2 Unit Bridging Course

Day 9 - The Quotient Rule of Differentiation

Jackie Nicholas

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We can form quotients of functions by dividing one function by another.

Here are some examples.

$$y = \frac{e^x}{x^2 - 1}$$

$$y = \frac{2t}{1 + 2t^3}$$

$$g(x) = \frac{x(x+1)}{x^2 + 3x + 1}$$

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If $y = \dfrac{u}{v}$, where $u = f(x)$ and $v = g(x)$, then

$$\frac{dy}{dx} = \frac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2}.$$

Alternatively, if $h(x) = \dfrac{f(x)}{g(x)}$ then

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

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Let $y = \dfrac{\overset{\substack{u \\ \downarrow}}{e^x}}{\underset{\substack{\uparrow \\ v}}{x^2 - 1}}$, then

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\frac{dy}{dx} &= \frac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2} \\
&= \frac{(x^2 - 1) \times e^x - e^x \times 2x}{(x^2 - 1)^2} \\
&= \frac{e^x(x^2 - 2x - 1)}{(x^2 - 1)^2}.
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\frac{dy}{dt} &= \frac{v \times \frac{du}{dt} - u \times \frac{dv}{dt}}{v^2} \\
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$$y = \frac{x(x+1)}{x^2+3x+1}.$$

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Practice questions

Differentiate the following functions:

(i) $\frac{x^2}{e^x+1}$

(ii) $\frac{x^3}{\sqrt{x-1}}$

(iii) $\frac{e^{x^2}}{1-x^3}$ (Hint: you need the chain rule too).

It is important that you follow the formula exactly and be careful of the $-$ sign.

Answers to practice questions

(i) $$\frac{(e^x+1)2x-x^2(e^x)}{(e^x+1)^2}$$

(ii) $$\frac{(x-1)^{\frac{1}{2}}(3x^2)-x^3\frac{1}{2}(x-1)^{-\frac{1}{2}}}{(x-1)}$$

(iii) $$\frac{(1-x^3)2xe^{x^2}-e^{x^2}(-3x^2)}{(1-x^3)^2}.$$