Optimization

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Finite-Dimensional Optimization

We are going to consider the following problem

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad x \in \Omega, \end{aligned} \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$, with $n \geq 1$.

Existence of solution

Weierstrass Extreme Value Theorem

If $f \in \mathbb{C}(\Omega)$ and $\Omega \subset \mathbb{R}^n$ is compact, then the problem (1) has a solution.

Definition

A point $x \in \mathbb{R}^n$ is said to be a **limit point** of $\Omega \subset \mathbb{R}^n$, if each neighborhood of x contains a point of Ω other than x.

Example

Let $\Omega = [0, 3) \cup \{4\}$. Is x a limit point of Ω?

a. $x = 0$

b. $x = 3$

c. $x = 2$

d. $x = 4$

Example

Let $\Omega = \{x = [x_1, x_2]^T : x_1^2 + x_2^2 \leq 1 \quad \text{and} \quad x_1 > 0\}$. Is x a limit point of Ω, if

a. $x = [0, 0]^T$;

b. $x = [1, 0]^T$.

Definition

A set $\Omega \subset \mathbb{R}^n$ is said to be **closed set** if it contains all its limit points.

Example

Check if the set Ω is a closed set, if

a. $\Omega = [0, 3)$;

b. $\Omega = [0, 3]$;

c. $\Omega = \{x = [x_1, x_2]^T : x_1^2 + x_2^2 \leq 1\}$;

d. $\Omega = \{x = [x_1, x_2]^T : x_1^2 + x_2^2 \leq 1 \quad \text{and} \quad x_1 > 0\}$.

Definition

A set $\Omega \subset \mathbb{R}^n$ is said to be **bounded** if there exists $M \in \mathbb{R}$ such that $||x|| \leq M$, for all $x \in \Omega$.

Example

Check if the set Ω is bounded, if

a. $\Omega = \{x = [x_1, x_2]^T : x_1^2 + x_2^2 \leq 1\}$;

b. $\Omega = \{x = [x_1, x_2]^T : x_1^2 + x_2^2 \geq 1 \quad \text{and} \quad x_1 > 0\}$.

Definition

A set $\Omega \subset \mathbb{R}^n$ is said to be **compact** if Ω is closed and bounded.

Example

Check if the set Ω is compact, if

a. $\Omega = [0, 3)$;

b. $\Omega = [0, 3]$;

c. $\Omega = \{x = [x_1, x_2, x_3]^T : x_1^2 + x_2^2 + x_3^2 \leq 1\}$;

d. $\Omega = \{x = [x_1, x_2]^T : x_1^2 + x_2^2 \geq 1 \quad \text{and} \quad x_1 > 0\}$.

Uniqueness of solution

Definition

A set $\Omega \subset \mathbb{R}^n$ is a **convex set** if $\alpha x + (1-\alpha)y \in \Omega$, $\forall x, y \in \Omega$ and $\forall \alpha \in [0, 1]$.

Example

Check if the set Ω is convex, if

a. $\Omega = \{x = [x_1, x_2]^T : x_1^2 + x_2^2 \leq 1\}$;

b. $\Omega = \{x = [x_1, x_2]^T : x_1^2 + x_2^2 \leq 1 \quad \text{and} \quad x_1 x_2 \geq 0\}$.

Definition

A function $f : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ is a **convex function** if Ω is a convex set and $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$, $\forall x, y \in \Omega$, $x \neq y$ and $\forall \alpha \in (0, 1)$.

If in the definition above we replace "$\leq$" with "$<$", then we have the definition of **strictly convex function.**

Definition

A function f is a **concave function** if $-f$ is convex.

Definition

A function f is a **strictly concave function** if $-f$ is strictly convex.

Example

Show that the linear function $f(x) = a^T x + b$, where $a, x \in \mathbb{R}^n$ and $b \in \mathbb{R}$, is convex and concave.

Example

Let $f(x) = x_1^2 + x_2^2 + ... + x_n^2$, $x \in \mathbb{R}^n$. Show that f is a strictly convex function.

Theorem

Let $f : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$. If f is a convex function and x_0 is a local minimum point of f over Ω, then x_0 is a global minimum point of f in Ω, i.e. $x_0 = arg\ min_{x \in \Omega} f(x)$.

Theorem

Let $f : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$. If f is a strictly convex function and x_0 is a local minimum point of f over Ω, then x_0 is the unique global minimum point of f on Ω, i.e. $x_0 = arg\ min_{x \in \Omega} f(x)$.