# 2 Unit Bridging Course

## Day 9 - The Derivative of a Composite Function

Jackie Nicholas

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# Composite functions

In Day 8 we learned to recognise a composite function or 'function of a function'.

Consider $y = (2x - 1)^7$.

$$\xrightarrow{x} \boxed{\ f\ } \xrightarrow{2x-1} \boxed{\ g\ } \xrightarrow{(2x-1)^7}$$

If $f(x) = 2x - 1$ and $g(u) = u^7$, we can write

$$(2x - 1)^7 = (f(x))^7 = g(f(x)).$$

$f(x) = 2x - 1$ is the 'inside' function as it sits inside the function $g$. $g$ is the 'outside' function.

# Composite functions

## Example

Consider $y = \sqrt{x^2 + 3}$.

$$\xrightarrow{x} \boxed{\quad f \quad} \xrightarrow{x^2+3} \boxed{\quad g \quad} \xrightarrow{\sqrt{x^2+3}}$$

If $f(x) = x^2 + 3$ and $g(u) = \sqrt{u}$ then

$$\sqrt{x^2 + 3} = \sqrt{f(x)} = g(f(x)).$$

Here, $f(x) = x^2 + 3$ is the inside function and $g$ is the ‘outside’ function.

# Differentiating composite functions

## The chain rule

If $y = g(f(x))$ and we let $u = f(x)$, so $y = g(u)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

Alternatively, in functional notation, if $h(x) = g(f(x))$, the derivative of $h$ is

$$h^{'}(x) = f^{'}(g(x))g^{'}(x).$$

This rule of differentiation is called the *chain* rule or composite function rule.

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## Example

Consider $y = (2x - 1)^7$.

We first identify the inside function as $(2x - 1)$ and let $u = 2x - 1$.

Then $y = u^7$.

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} && \longleftarrow \quad \text{the chain rule} \\
&= 7u^6 \times 2 && \longleftarrow \quad \text{differentiate} \\
&= 14(2x - 1)^6 && \longleftarrow \quad \text{substitute } u = 2x - 1 \text{ and tidy.}
\end{aligned}
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## Example

Consider $y = \sqrt{x^2+3} = (x^2+3)^{\frac{1}{2}}$.

Identify the inside function (in red).

Let $u = x^2 + 3$, then $y = \sqrt{u} = u^{\frac{1}{2}}$.

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\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} && \longleftarrow \quad \text{the chain rule} \\
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## Example

Consider $y = e^{x^2+1}$.

Let $u = x^2 + 1$, then $y = e^u$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= e^u \times 2x \\ &= 2x\, e^{x^2+1}. \end{aligned}$$

## Practice questions

Differentiate the following functions:

(i) $y = e^{x^3+1}$

(ii) $y = \sqrt{3 - x^2}$

(iii) $y = \dfrac{1}{(3x^3 - 1)^5}$.

## Answers to practice questions

(i) $3x^2e^{x^3+1}$

(ii) $-x^2(3-x^2)^{-\frac{1}{2}}$

(iii) $-45x^2(3x^3-1)^{-6}$.