own transitions.

2. A case of learning and teaching mathematics in Hong Kong

In this section, I will share my learning and teaching experiences in Hong Kong. The purpose of this sharing is to describe some forms which the gap between secondary school and university mathematics has taken over the years in Hong Kong. Hopefully the sharing may also bring out points and issues which are common to other places. The case of Hong Kong may be justified by the fact that Hong Kong is an international city, open to both Chinese and Western influences.

2.1. *The general situation*

I have taught at the same university in Hong Kong since the late 1970s. I took my first degree at another university in Hong Kong and my doctorate in the USA. When I was an undergraduate, there were two universities and a couple of tertiary institutions in Hong Kong. Since the mid-1990s, there have been eight universities. Roughly speaking, it is now easier to get into a university. Still, competition is keen. For example, in 2001, 92,578 school candidates took the Hong Kong Certificate of Education Examination, a public examination for all graduates from about 600 secondary schools. These students then had to fight for a place in about 400 secondary schools which also offered the two-year matriculation course to prepare students for the Hong Kong Advanced Level Examination. The result was that in 2003, 29,317 school candidates took the advanced level examination, and the universities admitted about 12,300 students into their degree programmes, plus about 5,700 students into their associate degree or higher diploma programmes. Thus, one can see the scale of the competition.

Yet there are complaints that student standards are not as high as before and the blame is often put on the 'less elitist' education system. Meanwhile, the universities eagerly devise schemes to grab the better students, usually by provisional offers conditional on some minimal requirements in the advanced level examination. Things are changing faster in the last few years as the Government proceeds to cut university funding drastically. Getting the 'best' students seems to be a safeguard for the university. The newest scheme is to admit the best students, labelled as 'pointed kids', letting them skip the second year of the matriculation course and the advanced level examination. 'Pointed kids' are defined by their results in the certificate of education examination. That criterion is questionable. The development of these 'pointed kids' remains to be seen.

The point here is that as school and university teachers, we are forced into various kinds of situations, due to social changes, public demands, examination systems, admission procedures, funding policies and school and university administration, which complicate the gap between secondary school and university education. Then as mathematics teachers, we may have to tackle these shifting circumstances through a better understanding of the more intrinsic and invariant nature of the gap in mathematics.

2.2. *A learner's difficulties*

I graduated from a very good secondary school. In school, we had a rigorous training in Euclidean plane geometry in which we learnt a systematic way of writing down what is given, what to prove and the theorem used in each step of a proof. We also learnt constructions by straight edge and compasses. Since we went quite far into the properties of circles, we were sufficiently impressed by the logical system, having forgotten its hazy beginning. In the matriculation course, we learnt calculus. That was more difficult. The hazy beginning was more extended. Limits remained mysterious. However, we were skilled in solving advanced level examination problems. These problems equipped us with sophisticated computational techniques, not only in calculus, but also in algebra and analytic geometry. For myself, the greatest difficulty was in advanced level applied mathematics, which seemed hazy all along.

Although my examination results were very good, I felt a difficult gap between secondary school and university mathematics. First, there was a crash course on logic and set theory which seemed to provide us with a sure foundation of modern mathematics. The difficulty lay not so much in the explicit use of the basic logical connectives, quantifiers, truth tables and set language, nor in the pleasant study of infinite sets, paradoxes and attempts at axiomatic rigour, but rather in the formal set axioms and the formalized proofs on transfinite induction, well ordering, cardinals and ordinals. These formalized axioms and proofs provided neither intuitive reasoning nor complete assurance. For, on the one hand I should not paraphrase, while on the other hand I was not working free of everyday language. Perhaps it is the unavoidable haziness at the beginning of any subject. In this case, it is the rigorous study of infinities and the related problem of a rigorous starting point for university mathematics. Both were formally laid out and appeared infinitely more rigorous than school mathematics. Yet haziness lingered.

In retrospect, this haziness should not have done too much harm, had I not been so worried about examinations, or had I not been so conscientious. With a secondary school mentality, I tried to prepare a perfect set of notes and tidy up all the arguments. I thought that was necessary to meet the professors' requirement of a firm foundation for my subsequent study. Obviously I failed, without the help of a completely formal language, of whatever order, and without any adequate mathematical maturity.

The demand for rigour and the fear of being examined on formal proofs left me somewhat uncertain and inhibited, even in relatively simpler matters. For example, in studying properties of determinants in linear algebra, I fussed over formal proofs of the properties of permutations. Feeling that I have not digested the proofs, I felt unsure about the signs of permutations, and subsequently, I felt I did not know determinants. It might have been just a matter of feeling. If I had played around with examples of permutations, I would have felt greater confidence and better understanding of how things work. In fact, it is easier to forget formal proofs than to forget the feelings we develop from examples. Yet, what would my professors say?

Another example was the number theory course I took in the first term. The emphasis on rigour overshadowed the obvious method of trial and error in learning the subject. I was discouraged, if not intimidated, by the clever and neat proofs. Too anxious to prepare myself for tricky examination problems, I never imagined that the masters did a lot of messy computations. Not only did I fail to acquire a good feeling for the subject, I actually developed a distaste for what I considered isolated number theoretical propositions.

The emphasis on rigour, however, paid off splendidly in analysis. For me, it was both an enlightenment and a challenge, and it kept alive my fascination for mathematics. The formulation of the completeness of the real line and the epsilon-delta reasoning dispelled the mystery surrounding limits. For example, I understood the Cauchy criterion only then. Continuity, differentiation and integration were put on a firmer footing than I had ever expected. Although the details, including Bolzano–Weierstrass and Heine–Borel theorems, were hard, they were instructive. With pictures to guide me, I even managed uniform continuity and uniform convergence quite easily.

2.3. *A teacher's difficulties*

Decades ago, the best students in Hong Kong aspired to study mathematics and some actually became outstanding mathematicians. However, mathematics gradually seemed to lose its charm. While other job-secure subjects became popular, the standard of students admitted by the mathematics department declined considerably. In more recent years, the mathematics department has tried very actively to attract better applicants through public talks and summer courses for secondary school students. Ironically, the prolonged economic downturn in Hong Kong since the late 1990s seems to have drawn students back to mathematics for its basic importance. The point here is that we have to be sensitive to the constantly changing standard and motivation of the students encountering the gap between school and university mathematics.

When the standard of incoming students was at its lowest around the mid-1990s, we started two term courses to bridge the gap. The course 'Essential Math I' goes over calculus and differential equations, strengthening computational techniques and gently introducing acceptable proofs. It is to ensure that all students get the necessary preparations for taking advanced calculus on functions of several variables and elementary analysis on the real line. In 'Essential Math II' we first cover the axiomatic method, and the logic and set theory necessary for undergraduate mathematics. Then we use numbers and functions as themes to introduce basic mathematical structures, techniques and intuition.

Essential Math I is more standard, even though different teachers may strike different balances between techniques and theory. Students actually should have done a lot of calculus computations in their matriculation course. Typically they do many years of past Hong Kong Advanced Level Examination papers in addition to textbook exercises. These questions can be quite tricky. It seems therefore paradoxical that the students often cannot handle some simple integration for example, as some colleagues complain. One main reason is that most students study for examinations and many believe that they can manage a topic by cramming for a few days. It is easy to blame them for bad study habits and attitudes. An alternative is to teach them better study habits and attitudes, but what are these? At this point, it would be best to instill in students the mathematical way of thinking: formulating a problem, working on the problem by means of trial and error, computation or geometric intuition, checking the solution, simply asking what has been done and why it works, and generalizing the method or technique.

In introducing students to the formal ways of mathematics in Essential Math II, we observe that the very basic ideas of 'undefined terms' and 'axioms', as well as the truth-table usage of 'if-then', which teachers find so simple and natural, perhaps out of familiarity, can be serious initial psychological blocks to students. We find it helpful to resolve the problem in the context of two axiomatic systems, using the classical example of Euclid's Elements and the modern, 'more perfect' example of axiomatic set theory. Paradoxes are discussed. Besides familiarizing students with set language, we insist on more substantial mathematics: infinite sets, counting techniques, the beginning of number theory, the real line, and the complex plane. We conclude Essential Math II by extending the basic elementary (rational, exponential and trigonometrical) functions to complex variable and plotting their graphs. This last topic is often omitted in complex analysis courses, but the complete picture of the elementary function is a main reason why complex numbers are needed.

There are difficulties with bridging courses. A course like Essential Math I may overlap with earlier or later calculus or analysis courses. A course like Essential Math II may be too broad, trying to connect too many things. On the other hand, even though some students may find them too hard or too easy, bridging courses do provide a good space to cultivate students' feelings for the mathematical way of thinking.

The best students, however, tend to skip Essential Math, eager to take more advanced courses. As the standard of incoming students improves, the following phenomenon occurs: the better students take heavy course loads even when they are not ready to do so. As a result, their grades, as well as their grasp and understanding of the basics, suffer. To alleviate the situation and to provide more individual guidance, Tutorial Seminars are offered for small group discussions on topics of algebra, geometry, analysis and applied mathematics. These tutorial seminars provide a flexible venue for developing independent reading, clear presentation, active questioning and problem solving. Teachers may suggest topics or books for study and conduct the seminars in their own styles. I only insist that students present clearly what they have studied and understand clearly what fellow students say. With gentle but persistent prodding, that requirement seems to have adequately taken care of the level of rigour needed. It also seems to have set the group thinking together more than competing with one another.

3. Discussion of issues and possibilities

Following the framework proposed in the introduction and drawing on examples from the personal sharing above, I would like to discuss some issues that underlie the gap between school and university mathematics. Generally speaking, for mathematics majors, it seems that the basic issues continue to be the old ones concerning the nature of mathematics. Although the issues are old to teachers, they are new to each student. Without crossing the gap successfully, students will continue to be plagued by it in their subsequent studies. On the other hand, it can be perceived that although the use of modern technology may be new to teachers, it has become familiar to students and it does open up possibilities, especially in easing abstract computations and developing geometric intuition.

3.1. *The formal language of mathematics: first formal acquaintance*

Mathematical concepts are precisely formulated, the deductions are strictly structured and the results are often codified in formulas and equations. This formalism is essential for the exactness, clarity and usefulness of mathematics, but it is difficult. The formal language of mathematics takes great care, concentration and patience to learn. Further, the learning process itself is not entirely formal. Hence, students have to feel the need and the fun to learn the formalism, while teachers need to reveal the subtleties and meanings behind the formalism, by examples, counterexamples, implications, applications, or mere repetitions. For the clarity of the formal definitions is often deceptive. Without informal explanations, that apparent clarity hides the deeper meanings which students need. Likewise, the clarity of a formal proof does not necessarily ensure a clear understanding of the reasoning behind it. All these are particularly significant at the school–university transition when students first handle the formalism.

At their first encounter with mathematical formalism, we may try to convince students that one of the most important aspects of mathematics training is the clarity of thought which they may apply in life. In particular, the training in mathematics formulation teaches us how to use words carefully and how to qualify our statements with required conditions; in short, how to be clear and accurate. These are applicable to everyday life. However, I often caution students that the clarity of thought in mathematics depends on, besides its almost perfect formulation, the restriction to mathematical structures and objects. Hence, they should not pompously challenge people in other subjects on definitions and axioms; rather, they should have understanding of the ways of thinking and reasoning in other disciplines.

To students first encountering university mathematics, the formal language of mathematics probably makes sense only in getting across concrete meaning and clear reasoning. Somehow teachers have to convey the meaning in the formalism. Otherwise the formal language is completely foreign and students may get nothing out of the lectures, no matter how polished they are. Then what clarity of thought can there be? Worse, some conscientious students may assume that things are not accessible in class. That would seriously hinder the development of their abilities to grasp an idea right away or to follow the general flow of an argument. On the contrary, we may also demonstrate to students by examples, such as in arguments to prove the parallel axiom, that no matter how eloquent and enthusiastic an explanation may be, it may not be easier without setting down the formal steps, and it may be wrong.

However formal a course has to be taught, we might have to strive at concrete understanding and clear thinking, and ask students to do likewise. In this connection, I would like to share two naive attempts. First, in introducing the axiomatic method to beginning students, I go over the early parts of Euclid's Elements and then show them Hilbert's Foundation of Geometry. I would ask them to imagine themselves at the time of Euclid studiously assimilating 'A point is that which has no part', and at the time of Hilbert conscientiously working on the new axiomatics. Then I ask the students to draw their own conclusions. Nowadays we may laugh at Euclid's sentence 'A line is length without breadth' as primitive, but then we would probably miss many of his ideas (see [5, 6]). We should wonder how scholars through the ages understood the primitive formulation and got at what Euclid meant, recognizing the depth of the 'fifth postulate'. On the other hand, we must realize that a lot of modern mathematics is used in Hilbert's axiomatics. How then can we understand the sophisticated formulation without making the underlying mathematical ideas explicit? Second, I show students a Bourbaki text and tell them how it is reportedly written through chaotic meetings and endless drafts. Then I ask the students to suggest a version of debate they may find comfortable with to draw out what lies under cold formalisms.

3.2. *The rigour of mathematics: out of errors*

As explained in section 2.2, it seems that each area of mathematics has its own hazy beginnings and its own games to play. While we must pay full attention to rigour in each case, we have to lead students through formal beginnings to substantial content in a rather short time. This is a crucial factor at the school–university transition, when students begin on different areas in mathematics.

The following words of Chevalley [7] in the preface of his textbook *Fundamental Concepts of Algebra* influenced me greatly, 'One of the important pedagogical problems which a teacher of beginners in mathematics has to solve is to impart to his students the techniques of rigorous mathematical reasoning: this is an exercise in rectitude of thought, of which it would be futile to disguise the austerity.' Since my undergraduate days, I held tightly onto these words, but I took too long on details, got stuck too often and saw through things too rarely. How I wish to make it less austere for the less able students as they embark on their study of mathematics!

After making sure that students have access to clear rigorous texts, I would try two things. First, I try to make things look more like a game of puzzles, tricks and surprises than a rigid sequence of deductions. I do so by words of mouth (see 3.4.3) and by the blackboard. The blackboard is a wonderful two-dimensional medium to de-linearize a discussion in class. For example, a proof is often not actually one linear sequence of implications from beginning to end, but consists of components with criss-cross arguments. Such a proof is better explained on the blackboard, and in my experience, more transparently displayed and checked for rigour. Also the computer could help algorithmically and graphically, enlivening and substantiating some highly rarefied rigour (see 3.3). Second, I emphasize learning by trial and error. I try to convince students that in learning anything deep, it is easy to misunderstand and make mistakes, perhaps exactly at the most subtle points. Thus, knowing what goes wrong strengthens our understanding of what is correct. In mathematics especially, its unambiguous formulation provides us with an effective explicit checking system and its abstractness frees us from values about which we would not easily detect or admit our mistakes. I ask beginning mathematics students to check these out for themselves as they go on. To say it in a slightly different way, I shift the emphasis from rectitude of thought to clarity of thought. I hope it is not futile.

3.3. *The abstractness of mathematics: out of concreteness*

While there seem no half-way houses in the standard formalism and rigour of mathematics, there are obvious levels of abstractness. It is extremely important to let students go through necessary preliminary levels. Otherwise the abstraction may not make sense. This is particularly relevant at the school–university juncture.

It may be illusory that an abstraction is self-contained. A prime example is the abstraction of a topology on a set. One needs nothing more than set language. Apparently the abstraction is self-contained. Indeed it is simpler than the lower-level abstractions of a distance on a set and the Euclidean distance on an n-dimensional vector space. However, at least pedagogically, it depends on the lower-level abstractions. It is doubtful whether a student can really grasp the significance of arbitrary unions and finite intersections without the necessary associations. The marvellous formulation of 'nearness without any distance' would simply ring hollow. All the directions and steps for deductions from that formulation would be missing. Throughout the mathematics student's career, hard jumps in levels of abstractness recur: from surfaces to manifolds, from zero sets of polynomials to varieties to schemes, ... In each case, mastery of preliminary levels is essential.

An appropriate choice of level of abstractness at the school–university transition is of special importance. However, this may not be possible if student backgrounds are too diverse. In this respect, bridging courses and tutorial seminars as described in section 2.3 would help. Otherwise, the problem has to be tackled as each abstraction is introduced. In such cases, it is customary to review preliminary levels in examples and exercises. However, without sufficient details, these examples and exercises can be very frustrating. It is therefore a challenge to us to pinpoint the crucial details in successive abstractions and render transitions from one level of abstraction to another naturally and easily.

In fact, as mathematics gets more inter-disciplinary, we may have more occasions to introduce abstractions from scratch to students of other disciplines. To do this, it may be useful to emphasize mathematical objects and phenomena in the formulation of mathematical concepts and structures. For example, integers and polynomials, their primes and irreducibles (objects and phenomena) should be emphasized in teaching rings and ideals (concepts and structures). This may be just a simple psychological twist, but it gives a more tangible hold. Moreover, it seems to be in line with the growing use of the computer. As the computer gives us pictures of geometric objects that are more intricate than we can dream of, and carries out algorithms for the most abstract algebraic computations, we get an unprecedented concrete feeling of abstract mathematics.

As mathematics teachers, we may respond to these situations by providing clear, self-contained yet motivated expositions, with rigorous proofs, supplemented by instructive, detailed but focused explanations. It is delightful to see that many wonderful texts have indeed been published which ease beginners' ways to more abstract expositions.

3.4. *Psychological versus logical barriers at the school–university transition*

The above features, namely formalism, rigour and abstractness, very much set mathematics apart as a purely logical subject, somewhat remote from reality in the minds of many students. This feeling of remoteness can be a serious hindrance to understanding. It may also reinforce the belief that mathematics is accessible only to people with special aptitudes. This is especially the case at the school–university transition when students meet 'superior others'. Thus we are inevitably led to psychological factors of the gap between school and university mathematics for students. These psychological factors include both personal and inter-personal factors, as the following discussion indicates.

3.4.1. Starting from where a student is. This may sound very idealistic but it may be the most important point in easing a student's gap. For the student's gap is where the student finds it to be. We have to accept students' ignorance, confusion, clumsiness, misunderstanding, silly questions, wrong answers and so on. We have to communicate to students in terms they can understand and lead them to understanding, clarity, accuracy, rigour, elegance, and so on. Probably we also learn from each student at least on pedagogical issues.

While we hear publicity about attracting and accepting the best students, we also hear ready laments about the less able students. What dissonance! It may be that our own mission lies in starting from where the students are and leading them as far as they can go. Who knows at the school–university transition how far that will be for any one of them?

3.4.2. The competitive spirit of mathematics. Many secondary school students now train for mathematics competitions or join special classes for the mathematically gifted. This injects a strong competitiveness into a subject which is already a performers' game, probably straining the class atmosphere further. Depending on their own philosophies, teachers may use or channel this competitive spirit. My concern is whether it is too competitive at such an early stage and how we may ease the way for more students at this transition. My inclination is to place more emphasis on wrestling with mathematics than on competing for recognition, for I guess the former develops students' interest and abilities in mathematics more than the latter.

We may try two things in class. First, we may turn, if appropriate, theorems into problems and proofs into solutions, describing alternatives or failed attempts, pointing out connections and relating historical anecdotes. Second, we may ask students to compete eventually with the masters, encouraging them to have a look at the classics. For there lies the profoundest competition.

3.4.3. Communication: teacher–student, student–student. Solitary pursuit is perhaps essential in mathematics, but discussions with others can open our eyes. Both involve interesting psychological elements that deserve study. We shall just make a few relevant remarks here.

First, beginning students may not be able to get good feelings of mathematics from formal lectures. Informal discussions between teacher and student that bring out intuitive reasoning and mental processes can be helpful and inspiring. Second, beginning students may labour too much on some minor problems. Informal discussions that neatly take care of the minor problems save time for more important things. Third, sharing insights and enlightenment among fellow students, not so much as to out-smart one another, would make an arduous course more worthwhile and enjoyable. Last but not least, could it be that we unintentionally intimidate our students by being too sure about the clarity of our explanations and too uncompromising in our arguments? Do we care more for students who are not mathematically talented, according to examination or competition results, or for our own judgement? Are we annoyed when students are not enthusiastic about what we find marvellous, or when they are outright distracted?

3.4.4. Assessment and accountability. On the first day of class, I always tell students that mathematics is like sports, at least in that one cannot learn mathematics without doing it. Therefore, I say, they must do exercises by themselves and make sure to learn how they succeed and how they fail. There is however a practical gap: routine exercises will count less and less, and students who have been trained for mathematics competitions are at a distinct advantage. The pace of university courses also being faster, assessment and examination pressure on students is not light.

As students have to face examinations, teachers and courses are also being evaluated. These are essential for assessing results and outcomes, and for seeking subsequent improvements. In practice, however, things get complicated for both students and teachers. On the one hand, it is difficult to weigh efforts, understanding, and performance. On the other hand, assessments are inevitably used in selections, ranking, and funding. Amid these complications, I can only say that the ultimate accountability lies in the full development of students' potentials, interests and abilities. This brings me back to students' needs and starting from where students are. Do I place myself in a vulnerable, indefinite position, especially when circumstances impose on us other accounts?

4. Concluding remarks

This paper is based on teaching practice and learning experiences in Hong Kong over a long period, during which circumstances have changed dramatically. The paper is an attempt to locate and discuss some invariant and intrinsic factors concerning the gap between school and university mathematics. This gap is crucial for students. The hope is that a better understanding of the inherent factors would enable us to ease the gap for more students, starting from where they are, whatever circumstances may happen to be.

In particular, we emphasize factors that are related to the nature of mathematics and to the mathematical way of thinking. Perhaps cold formal reasoning has dominated mathematics education to a fault, and it is enlightening to examine also the psychology and mental processes of mathematical thoughts (see, for example [8]) and its manifestations at the school–university juncture. Simply we have for observations young people of our days with various potentials, preparations, motivations and possibly 'different kinds of minds'. Each concept we introduce, each theorem we explain and each problem we assign is a stimulus for response. On a small scale, we can thus accumulate data on mental blocks and tunnel through them, on specific gaps and build bridges over them, as well as unexpected misunderstandings and unsuspected links. It is interesting to test such data on the psychology and mental processes of mathematical thinking related to pedagogy against work on the psychology and mental processes of mathematical thinking related to (a) epistemology [9], (b) the philosophy and foundation of mathematics [10, 11]), (c) neurobiology and cognitive science [12], or (d) the research and application of mathematics. By getting at the underlying mathematical reasoning, psychology and mental processes, we would not be so alarmed by gaps, mistakes, poor standards or lower rankings, but we would have a sure hand to help students through. In a collegial manner too, school and university teachers could work on innovations, assessments and evaluations that would really lead to a better mathematics education.

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