Basic Mathematics , Fall 2020

Karen Keryan
ASDS, YSU

September 23, 2020

Basis and Rank

Definition

Consider a set of vectors $\mathcal{A}$ in a vector space V. The set of all linear combinations of vectors in A is called the **span** *of $\mathcal{A}$ and is denoted by* $\mathrm{span}(\mathcal{A})$, *i.e.*

$$\mathrm{span}(\mathcal{A}) = \left\{ \sum_{i=1}^{k} \lambda_i \mathbf{x}_i;\ k \in \mathbb{N},\ \mathbf{x}_i \in \mathcal{A},\ \lambda_i \in R \right\}$$

If every vector $v \in V$ can be expressed as a linear combination of the vectors of $\mathcal{A}$, then $\mathcal{A}$ is called a **generating set** *of V and we say $\mathcal{A}$ spans the vector space V.*

Definition

A **basis** $\mathcal{B}$ *of a vector space* V *is a linearly independent subset of* V *that spans* V. *Equivalently,* $\mathcal{B}$ *is a basis of* V *if*

- $\mathcal{B}$ *is linearly independent*
- $\mathcal{B}$ *is a spanning set of* V*:* $\mathrm{span}(\mathcal{B}) = V$.

Definition

A generating set $\mathcal{A}$ *of* V *is called* **minimal** *if there exists no smaller set* $\tilde{A} \subset \mathcal{A} \subset V$ *that spans* V*.*

Proposition

The following statements are equivalent:

- $\mathcal{B}$ *is a basis of* V
- $\mathcal{B}$ *is a minimal generating(spanning) set of* V
- $\mathcal{B}$ *is a maximal linearly independent set of vectors in* V *, i.e., adding any other vector to this set will make it linearly dependent.*
- *Every vector* $\mathbf{x} \in V$ *is a unique linear combination of vectors from* $\mathcal{B}$*, i.e.*

$$\mathbf{x} = \sum_{i=1}^{k} \lambda_i \mathbf{b}_i = \sum_{i=1}^{k} \psi_i \mathbf{b}_i$$

and $\lambda_i, \psi_i \in \mathbb{R}, \mathbf{b}_i \in \mathcal{B}$*, it follows that* $\lambda_i = \psi_i,\ i = 1, \ldots, k$.

Example

- *In $\mathbb{R}^3$, the canonical/standard basis is*
$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- *Different bases in $\mathbb{R}^3$ are*
$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

- *The set*
$$\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right\}$$
is linearly independent, but not a generating set (and not a basis) of $\mathbb{R}^4$: For instance, the vector $[0,0,0,1]^T$ cannot be obtained by a linear combination of elements in $\mathcal{A}$.

Proposition

Let V be a vector space with a finite basis. Then every basis of V has the same number of elements.

Definition

The **dimension** *of V is the number of basis vectors, and it is denoted by* $\dim(V)$.

How to find a basis?

A basis of a subspace $U = \operatorname{span}[\mathbf{x}_1, \ldots, \mathbf{x}_m] \subset \mathbb{R}^n$ can be found by executing the following steps:

1. Write the spanning vectors as columns of a matrix A
2. Determine the row echelon form of A,
3. The spanning vectors associated with the pivot columns are a basis of U.

Example

Determine a basis for the subspace $U \subset \mathbb{R}^5$ spanned by the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} 0 \\ 2 \\ -1 \\ -1 \\ -2 \end{bmatrix}.$$

Rank

Definition

The number of linearly independent columns of a matrix $A \in \mathbb{R}^{m \times n}$ equals the number of linearly independent rows and is called the **rank** *of A and is denoted by $\mathrm{rk}(A)$ or $\mathrm{rank}(A)$.*

Definition

A matrix $A \in \mathbb{R}^{m \times n}$ has **full rank** *if its rank equals the largest possible rank for a matrix of the same dimensions, i.e. $\mathrm{rk}(A) = \min(m, n)$.*
A matrix is said to be **rank deficient** *if it does not have full rank.*

Proposition

- $\mathrm{rk}(A) = \mathrm{rk}(A^T)$ *for all* $A \in \mathbb{R}^{m\times n}$
- *The columns of* $A \in \mathbb{R}^{m\times n}$ *span a subspace* $U \subset \mathbb{R}^m$ *with* $\dim(U) = \mathrm{rk}(A)$. *Later, we will call this subspace the image or range.*
- *The rows of* $A \in \mathbb{R}^{m\times n}$ *span a subspace* $W \subset \mathbb{R}^n$ *with* $\dim(W) = \mathrm{rk}(A)$.
- *For all* $A \in \mathbb{R}^{n\times n}$ *holds:* A *is regular (invertible) if and only if* A *has full rank, i.e.* $\mathrm{rk}(A) = n$.
- *For all* $A \in \mathbb{R}^{m\times n}$ *and all* $\mathbf{b} \in \mathbb{R}^m$ *it holds that the SLE* $A\mathbf{x} = \mathbf{b}$ *has a solution if and only if* $\mathrm{rk}(A) = \mathrm{rk}(A|\mathbf{b})$, *where* $A|\mathbf{b}$ *denotes the augmented matrix.*
- *For* $A \in \mathbb{R}^{m\times n}$ *the dimension of the subspace of solutions for* $A\mathbf{x} = \mathbf{0}$ *is* $n - \mathrm{rk}(A)$. *Later, we will call this subspace the kernel or the null space.*

Example

Find the rank of the following matrices

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 3 \\ 3 & 7 & 9 \end{bmatrix}.$$

Linear Mappings

Definition

For vector spaces V, W, a mapping $T : V \to W$ is called a linear mapping (or vector space homomorphism/ linear transformation) if

$$T(a\mathbf{x} + b\mathbf{y}) = aT(\mathbf{x}) + bT(\mathbf{y}),\ \forall x, y \in V, a, b \in \mathbb{R}$$

or equivalently,

- $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$
- $T(a\mathbf{x}) = aT(\mathbf{x})$

Definition

Consider a transformation $T : V \to W$, where V, W can be arbitrary sets. Then T is called

- *injective, if for any $\mathbf{x}, \mathbf{y} \in V$ it follows that $T(\mathbf{x}) \neq T(\mathbf{y})$ if and only if $\mathbf{x} \neq \mathbf{y}$.*
- *surjective, if $T(V) = W$.*
- *bijective if it is injective and surjective.*

Definition

Let V, W be vector spaces.

- $T : V \to W$ *is called* **isomorphism**, *if it is linear and bijective*
- $T : V \to V$ *is called* **endomorphism**, *if T is linear*
- $T : V \to V$ *is called* **automorphism**, *if T is linear and bijective*
- *We define $id_V : V \to V$, $id_v(\mathbf{x}) = \mathbf{x}$ as the* **identity mapping** *in V.*

Example

Show that the mapping $T : \mathbb{R}^2 \to C$, given by $T(\mathbf{x}) = x_2 + ix_1$, is an isomorphism.

Definition

Vector spaces U and V are called isomorphic if there exists an isomorphism $T : U \to V$ and we write $U \cong V$

Theorem

Finite-dimensional vector spaces V and W are isomorphic if and only if $\dim(V) = \dim(W)$.

Example

Show that $R^{m\times n} \cong R^{mn}$.