Symmetric, Positive Definite Matrices

Definition

A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is called **positive definite**, *if*

$$\mathbf{x}^T A\mathbf{x} > 0, \quad \mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq \mathbf{0}$$

The matrix A is called positive semi-definite, if

$$\mathbf{x}^T A\mathbf{x} \geq 0, \quad \mathbf{x} \in \mathbb{R}^n.$$

Example

Which of the following matrices is positive definite?

$$A = \begin{bmatrix} 4 & -4 \\ -4 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -4 \\ -4 & 2 \end{bmatrix}$$

Proposition

Let V be an n-dimensional vector space with an inner product $\langle \cdot, \cdot \rangle$ and a basis $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$. Prove that

$$\langle \mathbf{x}, \mathbf{y} \rangle = [\mathbf{x}]_B^T A [\mathbf{y}]_B,$$

where $A_{ij} = \langle \mathbf{b}_i, \mathbf{b}_j \rangle$. Conclude that the matrix A is positive definite.

Theorem

For a real-valued, finite-dimensional vector space V and a basis B of V it holds that $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ is an inner product if and only if there exists a symmetric, positive definite matrix $A \in \mathbb{R}^{n \times n}$ with

$$\langle \mathbf{x}, \mathbf{y} \rangle = [\mathbf{x}]_B^T A [\mathbf{y}]_B,$$

Proposition

Let $A = (a_{ij})$ *be a positive definite matrix, then*

- $\ker(A) = \{\mathbf{0}\}$
- $a_{ii} > 0$

Any inner product induces a norm

$$\|x\| := \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$$

However, not every norm is induced by an inner product (e.g. Manhattan norm or ℓ_1 norm).

Example

Compute the norm of the vector $[2\ 2]^T$ w.r.t. the following inner products:

a) $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y}$

b) $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \mathbf{y}$

Distance and metric

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Definition

A function $d : V \times V \to \mathbb{R}$ *is metric, if*

- d *is positive definite, i.e.,* $d(\mathbf{x}, \mathbf{y}) \geq 0$ *for all* $\mathbf{x}, \mathbf{y} \in V$ *and* $d(\mathbf{x}, \mathbf{y}) = 0 \Leftrightarrow \mathbf{x} = \mathbf{y}$
- d *is symmetric, i.e.,* $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$ *for all* $\mathbf{x}, \mathbf{y} \in V$
- *Triangular inequality:* $d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$.

Proposition

Every norm (inner product) induces a metric:

$$d(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\| = \sqrt{\langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle}$$

By Cauchy-Schwarz inequality

$$-1 \leq \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \|\mathbf{y}\|} \leq 1.$$

Therefore, there exists a unique $w \in [0, \pi]$ with

$$\cos w = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \|\mathbf{y}\|}$$

The number w is the **angle** between the vectors $\mathbf{x}$ and $\mathbf{y}$.

Example

Find the angle between $\mathbf{x}$ *and* $\pm\mathbf{x}$.

Example

Find the angle between $\mathbf{x} = [1,\ 1]^T$ *and* $\mathbf{y} = [1, -1]^T$ *w.r.t. the inner products given by*

a) $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y}$

b) $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \mathbf{y}$