Optimization

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Numerical Algorithms for Constrained Minimization Problems

Penalty Methods

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & x \in \Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^n$.
Penalty Methods are procedures when constrained optimization problems are approximated by unconstrained optimization problem. The approximation is accomplished by adding to the objective function a term that prescribes a high cost for violation of constraints.

Specifically we consider the following unconstrained minimization problem

$$\text{minimize} \quad q(x, c) = f(x) + cP(x),$$

where $c \in \mathbb{R}$ is a positive constant and $P : \mathbb{R}^n \to \mathbb{R}$ is a given function. The constant c is called the penalty parameter and the function $P(x)$ is called the penalty function.

Definition

A function $P : \mathbb{R}^n \to \mathbb{R}$ is called a penalty function for the constrained minimization problem above, if it satisfies the following conditions

1. P is continuous.
2. $P(x) \geq 0$ for all $x \in \mathbb{R}^n$.
3. $P(x) = 0$ if and only if $x \in \Omega$.

We take the solution of

$$\text{minimize} \quad q(x,c) = f(x) + cP(x),$$

as an approximation to the solution of original problem. Of course, it may not be exactly equal to the true solution but we expect that the larger the value of the penalty parameter c, the closer the approximated solution will be to the true solution.

Assume $\{c_k\}$ is a sequence of positive numbers. Let's denote by x_k a solution of

$$\text{minimize} \quad q(x, c_k) = f(x) + c_k P(x).$$

Theorem

Suppose that f is a continuous function, $\{c_k\}$ is a sequence of strictly increasing positive numbers such that $\lim_{k\to\infty} c_k = +\infty$. If $\{x^{(k)}\}$ is a convergent sequence, then its limit is a solution to the original problem.

How to construct the penalty function?

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & h_i(x) = 0, \quad i = 1, \ldots m,\ m \le n. \end{aligned}$$

We will assume that f, h_i for $i = 1, \ldots m$ are continuous functions on $\mathbb{R}^n$.

$$P(x) = \sum_{i=1}^{m} |h_i(x)|$$

or

$$P(x) = \sum_{i=1}^{m} (h_i(x))^2 .$$

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0, \quad i = 1, \dots p. \end{aligned}$$

We will assume that f, g_i for $i = 1, \dots p$ are continuous functions on $\mathbb{R}^n$.

$$P(x) = \sum_{i=1}^{p} \max\{g_i(x), 0\}$$

or

$$P(x) = \sum_{i=1}^{p} \left(\max\{0, g_i(x)\}\right)^2 .$$

Example

Consider the following constrained minimization problem

$$\begin{aligned} \text{minimize} \quad & f(x) = \frac{100}{x^2+1} \\ \text{subject to} \quad & x \leq 2, \\ & x \geq -1. \end{aligned}$$

$$q(x,c) = \frac{100}{x^2+1} + c \cdot (\max\{x-2, 0\})^2 + c \cdot (\max\{-x-1, 0\})^2$$

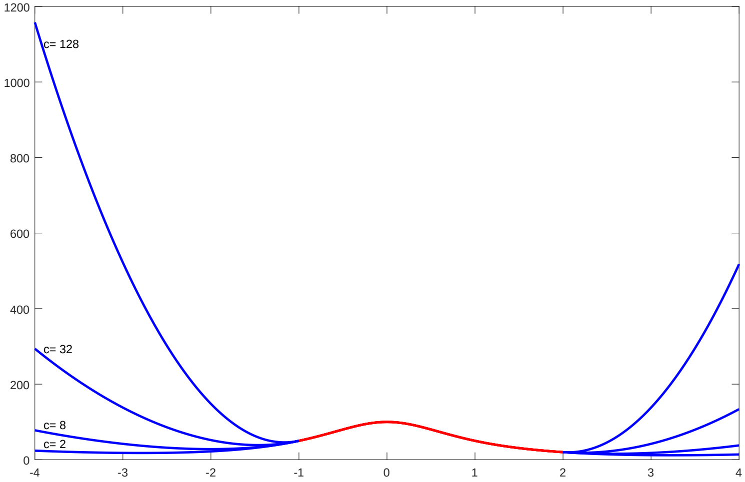

Figure: The graphs of $q(x, c)$ for different penalty parameters.

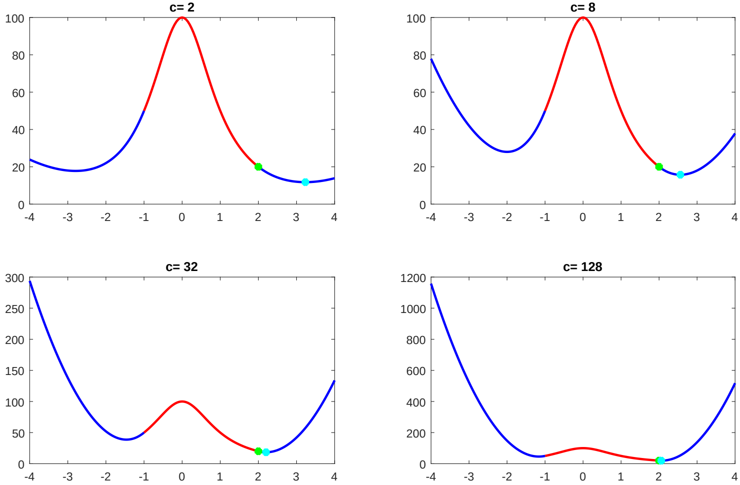

Figure: The green point is the solution of constrained minimization problem and the blue one is the minimizer of $q(x, c)$.

Example

Consider the following constrained optimization problem: find the minimum of $f(x_1, x_2) = (x_1 - 1)^2 + x_2^2$ subject to $x_1 + x_2 \leq -2$.

a. Solve this constrained minimization problem.

b. Consider the following Penalty function: for large $c > 0$,

$$q(x_1, x_2, c) = f(x_1, x_2) + c \cdot (\max\{0, x_1 + x_2 + 2\})^2 .$$

Assuming c is fixed, find the minimum point x^c of q;

c. Prove that x^c tends to the solution obtained in a., as $c \to +\infty$.

d Assume c is fixed and $c = 10$, we want to use the steepest descent method to minimize $q(x_1, x_2, c)$. As initial approximation take $x^{(0)} = [1, 0]^T$ and calculate $x^{(1)}$.