Basic Mathematics , Fall 2020

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Integration by substitution

If $f \circ g$ and g' are continuous on $[a, b]$, then

$$\int_a^b f(g(t)) \cdot g'(t) \mathrm{d}t \xlongequal{\text{g(t)=x}} \int_{g(a)}^{g(b)} f(x) \mathrm{d}x,$$

Example

Find $E[X]$ and $Var(X)$ when X is a normal random variable with parameters $\mu = 0$ and $\sigma = 1$.

Denote the cumulative distribution function of a standard normal random variable by $\Phi(x)$, that is

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy.$$

Example

Let X be a normal random variable with parameters μ and σ. Recall that the probability density of the normal distribution is $f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. Prove that the distribution function of X can be expressed as

$$F_X(a) = P\{X \leq a\} = \Phi\left(\frac{a-\mu}{\sigma}\right)$$

Example

Prove that $\Phi(-x) = 1 - \Phi(x)$.

Example

If X *is a normal random variable with parameters* $\mu = 4$ *and* $\sigma = 16$, *express in terms of the function* $\Phi(x)$
(a) $P\{4 < X < 8\}$;
(b) $P\{X > 0\}$;
(c) $P\{|X - 4| > 8\}$;

Example

Let X *be a uniform (0, 1) random variable. Compute* $E[X^n]$.

Definition

For a function $f : \mathbb{R}^n \to R$ (of n variables $x_1, \dots, x_n$) we define the **partial derivatives** *as*

$$\begin{aligned} f'_{x_1}(\mathbf{x}) = \frac{\partial f}{\partial x_1}(\mathbf{x}) &\quad = \lim_{h \to 0} \frac{f(x_1+h, x_2, \dots, x_n) - f(x_1, x_2, \dots, x_n)}{h} \\ &\vdots \\ f'_{x_n}(\mathbf{x}) = \frac{\partial f}{\partial x_n}(\mathbf{x}) &\quad = \lim_{h \to 0} \frac{f(x_1, x_2, \dots, x_n+h) - f(x_1, x_2, \dots, x_n)}{h} \end{aligned} .$$

The row vector

$$\nabla f = \operatorname{grad} f = \frac{df}{d\mathbf{x}} = \begin{bmatrix} \frac{\partial f(\mathbf{x})}{\partial x_1} & \frac{\partial f(\mathbf{x})}{\partial x_2} & \dots & \frac{\partial f(\mathbf{x})}{\partial x_n} \end{bmatrix} \in \mathbb{R}^{1 \times n},$$

is called the **gradient** *of f or the Jacobian.*

Differentiation Rules

If the functions f and g have partial derivatives, then

Sum Rule: $$\frac{\partial}{\partial x_i}(f(\mathbf{x}) + g(\mathbf{x})) = \frac{\partial f(\mathbf{x})}{\partial x_i} + \frac{\partial g(\mathbf{x})}{\partial x_i}$$

Product Rule: $$\frac{\partial}{\partial x_i}(f(\mathbf{x}) \cdot g(\mathbf{x})) = \frac{\partial f(\mathbf{x})}{\partial x_i} g(\mathbf{x}) + f(\mathbf{x}) \frac{\partial g(\mathbf{x})}{\partial x_i}$$

Example

Find the gradient of the following functions:

a) $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $f(x_1, x_2) = (2x_1 + 3x_2)^3$

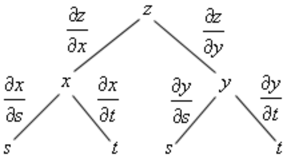
b) $f : \mathbb{R}^3 \to \mathbb{R}$ *defined by* $f(x, y, z) = e^{2x} + y^2 z^3$

Chain Rule

Let z be a function of two variables, x, y and each of these variables x, y be in turn functions of two variables, t, s. Then

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \end{bmatrix} = \nabla z \frac{\partial \mathbf{x}}{\partial t}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial s} \end{bmatrix} = \nabla z \frac{\partial \mathbf{x}}{\partial s}.$$



Example

Given $z(x,y) = x^2 + y^2$ *where* $x(r,t) = r\cos(t)$ *and* $y(r,t) = r\sin(t)$, *determine the value of* $\frac{\partial z}{\partial t}$ *and* $\frac{\partial z}{\partial r}$ *using the chain rule. Verify the results by expressing* z *as a function of* r, t *and computing the partial derivatives directly.*

In general, assume z is a function of n variables, $x_1, \dots, x_n$ and each of these variables are in turn functions of m variables, $t_1, t_2, \dots, t_m$. Then for any variable $t_i, i = 1, 2, \dots, m$ we have the following,

$$\frac{\partial z}{\partial t_i} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

Example

Find the partial derivatives $\frac{\partial z}{\partial t_i}$, $i = 1, 2, 3$ *of the function* $z(x,y)$ *where* $x = t_1 + 2t_2 + 4t_3$ *and* $y = t_1 - 3t_2 + 5t_3$.