2 Unit Bridging Course – Day 8

Applications of the Exponential Function

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The General Exponential Function

Previously, we looked at the exponential function

$$y = e^x$$

which had the special property that

$$\frac{dy}{dx} = e^x.$$

That is, the derivative of the exponential function is equal to itself.

Our aim in this module is to employ variations of the exponential function to model and study real-world phenomena.

The General Exponential Function (ct.)

Consider the function

$$f(x) = Ae^{kx},$$

where A and k are constants.

It turns out that:

$$f'(x) = k \times Ae^{kx} = k\,f(x),$$

so $f(x)$ and its rate of change $f'(x)$ differ by a constant multiple.

Terminology-wise, we say that f' is **proportional** to f.

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The General Exponential Function (ct.)

Since there exist many situations in the natural world where the rate of change of a quantity is proportional its size, f is well-positioned to act as a model for such phenomena.

Definition: The General Exponential Function

The function $f(x) = Ae^{kx}$ is called the **General Exponential Function**, with the property that f changes at a rate proportional to f itself.

That is, $f'(x) = k f(x)$ for all x.

A is called the **initial constant** and k is called the **proportionality constant**.

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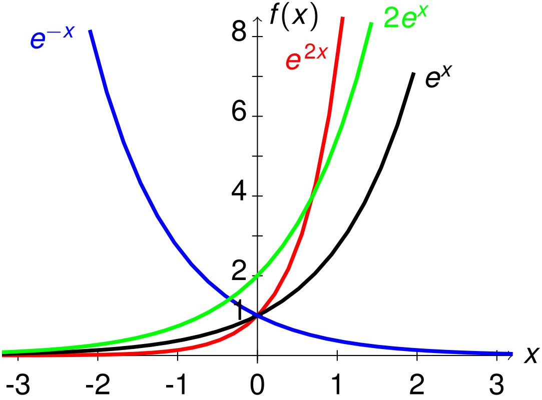
Graphically, k modifies the steepness and orientation of the exponential function while A primarily serves to shift its y-intercept.



Exponential Growth

When the proportionality constant k is positive, $f(x)$ is said to describe **exponential growth.**

A good example of this is population growth.

Example

The population, $P(t)$, of an outback town is growing exponentially according to the formula

$$P(t) = 1000\, e^{0.2t},$$

where t is the number of years after the year 2000.

Find the population in 2000, 2010, and estimate the population in 2020.

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Exponential Growth (cont.)

Notice that the formula is in the general form

$$P(t) = Ae^{kt},$$

where $A = 1000$ and $k = 0.2$.

Year 2000: Since t is the number of years after the year 2000, 2000 therefore corresponds to $t = 0$. Hence the population in the year 2000 is:

$$P(0) = 1000\,e^{0.2\times 0} = 1000\,e^{0} = 1000 \times 1 = 1000 = A.$$

That is, the **initial population** is equal to $A = 1000$, which is precisely the reason why A is termed the *initial* constant.

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Exponential Growth (cont.)

Year 2010: The year 2010 corresponds to $t = 10$. Hence the population in the year 2010 is:

$$P(10) = 1000\, e^{0.2 \times 10} = 1000\, e^{2} = 7389.$$

Hence, the population has increased 7-fold in 10 years.

Year 2020: Finally, we can use the model to predict what the population will be in the year 2020:

$$P(20) = 1000\, e^{0.2 \times 20} = 1000\, e^{4} = 54598.$$

That is, the population is expected to explode to roughly 55,000 in the subsequent 10 years.

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Exponential Decay

When the proportionality constant k is negative, $f(x)$ is said to describe **exponential decay**.

A good example of this is the decomposition of radioactive material, which decays at a rate proportional to its size.

Example

An amount of radioactive carbon A, measured in kilograms, is decaying exponentially according to the formula

$$A(t) = 25\, e^{-0.00012t},$$

where t is the number of years after the year 2000.

Find the initial amount, and the amount of radioactive carbon present in the year 3000.

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Exponential Decay (cont.)

Year 2000: The initial amount corresponds to $A(0)$, which we know is the initial constant $A = 25$. Hence initially, we have 25kg of radioactive carbon.

Year 3000: In the year 3000, the amount of radioactive carbon remaining is given by

$$A(1000) = 25\,e^{-0.00012\times 1000} = 25\,e^{-0.12} = 22.17.$$

That is, less than 3kg of the radioactive material has decomposed after a whole millenium, with almost 90% still remaining!

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Summary

- The *General Exponential Function* $f(x) = Ae^{kx}$ holds the property that f changes at a rate proportional to f itself. A is called the *initial constant* and k the *proportionality constant*.

- The formula $f(t) = Ae^{kt}$, for some quantity $f(t)$ at time t, serves as an excellent model for real-world phenomena following a pattern of *exponential growth* ($k > 0$) or *decay* ($k < 0$). Here, A represents the value of f at the time $t = 0$.

- Examples of exponential growth includes population growth, while examples of exponential decay includes the decomposition of radioactive material.