# 2 Unit Bridging Course

Day 9 - The Product Rule of Differentiation

THE UNIVERSITY OF
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Another way of combining functions to make new ones is by multiplying then together to form a product.

Examples:

$$y = (x + 1)(x^2 + 3)$$

$$f(x) = \sqrt{x}(x^3 - 3x^2 + 7)$$

$$h(t) = t\, e^t.$$

We can differentiate the first two examples by multiplying out the brackets (note that $\sqrt{x} = x^{\frac{1}{2}}$), but that method does not work for the third example.

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We need a rule to differentiate product functions in general.

### The Product rule

If $y = u \times v$, where $u = f(x)$ and $v = g(x)$, then

$$\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}.$$

Alternatively, If $h(x) = f(x) \times g(x)$ then

$$h'(x) = f(x)g'(x) + g(x)f'(x).$$

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## Example

Let $y = (x+1)(x^2+3)$, then

$$\underset{\underset{u}{\uparrow}}{\phantom{(x+1)}} \qquad \underset{\underset{v}{\uparrow}}{\phantom{(x^2+3)}}$$

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\begin{aligned}
\frac{dy}{dx} &= u \times \frac{dv}{dx} + v \times \frac{du}{dx} \\
&= (x+1) \times 2x + (x^2+3) \times 1 \\
&= 2x^2 + 2x + x^2 + 3 \\
&= 3x^2 + 2x + 3.
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Let $y = f(x) = \sqrt{x}(x^3 - 3x^2 + 7) = x^{\frac{1}{2}}(x^3 - 3x^2 + 7)$ , then

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Consider $y = h(t) = te^t$. Let $u = t$ and $v = e^t$.

Notice that $y$, $u$ and $v$ are functions of $t$ so the derivatives will be with respect to $t$.

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\begin{aligned}
\frac{dy}{dt} &= u \times \frac{dv}{dt} + v \times \frac{du}{dt} \\
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Practice questions

Differentiate the following functions:

  (i) $x^2 e^x$

 (ii) $x^3 \sqrt{x-1}$

(iii) $(x+1)e^{x^2}$ (Hint: you need the chain rule too).

Answers to practice questions

(i) $x^2 e^x + 2xe^x$

(ii) $x^3 \times \frac{1}{2}(x-1)^{-\frac{1}{2}} + 3x^2(x-1)^{\frac{1}{2}}$

(iii) $2x(x+1)e^{x^2} + e^{x^2}(1)$.