2 Unit Bridging Course – Day 6

Application of the second derivative: Curve sketching

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The second derivative: Curve sketching

In the last module, we saw how to use the second derivative to find points of inflexion.

We can also use the second derivative and concavity to determine the nature of stationary points.

Both can be used to help sketch the function.

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Second derivative test: Minimum

Recall: If $f''(x) > 0$ over an interval, then the function f is concave up over that interval.

Suppose that f has a stationary point at $x = p$, ie $f'(p) = 0$.

Then if $f''(p) > 0$, the curve is concave up at $x = p$, and f has a local minimum at p.

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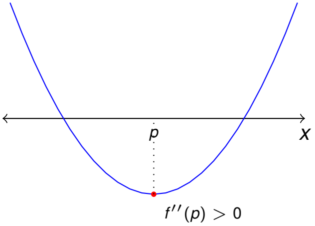

Second derivative test: Maximum

Recall: If $f''(x) < 0$ over an interval, then the function f is concave down over that interval.

Suppose that f has a stationary point at $x = p$, ie $f'(p) = 0$.

Then if $f''(p) < 0$, the curve is concave down at $x = p$, and f has a local maximum at p.

Second derivative test: Maximum

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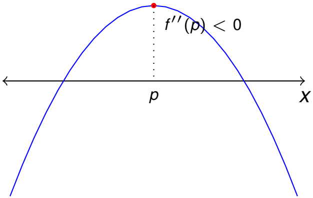

Second derivative test: Summary

- If $f'(p) = 0$ and $f''(p) > 0$, then the function f has a local minimum at p.
- If $f'(p) = 0$ and $f''(p) < 0$, then the function f has a local maximum at p.
- If $f'(p) = 0$ and $f''(p) = 0$, then the test tells us nothing.

So, if $f'(p) = 0$ and $f''(p) = 0$, then we must look at the first derivative to determine the nature of the stationary point like we did on Day 5.

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Example

Sketch the curve $y = f(x) = x^3 - 3x$.

First find any stationary points:

$$f'(x) = 3x^2 - 3 = 3(x-1)(x+1) = 0,$$

so the stationary points are $(1, -2)$ and $(-1, 2)$.

Next find the second derivative:

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Use the second derivative to test the stationary points:

$$f''(1) = 6,$$

since $f''(1) > 0$ the curve is concave up, hence $(1, -2)$ is a local minimum;

$$f''(-1) = -6$$

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Example continued

Notice that $f''(x) = 6x = 0$ when $x = 0$, this means we have a possible point of inflexion.

To test if this is a point of inflexion, we check the second derivative slightly before and after the point.

When

$$x < 0,\ f''(x) < 0$$

and when

$$x > 0,\ f''(x) > 0.$$

This means that the concavity of the curve is changing at $x = 0$, hence there is a point of inflexion at $(0, 0)$.

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The curve sketched: