Optimization

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Duality

The dual problem is constructed from the cost function and constraints. The solution of the dual problem can be obtained from the solution of the primal problem and vice versa. Solving an LP problem via its dual can be simpler in some cases.

Suppose we are given a linear programming problem of the form

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \geq b \\ & x \geq 0. \end{aligned}$$

We refer to the above as *primal problem*. We define the corresponding *dual problem* as

$$\begin{aligned} \text{maximize} \quad & w^T b \\ \text{subject to} \quad & w^T A \leq c^T \\ & w \geq 0. \end{aligned}$$

The form of duality defined above is called the *symmetric form of duality*.

Example

Form the dual LPP of the following problem

$$\begin{aligned} \text{minimize} \quad & 3x_1 + x_2 + 3x_3 \\ \text{subject to} \quad & 2x_1 + x_2 + x_3 \geq 2 \\ & x_1 + 2x_2 + 3x_3 \geq 5 \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0. \end{aligned}$$

To define the dual of an arbitrary linear programming problem we need to convert it into an equivalent problem of the primal form, then construct the corresponding dual problem.

Proposition. The dual of the dual problem is the primal problem.

Now let's consider an LP problem in standard form

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0. \end{aligned}$$

The corresponding *dual problem* will be

$$\begin{aligned} \text{maximize} \quad & w^T b \\ \text{subject to} \quad & w^T A \leq c^T. \end{aligned}$$

The form of duality above is referred to as the *asymmetric form of duality*.

Example

Form the dual LPP of the following problem

$$\begin{aligned} \text{minimize} \quad & 2x_1 + x_2 + 4x_3 \\ \text{subject to} \quad & x_1 + x_2 + 2x_3 = 3 \\ & 2x_1 + x_2 + 3x_3 = 5 \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0. \end{aligned}$$

Properties of Dual Problems

Proposition. If x_0 and w_0 are feasible solutions to primal and dual LP problems, respectively (either in the symmetric or asymmetric form), then

$$c^T x_0 \geq w_0^T b.$$

Theorem

Assume x_0 and w_0 are feasible solutions to primal and dual LP problems, respectively (either in the symmetric or asymmetric form). If $c^T x_0 = w_0^T b$, then x_0 and w_0 are optimal solutions to their respective problems.

Theorem

If the primal problem (either in the symmetric or asymmetric form) possesses an optimal solution, then so does the dual and the optimal values of their respective objective functions are equal.

Conclusions

(**P**)-the primal problem (**D**)-the dual problem

- (**P**) optimal $\iff$ (**D**) optimal
- (**P**) unbounded $\Longrightarrow$ (**D**) infeasible
- (**D**) unbounded $\Longrightarrow$ (**P**) infeasible
- (**P**) infeasible $\Longrightarrow$ (**D**) infeasible or unbounded
- (**D**) infeasible $\Longrightarrow$ (**P**) infeasible or unbounded

Theorem

Assume x^* and w^* are feasible solutions to the primal and dual LP problems, respectively (either in the symmetric or asymmetric form). x^* and w^* are optimal solutions to their respective problems if and only if:

1. $(c^T - {w^*}^T A)x^* = 0 \iff (c_i - {w^*}^T a_i)x_i^* = 0$, $i = \overline{1, n}$
2. ${w^*}^T(b - Ax^*) = 0 \iff w_j^*(b_j - a^j x^*) = 0$ $j = \overline{1, m}$.

a_i is the i-th column of matrix A.

a^j is the j-th row of matrix A.

Example

Solve the following minimization problem

$$
\begin{array}{rl}
\text{minimize} & 2x_1 + 3x_2 + 5x_3 + 2x_4 + 3x_5 \\
\text{subject to} & x_1 + x_2 + 2x_3 + x_4 + 3x_5 \geq 4 \\
& 2x_1 - 2x_2 + 3x_3 + x_4 + x_5 \geq 3 \\
& x_i \geq 0,\ i = \overline{1,5}.
\end{array}
$$