2 Unit Bridging Course – Day 8

Composite Functions

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The Composition of Two Functions

Consider two functions $f(x) = x + 1$ and $g(u) = u^2$.

$$\xrightarrow{x} \boxed{\quad f \quad} \xrightarrow{x+1}$$

$$\xrightarrow{u} \boxed{\quad g \quad} \xrightarrow{u^2}$$

A new function $(g \circ f)(x)$ can be formed from f and g by setting the output of f to be the input of g, giving $g(f(x)) = (x + 1)^2$.

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$g \circ f$ is read as "g composed with f", and the function $(g \circ f)(x) = g(f(x))$ is called a **composite function.**

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Identifying Composite Functions

Overall, you may think of composition as a process of creating a *new* function from two existing ones.

However, our main concern is the challenge of identifying when a given function is a composite function.

For instance, given the function $y = (x + 1)^2$ to begin with, how can you *show* that it is a composite function?

One way is by explicitly defining the function's two component functions f and g, followed by demonstrating that the function can be written in the required form $g(f(x))$.

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Identifying Composite Functions (cont.)

So by setting $f(x) = x + 1$ and $g(u) = u^2$, we have:

$$\begin{aligned} y &= (x+1)^2 \\ &= (f(x))^2 && \longleftarrow && \text{since } f(x) = x + 1 \\ &= g(f(x)). && \longleftarrow && \text{since } g(u) = u^2 \end{aligned}$$

Since $f(x) = x + 1$ is 'nested' inside of $y = (x + 1)^2$, $f(x)$ is called the **inner** or **inside function**, and g is called the **outer** or **outside function**.

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Example

Here's another example. Consider the function $y = e^{3x}$.

Since the $3x$ appears to be 'nested' within the exponential function, we should set $f(x) = 3x$ and $g(u) = e^u$. Then:

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One more example. Consider the function $y = \sqrt{x^2 - 2}$.

Since $x^2 - 2$ appears to be 'nested' within the square root function, we should set $f(x) = x^2 - 2$ and $g(u) = \sqrt{u}$. Then:

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Looking Ahead

The process of verifying a given function as a composite function may seem laborious, but there is a good reason for developing this skill.

Suppose you wanted to differentiate

$$y = (x+1)^2.$$

Expanding the right-hand side gives:

$$y = x^2 + 2x + 1,$$

and so

$$\frac{dy}{dx} = 2x + 2.$$

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But how about incrementing the power from 2 to 7 and trying to differentiate

$$y = (x+1)^7?$$

Unlike $y = (x+1)^2$, one would need to expend considerably more effort in expanding the right-hand side:

$$y = x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 7x + 1.$$

Hence:

$$\frac{dy}{dx} = 7x^6 + 42x^5 + 105x^4 + 140x^3 + 105x^2 + 42x + 7.$$

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However, it turns out that there is a special technique called the **Chain Rule** (to be studied in Day 9) which can be used to differentiate a composite function *provided* its inside and outside functions have been identified explicitly.

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Practice Questions

Here are several practice questions to train your proficiency in decomposing composite functions.

For the following, identify the inside function u and express y as a function of u:

- $y = (x^3 + 5)^6$
- $y = e^{3-x^2}$
- $y = \sqrt{x^2 + 4x - 1}$.

Solutions

- $u = x^3 + 5$, $y = u^6$;
- $u = 3 - x^2$, $y = e^u$;
- $u = x^2 + 4x - 1$, $y = \sqrt{u}$.

Summary

- Given two functions f and g, the *composite function* "g composed with f" is defined to be the function $g(f(x))$.
- Decomposing a given composite function into its inside and outside functions is a prerequisite for using the *Chain Rule* for differentiation (studied the next day).