Optimization

Lusine Poghosyan

YSU

October 23, 2020

Example

Determine whether the matrix A is positive definite (semidefinite), negative definite (semidefinite) or indefinite if

b.

$$A = \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix};$$

c.

$$A = \begin{pmatrix} 2 & 4 \\ 4 & 5 \end{pmatrix}.$$

Note. If one needs to show that $A \succeq 0$ ($A \preceq 0$), then one must show that $QF_A(y) \geq 0$ ($QF_A(y) \leq 0$), $\forall y \in \mathbb{R}^n$ and find an example of $y \neq 0$ such that $QF_A(y) = 0$.

Definition

Let $A = [a_{ij}]_{i,j=1}^{n}$ be $n \times n$ matrix. The leading principal minors are $\det A$ and the minors obtained by successively removing the last row and the last column. That is, the leading principal minors are

$$\Delta_1 = a_{11},\ \Delta_2 = \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},\ \Delta_3 = \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \ldots,$$

$\Delta_n = \det A$.

Theorem

Let $A = [a_{ij}]_{i,j=1}^{n}$ *be* $n \times n$ *symmetric matrix. The following three statements are equivalent*

- *A is positive definite;*
- *All eigenvalues of A are positive;*
- *All leading principal minors of A are positive (Sylvester's criterion).*

Definition

Let $A = \left[a_{ij}\right]_{i,j=1}^{n}$ be $n \times n$ matrix. The principal minors are $\det A$ itself and the determinants of matrices obtained by successively removing an i-th row and and i-th column. That is, the principal minors are

$$\det \begin{pmatrix} a_{i_1 i_1} & a_{i_1 i_2} & \dots & a_{i_1 i_p} \\ a_{i_2 i_1} & a_{i_2 i_2} & \dots & a_{i_2 i_p} \\ \vdots & & & \\ a_{i_p i_1} & a_{i_p i_2} & \dots & a_{i_p i_p} \end{pmatrix}, 1 \leq i_1 < i_2 < \cdots < i_p \leq n, p = 1, \dots, n.$$

Theorem

Let $A = \left[a_{ij}\right]_{i,j=1}^{n}$ be $n \times n$ symmetric matrix. The following three statements are equivalent

- *A is positive semidefinite;*
- *All eigenvalues of A are nonnegative;*
- *All principal minors of A are nonnegative.*

Example

Determine whether the matrix A is positive definite (semidefinite) if

a.

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix};$$

b.

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 9 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Theorem

Let $A = \left[a_{ij}\right]_{i,j=1}^{n}$ *be* $n \times n$ *symmetric matrix. The following three statements are equivalent*

- *A is negative definite;*
- *All eigenvalues of A are negative;*
- *All leading principal minors of even order are positive and of odd order negative (Sylvester's criterion).*

Theorem

Let $A = \left[a_{ij}\right]_{i,j=1}^{n}$ be $n \times n$ symmetric matrix. The following three statements are equivalent

- *A is negative semidefinite;*
- *All eigenvalues of A are nonpositive;*
- *All principal minors of even order are nonnegative and of odd order nonpositive.*

Example

Determine whether the matrix A is positive definite (semidefinite), negative definite (semidefinite) or indefinite if

a.

$$A = \begin{pmatrix} -1 & -2 & 0 \\ -2 & -5 & 1 \\ 0 & 1 & -4 \end{pmatrix};$$

b.

$$A = \begin{pmatrix} -1 & -3 & 0 \\ -3 & -9 & 0 \\ 0 & 0 & -2 \end{pmatrix};$$

Example

Determine whether the matrix A is positive definite (semidefinite), negative definite (semidefinite) or indefinite if

c.

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & -9 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Note. In order to show that some symmetric square matrix is indefinite it is enough to find a principal minor of even order that is negative or to find to principal minors of odd order such that one of them is positive and one of them is negative.