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THE MATHEMATICAL BEHAVIOR OF SIX SUCCESSFUL MATHEMATICS GRADUATE STUDENTS: INFLUENCES LEADING TO MATHEMATICAL SUCCESS

ABSTRACT. This study investigated the mathematical behavior of graduate students and the experiences that contributed to their mathematical development and success. Their problem-solving behavior was observed while completing complex mathematical tasks, and their beliefs were assessed by administering a written survey. These graduate students report that a mentor, most frequently a high school teacher, facilitated the development of their problem solving abilities and continued mathematical study. The mentors were described as individuals who provided challenging problems, encouragement, and assistance in learning how to approach complex problems. When confronted with an unfamiliar task, these graduate students exhibited exceptional persistence and high confidence. Their initial problem solving attempts were frequently to classify the problem as one of a familiar type, and they were not always effective in accessing recently taught information or monitoring their solution attempts, but were careful to offer only solutions that had a logical foundation. These results provide numerous insights into the complexities of using and extending one's mathematical knowledge and suggest that non-cognitive factors play a prominent role in a student's mathematical success.

1. INTRODUCTION

Students' beliefs and concept development have been established as significant factors in the process of learning mathematics (Schoenfeld, 1985, 1989b; Lester, Garafalo and Kroll, 1989; Philippou and Christou, 1998). Research on students' concept development has become prominent in the mathematics education literature, while research on non-cognitive factors continues to reside on the periphery of the field (McLeod, 1992). Prior research has substantiated the assumption that affective variables such as beliefs and attitudes toward mathematics play a central role in the development of students' problem solving abilities (Kroll, 1989; Schoenfeld, 1989b). However, little is known regarding the specific beliefs that interact most with students' cognitive processes when completing mathematical tasks.

Numerous studies have examined the mathematical beliefs of secondary and elementary children (e.g., Fennema and Peterson, 1985; Dossey et al., 1988; Lester et al. 1989; Garofalo, 1989; Mandler, 1989; Schoenfeld,

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1989b; Thompson and Thompson, 1989); while few studies (e.g., Schoenfeld, 1989b) have investigated the beliefs of university level students; and no study to date has investigated the beliefs of graduate level students. The present study explored the non-cognitive factors that play a prominent role in students' success and continued mathematical study by investigating the mathematical beliefs, behaviors and backgrounds of successful mathematics graduate students.

The mathematical behavior of six successful graduate students was investigated by examining the influences that have led to their continued study of mathematics. The mathematical methods of these same students were investigated in the context of completing complex mathematical tasks, and the mathematical beliefs of a larger group of 34 graduate students were assessed using the *Views About Mathematics Survey* (VAMS), a quantitative survey that broadly classifies students' mathematical views about knowing and learning mathematics. Finally, the views of these 34 graduate students were compared with the views of their undergraduate peers.

2. BACKGROUND

Schoenfeld (1989b) describes a framework for the analysis of mathematical behavior. His framework is composed of four components: resources, heuristics, control and beliefs. Resources are the mathematical facts and procedures potentially accessible to the problem solver. Heuristics are the broad range of general problem solving techniques (e.g., working backwards, drawing figures). Control refers to the global decisions regarding the selection and implementation of resources and strategies that determine the efficiency with which facts, techniques, and strategies are exploited (e.g., planning, monitoring, decision-making). Schoenfeld (1989b) contends that purely cognitive behavior is rare, and that performance of most intellectual tasks takes place within the context established by one's perspective regarding the nature of those tasks. Belief systems, says Schoenfeld (1989b), shape cognition and determine the perspective with which one approaches mathematics and mathematical tasks, even when one is not consciously aware of holding these beliefs. Beliefs prompt more deep-seated convictions than emotions and attitudes (Schoenfeld, 1989a), and are characterized by statements such as 'learning mathematics is mostly memorization' (p. 344) and 'some people are good at math and some just aren't' (p. 353). When observing high performing undergraduate students while attempting a problem that should have been easily solved, Schoenfeld (1989a) observed that their mathematical beliefs regarding the usefulness of mathematical knowledge appeared to be the overriding factor preventing them from producing a correct solution.

Most faculty would agree that successful graduate students have superior mathematical abilities and more expert mathematical beliefs than most students in an undergraduate course in mathematics. However, little has been known regarding the specific beliefs, behaviors and abilities that distinguish graduate students from their undergraduate peers, and very little information has been available regarding the influences that promote students' continued mathematical study. This study begins to explore these issues.

3. Methods

The students (subjects) for this study were 34 mathematics graduate students from two large, public universities in the United States. Each of these students had successfully completed four years of post-high school mathematics and had been admitted to a graduate program in mathematics. Although the students in this study were at various stages of advanced work in mathematics, only three had completed post-master's level coursework, and only five of the 34 intended to pursue a Ph.D. in mathematics.

The data was collected in two phases. Interviews were conducted with three male and three female mathematics graduate students who had just completed their first year of graduate level mathematics with a grade of 'A' in either complex analysis or abstract algebra.

The purpose of the research interview was to:

- identify the factors in graduate students' background that influenced their mathematical development and continued study of mathematics
- observe graduate students' behaviors while completing complex mathematical tasks

After the interview analysis was performed, *VAMS* was administered to the larger collection of 34 graduate students. In order to compare the views of graduate and undergraduate students, this instrument was also administered to 73 third semester calculus and 546 precalculus students, also enrolled in a large public university in the United States. Both the undergraduate and graduate students were taught in small classes of at most 35 students. The undergraduate students, completing the survey at the beginning of their course, had used a moderately reformed text during the prior semester with lecture being the primary mode of instruction. The use of hand held graphing calculators was encouraged for verifying and checking their homework, but was not allowed during the exams. This written instrument was administered in this study to:

- identify specific beliefs common to successful graduate students
- broadly classify graduate students' beliefs about knowing and learning mathematics
- identify differences in the beliefs held by three levels (i.e., precalculus, third semester calculus, graduate) of university students

Investigation of these graduate students' backgrounds, beliefs and problem solving behaviors, utilizing three data sources, allowed for triangulation of results and the generation of diverse information about these graduate students' affective and cognitive development.

4. THE QUALITATIVE ASSESSMENT

4.1. *The Research Interview*

Each of the six interviews was initiated by a prompt from the interviewer requesting that the student describe what had affected the development of her/his problem solving abilities, mathematical knowledge, and continued mathematical study. Following this phase of the interview, the student was asked to describe her/his solution approach to five different mathematical tasks. No time limit was imposed. Initially the student examined the problem independently, while the interviewer/author quietly observed. Once the student formulated a written response, he/she was prompted to describe her/his solution approach. The format of the interview was primarily unstructured, with the interviewer spontaneously reacting to the student's comments and responses. The interview tone was amiable and nonthreatening and efforts were made to make the student comfortable with providing candid responses. When appropriate the student was prompted to provide clarification or additional information. All interview subjects were paid a modest stipend for their participation. Interviews were taped and later transcribed by the author.

4.2. *Analysis of Background Data*

The analysis of the background interview data involved a systematic approach for discovering and categorizing the ideas conveyed by the interviewee. This was achieved using the Strauss and Corbin (1990) open and axial coding techniques. The first phase of the analysis, referred to as open coding, involved a process by which the content of the interview was carefully searched for discrete instances of students' expression of a concept or idea (e.g., asks questions, confident, supportive home environment). Once the main idea/phenomenon (e.g., mathematical success) was

TABLE I

Axial Coding Results – Interview B

Main Phenomenon	
Mathematical Success	
Causal conditions: • Repeat exposure to math content (teaching college algebra). • Early positive mathematical experience • Mentor (father, TA) **Specific dimensions of success:** • Intensity high • Duration intermittent **Context of mathematical success:** • Initiated at age 9 • Under conditions of high persistence, high enjoyment, working challenging problems **Strategies for success:** • Asks questions • Works problems twice • Seeks out help • Persistent • Worked regularly • Works with another student • Attempts hardest problem **Consequences:** • Confident • Persistent • Enjoys solving mathematics problems	**Properties of repeat exposure:** • Learned content that was forgotten • Learned to verbalize her ideas • Increased confidence **Properties of early positive mathematical experience:** • Worked on challenging problems **Properties of mentors:** • Posed good questions • Non-intimidating • Provided assistance in completing challenging problems • Engaged students in regular practice • Encouraged students to discuss problems and high availability of help **Intervening Conditions:** • Supportive home environment • Wants to excel **Negative experiences:** • Calculus went too fast • Intimidating high school teacher

identified, the identified concepts were grouped according to their properties. For example, during the interview from which Table I was generated, the student independently mentioned three items that had contributed to her success in mathematics (i.e., early positive mathematical experiences, mentors, repeat exposure to math content). These items were later grouped into a single category by their common characteristics. Once the categories were identified, they were given a name, 'causal conditions', to characterize their relationship to the main phenomenon, 'mathematical success'.

Since important properties were also described for each of these causal conditions, the properties for each condition were captured and emerged as a category. For example, the mentors were praised for posing good questions, creating a non-intimidating environment, providing assistance, etc., resulting in this collection of items being placed in a category named, 'properties of mentors'. It should be noted that the coding results for this collection of interviews is not unique, however the interpretation of the results of three independent analyses for this data produced similar results.

The results of performing open and axial coding (Table I) on one interview transcript revealed that this student's mathematical success was positively influenced by the encouragement and support she received from her father. Her father was described as someone who created a non-threatening environment, posed challenging problems and provided assistance in completing these problems. This student also revealed that she enjoys attempting challenging problems and is persistent in her solution attempts. She sometimes works with other students and is not intimidated to approach others with her questions. Even though this student reported that she enjoys mathematics and is confident in her mathematical abilities, she has also encountered negative experiences during her mathematics development. She reported having experienced an intimidating high school teacher and believes that her calculus courses moved too fast. An experience that she cited as having a positive influence on her mathematical development was the opportunity to teach college algebra. This experience provided her the opportunity to learn to verbalize her ideas and relearn specific content that had been forgotten, resulting in her increased mathematical confidence.

After performing the Strauss and Corbin coding procedures on the six interview transcripts, a combined axial coding was performed on the collection of axial coding results. The summary axial coding diagram (Table II) presents the categories and lists the interviewees for which each property was identified. These coding results provided a comprehensive summary of the contents of the collection of interviews.

4.3. *Summary of Background Results*

The graduate students in this study reported that they enjoy the challenge of attempting complex mathematical tasks and believe that they possess abilities and strategies that facilitate their problem solving success. They reported persisting when confronting difficulties while working challenging problems, indicated that they have confidence in their mathematical abilities and expressed a willingness to spend remarkably large amounts of time working mathematics problems. Their interest in studying mathematics was initiated prior to enrolling in the university and was facilitated by

TABLE II

Summary Axial Coding Results

Phenomenon: Success in mathematics	
Causal conditions:	**Interviewee**
• Mentor (5 of 6)	A,B,C,E,F
☐ High school teacher (4 of 6)	A,C,E,F
☐ Other mentor, father/brother (2 of 6)	A,B
• Repeat experience with the same content (4 of 6)	A,B,D,F
Properties of mentor:	
• Encouraged students to reflect and attempt complex problems	A,C,E,F
• Provided exposure to challenging problems and constructive activities	A,B,C,E,F
• Posed good questions	B,E,F
• Non-intimidating/Encouraged to ask questions	A,B,E,F
• Assisted in learning problem solving approaches	A,B,E,F
• Required work/practice	B,C,F
Strategies for success:	
• Persists in completing complex problems	A,B,C,D,E,F
• Believes can conquer challenging problems	A,B,C,D,E,F
Context of success:	
• Initiated prior to enrolling in the university	A,B,C,E,F
• Under conditions of high enjoyment	A,B,E
• A supportive environment	A,B,F
Intervening conditions:	
• Focused practice; learned to work lots of problems	A,B,C,E,F
• Encouraged to ask probing questioning	A,B,E
• Opportunities to verbalize mathematical ideas	A,B,D,F
Consequences:	
• Confident	A,B,C,D,E,F
• Persistent	A,B,C,D,E,F
• Able to solve complex problems	A,B,C,D,E,F
• Currently enjoys engaging in problem solving	A,B,C,D,E,F
Negative experiences:	
• Calculus went too fast	B,F
• Intimidating mentor/teacher (h.s. teacher)	B,F
• Completed course with a good grade, but possessed little understanding of the content	B,D,F
• Content was not interesting/no motivation to study	D,E,F

Note that the phenomenon ‘success in mathematics’ is defined by the criteria for participating in the interview (i.e., completion of a two course sequence in graduate level mathematics with a grade of ‘A’).

contact with a mentor, most commonly a high school mathematics teacher. When prompted to describe attributes of the mentor that were exceptional, Student E responded that her high school teacher had made math interesting and had taught her how to think mathematically, study for exams, and read mathematical texts. Student A described his big brother as one who stimulated his interest in thinking about challenging problems and student B (Table I) described her father as an individual who challenged her, beginning at age 9, to work thought provoking and challenging problems. These mentors were also frequently described as individuals who required regular work and provided incentive for working 'lots of problems'. They created a non-intimidating environment where students were encouraged to pose questions until they acquired understanding. Student E described her high school math teacher as someone who, 'made you feel every question is valid and she didn't put you down . . . she allowed you to search until you really understood'.

Persistence was the trait most frequently cited as facilitating these graduate students' mathematical success. When prompted to describe what had contributed most to her mathematical success, student B responded, 'I do above and beyond anything that is required. I do the assigned problems and then try some of the harder problems on my own'. Student D, when asked if he works hard in the mathematics courses, answered, 'yes, I do now, I always try to figure things out if they don't make sense. I put in whatever amount of time that it takes to understand things . . ., graduate school has taught me how to read math texts on my own. In order to do well in graduate classes, I've had to learn how to sort things out without the help of the teacher'. As reported by the students in the interview, their persistence was motivated by different factors: students D and F, their competitive personalities; student E, her curiosity and joy of doing mathematics; student C, her trained discipline instilled by a high school math teacher; student B, her desire to do well and gain control of the many things she had forgotten; and student A, his genuine interest in mathematics and solving difficult problems.

These graduate students indicated that repeat experiences with the same content had contributed both to their increased mathematical confidence and enjoyment, and to their acquisition of a deeper understanding of specific mathematics content. Three of the six interview subjects had re-enrolled in a course they had already taken. According to these students, this created an opportunity to genuinely understand the concepts rather than superficially learn what was necessary to acquire her/his desired grade. For student D, retaking first semester calculus provided his first high grade in a math course. Subsequently, he possessed an expectation for mathematics to 'make sense'. Two of these interview subjects also indicated that on at least one occasion they had received an 'A', despite having little understanding of the material taught in the course. This statement was in tandem with the admission that, during the course, memorization had frequently been substituted for the pursuit of understanding.

Even though each of these individuals have been successful in mathematics, as confirmed by their successful completion of graduate courses in mathematics, some of the interview subjects (Table II) reported having negative experiences during periods of their mathematical development. In fact, three of the interview subjects reported having low motivation and little interest in studying mathematics at some point during high school. These students also revealed they had, at times, experienced intimidating mentors who had made them feel uncomfortable in posing questions.

4.4. *The Behavior of Graduate Students when Attempting Complex Mathematical Tasks*

Analysis of the interview data generated during problem solving involved multiple readings of each transcript, with specific attention given to the identification and classification of response types and problem-solving behaviors. Once common responses were identified, the interviews were reread to tally the number of students that provided the previously identified response types and behaviors. This data was then analyzed using Schoenfeld's (1989b) theoretical model. In addition to observing students' use of control, heuristics, and their tendency to access their known resources, students self-reported beliefs were compared with beliefs exhibited during problem solving.

Even though each of these students exhibited high confidence and remarkable persistence in their solution attempts, they varied with the ease in which they constructed a response to particular problems. While attempting these problems, they were sometimes ineffective in accessing recently taught information and monitoring their solution attempts. When attempting an unfamiliar problem each interviewee sometimes struggled to make sense of the problem and demonstrated efforts to try to classify the problem as one of a familiar type. At the same time, each of these graduate students appeared to exhibit an expectation that they would eventually 'solve' each of these problems, with their persistence frequently being the critical factor contributing to their solution attainment. The responses constructed by these graduate students, although not always correct, appeared to have a logical foundation. A summary of the collective responses for representative problems are presented, followed by a description of the various behaviors observed.

Problem 1

Assume F(x) is any quadratic function.
True or False: $F\left(\frac{x+y}{2}\right) < \frac{F(x)+F(y)}{2}$
Justify your answer

Only one of the six graduate students completed this problem with little hesitation, providing an algebraic argument for his justification. The other five students were slow to devise a solution approach, resulting in the interviewer providing some prompts. After various amounts of time (i.e., 6 to 23 minutes) spent in their struggle five of these six graduate students constructed a parabola and initiated efforts to understand what each expression represented in the context of the graph. Each of these graduate students were slow to recognize that (x + y)/2 is the average of two inputs, and [F(x) + F(y)]/2 corresponds to the average of two outputs. Once this aspect of making sense of the problem was mastered, they quickly began to construct a variety of graphs to explore specific situations and determine the circumstances under which the inequality was true. The remaining graduate student chose to define F algebraically and proceeded to attempt an algebraic solution to the problem. Even though this student never arrived at a correct solution (due to a careless error), he persisted for 28 minutes, making no attempt to determine if his algebraic solution made sense in the context of a graph. In reviewing the interview transcripts for these students, it appeared that once an approach brought about progress, each student proceeded with that approach until a response was formulated, with little concern regarding the effectiveness of the approach. Also noteworthy was the intensity and persistence with which each of these students pursued a solution to this problem. In fact, not once during these interviews did any of these graduate students express frustration or show any indication of abandoning their attempts. Even though these graduate students appeared to be equipped with the knowledge (resources) to interpret the statement of the question, their inefficient control decisions appeared to be a major obstacle during their solution attempts. However, their strong persistence and high confidence appeared to facilitate their eventual success in constructing their solutions.

Problem 2

Tom sees a ladder against a wall (in an almost vertical position). He pulls the base of the ladder away from the wall by a certain amount and then again by the same amount and then again by the same amount, and so forth. Each time he does this he records the distances by which the top of the ladder drops down. Do

VAMS Taxonomy

Epistemological

Structure: Mathematics is a coherent body of knowledge about relationships and patterns contrived by careful investigation
-rather than a collection of isolated facts and algorithms.

Methodology: The methods of mathematics are systematic and generic
-rather than idiosyncratic and situation specific.

Mathematical modeling for problem solving involves more
-than selecting formulas for number crunching.

Mathematicians use technology more to enhance their ways of solving problems
-than to allow them to get quick easy solutions.

Validity: Mathematical knowledge is validated by logical proofs
-rather than by correspondence to the real world.

Mathematical knowledge is tentative and refutable
-rather than absolute and final.

Pedagogical

Learnability: Mathematics is learnable by anyone willing to make the effort
-rather than by a few talented people.

Achievement depends more on persistent effort
-than on the influence of teacher or textbook.

Critical Thinking: For meaningful understanding of mathematics, one needs to:
Concentrate more on the systematic use of general thought processes
-rather than on memorizing isolated facts and algorithms.

Examine situations in many ways, and not feel intimidated by committing mistakes
-rather than follow a single approach from an authoritative source.

Look for discrepancies in one's own knowledge
-instead of just accumulating new information;

Reconstruct new knowledge in one's own way
-instead of memorizing it as given.

Personal relevance: Mathematics and related technology are relevant to everyone's life
-rather than being of exclusive concern to mathematicians.

Mathematics should be studied more for personal benefit
-than for just fulfilling curriculum requirements.

Figure 1. VAMS Taxonomy dimensions and sub-dimensions.

the amounts by which the top of the ladder drops down remain constant as Tom repeats this step; or do they get bigger, or do they get smaller? EXPLAIN.

Two of the six graduate students attempting this problem moved directly to a correct response. These successful students defined the Pythagorean relationship and proceeded to evaluate the first derivative and analyze this function. Each of the remaining four students continued to persist until they had arrived at a correct solution, with their completion times ranging from 8 to 26 minutes. These interviewees initially used a physical model to visualize the problem. Three of these students placed their pencils against a book and pulled the bottom out, simulating the problem situation. These students appeared to be assessing the changing rate at which the tops of their pencils fell. The other student placed his forearms together, and in slow motion, slid one arm down by moving his elbow away from the other. This student appeared to move the elbow out by equal increments, concurrently assessing how far the top had dropped. Once these students intuitively determined a response, they were prompted to justify their answers. Two of these students, with some hesitation, proceeded to define a Pythagorean relationship and compute a derivative. One student justified his response by saying, ‘It is just a little relative rate problem. I set up this equation and found dy/dt. As x gets bigger, the numerator gets bigger and the denominator gets smaller. So the whole expression gets bigger’. The remaining two students set up the Pythagorean relationship and numerically explored the event using a table, both completing the problem in approximately 25 minutes. One of these students proclaimed, ‘It gets larger, according to the pencil test. OK, the pencil must be right. I see now what I was doing wrong, now the numbers work out too’. This student indicated that as a returning student many of the basics of calculus were not accessible as she had not taken a beginning calculus course in many years. She stated during her interview, ‘I don’t seem to have certain things at my finger tips. There is something to having factual information accessible. I knew when I started taking graduate level math, that was my biggest stumbling block. I had to look everything up. I was frequently going back to the basic calculus book when I couldn’t figure out what to do in analysis. It was frustrating and time consuming, but usually I was able to see it after this’. It was also interesting to note that the two students providing a correct response with little hesitation had recently taught calculus. In fact, one of these students revealed that this problem was similar to the problems she had recently been teaching. The approaches and completion time of these students varied, with each student persisting and eventually providing a correct solution, although they appeared non-reflective regarding the effectiveness of their solution attempts. It was surprising that four of these six graduate students needed prompting, with two students continuing with a numeric approach, not recognizing to compute a derivative. Both of these responses represent inefficient use of control during problem solving.

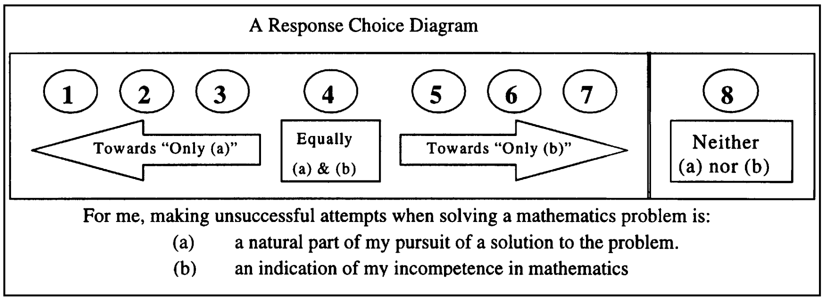

Figure 2. A response choice diagram and sample VAMS item.
A response of 1 indicates a belief that making unsuccessful attempts when solving a mathematics problem is a natural part of the student's pursuit of a solution to the problem and not an indication of incompetence in mathematics. (See Appendix A for a complete description of the response options.)

5. THE QUANTITATIVE ASSESSMENT

5.1. *The Quantitative Instrument (VAMS)*

VAMS was designed to assess students' views about knowing and learning mathematics (Carlson, 1997). Its design followed that of the widely used Views About Science Survey (VASS) (Halloun, 1997). The VAMS taxonomy is grouped in two broad dimensions and six sub-dimensions that describe the aspects of students' beliefs assessed in the VAMS instrument. The epistemological dimension pertains to the structure of mathematical knowledge, the validity of mathematical knowledge, and the methods of mathematics. The pedagogical dimension pertains to the learnability of mathematics, the role of critical thinking, and personal relevance of mathematics. The VAMS taxonomy (Figure 1) is presented in the form of contrasting views in each of the six dimensions. The primary view, referred to in this article as the *expert view*, corresponds to the view most commonly held among mathematicians. The opposing view, referred to as the *folk view*, is the view often attributed to the lay community and is widespread among mathematics students (Carlson, 1997). The VAMS taxonomy is presented so that the first view expressed tends more toward the expert view.

Individual survey items consist of a statement followed by two contrasting alternatives which respondents are asked to balance on an eight-point scale (Figure 2). Respondents can select either alternative exclusively (options 1 or 7), or a weighted combination of the two alternatives (options 2, 3, 4, 5, or 6) or neither (option 8). By providing two clearly stated benchmarks, ambiguity for specific items is reduced and reliability for specific items is increased. This design, known as the Contrasting Alternative Design (CAD), produces instruments which are more valid, reliable and

TABLE III

VAMS item and results

For me, solving a problem that involves mathematical reasoning is: (a) an enjoyable experience. (b) a frustrating experience.		
Expert View: Options 1,2	Mixed View: Options 3,4	Folk View: Options 5-7
76%	24%	0%

applicable to large populations (Halloun, 1997). Offering explicit choices may avoid the diversity of views expressed when asked open-ended questions such as, 'how do you define mathematics?' (Mura, 1995, p. 387). Figure 2 shows the response choice diagram and a typical CAD item from VAMS. A more detailed description of the instrument development can be found in Carlson et al. (1999).

For each VAMS item, students' responses are classified as 'expert', 'mixed' or 'folk'. These classifications were determined by administering VAMS to a community of 16 mathematicians. The range of responses for the expert classification was determined by the criteria that 15 of the 16 mathematicians provided a response in that response range. For example, on the VAMS item in Figure 2, 15 of the 16 mathematicians responded with a 1 or 2, resulting in the classification of both 1 and 2 to the 'expert' view. It followed from further analysis of the mathematicians' responses and preliminary student data that 3 and 4 were assigned to the 'mixed' classification and 5–7 were assigned to the 'folk' classification for this item. A similar analysis was performed on all 33 VAMS items.

Further analysis of the mathematicians responses and preliminary student data facilitated the classification of each student's overall VAMS score into four broad profiles: expert, upper transitional, lower transitional and folk. In order to receive an expert profile classification, the student needed to have an expert view classification on 16 of 27 VAMS items. This was the fewest number of expert views expressed by any of the 16 mathematicians. A more detailed description of the VAMS profiling strategy is found elsewhere (Carlson et al., 1999).

5.2. *VAMS Results*

Results from selected items on the VAMS form are reported in Tables III-VII. In each table, the range of choices that were classified as expert, mixed and folk are provided, and the percentages of the 34 graduate students possessing each of these three views are displayed.

TABLE IV

VAMS item and results

When I experience a difficulty while studying mathematics:
(a) I immediately seek help, or give up trying.
(b) I try hard to figure it out on my own.

Expert View: Options 6,7	Mixed View: Option 5	Folk View: Options 1-4
82%	6%	12%

TABLE V

VAMS item and results

In mathematics, it is important for me to:
(a) memorize technical terms and mathematical formulas.
(b) learn ways to organize information and use it.

Expert View: Options 4-7	Mixed View: Options 3	Folk View: Options 1,2
94%	6%	0%

The results reveal that this collection of graduate students enjoy solving problems that require mathematical reasoning (Table III), as revealed by the fact that 76% of these graduate students were classified as having an ‘expert’ view and 0% had a folk view on this item (Table III).

These graduate students also reported applying methods that align with the methods used by mathematicians. They indicated that they persist when experiencing difficulty while studying mathematics (Table IV) and believe that mathematics requires more organizing and using information, than memorizing technical terms (Table V).

Even though the majority of these students possess expert views regarding the nature of mathematics (e.g., Tables VI and VII), their views on these items were not as strongly aligned with those of mathematicians as on those items assessing the methods of mathematics (e.g., Tables III-V). Only 65% of these graduate students report believing the Pythagorean theorem is true because it has been proven by logical argument (Table VI) and only 56% believe that mathematical formulas are primarily useful for expressing meaningful relationships among variables (Table VII).

These results provided insight into how these graduate students’ views compare with those of mathematicians. As revealed by their VAMS responses, these graduate students’ views about the methods of mathematics appear to be closely aligned with those of mathematicians, while their views about the nature of mathematics appear to be in transition.

TABLE VI

VAMS item and results

The relationship among the sides of a right triangle expressed in the Pythagorean theorem is true because it has been:
(a) proven by a logical argument.
(b) verified by measurement.

Expert View: Options 1,2	Mixed View: Options 3,4	Folk View: Options 5-7
65%	32%	3%

TABLE VII

VAMS item and results

Mathematical formulas:
(a) express meaningful relationships among variables.
(a) provide ways to get numerical answers to problems.

Expert View: Options 1-3	Mixed View: Options 4	Folk View: Options 5-7
56%	41%	3%

5.3. *Comparison of Views across Course Level*

The VAMS responses of these 34 graduate students were compared with the responses of 73 third semester calculus students and 546 precalculus students at the beginning of their respective courses. The students in this study (both graduate and undergraduate) were enrolled in a large public university in the United States. Comparing the response patterns of these three groups revealed a dramatic difference in the overall mathematical views of these groups, as assessed by VAMS.

Using the VAMS profile classification strategy (Carlson et al., 1999) each of these three groups was classified into one of the four VAMS profiles: expert, upper transitional, lower transitional and folk. A comparison of the relative profiles of these three groups revealed that 82% of the graduate students were classified as expert, while only 23% of third semester calculus students and 9% of precalculus students were assigned this profile (Figure 4).

While the graduate students had fewer classifications into the two lower profiles than the other two groups, it was surprising to find that this many (6 of 34) graduate students held these lower profile classifications. Even

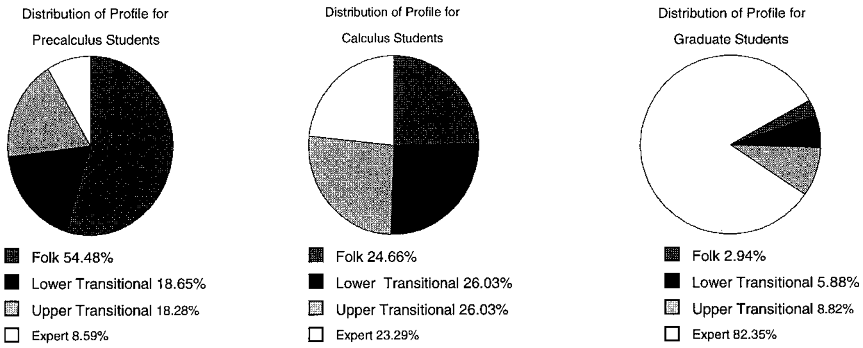

Figure 4. VAMS profile comparisons.

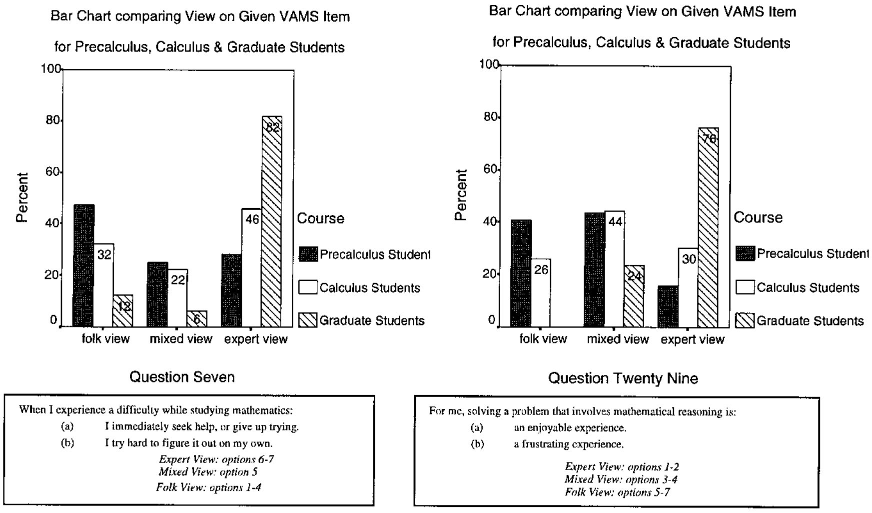

Figure 5. Comparisons for two VAMS items.

though third semester calculus students hold more expert views than pre-calculus students, they still hold many views that differ from those of mathematicians, as revealed by the fact that a majority (51%) of the third semester calculus students had a folk or lower transitional profile.

Student responses on individual VAMS items were investigated to identify the VAMS items where the differences in the responses of the three groups was the greatest. Comparison of these students' VAMS responses revealed that these graduate students report much greater persistence when confronting a difficulty while studying mathematics and greater enjoyment when completing mathematical tasks than their undergraduate peers (Figure 5).

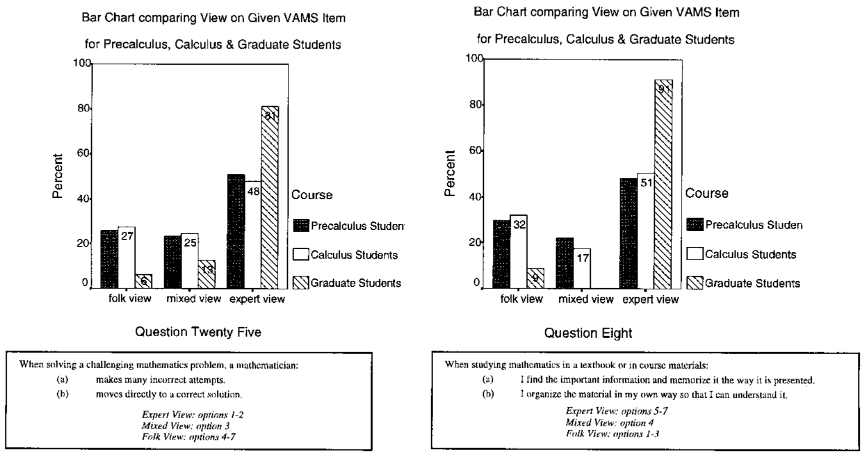

Figure 6. Comparisons for two VAMS items.

These graduate students recognize that mathematicians may make many repeated attempts when solving complex mathematics problems. In addition, they are more likely than their undergraduate peers to attempt generic problem solving approaches (Figure 6).

In addition to assessing students' views about mathematics, VAMS also prompted students to assess their confidence with respect to their overall mathematical abilities using a four point scale (i.e., *excellent, good, average, and weak*). These graduate students report having high self-confidence, as revealed by the fact that 86% of the surveyed graduate students rated their confidence in their mathematical abilities as excellent or good. These results revealed that many attributes (e.g., perseverance, confidence) observed during the interviews were also prevalent among a larger group of graduate students.

6. CONCLUSIONS

The following are general conclusions reached as a result of this research.

- Several mathematical beliefs appeared to be widespread and persistent in the graduate students in this study, suggesting aspects of students' beliefs that are crucial for continued mathematical study. Specific beliefs surfaced as strong in all three data sources. Included among these beliefs are the belief that: mathematics involves a process that may include many incorrect attempts; problems that involve mathematical reasoning are enjoyable; individual effort is needed when confronting a difficulty; students should be expected to 'sort out' information on their own; and persistence will eventually result in a solution to a problem.

- Persistence appears to be a necessary trait for successful completion of complex mathematical tasks and continued mathematical study. This observation is stronger than what has been reported in the literature. Schoenfeld (1989b) and Lester et al. (1989) report that persistence is a valuable trait, but only if accompanied by appropriate monitoring behaviors. From performing both the quantitative and qualitative analysis, it was observed that successful graduate students all possess exceptionally strong persistence. This result does not dispute the Schoenfeld (1989b) and Lester et al. (1989) findings that appropriate monitoring behaviors are also needed during problem solving; however it does suggest that, in the absence of persistence with complex mathematical problem solving, students will likely discontinue their study of mathematics.
- The interview subjects reported that a mentor (most commonly a high school math teacher) played an important role in developing their problem solving abilities. The mentors were praised for posing challenging problems, providing assistance in learning to approach challenging problems, requiring regular work, providing opportunity for students to verbalize their mathematical ideas, and making students feel comfortable in posing questions.
- These graduate students' control decisions (i.e., global choices) during problem solving appeared weak. Known mathematical knowledge (i.e., resources) was frequently not accessed, general problem solving strategies were frequently not employed, and little reflection was observed during problem solving. These results are consistent with Schoenfeld's (1989b) findings when observing talented undergraduate students while attempting to complete an unfamiliar problem.
- These graduate students reported and displayed an expectation that they could complete complex mathematical tasks, and were careful during their problem solving attempts to offer only responses that appeared to have a logical foundation. These traits have not been widely observed in high-performing undergraduate students in mathematics (Schoenfeld, 1989b; Monk, 1992; Carlson, 1998).

7. DISCUSSION

The process of sorting through possible solution approaches and accessing the appropriate mathematical knowledge at the right moment when solving a unique problem appears to be extremely complex. This raises many questions regarding the experiences that are needed to promote development of the non-cognitive abilities that appear to be essential for extending a student's problem solving effectiveness. These graduate students reported believing they primarily invoke expert mathematical methods and behaviors. However, the interview subjects did not always appear to fully operate within the context of these professed beliefs. The difficulties which these graduate students experienced are not surprising if one considers the magnitude of options available when confronting an unfamiliar problem. It seems natural that students initially attempt to classify an unfamiliar problem into a familiar category of problems, as observed during the interviews. It is also reasonable that a student attempt to employ familiar techniques when confronting a complex problem. However, for the interview subjects in this study, difficulties arose during problem solving when the student was unable to retrieve known facts or failed to consider a wide range of options when determining their solution approach (i.e., inefficient control decisions). When students in this study found themselves in 'unfamiliar territory' they appeared to blindly pursue the first familiar approach that surfaced, with little reflection and monitoring during their solution attempts. Even though these results suggest that graduate students hold beliefs that are more expert than undergraduate mathematics students, their control decisions and beliefs appear to be tenuous and in evolution. Their continued development would be interesting to monitor.

Even though the qualitative and quantitative results were sometimes inconsistent, much of what was observed during the interviews was verified as pervasive among the larger group of graduate students completing VAMS. These surveyed graduate students both reported and were observed possessing strong persistence and high mathematical confidence. They reported and demonstrated an expectation for difficult mathematics problems to require an investment of time, and both reported and were observed enjoying problems that involve mathematical reasoning. These combined results support the conclusion that these behaviors and beliefs are widespread among successful graduate students and are crucial for a student's continued mathematical study. Therefore, it is suggested that future studies investigate the complexities of acquiring these essential beliefs and behaviors.

These graduate students' mathematical success and problem solving abilities have been positively affected by numerous factors. They have received 'coaching' from a mentor; have been asked to work challenging problems; are aware that complex problems require time; are not inhibited to attempt to 'sort out' information on their own; and believe they can eventually arrive at a correct solution to most problems. Repeat exposure to the same content appeared to boost these students' confidence and enjoyment, as well as their understanding of concepts.

This study raises many questions regarding the interaction between students' professed beliefs and their actual behaviors when completing mathematical tasks. How aligned are they really? How do control decisions interact with students' beliefs? Is it realistic to expect that students are eventually able to operate fully within the context of their professed beliefs? Many questions remain, prompting the need for further investigation of students' problem solving behaviors.

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