Mathematics Learning Centre

The University of Sydney

The second derivative and points of inflection

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The second derivative

The second derivative, $\frac{d^2y}{dx^2}$, of the function $y = f(x)$ is the derivative of $\frac{dy}{dx}$. $\frac{dy}{dx}$ is a function of x which describes the slope of the curve. If we take its derivative, $\frac{d^2y}{dx^2}$, and find the values of x for which $\frac{d^2y}{dx^2}$ is positive or negative, we determine where $\frac{dy}{dx}$ is increasing or decreasing. This can be used to work out the concavity of the curve or how the curve bends.

Concave up

The following curves are examples of curves which are *concave up*; that is they bend up or open upwards like a cup. The tangents to the curve sit underneath the curve.

Notice that for both of these curves the slope of the tangents to the curve increase as x increases.

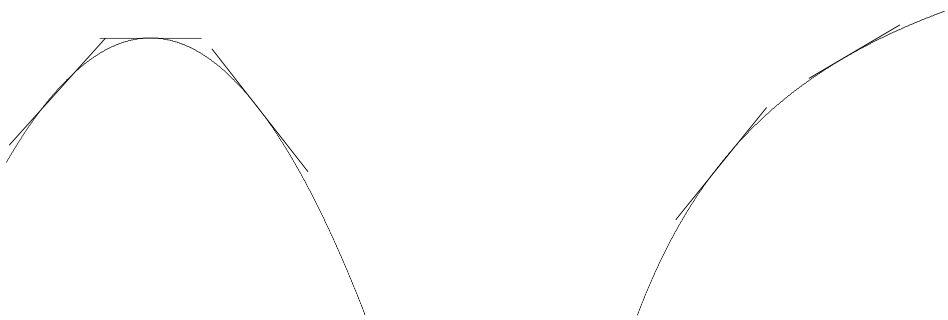
We say that a curve is *concave up* on an interval I when

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) > 0 \qquad \text{for all } x \text{ in } I.$$

Since $\frac{d}{dx}(\frac{dy}{dx}) > 0$, we know that $\frac{dy}{dx}$ is increasing and the function itself must be concave up on the interval I.

Concave down

The following curves are examples of curves which are *concave down*; that is they bend down or open downwards like a mound. The tangents to the curve sit on top of the curve.



Notice that for both of these curves the slope of the tangents to the curve decrease as x increases.

We say that a curve is *concave down* on an interval I when

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) < 0 \qquad \text{for all } x \text{ in } I.$$

Since $\frac{d}{dx}(\frac{dy}{dx}) < 0$, we know that $\frac{dy}{dx}$ is a decreasing function and the function $y = f(x)$ itself must be concave down.

Points of inflection

A point of inflection occurs at a point where $\frac{d^2y}{dx^2} = 0$ **AND** there is a change in concavity of the curve at that point.

For example, take the function $y = x^3 + x$.

$$\frac{dy}{dx} = 3x^2 + 1 > 0 \quad \text{for all values of } x \qquad \text{and} \qquad \frac{d^2y}{dx^2} = 6x = 0 \quad \text{for } x = 0.$$

This means that there are no stationary points but there is a *possible* point of inflection at $x = 0$.

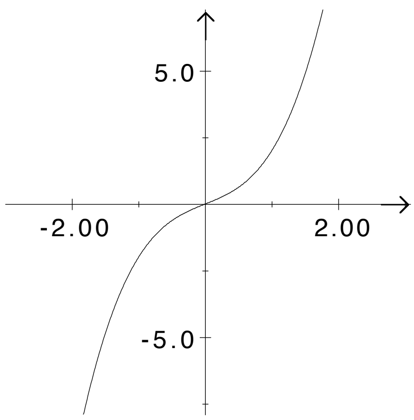
Since

$$\frac{d^2y}{dx^2} = 6x < 0 \quad \text{for } x < 0, \qquad \text{and} \qquad \frac{d^2y}{dx^2} = 6x > 0 \quad \text{for } x > 0$$

the concavity changes at $x = 0$ and so $x = 0$ is a point of inflection.

We can construct a table that summarises this information about the second derivative.

x	< 0	0	> 0
y''	$-ve$	0	$+ve$
y	concave down	0	concave up



The graph of $y = x^3 + x$.

Point of inflection that is a stationary point

The third kind of stationary point is a point of inflection. Since it is a stationary point, $\frac{dy}{dx} = 0$. Since it is also a point of inflection $\frac{d^2y}{dx^2} = 0$ and there is a change of concavity of the curve at this point. A point of inflection that is also a stationary point is sometimes called a horizontal point of inflection as the tangent to the curve when $\frac{dy}{dx} = 0$ is horizontal.

For example, let $y = x^3 - 3x^2 + 3x - 1$.

Since

$$\frac{dy}{dx} = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x-1)^2 = 0 \quad \text{when } x = 1,$$

there is a stationary point at $x = 1$.

We use $x = 1$ to divide the real line into two intervals; $x < 1$ and $x > 1$, and look at the sign of $\dfrac{dy}{dx} = 3(x-1)^2$ in both of these intervals (using $x = 0$ and $x = 2$ as test values perhaps).

x	< 1	1	> 1
y'	$+ve$	0	$+ve$
y	$\nearrow$	0	$\nearrow$

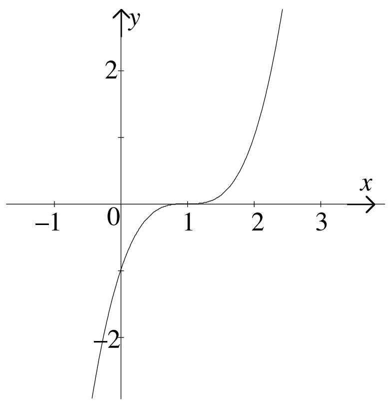
So, the stationary point is neither a maximum nor a minimum.

We confirm that it is a point of inflection (and not some other animal) by looking at the second derivative.

$$\frac{d^2y}{dx^2} = 6x - 6 = 6(x-1) = 0 \quad \text{when } x = 1.$$

x	< 1	1	> 1
y''	$-ve$	0	$+ve$
y	concave down	0	concave up

This confirms that there is a change of concavity at $x = 1$, and so there is a point of inflection at $x = 1$.



The graph of $y = x^3 - 3x^2 + 3x - 1$.