Mathematics Learning Centre

The University of Sydney

Optimisation using calculus

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1 What is differential calculus used for?

1.1 Introduction

The development of mathematics stands as one of the most important achievements of humanity, and the development of the calculus, both the differential calculus and integral calculus is one of most important achievements in mathematics. The practical applications of differential calculus are so wide ranging that it would be impossible to mention them all here. Suffice to say that differential calculus is an indispensable tool in *every* branch of science and engineering.

In elementary mathematics there are two main applications of differential calculus. One is to help in sketching curves, and the other is in optimisation problems. In this section we will give a brief introduction to how differential calculus is used in optimisation problems.

1.2 Optimisation problems

There are many practical situations in which we would like to make a quantity as small as we possibly can or as large as we possibly can.

For example, a manufacturer of bicycles trying to decide how much to charge for a model of bicycle would think that if he charges too little for the bicycles then he will probably sell a lot of bicycles but that he won't make much profit because the price is too low, and that if he charges too much for the bicycle then he won't make much profit because not many people will buy his bicycles. The manufacturer would like to find just the right price to charge to *maximise* his profit.

Similarly a farmer might realise that if she uses too little fertiliser on her crops then her yield will be very low, and if she uses too much fertiliser then she will poison the soil and her yield will be low. The farmer might like to know just how much fertiliser to use to *maximise* the crop yield.

A manufacturer of sheet metal cans that are meant to hold one litre of liquid might like to know just what shape to make the can so that the amount of sheet metal that is used is a *minimum*.

These are all examples of optimisation problems.

If we were to draw a graph of the profit versus price for the bicycle manufacturer mentioned above then finding the maximum profit is equivalent to finding the highest point on the graph. Similarly a minimisation problem may be thought of geometrically as finding the lowest point on the graph of a funcion.

1.2.1 Optimisation

To maximise a function $f(x)$ in a certain region of the x values, we are looking for the greatest value that $f(x)$ can possibly take for x in the region that we are interested in. This may or may not be at a stationary point.

Figure 1 illustrates this. In this figure, we are looking for the maximum and minimum of the function in the region $2 \leq x \leq 7$. In this region there are two stationary points, one a local maximum and one a local minimum. However notice that the maximum value of the function does not occur at the local maximum, but at the endpoint of the region, ie where $x = 7$. This point is not a stationary point, but it is still the maximum value of the function for $2 \leq x \leq 7$ because we are ignoring any x which is bigger than 7. On the other hand, in this case the minimum value of this function for $2 \leq x \leq 7$ is found at a stationary point. Now we are in a position to tell you exactly how to find the maximum or minimum of a function.

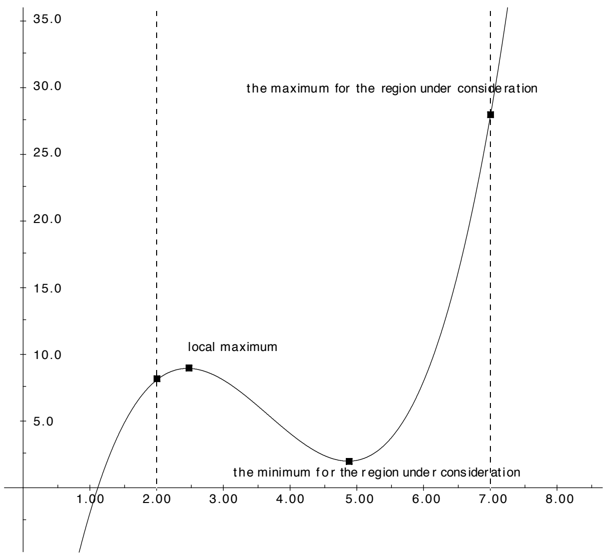

Figure 1: The maximum is found at the endpoint of the region under consideration, and not at a stationary point. The minimum is found inside the region under consideration at a stationary point.

The location of maxima and minima

A function $f(x)$ may or may not have a maximum or minimum value in a particular region of x values. However, if they do exist the maximum and the minimum values must occur at one of three places:

1. At the endpoints (if they exist) of the region under consideration.
2. Inside the region at a stationary point.
3. Inside the region at a point where the derivative does not exist.

Notes

1. It is easy to find an example of a function which has no maximum or minimum in a particular region. For example the function $f(x) = x$ has neither a maximum nor a minimum value for $-\infty < x < \infty$. Its graph simply keeps increasing as the values of x increase. Referring to Point 1 above, if for example the region under consideration was $-\infty < x < \infty$ then this region has no endpoints. As another example, the region $x \geq 1$ has only one endpoint, $x = 1$.

2. A note about Point 3 above: we will not treat points where the derivative does not exist. However you should be aware that there may be such points, and that the maximum or minimum may be found at one. For more information consult a more comprehensive calculus text.

Now that we know exactly where the maxima or minima can occur, we can give a procedure for finding them.

Procedure for finding the maximum or minimum values of a function.

1. Find the endpoints of the region under consideration (if there are any).
2. Find all the stationary points in the region.
3. Find all points in the region where the derivative does not exist.
4. Substitute each of these into the function and see which gives the greatest (or smallest) function value.

Example

Find the minimum value and the maximum value of the function $f(x) = x^2e^x$ for $-4 \le x \le 1$.

Solution

We will follow the procedure outlined above. The endpoints are -4 and 1. Differentiating we obtain $f'(x) = x^2e^x + 2xe^x = x(x+2)e^x$. Setting $f'(x) = 0$ and solving we get stationary points at $x = 0$ and $x = -2$. There are no points where the derivative does not exist. Therefore the maximum and minimum values will be found at one of the points $x = -4, -2, 0, 1$. Substituting we obtain $f(-4) \approx 0.29$, $f(-2) \approx 0.54$, $f(0) = 0$ and $f(1) = e \approx 2.7$. Therefore the maximum value occurs at $x = 1$ and is equal to e, and the minimum value occurs at $x = 0$ and is 0.

Example

Find the maximum and minimum values of the function $g(t) = \frac{1}{3}t^3 - t + 2$ for $0 \le t \le 3$.

Solution

The endpoints are $t = 0$ and $t = 3$. Differentiating and equating to zero we get $g'(t) = t^2 - 1 = (t-1)(t+1) = 0$ so the stationary points are at $t = -1, 1$. Since -1 is not in the region, the possible locations of the maximum and the minimum are $t = 0, 1, 3$. Substituting into g we obtain $g(0) = 2$, $g(1) = \frac{4}{3}$ and $g(3) = 8$. The maximum is therefore $g(3) = 8$ and the minimum is $g(1) = \frac{4}{3}$.

Example

A farmer is to make a rectangular paddock. The farmer has 100 metres of fencing and wants to make the rectangle that will enclose the greatest area. What dimensions should the rectangle be?

Solution

There are many rectangular paddocks that can be made with 100 metres of fencing. If we call one side of the rectangle x, then because the perimeter is 100, the other side of the rectangle is $50 - x$. This is illustrated in Figure 2.

The area of the paddock is then $A(x) = x(50 - x)$. We must maximise the function $A(x)$ for $0 \le x \le 50$ (since the sides of the rectangle cannot have negative length). Now

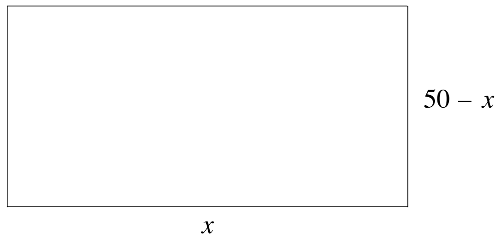

Figure 2: Rectangular paddock with perimeter 100m.

$\frac{dA}{dx} = 50 - 2x$ which is zero when $x = 25$. Thus $x = 25$ is the only stationary point and the maximum is found at one of the points $x = 0, 25, 50$. Substituting these values into $A(x)$ we find that the maximum occurs when $x = 25$. The rectangular paddock with the maximum area is a square.

Exercises

1. Find the maximum and the minimum of the function $f(x) = x^4 - 2x^2$ for $-1 \le x \le 2$

2. Maximise the function $g(t) = te^{-t^2}$ for $-2 < t < 2$.

3. Find the minimum value of $h(u) = 2u^3 + 3u^2 - 12u + 5$ in the region $-3 \le u \le 2$.

4. A farmer wishes to make a rectangular chicken run using an existing wall as one side. He has 16 metres of wire netting. Find the dimensions of the run which will give the maximum area. What is this area?

Solutions to exercises

1. $f'(x) = 4x^3 - 4x$ so $f'(x) = 0$ at $x = 0, \pm 1$ and the maxima and minima must occur at the points $x = -1, 0, 1, 2$. Substituting these values into $f(x)$ we find that the maximum occurs at $x = 2$ and the minimum occurs at $x = -1$ and at $x = 1$.

2. $g'(t) = (1 - 2t^2)e^{-t^2}$. Setting this equalto zero and solving we find that the stationary points are at $t = \pm\frac{1}{\sqrt{2}}$ and the maximum must occur at one of the points $t = -2, \pm\frac{1}{\sqrt{2}}, 2$. Substituting into $g(t)$ we find that the maximum value occurs at $t = \frac{1}{\sqrt{2}}$.

3. $h'(u) = 6u^2 + 6u - 12 = 6(u^2 + u - 2)$. The stationary points are at $u = -2, 1$ and the minimum value occurs at one of the points $u = -3, -2, 1, 2$. Substituting into $h(u)$ we find that the minimum occurs at $u = 1$.

4. If we let the side of the run that is opposite the existing wall have length x, then the other side of the run has length $8 - x/2$.

The area of the run is $A(x) = x(8 - x/2)$ and we must maximise this function in the region $0 \leq x \leq 16$. Differentiating give $A'(x) = 8 - x$ so the only stationary point is at $x = 8$. The maximum occurs at one of $x = 0, 8, 16$. Substituting, we see that the maximum occurs when $x = 8$, giving an area of 32 square metres.

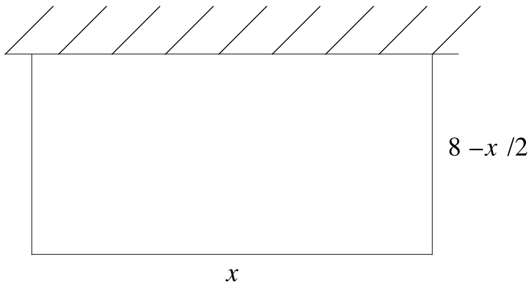

Figure 3: A chicken run built against the side of an existing wall, with 16 metres of netting.