Optimization

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Theorem

If $\Omega \subset \mathbb{R}^n$ is an open convex set and f is twice continuously differentiable, then f is convex if and only if

$$\nabla^2 f(x) \succeq 0, \quad \forall x \in \Omega.$$

Theorem

If $\Omega \subset \mathbb{R}^n$ is an open convex set and f is a twice continuously differentiable function such that $\nabla^2 f(x) \succ 0, \forall x \in \Omega$, then f is strictly convex.

Theorem

If $\Omega \subset \mathbb{R}^n$ is an open convex set and f is twice continuously differentiable, then f is concave if and only if

$$\nabla^2 f(x) \preceq 0, \quad \forall x \in \Omega.$$

Theorem

If $\Omega \subset \mathbb{R}^n$ is an open convex set and f is a twice continuously differentiable function such that $\nabla^2 f(x) \prec 0, \forall x \in \Omega$, then f is strictly concave.

Example

Check whether f is convex (strictly convex), concave (strictly concave) on Ω if

a. $f(x_1, x_2, x_3) = x_1^2 + 3x_1x_2 + 4x_2^2 + x_3^2 - x_1x_3$, $\Omega = \mathbb{R}^3$;

b. $f(x_1, x_2) = -x_1^4 + 2x_1x_2 - x_2^4 - x_1^2 - x_2^2$, $\Omega = \mathbb{R}^2$;

c. $f(x_1, x_2) = e^{x_1x_2}$, $\Omega = \mathbb{R}^2$;

d. $f(x_1, x_2) = x_1^3 + x_2^3$, $\Omega = \mathbb{R}^2$.

Unconstrained Optimization

Conditions for Local Minimizers

$$\begin{aligned} &\textit{minimize} \quad f(x) \\ &\textit{subject to} \quad x \in \Omega, \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$, with $n \geq 1$.

Theorem (First-Order Necessary Conditions (FONC))

Assume f is a continuously differentiable function in Ω. If x^* is a local minimizer (maximizer) of f over Ω and x^* is an interior point of Ω, then

$$\nabla f(x^*) = 0.$$

Definition

We call x^* a **stationary point** if $\nabla f(x^*) = 0$.

Example

Consider the problem

$$\begin{aligned} \textit{minimize} \quad & x_1^2 + 0.5x_2^2 + 3x_2 + 4.5 \\ \textit{subject to} \quad & x \in \mathbb{R}^2. \end{aligned}$$

Find the points which satisfy the first-order necessary condition (FONC) for a local minimizer.

Theorem (Second-Order Necessary Conditions (SONC))

Assume f is twice continuously differentiable in Ω. If x^* is a local minimizer (maximizer) of f over Ω and x^* is an interior point of Ω, then $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ is positive (negative) semidefinite.

Definition

A **saddle point** is a stationary point which is not a local extremum.

Example

Show that $x^* = (0, 0)^T$ is a saddle point for the function $f(x_1, x_2) = x_1^2 + 8x_1x_2 + x_2^2$.

Theorem (Second-Order Sufficient Conditions (SOSC))

Assume f is twice continuously differentiable in Ω. If x^* is an interior point of Ω such that $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ is positive (negative) definite, then x^* is a strict local minimizer (maximizer) of f.

Example

Find all stationary points of f and check if these points are local maximum, minimum or saddle points for that function if

a. $f(x_1, x_2) = 4x_1^4 + x_2^4 + 4x_1x_2;$

b. $f(x_1, x_2) = \frac{1}{3}x_1^3 - 4x_1 + \frac{1}{3}x_2^3 - 16x_2;$

c. $f(x_1, x_2, x_3) = 3x_1^3 - 9x_1 + x_2^3 + x_3^3 - 6x_3^2 - 10.$