2 Unit Bridging Course – Day 8

Exponential Functions

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Recall from the previous day that given the expression x^n:

- n is called the *index* or *power* of x;
- x is called the *base*.

However, the word **exponent** is often used instead of index or power. Functions where the independent variable is in the index (e.g. 3^x, 10^{2x}) are called **exponential functions.**

It turns out that exponential functions describe a variety of real-world phenomena – from population models and financial growth to the dynamics of heat transfer and decay of radioactive isotopes.

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The function $y = 2^x$

Example

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Can you think of some important properties of this function?
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The function $y = 2^x$ (cont.)

Firstly, notice that 2 raised to any power is always a positive number. E.g. $2^2 = 4$, $2^3 = 8$, etc.

In fact, even when 2 is raised to the power of a negative number, the index laws tell you that the result is another positive number! For instance,

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8}.$$

Therefore, the first important property of 2^x is that **the function value is never negative.** In other words, the graph sits entirely above the x-axis.

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The function $y = 2^x$ (cont.)

Secondly, the **function is always increasing.** This pattern can be observed from a table of values:

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Thirdly, notice how as $x \to -\infty$, the y-value inches ever closer to the x-axis (but will never touch it). That's because as x becomes more and more negative, the y-value becomes a smaller and smaller fraction, but never reaches zero.

Terminology-wise, the x-axis is called an '**asymptote**' for the graph.

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The function $y = 2^x$ (cont.)

Fourthly, the graph isn't just increasing, it's increasing **exponentially**:

- $2^0 = 1$
- $2^1 = 2$
- $2^2 = 2 \times 2 = 4$
- $2^3 = 2 \times 2 \times 2 = 8$, etc.

That is, as you steadily increase x, the corresponding y values are increasingly *very very* rapidly. When x is only 10 for instance, the y-value is already a massive $2^{10} = 1024$!

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The function $y = 2^x$ (cont.)

Finally, since $2^0 = 1$, the graph cuts the y-axis at $(0, 1)$.

In fact, since the graph doesn't the x-axis at all (because the x-axis is an asymptote), the point $(0, 1)$ represents the sole point of intersection between the graph and either axis.

In summary:

- The function $y = 2^x$ is always positive.
- It's increasing 'exponentially'.
- It approaches the x-axis as $x \to -\infty$.
- It cuts the y-axis at $(0, 1)$.

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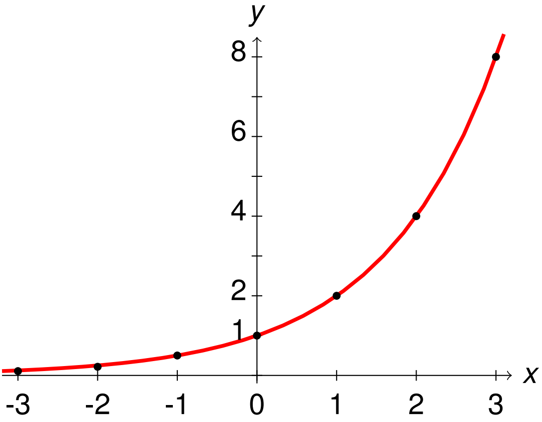
The function $y = 2^x$ (cont.)

Putting all this together, our graph for $y = 2^x$ looks like this:



Practice Question

Can you think of some properties of the exponential function

$$y = 3^x?$$

The function $y = 3^x$ (cont.)

Answer

It turns out that all the properties described for $y = 2^x$ above holds for $y = 3^x$!

- Like the number 2, 3 raised to the power of anything is always positive, so 3^x is always positive, just like 2^x.
- As with 2^x, 3^x increases exponentially.
- As with 2^x, the x-axis is an asymptote for 3^x.
- From the previous day on index laws, we know that $a^0 = 1$ for any number $a \neq 0$, so 3^0 also equals 1. Therefore $y = 3^x$ also cuts the y-axis at $(0, 1)$.

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The function $y = 3^x$ (cont.)

This means that the graph of $y = 3^x$ should be very similar in shape and form to the graph of $y = 2^x$.

In fact, the only difference between the two graphs is their *steepness*. Because 3 is a higher base than 2, $y = 3^x$ will increase quicker than $y = 2^x$. That is, 3^x has a *steeper graph* than 2^x.

To see this more clearly, observe the table of values below:

x	-3	-2	-1	0	1	2	3
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
3^x	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27

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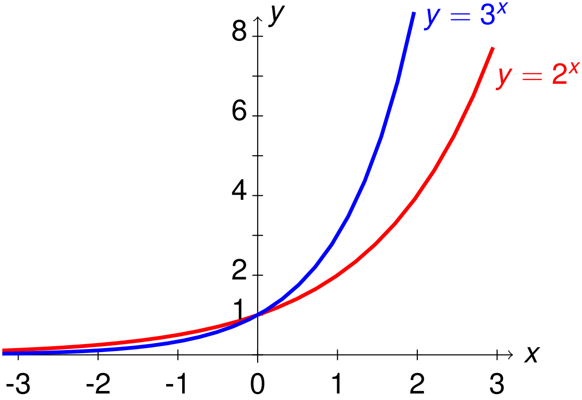
The function $y = 3^x$ (cont.)

Here are both graphs shown together. Observe the similarity in *shape* for both graphs but their difference in *steepness*.



The Exponential Function e^x

Generally, all exponential functions of the form $y = a^x$ (for any number $a > 1$) share the same shape, with the major difference being their steepness. For instance:

- The gradient of $y = 2^x$ at $(0, 1)$ is approximately 0.69.
- The gradient of $y = 3^x$ at $(0, 1)$ is approximately 1.1.

Note this means that somewhere between 2 and 3 lies a special value e so that e^x has a gradient of $e^0 = 1$ at $x = 0$, i.e. e^x equals the value of its own gradient at the point $(0, 1)$!

It turns out that e is roughly equal to 2.72.

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The Exponential Function e^x (cont.)

However, what makes $y = e^x$ truly remarkable is that it is equal to its gradient for ***all*** x-values, not just at $x = 0$!

Definition: The Exponential Function

The function $y = e^x$ is called *The* **Exponential Function**, which has the special property that

$$\frac{dy}{dx} = e^x \text{ for all } x.$$

The number e itself is a special non-terminating number called **Euler's constant** roughly equal to 2.7183 (correct to 4 decimal places).

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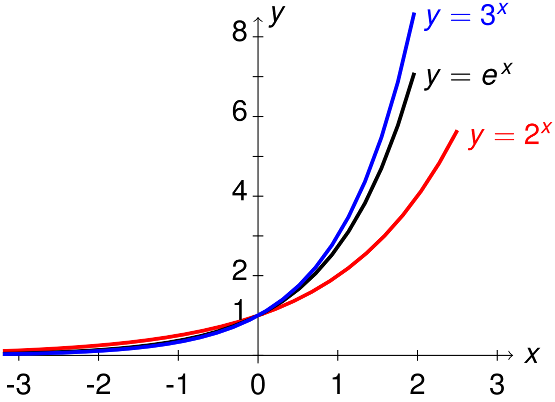
Here are the graphs for all three functions plotted together.



Summary

- The term *exponent* is another word for ‘index’ or ‘power’.
- Functions where the independent variable is in the exponent are called *exponential functions*.
- Exponential functions of the form $y = a^x$ (for $a > 1$) share the same shape but differ in their steepness.
- $y = e^x$ (where $e \approx 2.72$) is called *the* exponential function and has the special property that $\frac{dy}{dx} = e^x$.
- Exponential functions describe many real-world phenomena.