ASDS Statistics, YSU, Fall 2020

Lecture 13

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14 Oct 2020

Contents

Properties of the Sample Correlation Coefficient

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Say, if x is a Dataset of heights of some persons, in centimeters, y their weights in grams, and if x' will be the same heights Dataset using meters as units, and y' will be the weights in Kg-s, then $cov(x, y)$ and $cov(x', y')$ will not be the same, but $cor(x, y) = cor(x', y')$.

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- If $|cov(x, y)| > |cov(z, t)|$, we cannot state that the relationship between x and y is stronger than the relationship between z and t. But if $|cor(x, y)| > |cor(z, t)|$, we can.

So it is not easy to interpret the magnitude of the covariance, but the magnitude of the correlation coefficient is the strength of the linear relationship.

Pros/Cons of Sample Covariance and Correlation Coefficient

- An important drawback of the Sample Correlation Coefficient is that it is sensitive to outliers.

Covariance and Correlation Coefficient, again

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$$cov(x, y) > 0, \quad \text{equivalently, if} \quad cor(x, y) > 0,$$

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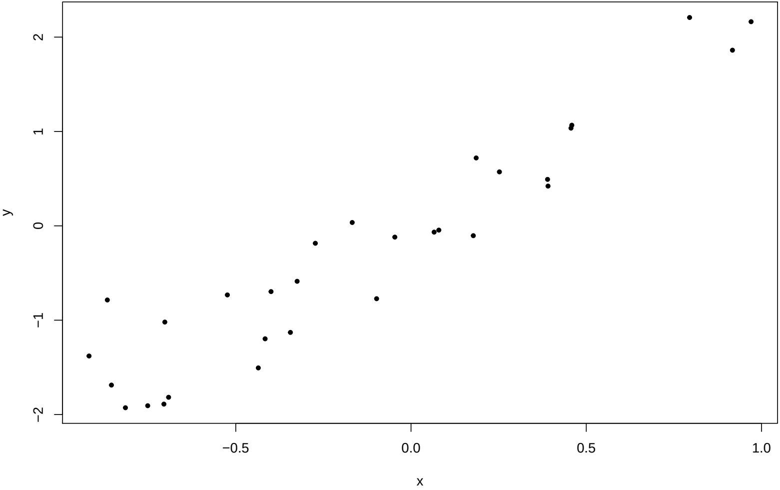
- The magnitude of the Correlation Coefficient shows the strength of the Linear Relationship.

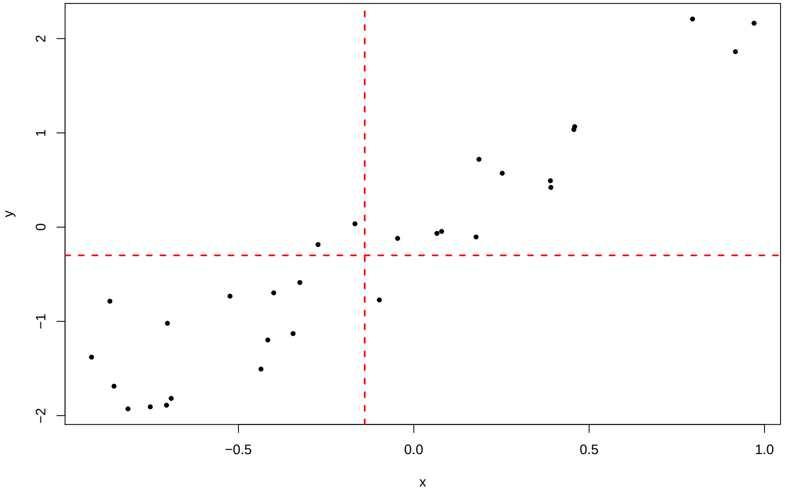
Explanation

Here is a Bivariate Dataset (x, y) with $cov(x, y) > 0$:



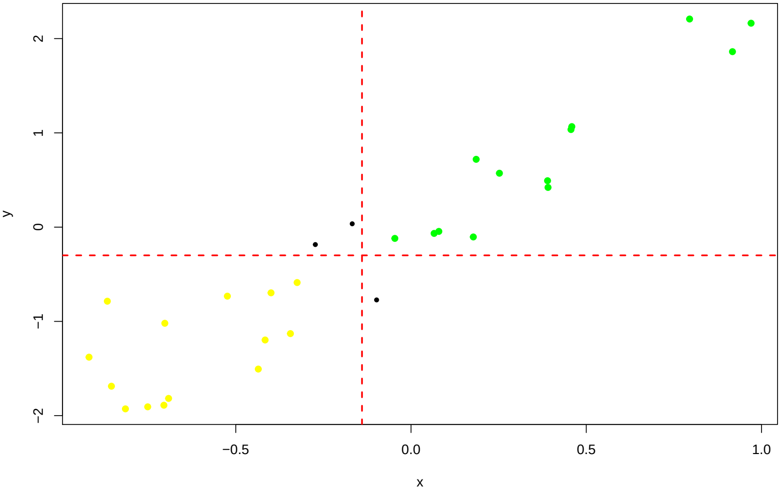
Explanation

Now we add a vertical line through $\bar{x}$ and a horizontal line through $\bar{y}$



Explanation

We color the points in the first and third quadrants:



Explanation

The points in the 1st quadrant (of the dotted coordinate system, with the center at $(\bar{x}, \bar{y})$), green points, satisfy

$$x_k > \bar{x} \quad \textit{and} \quad y_k > \bar{y},$$

so

$$(x_k - \bar{x}) \cdot (y_k - \bar{y}) > 0,$$

so green points contribute positive terms to

$$cov(x, y) = \frac{1}{n} \cdot \sum_{k=1}^{n} (x_k - \bar{x}) \cdot (y_k - \bar{y}).$$

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Similarly, Points in the 3rd quadrant, yellow points, again contribute positive terms to $cov(x, y)$, since in this case

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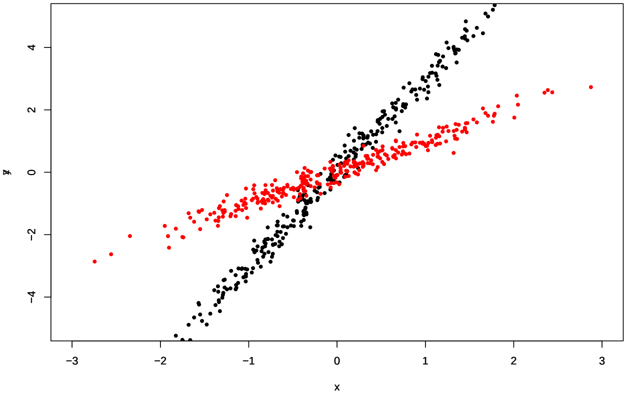
In the same way, the points in the 2nd and 4th quadrants give negative terms to $cov(x, y)$, as in this case $(x_k - \bar{x}) \cdot (y_k - \bar{y}) < 0$. And positive covariance means that the terms for points in the 1st and 3rd quadrants dominate to the ones from 2nd and fourth ones.

Note: Of course, we can have a negative trend and just one strong outlier in the 1st quadrant resulting in a positive covariance.

Example

For which of the following pairs the Correlation is higher ((x, y) pairs are in black, and (x, z) pairs are in red)?



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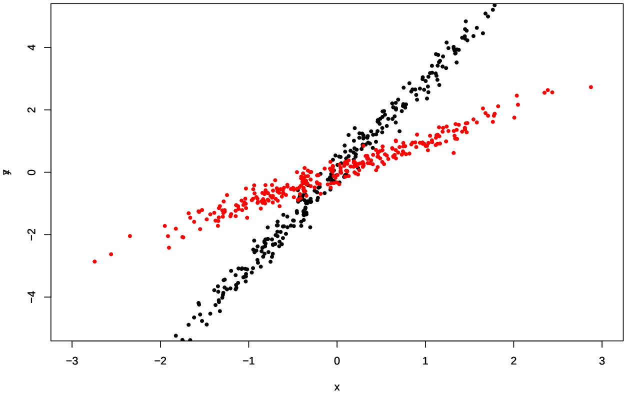


```r
c(cor(x,y), cor(x,z))
```

```
## [1] 0.9949983 0.9775781
```

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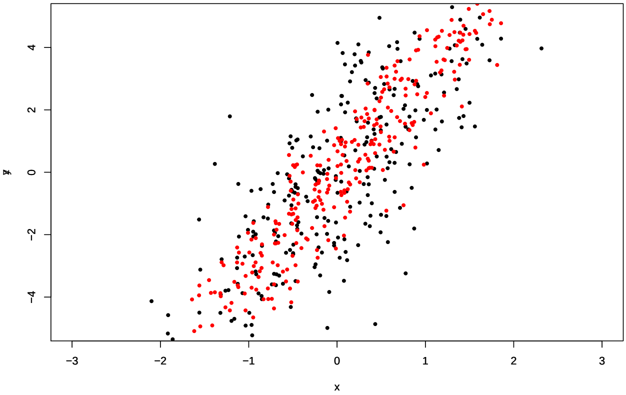

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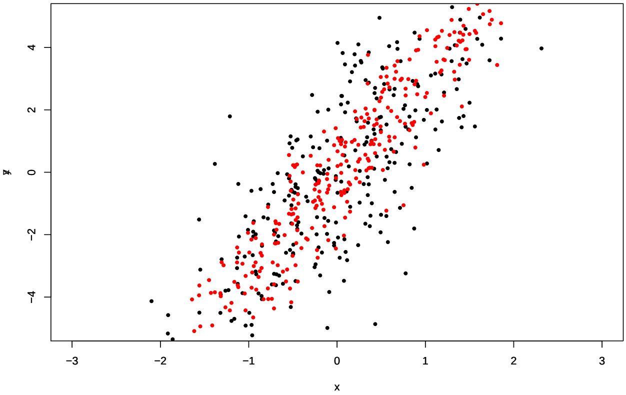


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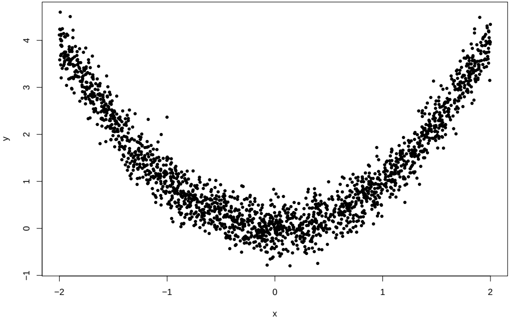
Note: We will talk about this and about the relationship of slope with the Correlation Coefficient during the Linear Regression lectures.

Correlation is a Measure of Linear Relationship

```
x <- runif(2000, -2,2)
y <- x^2 + 0.3*rnorm(2000)
plot(x,y, pch = 20)
```




```
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```

```
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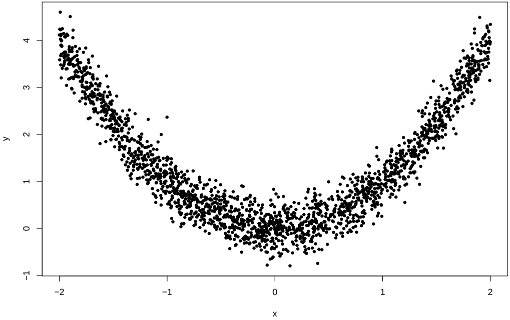


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See more at Wiki

Another Relationship between the Correlation and Covariance

Assume we have two datasets x and y of the same size. We standardize them, i.e., we consider

$$\frac{x-\bar{x}}{s_x}, \qquad \frac{y-\bar{y}}{s_y},$$

then the Correlation Coefficient is just the Covariance between these standardized daatasets:

$$cor(x,y) = cov\left(\frac{x-\bar{x}}{s_x}, \frac{y-\bar{y}}{s_y}\right).$$

Supplements, Other Measures of Correlation

- if working with several variables, we can calculate pairwise Correlations (Correlation Matrix) and plot the HeatMap

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- If working with multiple variables, one can calculate the Multiple correlation
- One can interpret the Correlation Coefficient as a Cosine of the angle between the r.v.s (or observations), see Wiki
- There are other measures of Association between variables, such as Rank Correlations, say, Kendal's τ

In **R**, the `cor` function has a parameter *method*, where you can change the Correlation Coefficient type.

Correlation is not Causation

- Some Examples: Spurious Correlations

Anscombe Quartet

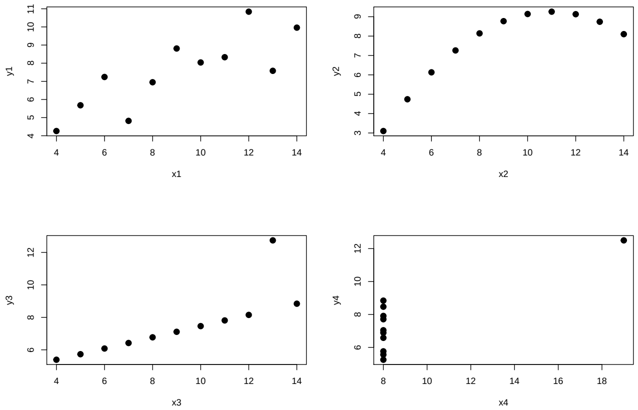
See Wiki

```
anscombe
```

```
##    x1 x2 x3 x4    y1   y2    y3    y4
## 1  10 10 10  8  8.04 9.14  7.46  6.58
## 2   8  8  8  8  6.95 8.14  6.77  5.76
## 3  13 13 13  8  7.58 8.74 12.74  7.71
## 4   9  9  9  8  8.81 8.77  7.11  8.84
## 5  11 11 11  8  8.33 9.26  7.81  8.47
## 6  14 14 14  8  9.96 8.10  8.84  7.04
## 7   6  6  6  8  7.24 6.13  6.08  5.25
## 8   4  4  4 19  4.26 3.10  5.39 12.50
## 9  12 12 12  8 10.84 9.13  8.15  5.56
## 10  7  7  7  8  4.82 7.26  6.42  7.91
## 11  5  5  5  8  5.68 4.74  5.73  6.89
```

Anscombe Quartet

```
rm(x1, x2, x3, x4, y1, y2, y3, y4); attach(anscombe)
par(mfrow=c(2,2))
plot(y1~x1, pch=19, cex=1.4); plot(y2~x2, pch=19, cex=1.4);
plot(y3~x3, pch=19, cex=1.4); plot(y4~x4, pch=19, cex=1.4);
```



Anscombe Quartet

```
c(mean(x1), mean(x2), mean(x3), mean(x4))
```

```
## [1] 9 9 9 9
```

```
c(mean(y1), mean(y2), mean(y3), mean(y4))
```

```
## [1] 7.500909 7.500909 7.500000 7.500909
```

```
c(var(x1), var(x2), var(x3), var(x4))
```

```
## [1] 11 11 11 11
```

```
c(var(y1), var(y2), var(y3), var(y4))
```

```
## [1] 4.127269 4.127629 4.122620 4.123249
```

```
c(cor(x1,y1), cor(x2,y2), cor(x3,y3), cor(x4,y4))
```

```
## [1] 0.8164205 0.8162365 0.8162867 0.8165214
```

Anscombe Quartet

Moral: Just calculating numbers (summary statistics) is not enough, visualize your Data if possible.

Reminder on Random Variables and Distributions

Random Variables

Everything starts at the Probability Space (Experiment, Model): we are given

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Definition: Any (measurable) function $X : \Omega \to \mathbb{R}$ is called a r.v. on the Probability Space $(\Omega, \mathbb{P})$.

So $X = X(\omega)$, but usually we forget about ω, and use X.

Main Complete Characteristics of a r.v.

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So for a Continuous r.v., another complete characteristic, besides the CDF, is its PDF.