Mathematics Learning Centre

The University of Sydney

Derivatives of trigonometric functions

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1 Derivatives of trigonometric functions

To understand this section properly you will need to know about trigonometric functions. The Mathematics Learning Centre booklet *Introduction to Trigonometric Functions* may be of use to you.

There are only two basic rules for differentiating trigonometric functions:

$$\begin{aligned}\frac{d}{dx}\sin x &= \cos x\\ \frac{d}{dx}\cos x &= -\sin x.\end{aligned}$$

For differentiating all trigonometric functions these are the only two things that we need to remember.

Of course all the rules that we have already learnt still work with the trigonometric functions. Thus we can use the product, quotient and chain rules to differentiate functions that are combinations of the trigonometric functions.

For example, $\tan x = \frac{\sin x}{\cos x}$ and so we can use the quotient rule to calculate the derivative.

$$\begin{aligned}f(x) &= \tan x = \frac{\sin x}{\cos x},\\ f'(x) &= \frac{\cos x.(\cos x) - \sin x.(-\sin x)}{(\cos x)^2}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos x} = \frac{1}{\cos^2 x} \qquad (\text{since } \cos^2 x + \sin^2 x = 1)\\ &= \sec^2 x\end{aligned}$$

Note also that

$$\frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = 1 + \tan^2 x$$

so it is also true that

$$\frac{d}{dx}\tan x = \sec^2 x = 1 + \tan^2 x.$$

Example

Differentiate $f(x) = \sin^2 x$.

Solution

$f(x) = \sin^2 x$ is just another way of writing $f(x) = (\sin x)^2$. This is a composite function, with the outside function being $(\cdot)^2$ and the inside function being $\sin x$.

By the chain rule, $f'(x) = 2(\sin x)^1 \times \cos x = 2 \sin x \cos x$. Alternatively using the other method and setting $u = \sin x$ we get $f(x) = u^2$ and

$$\frac{df(x)}{dx} = \frac{df(x)}{du} \times \frac{du}{dx} = 2u \times \frac{du}{dx} = 2 \sin x \cos x.$$

Example

Differentiate $g(z) = \cos(3z^2 + 2z + 1)$.

Solution

Again you should recognise this as a composite function, with the outside function being $\cos(\cdot)$ and the inside function being $3z^2 + 2z + 1$. By the chain rule $g'(z) = -\sin(3z^2 + 2z + 1) \times (6z + 2) = -(6z + 2)\sin(3z^2 + 2z + 1)$.

Example

Differentiate $f(t) = \frac{e^t}{\sin t}$.

Solution

By the quotient rule

$$f'(t) = \frac{e^t \sin t - e^t \cos t}{\sin^2 t} = \frac{e^t(\sin t - \cos t)}{\sin^2 t}.$$

Example

Use the quotient rule or the composite function rule to find the derivatives of $\cot x$, $\sec x$, and $\operatorname{cosec} x$.

Solution

These functions are defined as follows:

$$\begin{aligned}
\cot x &= \frac{\cos x}{\sin x} \\
\sec x &= \frac{1}{\cos x} \\
\csc x &= \frac{1}{\sin x}.
\end{aligned}$$

By the quotient rule

$$\frac{d\cot x}{dx} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x}.$$

Using the composite function rule

$$\frac{d\sec x}{dx} = \frac{d(\cos x)^{-1}}{dx} = -(\cos x)^{-2} \times (-\sin x) = \frac{\sin x}{\cos^2 x}.$$

$$\frac{d\csc x}{dx} = \frac{d(\sin x)^{-1}}{dx} = -(\sin x)^{-2} \times \cos x = -\frac{\cos x}{\sin^2 x}.$$

Exercise 1

Differentiate the following:

a.	$\cos 3x$	**b.**	$\sin(4x+5)$	**c.**	$\sin^3 x$	**d.**	$\sin x \cos x$	**e.**	$x^2 \sin x$
f.	$\cos(x^2+1)$	**g.**	$\dfrac{\sin x}{x}$	**h.**	$\sin \dfrac{1}{x}$	**i.**	$\tan(\sqrt{x})$	**j.**	$\dfrac{1}{x} \sin \dfrac{1}{x}$

Solutions to Exercise 1

a. $\dfrac{d}{dx}\cos 3x = -3\sin 3x$

b. $\dfrac{d}{dx}\sin(4x+5) = 4\cos(4x+5)$

c. $\dfrac{d}{dx}\sin^3 x = 3\sin^2 x\cos x$

d. $\dfrac{d}{dx}\sin x\cos x = \cos^2 x - \sin^2 x$

e. $\dfrac{d}{dx}x^2\sin x = 2x\sin x + x^2\cos x$

f. $\dfrac{d}{dx}\cos(x^2+1) = -2x\sin(x^2+1)$

g. $\dfrac{d}{dx}\left(\dfrac{\sin x}{x}\right) = \dfrac{x\cos x - \sin x}{x^2}$

h. $\dfrac{d}{dx}\sin\dfrac{1}{x} = -\dfrac{1}{x^2}\cos\dfrac{1}{x}$

i. $\dfrac{d}{dx}\tan\sqrt{x} = \dfrac{1}{2\sqrt{x}}\sec^2\sqrt{x}$

j. $\dfrac{d}{dx}\left(\dfrac{1}{x}\sin\dfrac{1}{x}\right) = -\dfrac{1}{x^2}\sin\dfrac{1}{x} - \dfrac{1}{x^3}\cos\dfrac{1}{x}$