2 Unit Bridging Course - Day 11

The Logarithm Function

Collin Zheng

The Logarithm Function $y = \ln x$

Recall from the previous module that if $y = e^x$, then

$$x = \ln y.$$

Hence the **Logarithm Function**

$$y = \ln x \qquad (\text{for all } x > 0)$$

is precisely the inverse function of the exponential function.

That is,

$$\text{If } f(x) = e^x, \text{ then } f^{-1}(x) = \ln(x)$$

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The Logarithm Function $y = \ln x$ (cont.)

Recall the *Cancellation Property* for mutually inverse functions f and f^{-1}:

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x.$$

Setting $f(x) = e^x$ and $f^{-1}(x) = \ln(x)$, we thus have the following two identities:

$$\ln(e^x) = x$$

$$e^{\ln x} = x \qquad \text{for all } x > 0$$

That is, 'exponentiating' and 'logging' are mutually inverse operations and therefore cancel each other out.

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Example

We can use this cancelling phenomenon to simplify $e^{2\ln x}$:

$$\begin{aligned} e^{2\ln x} &= e^{\ln x^2} && \longleftarrow && \text{using log law \#3} \\ &= x^2. && \longleftarrow && \text{using cancellation property} \end{aligned}$$

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The Logarithm Function $y = \ln x$ (cont.)

We can also exploit the cancelling phenomenon to help solve equations.

Example

For instance, suppose we want to solve $e^{2x-3} = 7$.

Logging both sides, we get

$$2x - 3 = \ln 7.$$

Isolating x, we hence have

$$x = \frac{\ln 7 + 3}{2} \approx 2.47.$$

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Example

A final example – recall our population model from Day 8. That is, the population, $P(t)$, of an outback town is growing exponentially according to the formula

$$P(t) = 1000\, e^{0.2t},$$

where t is the number of years after the year 2000.

Recall that the model gave a population of 1000 in the year 2000, 7389 in 2010, and an estimate of roughly 55000 in 2020.

When will the population reach 1,000,000 people?

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The Logarithm Function $y = \ln x$ (cont.)

We must set $P(t)$ to be 1000000 and solve for t in the equation

$$1000000 = 1000\, e^{0.2t}, \quad \text{i.e.} \quad 1000 = e^{0.2t}.$$

Logging both sides, we get

$$\ln 1000 = 0.2t,$$

so

$$t = \frac{\ln 1000}{0.2} \approx 34.5.$$

That is, the population should reach 1 million by the middle of the year 2034.

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Practice Questions

- Simplify $e^{\frac{1}{2}\ln(x+9)}$.
- Simplify $\ln\left(e^{2x+1}\right)$.
- Solve $e^{-\ln x} = 2$.
- Solve $\ln 3x = 2$.
- Solve $\ln x^4 - \ln x = 0$.

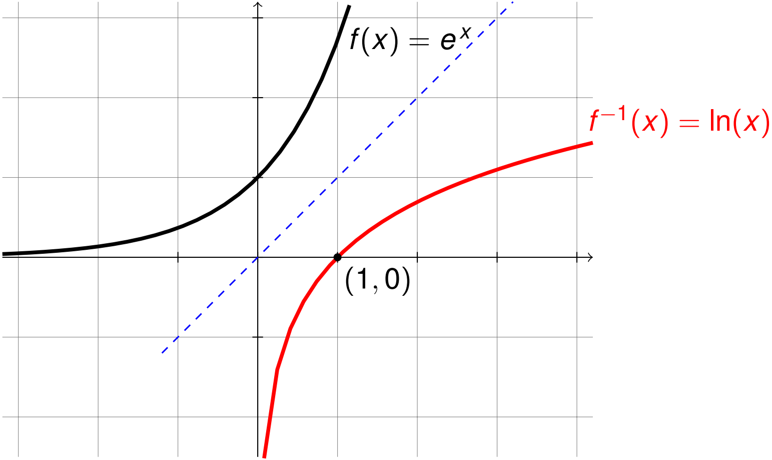
The Logarithm Function $y = \ln x$ (cont.)

Answers

- $\sqrt{x+9}$.
- $2x+1$.
- $\dfrac{1}{2}$.
- $\dfrac{e^2}{3}$.
- 1.

The graph of $y = \ln x$

Finally, recall that the graphs of two mutually inverse functions f and f^{-1} are symmetric about the diagonal line $y = x$:



Summary

- $y = e^x$ and $y = \ln x$ are mutually inverse functions. Hence $\ln(e^x) = x$ and $e^{\ln x} = x$.
- The graph of $y = \ln x$ is the reflection of the graph of $y = e^x$ about the diagonal $y = x$.