ASDS Statistics, YSU, Fall 2020
Lecture 21

Michael Poghosyan

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Contents

Risk, Mean Squared Error of the Estimator

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$$MSE(\hat{\theta}_n, \theta) = Risk(\hat{\theta}_n, \theta) = \mathbb{E}_\theta[(\hat{\theta}_n - \theta)^2].$$

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Definition: We say that the estimator $\hat{\theta}_n^1$ of θ is **preferable** to $\hat{\theta}_n^2$, another estimator of θ, if

$$MSE(\hat{\theta}_n^1, \theta) \leq MSE(\hat{\theta}_n^2, \theta), \qquad \forall \theta \in \Theta,$$

and there exists a θ s.t. $MSE(\hat{\theta}_n^1, \theta) < MSE(\hat{\theta}_n^2, \theta)$.

Useful Identity

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$$\mathbb{E}(X^2) = Var(X) + \left[\mathbb{E}(X)\right]^2.$$

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Solution: OTB: Use the fact that for $X \sim Pois(\lambda)$,

$$\mathbb{E}(X) = \lambda = Var(X).$$

Best MSE Estimators

Now, having the idea of the Error of estimation for an Estimator, we can try to find the best Estimator in the sense of MSE, i.e., we can try to solve

Find $\hat{\theta}$ among all Estimators that minimizes $MSE(\hat{\theta}, \theta)$.

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So we move forward considering other Properties of Estimators making them useful in the estimation process.

Bias, Biased and Unbiased Estimators

Definition: The **Bias** of Estimator $\hat{\theta}$ of θ is

$$Bias(\hat{\theta}, \theta) = \mathbb{E}_\theta(\hat{\theta} - \theta) = \mathbb{E}_\theta(\hat{\theta}) - \theta.$$

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$$\mathbb{E}_\theta\left(\hat{\theta}\right) = \theta \qquad \text{for any} \qquad \theta \in \Theta.$$

Note: We require the above hold **for any parameter value** since, if, say, it is correct for *some* values of θ, then it can happen that the true value of our unknown θ is exactly that value, for whihc we do not have the equality.

Bias and Unbiasedness

- The idea of Unbiased Estimator is the following: if we will calculate Estimates many-many times using our Unbiased Estimator, and then average the obtained Estimates, we will obtain the (almost exact, exact when many-many$\to \infty$) value of our Parameter.

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- Bias can be interpreted, in some sense, as the *accuracy* of the Estimator.

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We use size n Random Sample to Model the situation:

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We use size *n* Random Sample to Model the situation:

$$X_1, X_2, ..., X_n \sim \mathcal{F}_\mu.$$

Now, we choose several Estimators to estimate μ:

$$\hat{\mu}_1 = X_1, \qquad \hat{\mu}_2 = \frac{X_1 + X_3}{2}, \qquad \hat{\mu}_3 = \frac{X_1 + X_4}{10},$$

$$\hat{\mu}_4 = \overline{X}_n = \frac{X_1 + X_2 + ... + X_n}{n}.$$

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Problem: Check if each Estimator is Biased or Unbiased.

Solution: OTB

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UnBiased Estimator Case
We consider the Poisson Model:

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Easy to se that $\hat{\lambda}$ is an Unbiased Estimator for λ (OTB!).

# Example, cont'd

Now, the code

- ▶ observing once: ganarating a Sample just once and calculating one Estimate:

```
lambda <- 3.21
x <- rpois(10, lambda = lambda)
lambda.hat <- mean(x)
lambda.hat
```

```
## [1] 2.9
```

▶ observing many times: ganarating Samples many times, calculating Estimates, and then averaging:

```r
lambda <- 3.21; n <- 10; m <- 2000
x <- rpois(n*m, lambda = lambda)
x <- as.data.frame(matrix(x, ncol = m))
lambda.hats <- sapply(x, mean)
boxplot(lambda.hats, horizontal = T);
abline(v = lambda, col="red", lwd = 2, lty = 2)
```
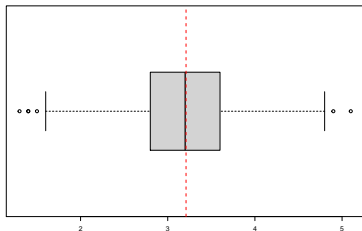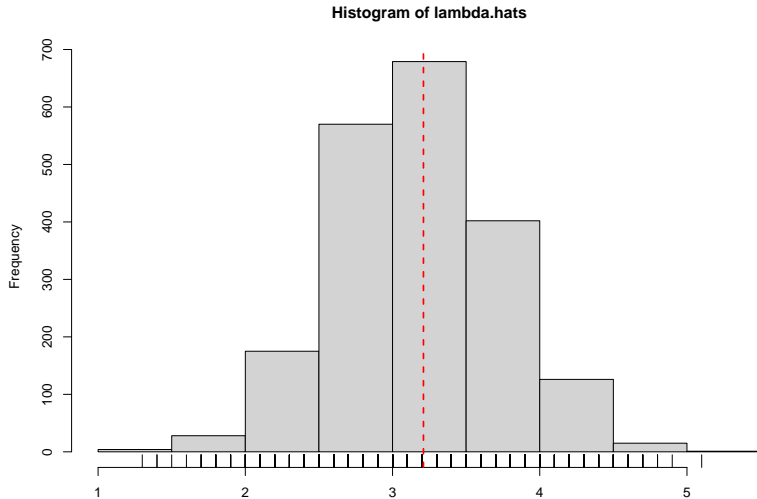


```r
mean(lambda.hats)
```

```
## [1] 3.2223
```

With a Histogram:

```
hist(lambda.hats)
rug(lambda.hats)
abline(v = lambda, col="red", lwd = 2, lty = 2)
```


Histogram of lambda.hats

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Biased Estimator Case

Say, let us consider the Exponential Model:

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$$\hat{\lambda} = \frac{X_1 + X_2 + ... + X_{10}}{10}.$$

Easy to see that $\hat{\lambda}$ is an Biased Estimator for λ (OTB!).

Example, cont'd

Now, the code:

- observing once: generating a Sample just once and calculating one Estimate:

```
lambda <- 0.3
x <- rexp(10, rate = lambda)
lambda.hat <- mean(x)
lambda.hat
```

```
## [1] 1.693733
```

► observing many times: ganarating Samples many times, calculating Estimates, and then averaging:

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lambda <- 0.3; n <- 10; m <- 2000
x <- rexp(n*m, rate = lambda)
x <- as.data.frame(matrix(x, ncol = m))
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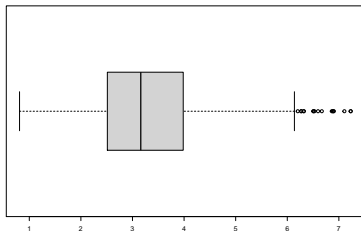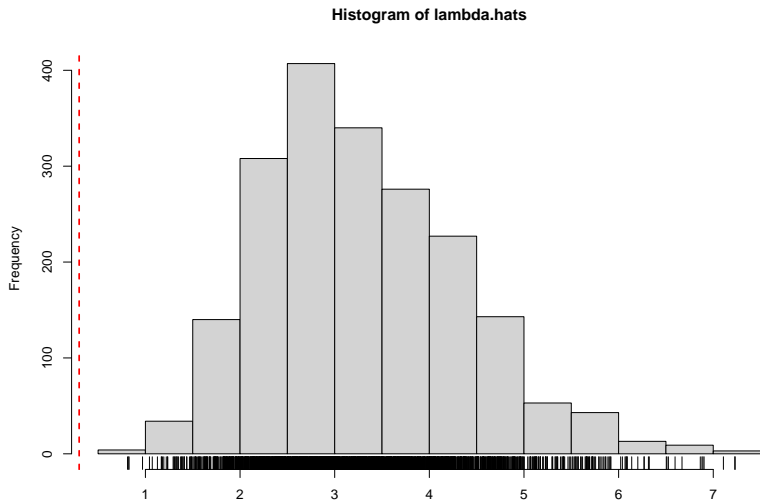


```
mean(lambda.hats)
```

```
## [1] 3.289266
```

With a Histogram:

```
hist(lambda.hats)
rug(lambda.hats)
abline(v = lambda, col="red", lwd = 2, lty = 2)
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Histogram of lambda.hats

Example

Example: Assume we have a Random Sample for a some Distribution with the Mean μ and Variance σ^2:

$$X_1, X_2, ..., X_n \sim \mathcal{F}_{\mu, \sigma^2},$$

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We consider the following Estimators:

$$\hat{\mu} = \overline{X}_n = \frac{X_1 + X_2 + ... + X_n}{n}$$

and

$$\widehat{\sigma^2} = \frac{\sum_{k=1}^{n}(X_k - \overline{X}_n)^2}{n} \qquad \text{and} \qquad \widehat{\sigma^2} = S^2 = \frac{\sum_{k=1}^{n}(X_k - \overline{X}_n)^2}{n-1}$$

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Let us see (OTB) which ones are Biased and which ones are not.

Asymptotic Unbiasedness

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Example: Say, for the Mean μ of the Population,

$$\hat{\mu} = \frac{X_1 + X_2 + ... + X_n}{n+1}$$

is a Biased, but Asymptotically Unbiased Estimator. OTB, please!