Basic Mathematics, Fall 2020

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Definition

Two vectors $\mathbf{x}$ *and* $\mathbf{y}$ *are* **orthogonal** *if and only if* $\langle \mathbf{x}, \mathbf{y} \rangle = 0$, *and we write* $x \perp y$. *If additionally* $\|\mathbf{x}\| = 1 = \|\mathbf{y}\|$, *i.e., the vectors are unit vectors, then* $\mathbf{x}$ *and* $\mathbf{y}$ *are* **orthonormal**.

Remark

Orthogonality is the generalization of the concept of perpendicularity to bilinear forms that do not have to be the dot product. Geometrically, we can think of orthogonal vectors as having a right angle with respect to a specific inner product.

Definition

A square matrix $A \in \mathbb{R}^{n \times n}$ *is an orthogonal matrix if its columns are orthonormal, i.e.*

$$A^T A = I$$

Note that for orthonormal matrix $AA^T = I$ and $A^{-1} = A^T$.

Proposition

Transformations with orthogonal matrix of transformation preserve the length and the angle w.r.t. to the dot product, i.e.

$$\|A\mathbf{x}\| = \|\mathbf{x}\|$$

and

$$w_{\mathbf{x},\mathbf{y}} = w_{A\mathbf{x},A\mathbf{y}}$$

Proof.

$$\|A\mathbf{x}\|^2 = A\mathbf{x} \cdot A\mathbf{x} = (A\mathbf{x})^T A\mathbf{x} = \mathbf{x}^T A^T A\mathbf{x} = \mathbf{x}^T I\mathbf{x} = \|\mathbf{x}\|^2$$

and

$$\cos w_{A\mathbf{x},A\mathbf{y}} = \frac{(A\mathbf{x})^T A\mathbf{y}}{\|A\mathbf{x}\|\|A\mathbf{y}\|} = \frac{\mathbf{x}^T A^T A\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} = \frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} = \cos w_{\mathbf{x},\mathbf{y}}.$$

□

Orthonormal Basis

Definition

Consider an n-dimensional vector space V and a basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ of V. The basis B is called an **orthogonal basis**, *if*

$$\langle \mathbf{b}_i, \mathbf{b}_j \rangle = 0, \ i \neq j$$

If additionally the vectors $\mathbf{b}_i$ are unit, then the basis B is called an **orthonormal basis (ONB).**

Example

The canonical/standard basis for a Euclidean vector space $\mathbb{R}^n$ is an orthonormal basis, where the inner product is the dot product of vectors.

Example

Check that the vectors

$$\mathbf{b}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}, \quad \mathbf{b}_2 = \frac{1}{2}\begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}$$

form an orthonormal basis for $\mathbb{R}^2$.

Inner Product of Functions

For any two continuous functions $u, v : [ab] \to \mathbb{R}$ we can define the inner product as follows

$$\langle u, v\rangle = \int_a^b u(x)v(x)dx.$$

The inner product induces norm and metric on the vector space $C[a, b]$.
We will say the function u, v are orthogonal, if $\int_a^b u(x)v(x)dx = 0$.

Example

Prove that the collection of functions $\{1, \sin(x), \cos(x), \sin(2x), \cos(2x), ...\}$ *is orthogonal on* $[-\pi, \pi]$.

Projection operators

Definition

Let V be a vector space and $W \subset V$ a subspace of V. A linear mapping $\pi : V \to W$ is called a **projection** *if $\pi^2 = \pi \circ \pi = \pi$.*

Remark

The transformation matrix P_π of a projection π is called projection matrix and satisfies $P_\pi^2 = P_\pi$.

Projection onto 1-Dimensional Subspaces (Lines)

Let $U \subset \mathbb{R}^n$ be a line passing through the origin parallel to a vector $\mathbf{b} \in \mathbb{R}^n$. Then:

- $U = \operatorname{span} \mathbf{b}$
- $\pi_U(\mathbf{x}) \in U$, hence $\pi_U(\mathbf{x}) = \lambda \mathbf{b}$ for some $\lambda \in \mathbb{R}$
- $\pi_U(\mathbf{x})$ is closest to $\mathbf{x}$ in U, and hence $\pi_U(\mathbf{x}) - \mathbf{x} \perp \mathbf{b} \Leftrightarrow \langle \pi_U(\mathbf{x}) - \mathbf{x}, \mathbf{b} \rangle = 0$
- $\pi_U(\mathbf{x}) = \lambda \mathbf{b} = \dfrac{\langle \mathbf{x}, \mathbf{b} \rangle}{\|\mathbf{b}\|^2} \mathbf{b}$
- $\|\pi_U(\mathbf{x})\| = |\cos w| \|\mathbf{x}\|$, where w is the angle between $\mathbf{x}$ and $\mathbf{b}$.
- With the dot product as inner product the projection matrix P_π satisfying the condition $\pi_U(\mathbf{x}) = P_\pi \mathbf{x}$ is given by

$$P_\pi = \frac{\mathbf{b}\mathbf{b}^T}{\|\mathbf{b}\|^2} = \frac{\mathbf{b}\mathbf{b}^T}{\mathbf{b}^T\mathbf{b}}.$$

Clearly, $\mathbf{b}\mathbf{b}^T$ is a symmetric matrix.

Example

Find the projection matrix P_π *onto the line through the origin spanned by* $\mathbf{b} = [1,\ 1,\ -1]^T$. *Note that* $\mathbf{b}$ *is a direction and a basis of the one-dimensional subspace (line through origin). Find also the projection of the vector* $\mathbf{x} = [-1,\ 2,\ -2]^T$

Projection onto General Subspaces

Let $U \subset \mathbb{R}^n$ be a subspace with $\dim(U) = m$.
Assume $(\mathbf{b}_1, \ldots, \mathbf{b}_m)$ is a basis for U. Then:

- $\pi_U(\mathbf{x}) = \sum_{i=1}^{m} \lambda_i \mathbf{b}_i = B\boldsymbol{\lambda}$, where $B = [\mathbf{b}_1, \ldots, \mathbf{b}_m] \in \mathbb{R}^{n\times m}$, $\boldsymbol{\lambda} = [\lambda_1, \ldots, \lambda_m]^T \in \mathbb{R}^m$.
- The condition $\mathbf{x} - \pi_U(\mathbf{x}) \perp U$ is equivalent to $\langle \mathbf{b}_i, \mathbf{x} - \pi_U(\mathbf{x})\rangle = \mathbf{b}_i^T(\mathbf{x} - \pi_U(\mathbf{x})) = 0$ for $i = 1, \ldots, m$. Hence

$$\begin{bmatrix} \mathbf{b}_1^T \\ \vdots \\ \mathbf{b}_m^T \end{bmatrix} [\mathbf{x} - B\boldsymbol{\lambda}] = \mathbf{0} \Leftrightarrow B^T(\mathbf{x} - B\boldsymbol{\lambda}) = \mathbf{0} \Leftrightarrow B^T B\boldsymbol{\lambda} = B^T\mathbf{x}.$$

Therefore

$$\boldsymbol{\lambda} = (B^T B)^{-1} B^T \mathbf{x}.$$

The matrix $(B^T B)^{-1} B^T$ is also called **pseudo-inverse** of B, which can be computed for non-square matrices B.

- By $\pi_U(\mathbf{x}) = B\boldsymbol{\lambda}$ we have the projection $\pi_U(\mathbf{x})$ is computed by

$$\pi_U(\mathbf{x}) = B(B^T B)^{-1} B^T \mathbf{x}$$

- The projection matrix P_π satisfying $P_\pi \mathbf{x} = \pi_U(\mathbf{x})$ is given by

$$P_\pi = B(B^T B)^{-1} B^T.$$

Remark

In the case $\dim(U) = 1$, *the matrix* $B^T B$ *is just a real number, hence* $P_\pi = \dfrac{BB^T}{B^T B}$

Remark

If the basis $(\mathbf{b}_1, \ldots, \mathbf{b}_m)$ *is an ONB, i.e.* $\mathbf{b}_1, \ldots, \mathbf{b}_m$ *are unit and orthogonal vectors, then the projection equation simplifies greatly to*

$$\pi_U(\mathbf{x}) = BB^T \mathbf{x}$$

Projections allow us to look at situations where we have a linear system $A\mathbf{x} = \mathbf{b}$ without a solution. Recall that this means that $\mathbf{b} \notin \operatorname{col}(A)$. Given that the linear equation cannot be solved exactly, we can find an *approximate solution.*

The idea is to find the vector in $\operatorname{col}(A)$ *that is closest to* $\mathbf{b}$, *i.e., we compute the orthogonal projection of* $\mathbf{b}$ *onto the subspace spanned by the columns of* A. *This problem arises often in practice, and the solution is called the* **least squares solution**.

Example

For a subspace $U = \operatorname{span}[\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}] \subset \mathbb{R}^3$ *and* $\mathbf{x} = \begin{bmatrix}3\\0\\3\end{bmatrix}$ *find the coordinates* $\boldsymbol{\lambda}$ *of* $\mathbf{x}$ *in terms of the basis of the subspace* U, *the projection* $\pi_U(\mathbf{x})$, *the distance* $d(\mathbf{x}, U)$ *and the projection matrix* P_π.