2 Unit Bridging Course – Day 5

Applications of the derivative: I

Jackie Nicholas & Emi Tanaka

THE UNIVERSITY OF SYDNEY

Solving optimisation problems

We can apply our knowledge of derivatives to solve optimisation problems.

Those are problems where we want to find the optimal solution – a maximum or a minimum – to a given situation.

For example:

- a farmer might want to know just how much fertiliser to use to maximise the yield of a crop; or
- a manufacturer of 1000 litre cylindrical steel rainwater tanks might want to know what the radius of the tank should be to minimise the amount of steel used.

Solving optimisation problems

We can apply our knowledge of derivatives to solve optimisation problems.

Those are problems where we want to find the optimal solution – a maximum or a minimum – to a given situation.

For example:

- a farmer might want to know just how much fertiliser to use to maximise the yield of a crop; or
- a manufacturer of 1000 litre cylindrical steel rainwater tanks might want to know what the radius of the tank should be to minimise the amount of steel used.

Solving optimisation problems

We can apply our knowledge of derivatives to solve optimisation problems.

Those are problems where we want to find the optimal solution – a maximum or a minimum – to a given situation.

For example:

- a farmer might want to know just how much fertiliser to use to maximise the yield of a crop; or
- a manufacturer of 1000 litre cylindrical steel rainwater tanks might want to know what the radius of the tank should be to minimise the amount of steel used.

Solving optimisation problems

In this course we will limit our attention to relatively simple examples of optimisation.

To find the optimal solution for function y, we first find the values of x for which $\frac{dy}{dx} = 0$, ie find a stationary point.

Then, we investigate the nature of the stationary point to determine whether or not we have a maximum or a minimum.

Solving optimisation problems

In this course we will limit our attention to relatively simple examples of optimisation.

To find the optimal solution for function y, we first find the values of x for which $\dfrac{dy}{dx} = 0$, ie find a stationary point.

Then, we investigate the nature of the stationary point to determine whether or not we have a maximum or a minimum.

Solving optimisation problems

In this course we will limit our attention to relatively simple examples of optimisation.

To find the optimal solution for function y, we first find the values of x for which $\frac{dy}{dx} = 0$, ie find a stationary point.

Then, we investigate the nature of the stationary point to determine whether or not we have a maximum or a minimum.

Maximum stationary point

We know from Day 4 that one of the two types of stationary point is a turning point.

Our task now is to learn to distinguish between the two types of turning points; a maximum and a minimum.

Notice that for a maximum, the sign of the gradient goes from positive to negative as we go left to right through the stationary point.

Maximum stationary point

We know from Day 4 that one of the two types of stationary point is a turning point.

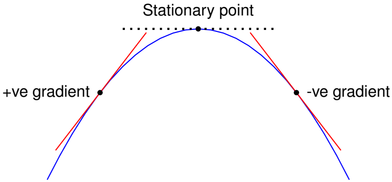
Our task now is to learn to distinguish between the two types of turning points; a maximum and a minimum.

Notice that for a maximum, the sign of the gradient goes from positive to negative as we go left to right through the stationary point.

Maximum stationary point

We know from Day 4 that one of the two types of stationary point is a turning point.

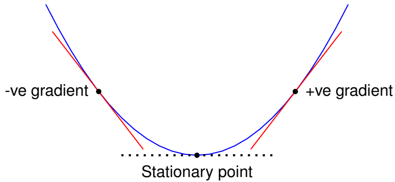
Our task now is to learn to distinguish between the two types of turning points; a maximum and a minimum.



Notice that for a maximum, the sign of the gradient goes from positive to negative as we go left to right through the stationary point.

Minimum stationary point

Similarly for a minimum turning point, the sign of the derivative goes from negative to positive as we go left to right through the stationary point.

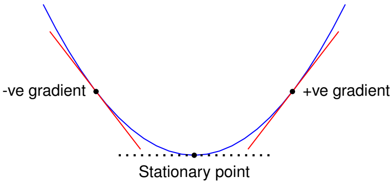


We can write this as follows:

Suppose that f has a stationary point at $x = a$, if f' changes from negative to positive at a, then f has a local minimum at a.

Minimum stationary point

Similarly for a minimum turning point, the sign of the derivative goes from negative to positive as we go left to right through the stationary point.



We can write this as follows:

Suppose that f has a stationary point at $x = a$, if f' changes from negative to positive at a, then f has a local minimum at a.

Example

If a baker sells bread for x dollars each ($x \geq 0$), the profit gained is given by $30x - x^3$. Find the price of bread to give maximum profit.

Let P = profit $= 30x - x^3$.
The maximum or minimum value of P occurs when

$$P'(x) = 30 - 3x^2 = 0.$$

Solve for x to get $x = \sqrt{10} \approx 3.16$.

Example

If a baker sells bread for x dollars each ($x \geq 0$), the profit gained is given by $30x - x^3$. Find the price of bread to give maximum profit.

Let P = profit $= 30x - x^3$.
The maximum or minimum value of P occurs when

$$P'(x) = 30 - 3x^2 = 0.$$

Solve for x to get $x = \sqrt{10} \approx 3.16$.

Example

If a baker sells bread for x dollars each ($x \geq 0$), the profit gained is given by $30x - x^3$. Find the price of bread to give maximum profit.

Let P = profit $= 30x - x^3$.
The maximum or minimum value of P occurs when

$$P'(x) = 30 - 3x^2 = 0.$$

Solve for x to get $x = \sqrt{10} \approx 3.16$.

Example

If a baker sells bread for x dollars each ($x \geq 0$), the profit gained is given by $30x - x^3$. Find the price of bread to give maximum profit.

Let $P = \text{profit} = 30x - x^3$.
The maximum or minimum value of P occurs when

$$P'(x) = 30 - 3x^2 = 0.$$

Solve for x to get $x = \sqrt{10} \approx 3.16$.

Example continued

We now must confirm that we do have a maximum when $x = \sqrt{10} \approx 3.16$.

If $x < \sqrt{10}$, say $x = 3$, $P'(3) = 30 - 3(3)^2 = 3 > 0$.

If $x > \sqrt{10}$, say $x = 4$, $P'(4) = 30 - 3(4)^2 = -18 < 0$.

P' changes from positive to negative, so P has a local maximum at \$3.16.

The maximum profit is $P(3.16) = 30 \times 3.16 - 3.16^3 =$\$63.25.

Example continued

We now must confirm that we do have a maximum when $x = \sqrt{10} \approx 3.16$.

If $x < \sqrt{10}$, say $x = 3$, $P'(3) = 30 - 3(3)^2 = 3 > 0$.

If $x > \sqrt{10}$, say $x = 4$, $P'(4) = 30 - 3(4)^2 = -18 < 0$.

P' changes from positive to negative, so P has a local maximum at \$3.16.

The maximum profit is $P(3.16) = 30 \times 3.16 - 3.16^3 =$\$63.25.

Example continued

We now must confirm that we do have a maximum when $x = \sqrt{10} \approx 3.16$.

If $x < \sqrt{10}$, say $x = 3$, $P'(3) = 30 - 3(3)^2 = 3 > 0$.

If $x > \sqrt{10}$, say $x = 4$, $P'(4) = 30 - 3(4)^2 = -18 < 0$.

P' changes from positive to negative, so P has a local maximum at \$3.16.

The maximum profit is $P(3.16) = 30 \times 3.16 - 3.16^3 =$\$63.25.

Example continued

We now must confirm that we do have a maximum when $x = \sqrt{10} \approx 3.16$.

If $x < \sqrt{10}$, say $x = 3$, $P'(3) = 30 - 3(3)^2 = 3 > 0$.

If $x > \sqrt{10}$, say $x = 4$, $P'(4) = 30 - 3(4)^2 = -18 < 0$.

P' changes from positive to negative, so P has a local maximum at \$3.16.

The maximum profit is $P(3.16) = 30 \times 3.16 - 3.16^3 =$\$63.25.

THE UNIVERSITY OF SYDNEY

Example continued

We now must confirm that we do have a maximum when $x = \sqrt{10} \approx 3.16$.

If $x < \sqrt{10}$, say $x = 3$, $P'(3) = 30 - 3(3)^2 = 3 > 0$.

If $x > \sqrt{10}$, say $x = 4$, $P'(4) = 30 - 3(4)^2 = -18 < 0$.

P' changes from positive to negative, so P has a local maximum at \$3.16.

The maximum profit is $P(3.16) = 30 \times 3.16 - 3.16^3 =$\$63.25.

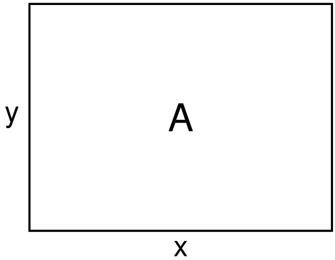
Example 2

Example

A farmer wants to make a rectangular paddock. The farmer has 100m of fencing and wants to make a paddock that will enclose the greatest area. What dimensions should the rectangle be?

Example

A farmer wants to make a rectangular paddock. The farmer has 100m of fencing and wants to make a paddock that will enclose the greatest area. What dimensions should the rectangle be?



Example 2

- Step 1: Define our variables.

$$\begin{aligned} \text{Let } x &= \text{the length of one side of rectangle,} \\ y &= \text{the length of the other side of the rectangle} \\ \text{and } A &= \text{the area of the rectangular paddock.} \end{aligned}$$

- Step 2: Write down the function to be maximised in terms of our variables.

$$A = xy.$$

- Step 3: Find any relationship between the our variables.

  Here we know the perimeter of the rectangle is 100m, so

$$2x + 2y = 100.$$

Example 2

- Step 1: Define our variables.

$$\begin{aligned} \text{Let } x &= \text{the length of one side of rectangle,} \\ y &= \text{the length of the other side of the rectangle} \\ \text{and } A &= \text{the area of the rectangular paddock.} \end{aligned}$$

- Step 2: Write down the function to be maximised in terms of our variables.

$$A = xy.$$

- Step 3: Find any relationship between the our variables.

Here we know the perimeter of the rectangle is 100m, so

$$2x + 2y = 100.$$

Example 2

- Step 1: Define our variables.

$$\begin{aligned} \text{Let } x &= \text{the length of one side of rectangle,} \\ y &= \text{the length of the other side of the rectangle} \\ \text{and } A &= \text{the area of the rectangular paddock.} \end{aligned}$$

- Step 2: Write down the function to be maximised in terms of our variables.

$$A = xy.$$

- Step 3: Find any relationship between the our variables.

  Here we know the perimeter of the rectangle is 100m, so

$$2x + 2y = 100.$$

Example 2 continued

- Step 4: Write one of the variable in Step 3 in terms of the other.

$$
\begin{aligned}
2x + 2y &= 100 \\
2y &= 100 - 2x \\
y &= 50 - x.
\end{aligned}
$$

- Step 5: Substitute the result of the previous step into the function to be maximised.

$$
\begin{aligned}
A &= xy \\
&= x(50 - x) \\
&= 50x - x^2.
\end{aligned}
$$

Example 2 continued

- Step 4: Write one of the variable in Step 3 in terms of the other.

$$\begin{aligned} 2x + 2y &= 100 \\ 2y &= 100 - 2x \\ y &= 50 - x. \end{aligned}$$

- Step 5: Substitute the result of the previous step into the function to be maximised.

$$\begin{aligned} A &= xy \\ &= x(50 - x) \\ &= 50x - x^2. \end{aligned}$$

Example 2 continued

- Step 6: Differentiate the function, set the derivative to zero, and solve.

$$\frac{dA}{dx} = 50 - 2x = 0,$$

so $\qquad 50 - 2x = 0$, ie $x = 25$.

- Step 7: Confirm that we have a maximum or a minimum.

If $x < 25$, say $x = 24$, $\frac{dA}{dx}(24) = 50 - 2(24) > 0$.

If $x > 25$, say $x = 26$, $\frac{dA}{dx}(26) = 50 - 2(26) < 0$.

$\frac{dA}{dx}$ changes from positive to negative, so A has a local maximum when $x = 25$.

Example 2 continued

- Step 6: Differentiate the function, set the derivative to zero, and solve.

$$\frac{dA}{dx} = 50 - 2x = 0,$$

so $\qquad 50 - 2x = 0$, ie $x = 25$.

- Step 7: Confirm that we have a maximum or a minimum.

If $x < 25$, say $x = 24$, $\frac{dA}{dx}(24) = 50 - 2(24) > 0$.

If $x > 25$, say $x = 26$, $\frac{dA}{dx}(26) = 50 - 2(26) < 0$.

$\frac{dA}{dx}$ changes from positive to negative, so A has a local maximum when $x = 25$.

Example 2 continued

- Step 6: Differentiate the function, set the derivative to zero, and solve.

$$\frac{dA}{dx} = 50 - 2x = 0,$$

so $\qquad 50 - 2x = 0$, ie $x = 25$.

- Step 7: Confirm that we have a maximum or a minimum.

If $x < 25$, say $x = 24$, $\dfrac{dA}{dx}(24) = 50 - 2(24) > 0$.

If $x > 25$, say $x = 26$, $\dfrac{dA}{dx}(26) = 50 - 2(26) < 0$.

$\dfrac{dA}{dx}$ changes from positive to negative, so A has a local maximum when $x = 25$.

Example 2 continued

- Step 6: Differentiate the function, set the derivative to zero, and solve.

$$\frac{dA}{dx} = 50 - 2x = 0,$$

so $\quad 50 - 2x = 0$, ie $x = 25$.

- Step 7: Confirm that we have a maximum or a minimum.

If $x < 25$, say $x = 24$, $\dfrac{dA}{dx}(24) = 50 - 2(24) > 0$.

If $x > 25$, say $x = 26$, $\dfrac{dA}{dx}(26) = 50 - 2(26) < 0$.

$\dfrac{dA}{dx}$ changes from positive to negative, so A has a local maximum when $x = 25$.

Example 2 continued

- Step 6: Differentiate the function, set the derivative to zero, and solve.

$$\frac{dA}{dx} = 50 - 2x = 0,$$

so $\qquad 50 - 2x = 0$, ie $x = 25$.

- Step 7: Confirm that we have a maximum or a minimum.

If $x < 25$, say $x = 24$, $\dfrac{dA}{dx}(24) = 50 - 2(24) > 0$.

If $x > 25$, say $x = 26$, $\dfrac{dA}{dx}(26) = 50 - 2(26) < 0$.

$\dfrac{dA}{dx}$ changes from positive to negative, so A has a local maximum when $x = 25$.

Example 2 continued

- Step 8: Answer the question.

  Since $x = 25$, $y = 50 - x = 25$, so the farmer should make a square paddock with each side 25m long.

In this question, we had implicit bounds on the possible values of x, ie $0 \leq x \leq 50$. Can you think why?

When we have such bounds, we should also check the value of A at the endpoints of the interval.

$A(0) = A(50) = 0$ which gives us a minimum value for A.

Example 2 continued

- Step 8: Answer the question.

  Since $x = 25$, $y = 50 - x = 25$, so the farmer should make a square paddock with each side 25m long.

In this question, we had implicit bounds on the possible values of x, ie $0 \leq x \leq 50$. Can you think why?

When we have such bounds, we should also check the value of A at the endpoints of the interval.

$A(0) = A(50) = 0$ which gives us a minimum value for A.

Example 2 continued

- Step 8: Answer the question.

  Since $x = 25$, $y = 50 - x = 25$, so the farmer should make a square paddock with each side 25m long.

In this question, we had implicit bounds on the possible values of x, ie $0 \leq x \leq 50$. Can you think why?

When we have such bounds, we should also check the value of A at the endpoints of the interval.

$A(0) = A(50) = 0$ which gives us a minimum value for A.

Example 2 continued

- Step 8: Answer the question.

  Since $x = 25$, $y = 50 - x = 25$, so the farmer should make a square paddock with each side 25m long.

In this question, we had implicit bounds on the possible values of x, ie $0 \leq x \leq 50$. Can you think why?

When we have such bounds, we should also check the value of A at the endpoints of the interval.

$A(0) = A(50) = 0$ which gives us a minimum value for A.

Summarising the steps

- Step 1: Draw a diagram if appropriate and define your variables.
- Step 2: Write down the function to be maximised (or minimised) in terms of your variables.
- Step 3: Find any relationship between your variables.
- Step 4: Write one of the variables in Step 3 in terms of the other.
- Step 5: Substitute the result into the function you are maximising.
- Step 6: Differentiate, set the derivative to zero and solve.
- Step 7: Check for a maximum (or a minimum).
- Step 8: Finally answer the question which was asked.

Summarising the steps

- Step 1: Draw a diagram if appropriate and define your variables.
- Step 2: Write down the function to be maximised (or minimised) in terms of your variables.
- Step 3: Find any relationship between your variables.
- Step 4: Write one of the variables in Step 3 in terms of the other.
- Step 5: Substitute the result into the function you are maximising.
- Step 6: Differentiate, set the derivative to zero and solve.
- Step 7: Check for a maximum (or a minimum).
- Step 8: Finally answer the question which was asked.

Summarising the steps

- Step 1: Draw a diagram if appropriate and define your variables.
- Step 2: Write down the function to be maximised (or minimised) in terms of your variables.
- Step 3: Find any relationship between your variables.
- Step 4: Write one of the variables in Step 3 in terms of the other.
- Step 5: Substitute the result into the function you are maximising.
- Step 6: Differentiate, set the derivative to zero and solve.
- Step 7: Check for a maximum (or a minimum).
- Step 8: Finally answer the question which was asked.

Summarising the steps

- Step 1: Draw a diagram if appropriate and define your variables.
- Step 2: Write down the function to be maximised (or minimised) in terms of your variables.
- Step 3: Find any relationship between your variables.
- Step 4: Write one of the variables in Step 3 in terms of the other.
- Step 5: Substitute the result into the function you are maximising.
- Step 6: Differentiate, set the derivative to zero and solve.
- Step 7: Check for a maximum (or a minimum).
- Step 8: Finally answer the question which was asked.

Summarising the steps

- Step 1: Draw a diagram if appropriate and define your variables.
- Step 2: Write down the function to be maximised (or minimised) in terms of your variables.
- Step 3: Find any relationship between your variables.
- Step 4: Write one of the variables in Step 3 in terms of the other.
- Step 5: Substitute the result into the function you are maximising.
- Step 6: Differentiate, set the derivative to zero and solve.
- Step 7: Check for a maximum (or a minimum).
- Step 8: Finally answer the question which was asked.

Summarising the steps

- Step 1: Draw a diagram if appropriate and define your variables.
- Step 2: Write down the function to be maximised (or minimised) in terms of your variables.
- Step 3: Find any relationship between your variables.
- Step 4: Write one of the variables in Step 3 in terms of the other.
- Step 5: Substitute the result into the function you are maximising.
- Step 6: Differentiate, set the derivative to zero and solve.
- Step 7: Check for a maximum (or a minimum).
- Step 8: Finally answer the question which was asked.

Summarising the steps

- Step 1: Draw a diagram if appropriate and define your variables.
- Step 2: Write down the function to be maximised (or minimised) in terms of your variables.
- Step 3: Find any relationship between your variables.
- Step 4: Write one of the variables in Step 3 in terms of the other.
- Step 5: Substitute the result into the function you are maximising.
- Step 6: Differentiate, set the derivative to zero and solve.
- Step 7: Check for a maximum (or a minimum).
- Step 8: Finally answer the question which was asked.

Summarising the steps

- Step 1: Draw a diagram if appropriate and define your variables.
- Step 2: Write down the function to be maximised (or minimised) in terms of your variables.
- Step 3: Find any relationship between your variables.
- Step 4: Write one of the variables in Step 3 in terms of the other.
- Step 5: Substitute the result into the function you are maximising.
- Step 6: Differentiate, set the derivative to zero and solve.
- Step 7: Check for a maximum (or a minimum).
- Step 8: Finally answer the question which was asked.