# 2 Unit Bridging Course – Day 7

## Index Laws II – Examples and the power law for differentiation

Clinton Boys

THE UNIVERSITY OF SYDNEY

Let's see some examples of how to manipulate expressions involving indices.

### Example

Simplify

$$\frac{\sqrt[3]{(x^2)}}{x}.$$

The best way to attack problems like this is by working from the inside out. This will be more useful the more complicated the problem.

Let's see some examples of how to manipulate expressions involving indices.

## Example

Simplify

$$\frac{\sqrt[3]{(x^2)}}{x}.$$

The best way to attack problems like this is by working from the inside out. This will be more useful the more complicated the problem.

Let's see some examples of how to manipulate expressions involving indices.

## Example

Simplify

$$\frac{\sqrt[3]{(x^2)}}{x}.$$

The best way to attack problems like this is by working from the inside out. This will be more useful the more complicated the problem.

Example (continued)

Remember that $\sqrt[3]{y} = y^{1/3}$ for any number $y$. So

$$\sqrt[3]{x^2} = (x^2)^{1/3} = x^{2\times 1/3} = x^{2/3}$$

by the second index law.

Now by the third index law,

$$\frac{\sqrt[3]{x^2}}{x} = \frac{x^{2/3}}{x} = x^{2/3-1} = x^{-1/3},$$

since $x = x^1$.

## Example (continued)

Remember that $\sqrt[3]{y} = y^{1/3}$ for any number $y$. So

$$\sqrt[3]{x^2} = (x^2)^{1/3} = x^{2\times 1/3} = x^{2/3}$$

by the second index law.

Now by the third index law,

$$\frac{\sqrt[3]{x^2}}{x} = \frac{x^{2/3}}{x} = x^{2/3-1} = x^{-1/3},$$

since $x = x^1$.

## Example

Simplify

$$\frac{1}{x^{-2}\sqrt{x^{-1}}}.$$

Let's simplify the bottom first. $\sqrt{x^{-1}} = (x^{-1})^{1/2} = x^{-1/2}$ (by the second index law). Multiplying this by $x^{-2}$ gives that the bottom of our expression is

$$x^{-2}x^{-1/2} = x^{-2-1/2} = x^{-5/2}$$

by the first index law.

## Example

Simplify

$$\frac{1}{x^{-2}\sqrt{x^{-1}}}.$$

Let's simplify the bottom first. $\sqrt{x^{-1}} = (x^{-1})^{1/2} = x^{-1/2}$ (by the second index law). Multiplying this by $x^{-2}$ gives that the bottom of our expression is

$$x^{-2}x^{-1/2} = x^{-2-1/2} = x^{-5/2}$$

by the first index law.

## Example

Simplify

$$\frac{1}{x^{-2}\sqrt{x^{-1}}}.$$

Let's simplify the bottom first. $\sqrt{x^{-1}} = (x^{-1})^{1/2} = x^{-1/2}$ (by the second index law). Multiplying this by $x^{-2}$ gives that the bottom of our expression is

$$x^{-2}x^{-1/2} = x^{-2-1/2} = x^{-5/2}$$

by the first index law.

Example (continued)

Now we know that

$$\frac{1}{x^a} = x^{-a}$$

for any number $x$ (this was a consequence of the third index law). In this example, $a = -5/2$ and so

$$\frac{1}{x^{-2}\sqrt{x^{-1}}} = \frac{1}{x^{-5/2}} = x^{5/2}.$$

Example (continued)

Now we know that

$$\frac{1}{x^a} = x^{-a}$$

for any number $x$ (this was a consequence of the third index law). In this example, $a = -5/2$ and so

$$\frac{1}{x^{-2}\sqrt{x^{-1}}} = \frac{1}{x^{-5/2}} = x^{5/2}.$$

## Practice questions

Simplify the following expressions using the index laws:

(i) $x^{\frac{1}{4}} \times x^{-\frac{5}{4}} \div x^{-2}$

(ii) $\dfrac{x^6 \sqrt[2]{x^6}}{x^3}$

(iii) $\dfrac{x^{-1}\sqrt{x^5}}{x^{\frac{5}{3}}}$

(iv) $\dfrac{\sqrt[3]{x^{-\frac{5}{2}} x^{\frac{3}{4}} x^{-1}}}{\sqrt{x}}$.

## Answers

(i) $x$

(ii) $x^6$

(iii) $x^{-\frac{1}{6}}$

(iv) $x^{-\frac{25}{12}}$

# Power law for differentiation: revisited

Recall on Day 5 we saw the power law for differentiating polynomials:

$$\frac{d}{dx}x^n = nx^{n-1},$$

where $n$ was a positive integer.

With our knowledge of the index laws, we can now apply the power law to situations where $n$ is a negative number, or a fraction.

# Power law for differentiation: revisited

Recall on Day 5 we saw the power law for differentiating polynomials:

$$\frac{d}{dx}x^n = nx^{n-1},$$

where $n$ was a positive integer.

With our knowledge of the index laws, we can now apply the power law to situations where $n$ is a negative number, or a fraction.

# Power law for differentiation: revisited

**Power law for differentiation**

If $n$ is any number,

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Example

$$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2})$$

**Example**

$$
\begin{aligned}
\frac{d}{dx}(\sqrt{x}) &= \frac{d}{dx}(x^{1/2}) \\
&= \frac{1}{2}x^{1/2-1}
\end{aligned}
$$

### Example

$$\begin{aligned}\frac{d}{dx}(\sqrt{x}) &= \frac{d}{dx}(x^{1/2}) \\ &= \frac{1}{2}x^{1/2-1} \\ &= \frac{1}{2}x^{-1/2}.\end{aligned}$$

# Examples

## Example

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1})$$

Example

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{x}\right) &= \frac{d}{dx}(x^{-1}) \\ &= (-1)x^{-1-1} \end{aligned}$$

## Example

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{x}\right) &= \frac{d}{dx}(x^{-1}) \\ &= (-1)x^{-1-1} \\ &= -x^{-2}\end{aligned}$$

## Example

$$
\begin{aligned}
\frac{d}{dx}\left(\frac{1}{x}\right) &= \frac{d}{dx}(x^{-1}) \\
&= (-1)x^{-1-1} \\
&= -x^{-2} \\
&= \frac{-1}{x^2}.
\end{aligned}
$$

## Example

If $f(x)$ is the function

$$f(x) = 3x^2 + 1 - \frac{3}{x^2},$$

find $f'(x)$.

We can differentiate the expression term-by-term using the power law. The first term is easy –

$$\frac{d}{dx}(3x^2) = 6x.$$

## Example

If $f(x)$ is the function

$$f(x) = 3x^2 + 1 - \frac{3}{x^2},$$

find $f'(x)$.

We can differentiate the expression term-by-term using the power law. The first term is easy –

$$\frac{d}{dx}(3x^2) = 6x.$$

## Example

If $f(x)$ is the function

$$f(x) = 3x^2 + 1 - \frac{3}{x^2},$$

find $f'(x)$.

We can differentiate the expression term-by-term using the power law. The first term is easy –

$$\frac{d}{dx}(3x^2) = 6x.$$

## Example (continued)

The middle term is just the constant 1, whose derivative is zero.

For the final term, we need to use the index laws:

Hence

$$f'(x) = 6x - 6x^{-3} = 6\left(x - \frac{1}{x^3}\right).$$

# Examples

## Example (continued)

The middle term is just the constant 1, whose derivative is zero.

For the final term, we need to use the index laws:

$$\frac{d}{dx}\left(\frac{3}{x^2}\right) = 3\frac{d}{dx}\left(\frac{1}{x^2}\right)$$

Hence

$$f'(x) = 6x - 6x^{-3} = 6\left(x - \frac{1}{x^3}\right).$$

# Examples

## Example (continued)

The middle term is just the constant 1, whose derivative is zero.

For the final term, we need to use the index laws:

$$\begin{aligned} \frac{d}{dx}\left(\frac{3}{x^2}\right) &= 3\frac{d}{dx}\left(\frac{1}{x^2}\right) \\ &= 3\frac{d}{dx}(x^{-2}) \end{aligned}$$

Hence

$$f'(x) = 6x - 6x^{-3} = 6\left(x - \frac{1}{x^3}\right).$$

## Example (continued)

The middle term is just the constant 1, whose derivative is zero.

For the final term, we need to use the index laws:

$$\begin{aligned}\frac{d}{dx}\left(\frac{3}{x^2}\right) &= 3\frac{d}{dx}\left(\frac{1}{x^2}\right)\\ &= 3\frac{d}{dx}(x^{-2})\\ &= 3((-2)\times x^{-2-1})\end{aligned}$$

Hence

$$f'(x) = 6x - 6x^{-3} = 6\left(x - \frac{1}{x^3}\right).$$

## Example (continued)

The middle term is just the constant 1, whose derivative is zero.

For the final term, we need to use the index laws:

$$\begin{aligned}\frac{d}{dx}\left(\frac{3}{x^2}\right) &= 3\frac{d}{dx}\left(\frac{1}{x^2}\right)\\ &= 3\frac{d}{dx}(x^{-2})\\ &= 3((-2)\times x^{-2-1})\\ &= -6x^{-3}.\end{aligned}$$

Hence

$$f'(x) = 6x - 6x^{-3} = 6\left(x - \frac{1}{x^3}\right).$$

## Example (continued)

The middle term is just the constant 1, whose derivative is zero.

For the final term, we need to use the index laws:

$$\begin{aligned}
\frac{d}{dx}\left(\frac{3}{x^2}\right) &= 3\frac{d}{dx}\left(\frac{1}{x^2}\right) \\
&= 3\frac{d}{dx}(x^{-2}) \\
&= 3((-2) \times x^{-2-1}) \\
&= -6x^{-3}.
\end{aligned}$$

Hence

$$f'(x) = 6x - 6x^{-3} = 6\left(x - \frac{1}{x^3}\right).$$

## Practice questions

For each function below, find $f'(x)$:

(i) $f(x) = x - \frac{1}{x}$.

(ii) $f(x) = \sqrt[3]{x}$ (*Hint:* rewrite as $x^{1/3}$).

(iii) $f(x) = 2x^3 - x^{-4}$.

(iv) $f(x) = 2\sqrt[3]{x^4} + x\sqrt{x^{-1}}$ (*Hint:* first simplify using the index laws).

## Answers

(i) $f'(x) = 1 + \frac{1}{x^2}$

(ii) $f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$

(iii) $f'(x) = 6x^2 + 4x^{-5}$

(iv) $f(x) = 2x^{4/3} + x^{1/2}$, so

$$f'(x) = \frac{8}{3}x^{1/3} + \frac{1}{2}x^{-1/2} = \frac{8}{3}\sqrt[3]{x} + \frac{1}{2\sqrt{x}}.$$