ASDS Statistics, YSU, Fall 2020

Lecture 25

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- The Method of Maximum Likelihood Estimation

The Maximum Likelihood Method

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Ok, let's do some calculations.

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So our guess was to **select the value of p giving the highest likelihood to our outcome**.

Idea of the Maximum Likelihood Method

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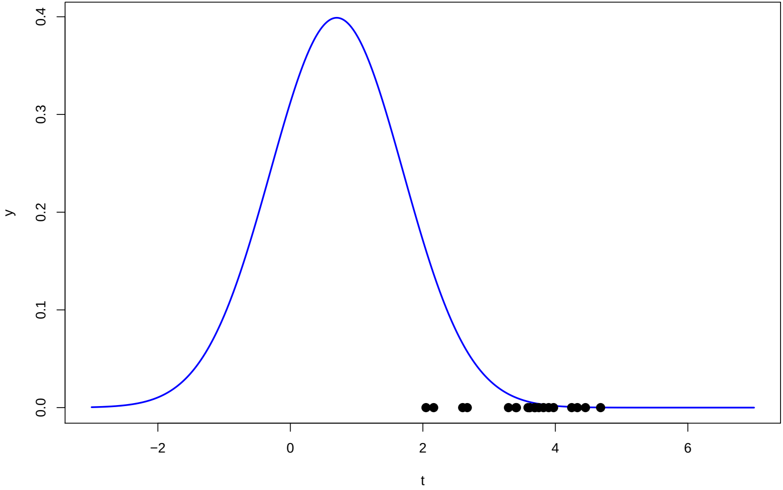
Idea of Maximum Likelihood Estimation: We choose that value of our parameter, under which **our Observation is the most Probable**.

Idea of the MLE

Guess which Distribution makes our Data most Probable:



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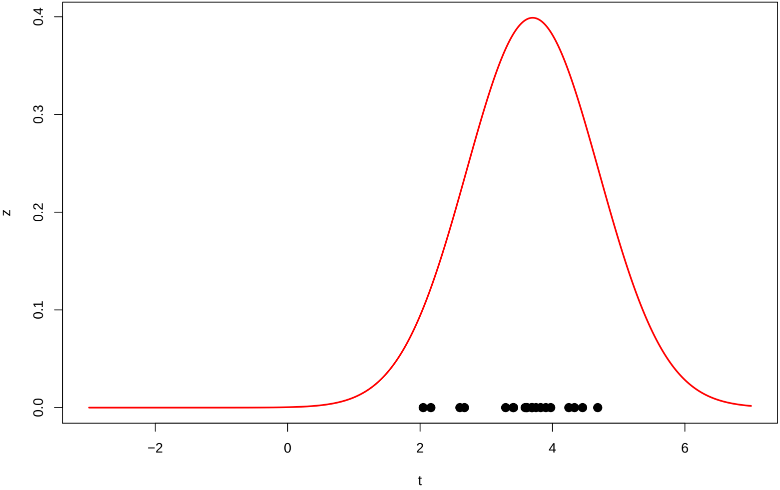

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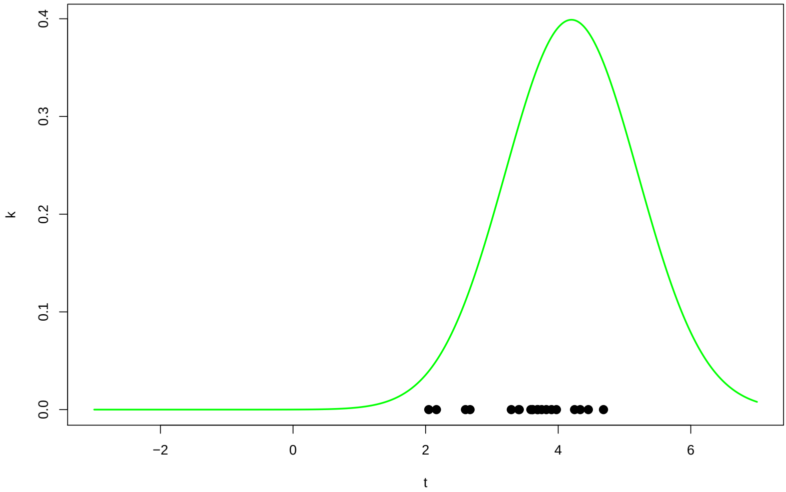

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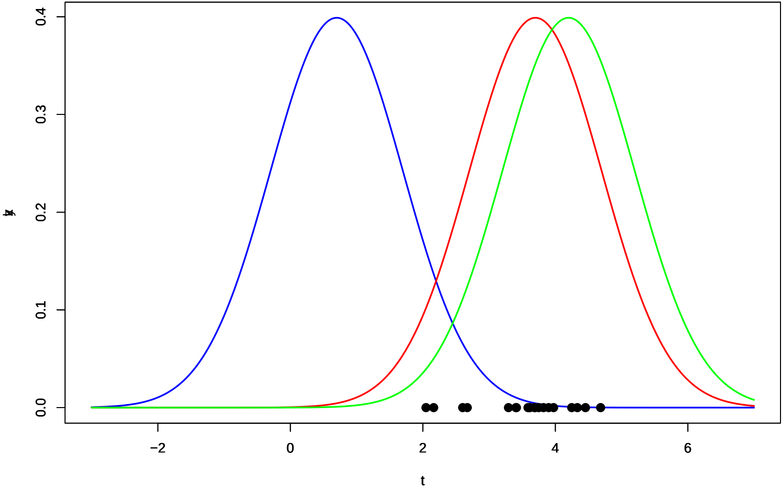

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Again, assume we have an Observation $x: \ x_1, ..., x_n$, from one of the Distributions of Parametric Family $\mathcal{F}_\theta$, with the PD(M)F $f(x|\theta)$, $\theta \in \Theta$.

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And the Maximum Likelihood Method is saying: **choose that value of θ, under which it is most likely to get** $X_1, X_2, ..., X_n$.

Likelihood

Definition: The **Likelihood Function** for the above Model and Random Sample is the Joint PD(M)F of $X_1, \ldots, X_n$, **considered as a function of the parameter** θ, and **calculated at the Random Sample**, i.e., it is given by[1]

$$\mathcal{L}(\theta) = \mathcal{L}_n(X_1, \ldots, X_n|\theta) = f(X_1|\theta) \cdot f(X_2|\theta) \cdot \ldots \cdot f(X_n|\theta), \qquad \theta \in \Theta.$$

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The **Log-Likelihood Function** is the function

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Also we define the **Negative Log-Likelihood Function** to be

$$-\ell(\theta) = -\ln \mathcal{L}(\theta).$$

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And in the case if we have an Observation $x: \ x_1, x_2, ..., x_n$ from the above Model (from one of the Distributions of that Model), the **Maximum Likelihood Estimate** (again **MLE**) of the parameter θ is the value of $\hat{\theta}^{MLE}$ on our Observation.

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Example: Find the MLE Estimator for λ in the *Exp*(λ) Model.

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Example: Find the MLE for p in the $Bernoulli(p)$ Model.

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Example: Find the MLE Estimator for λ in the $Exp(\lambda)$ Model.

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$$0, 1, 1, 2, 1, 0, 0, 1, 1$$

from the following Model:

X	0	1	2
$\mathbb{P}(X = x)$	$\frac{\theta}{10}$	$\frac{\theta}{5}$	$1 - \frac{3\theta}{10}$,

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where $\theta \in [0, \frac{10}{3}]$. Find the MLE Estimator and MLE Estimate for θ.

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