Basic Mathematics , Fall 2020

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Definition

Let V be a set on which addition and scalar multiplication have been defined. If the following axioms hold for all $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and $d \in \mathbb{R}$ then V is called a **vector space** *and its elements are called* **vectors.**

$$
\begin{array}{ll}
1. \mathbf{u}+\mathbf{v} \ \textit{ is in } V. & \textit{Closure under addition} \\
2. \mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u} & \textit{Commutativity} \\
3. (\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w}) & \textit{Associativity} \\
4. \exists \mathbf{0} \in V, (\textit{called a } \textbf{zero vector}), & s.t. \mathbf{u}+\mathbf{0}=\mathbf{u} \\
5. \forall \mathbf{u} \in V, \exists -\mathbf{u} \in V & s.t. \ \mathbf{u}+(-\mathbf{u})=\mathbf{0} \\
6. c\mathbf{u} \in V & \textit{Closure under scalar mult.} \\
7. c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v} & \textit{Distributivity} \\
8. (c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u} & \textit{Distributivity} \\
9. c(d\mathbf{u})=(cd)\mathbf{u} & \\
10. 1\mathbf{u}=\mathbf{u} &
\end{array}
$$

Exercise

Prove that if V is a vector space then $0\mathbf{u} = \mathbf{0}$.

Example

For any $n \geq 1$, $\mathbb{R}^n$ is a vector space with the usual operations of addition and scalar multiplication.

Example

For any natural m and n, the set of all $m \times n$ matrices $R^{m\times n}$ forms a vector space with the usual operations of matrix addition and matrix scalar multiplication. Here the "vectors" are actually matrices.

Example

The set $\mathbb{Z}$ of integers with the usual operations is **not** *a vector space.*

Example

Let $\mathcal{P}_3$ denote the set of all polynomials of degree 3 or less with real coefficients. Define addition and scalar multiplication in the usual way. If

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3,\ q(x) = b_0 + b_1x + b_2x^2 + b_3x^3$$

are in $\mathcal{P}_3$, then

$$p(x) + q(x) =$$

$$= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3$$

is also in $\mathcal{P}_3$. If c is a scalar, then

$$cp(x) = ca_0 + ca_1x + ca_2x^2 + ca_3x^3.$$

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$$cp(x) = ca_0 + ca_1 x + ca_2 x^2 + ca_3 x^3.$$

In general, for any fixed $n \geq 0$, the set $\mathcal{P}_n$ of all polynomials of degree less than or equal to n is a vector space, as is the set $\mathcal{P}$ of all polynomials.

Subspaces

Definition

A subset W of a vector space V is called a subspace of V if W is itself a vector space with the same scalars, addition, and scalar multiplication as V.

Example

A line through the origin is a subspace of $\mathbb{R}^2$.

Theorem

Let V be a vector space and let W be a nonempty subset of V. Then W is a subspace of V if and only if the following conditions hold:

a. If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u} + \mathbf{v}$ is in W.

b. If $\mathbf{u}$ is in W and c is a scalar, then $c\mathbf{u}$ is in W.

Subspaces

Let us have a look at some subspaces.

- For every vector space V the trivial subspaces are V itself and $\{0\}$.
- The solution set of a homogeneous SLE $A\mathbf{x} = \mathbf{0}$ with n unknowns $\mathbf{x} = [x_1, \ldots, x_n]$ is a subspace of $\mathbb{R}^n$.
- The solution of an inhomogeneous SLE $A\mathbf{x} = \mathbf{b}; \mathbf{b} \neq \mathbf{0}$ is not a subspace of $\mathbb{R}^n$.
- The intersection of arbitrarily many subspaces is a subspace itself.

Example

Which of the following is a subspace of R^2?

1. *a line passing through the origin*
2. *two distinct lines passing through the origin*
3. *a unit circle*
4. *a line not passing through the origin*

Linear Independence

Definition

A vector $\mathbf{v} \in V$ *is called linear combination of vectors* $\mathbf{x}_1, \ldots, \mathbf{x}_k \in V$*, if there are scalars* $c_1, \ldots, c_k \in \mathbb{R}$ *so that*

$$\mathbf{v} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots + c_k\mathbf{x}_k = \sum_{i=1}^{k} c_i\mathbf{x}_i.$$

Definition

The vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k \in V$ *are called* **linearly dependent***, if there is a non-trivial linear combination, such that*

$$\sum_{i=1}^{k} c_i\mathbf{x}_i = \mathbf{0}$$

with at least one $c_i \neq 0$*. If only the trivial solution exists, i.e.,* $c_1 = \ldots = c_k = 0$*, then the vectors* $\mathbf{x}_1, \ldots, \mathbf{x}_k$ *are* **linearly independent.**

When vectors are linearly dependent

- If at least one of the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$ is $\mathbf{0}$ then they are linearly dependent.
- The vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are linearly dependent if and only if (at least) one of them is a linear combination of the others. In particular, if one vector is a multiple of another vector, i.e., $\mathbf{x}_i = c\mathbf{x}_j, c \in \mathbb{R}$, then the set $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is linearly dependent.

How to check linear independence?

A way of checking whether vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are linearly independent is to write all vectors as columns of a matrix A. Gaussian elimination yields a matrix in (reduced) row echelon form

- The pivot columns indicate the vectors, which are linearly independent.
- The non-pivot columns can be expressed as linear combinations of the pivot columns on their left.

Example

Are vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\3\\5\end{bmatrix}$ *linearly dependent?*

Let $\mathbf{b}_1, \dots, \mathbf{b}_k$ be linearly independent vectors and

$$\mathbf{x}_j = \sum_{i=1}^{k} c_{ij}\mathbf{b}_i, \;\; j = 1, \dots, m.$$

Equivalently

$$\mathbf{x}_j = B\mathbf{c}_j, \;\; \text{where } B = [\mathbf{b}_1, \dots, \mathbf{b}_k], \;\; \mathbf{c}_j = \begin{bmatrix} c_{1j} \\ \vdots \\ c_{kj} \end{bmatrix}$$

Proposition

The vectors $\mathbf{x}_1, \dots, \mathbf{x}_m$ are linearly independent **iff** *the columns $\mathbf{c}_1, \dots, \mathbf{c}_m$ are linearly independent.*

Corollary

In a vector space V, m linear combinations of k vectors $\mathbf{b}_1, \dots, \mathbf{b}_k$ are linearly dependent if $m > k$.

Example

Consider linearly independent vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \in \mathbb{R}^n$ *Let*

$$\begin{array}{ccccccc} \mathbf{x}_1 & = & -\mathbf{b}_1 & + & 2\mathbf{b}_2 & + & 3\mathbf{b}_3 \\ \mathbf{x}_2 & = & \mathbf{b}_1 & + & \mathbf{b}_2 & - & 2\mathbf{b}_3 \\ \mathbf{x}_3 & = & -\mathbf{b}_1 & + & 5\mathbf{b}_2 & + & 4\mathbf{b}_3 \end{array}$$

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Are the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{R}^n$ *linearly independent? Let*

$$\begin{array}{ccccccc} \mathbf{y}_1 & = & \mathbf{b}_1 & - & 2\mathbf{b}_2 & + & 3\mathbf{b}_3 \\ \mathbf{y}_2 & = & \mathbf{b}_1 & + & \mathbf{b}_2 & - & 3\mathbf{b}_3 \\ \mathbf{y}_3 & = & 2\mathbf{b}_1 & + & \mathbf{b}_2 & - & 3\mathbf{b}_3 \end{array}$$

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