Theorem

For a linear mapping $T : V \to W$, ordered bases

$$B = (\mathbf{b}_1, \ldots, \mathbf{b}_n), \quad \tilde{B} = (\tilde{\mathbf{b}}_1, \ldots, \tilde{\mathbf{b}}_n)$$

of V and

$$C = (\mathbf{c}_1, \ldots, \mathbf{c}_m), \quad \tilde{C} = (\tilde{\mathbf{c}}_1, \ldots, \tilde{\mathbf{c}}_m)$$

of W, and a transformation matrix A_T of T with respect to B and C, the corresponding transformation matrix $\tilde{A}_T$ with respect to the bases $\tilde{B}$ and $\tilde{C}$ is given as

$$\tilde{A}_T = L^{-1} A_T S$$

Here, $S \in \mathbb{R}^{n\times n}$ $(L \in \mathbb{R}^{m\times m})$ is the transformation matrix of id_V (id_W) that maps coordinates with respect to $\tilde{B}(\tilde{C})$ onto coordinates with respect to $B(C)$.

Definition

Matrices $A, \tilde{A} \in \mathbb{R}^{m\times n}$ *are* **equivalent** *if there exist regular matrices* $P \in \mathbb{R}^{n\times n}$ *and* $Q \in \mathbb{R}^{m\times m}$ *such that*

$$\tilde{A} = Q^{-1}AP$$

Definition

Matrices $A, \tilde{A} \in \mathbb{R}^{n\times n}$ *are* **similar** *if there exists a regular matrix* $Q \in \mathbb{R}^{n\times n}$ *such that*

$$\tilde{A} = Q^{-1}AQ.$$

Remark

Similar matrices are always equivalent. However, equivalent matrices are not necessarily similar.

Image and Kernel

Definition

For $T : V \to W$, we dfine the kernel/null space

$$\ker(T) := T^{-1}(\mathbf{0}_W) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0_W}\}$$

and the image/range

$$\mathrm{Im}(T) := T(V) = \{\mathbf{w} \in W | \exists \mathbf{v} \in V : T(\mathbf{v}) = \mathbf{w}\}$$

We also call V and W also the domain and codomain of T, respectively

Proposition

- $T(\mathbf{0}_V) = \mathbf{0}_W$ *and, therefore,* $\mathbf{0}_V \in \ker(T)$
- $\mathrm{Im}(T) \subset W$ *is a subspace of* W, *and* $\ker(T) \subset V$ *is a subspace of* V
- T *is injective (one-to-one) if and only if* $\ker(T) = \{0\}$

Definition

Let $A \in \mathbb{R}^{m\times n}$ *be a matrix with columns* $\mathbf{a}_1, \ldots, \mathbf{a}_n$, *then*

- $\mathrm{span}[\mathbf{a}_1, \ldots, \mathbf{a}_n] \subset \mathbb{R}^m$ *is called* **column space**
- $\{A\mathbf{x} = 0; \mathbf{x} \in \mathbb{R}^n\} \subset \mathbb{R}^n$ *is called* **null space**

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping with its transformation matrix $A \in \mathbb{R}^{m\times n}$, i.e. $T(\mathbf{x}) = A\mathbf{x}$.

- If $A = [\mathbf{a}_1, \ldots, \mathbf{a}_n]$, then $\mathrm{Im}(T) = \{A\mathbf{x}; \mathbf{x} \in \mathbb{R}^n\}$
  $= \{x_1\mathbf{a}_1 + \ldots + x_n\mathbf{a}_n; x_1, \ldots, x_n \in \mathbb{R}^n\} = \mathrm{span}[\mathbf{a}_1, \ldots, \mathbf{a}_n]$
- The image $\mathrm{Im}(T)$ is a subspace of $\mathbb{R}^m$
- $\mathrm{rk}(A) = \dim(\mathrm{Im}(T))$
- The kernel $\ker(T)$ is a subspace of $\mathbb{R}^n$
- The kernel $\ker(T)$ is the null space of A

Example

Find a basis for image and kernel of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ *given by*

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 - x_3 \\ x_2 + 3x_3 \end{bmatrix}$$

Rank-Nullity Theorem

For vector spaces U, V and a linear mapping $T : U \to V$ it holds that

$$\dim(\ker(T)) + \dim(\mathrm{Im}(T)) = \dim(U),$$

equiavalently for a matrix $A \in \mathbb{R}^{n \times m}$

$$\mathrm{nullity}\,(A) + \mathrm{rank}(A) = m,$$

where $\mathrm{nullity}\,(A)$ is the dimension of the null space $\mathrm{null}\,(A)$.

Definition

Given a vector space V , a **norm** *on* V *is a function* $\|\cdot\| : V \to [0, +\infty)$ *with the following properties: For all* $\mathbf{x}, \mathbf{y} \in V$*, and all* $\lambda \in \mathbb{R}$

- $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ *(triangle inequality)*
- $\|\lambda\mathbf{x}\| = |\lambda|\|\mathbf{x}\|$ *(absolutely homogeneous)*
- $\|\mathbf{x}\| \geq 0$ *and* $\|\mathbf{x}\| = 0$ *iff* $\mathbf{x} = 0$ *(positive definite)*

Example

The **Manhattan norm** *on* $\mathbb{R}^n$ *is defined for* $\mathbf{x} \in \mathbb{R}^n$ *as*

$$\|\mathbf{x}\|_1 := \sum_{i=1}^{n} |x_i|$$

The Manhattan norm is also called ℓ_1 **norm**.

Example

The length of a vector $\mathbf{x} \in \mathbb{R}^n$ *is given by*

$$\|\mathbf{x}\|_2 := \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{\mathbf{x}^T \mathbf{x}}.$$

This norm is called the **Euclidean norm.** *The Euclidean norm is also called* ℓ_2 **norm**.

Inner Products

Definition

A function $f : V \times V \to \mathbb{R}$ *is called* **bilinear** *if*

$$f(a\mathbf{x} + b\mathbf{y}, \mathbf{z}) = af(\mathbf{x}, \mathbf{z}) + bf(\mathbf{y}, \mathbf{z})$$

$$f(\mathbf{x}, a\mathbf{y} + b\mathbf{z}) = af(\mathbf{x}, \mathbf{y}) + bf(\mathbf{x}, \mathbf{z})$$

Definition

A function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ *is called* **inner product on** V**,** *if*

- $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ *(symmetric)*
- $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ *and* $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ *iff* $\mathbf{x} = 0$ *(positive definite)*
- $\langle \cdot, \cdot \rangle$ *is a bilinear function*

Example

The scalar product/dot product in $\mathbb{R}^n$, *which is given by*

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y} = \sum_{i=1}^{n} x_i y_i.$$

The dot product is an inner product.
$\mathbb{R}^n$ *with the dot product is called* **Euclidean vector space.**

Example

If we define for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$

$$\langle \mathbf{x}, \mathbf{y} \rangle := x_1 y_1 - (x_2 y_1 + x_1 y_2) + 2 x_2 y_2$$

then $\langle \cdot, \cdot \rangle$ *is an inner product but different from the dot product.*

Symmetric, Positive Definite Matrices

Definition

A symmetric matrix $A \in \mathbb{R}^{n\times n}$ *is called* **positive definite**, *if*

$$\mathbf{x}^T A\mathbf{x} > 0, \quad \mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq \mathbf{0}$$

The matrix A *is called positive semi-definite, if*

$$\mathbf{x}^T A\mathbf{x} \geq 0, \quad \mathbf{x} \in \mathbb{R}^n.$$

Example

Which of the following matrices is positive definite?

$$A = \begin{bmatrix} 4 & -4 \\ -4 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -4 \\ -4 & 2 \end{bmatrix}$$

Proposition

Let V be an n-dimensional vector space with an inner product $\langle\cdot,\cdot\rangle$ and a basis $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$. Prove that

$$\langle \mathbf{x}, \mathbf{y} \rangle = [\mathbf{x}]_B^T A [\mathbf{y}]_B,$$

where $A_{ij} = \langle \mathbf{b}_i, \mathbf{b}_j \rangle$. Conclude that the matrix A is positive definite.

Theorem

For a real-valued, finite-dimensional vector space V and a basis B of V it holds that $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ is an inner product if and only if there exists a symmetric, positive definite matrix $A \in \mathbb{R}^{n \times n}$ with

$$\langle \mathbf{x}, \mathbf{y} \rangle = [\mathbf{x}]_B^T A [\mathbf{y}]_B,$$