Basic Mathematics , Fall 2020

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Singular Value Decomposition (SVD)

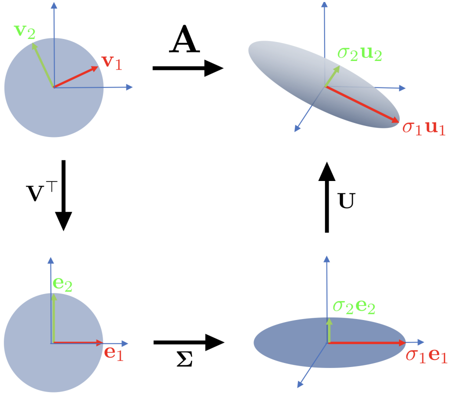
Theorem

Let $A \in \mathbb{R}^{m \times n}$ *be a rectangular matrix of rank* r, *with* $r \in [0, \min(m,n)]$. *The Singular Value Decomposition or SVD of* A *is a decomposition of* A *of the form*

$$A = U\Sigma V^T$$

where $U \in \mathbb{R}^{m \times m}$ *is an orthogonal matrix of column vectors* $\mathbf{u}_i$, *and* $V \in \mathbb{R}^{n \times n}$ *is an orthogonal matrix of column vectors* $\mathbf{v}_j$ *and* Σ *is an* $m \times n$ *matrix with* $\Sigma_{ii} = \sigma_i > 0$ *and* $\Sigma_{ij} = 0,\ i \neq j$. *The SVD is always possible for any matrix* A.

The σ_i are called the singular values, and by convention the singular values are ordered, i.e., $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r \geq 0$
$\mathbf{u}_i$ are called the left-singular vectors and $\mathbf{v}_j$ are called the right-singular vectors.



$\mathbf{v}_2$
$\mathbf{v}_1$
$\mathbf{A}$
$\sigma_2\mathbf{u}_2$
$\sigma_1\mathbf{u}_1$
$\mathbf{V}^\top$
$\mathbf{U}$
$\mathbf{e}_2$
$\mathbf{e}_1$
Σ
$\sigma_2\mathbf{e}_2$
$\sigma_1\mathbf{e}_1$

Σ has a diagonal submatrix that contains the singular values and needs additional zero vectors that increase the dimension.

if $m > n$

$$\Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & \sigma_n \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix}$$

if $m < n$

$$\Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 & 0 & \dots & 0 \\ 0 & \ddots & 0 & 0 & \dots & 0 \\ 0 & \dots & \sigma_n & 0 & \dots & 0 \end{bmatrix}$$

The singular value matrix Σ must be of the same size as A

Example

$$A = \begin{pmatrix} -1 & \frac{1}{\sqrt{2}} & 1 \\ -1 & -\frac{1}{\sqrt{2}} & 1 \end{pmatrix} = U\Sigma V^T$$

$$= \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & -1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \end{pmatrix}$$

Example

$$A = \begin{pmatrix} -\frac{3\sqrt{3}}{4} & -\frac{3}{4} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{9}{4} & -\frac{3\sqrt{3}}{4} \end{pmatrix} = U\Sigma V^T$$

$$= \begin{pmatrix} -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & -1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

Remark

The eigenvalue decomposition of a symmetric matrix

$$S = S^T = PDP^T$$

is a special case of the SVD

$$S = U\Sigma V^T,$$

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where $U = V = P$ *and* $\Sigma = D$.

Let $r = \mathrm{rk}(A)$.

Remark

- *The columns of U (m by m) are eigenvectors of AA^T,*
- *the columns of V (n by n) are eigenvectors of A^TA.*
- *The r singular values on the diagonal of Σ (m by n) are the square roots of the nonzero eigenvalues of both AA^T and A^TA.*

Remark

U and V give orthonormal bases for all four fundamental subspaces:

- *first r columns of U: column space of A*
- *last $m - r$ columns of U: nullspace of A^T*
- *first r columns of V: row space of A*
- *last $n - r$ columns of V: nullspace of A*

Remark

When A multiplies a column $\mathbf{v}_i$ of V, it produces σ_i times a column of U.

$$AV = U\Sigma \Leftrightarrow A\mathbf{v}_i = \sigma_i\mathbf{u}_i$$

Geometric interpretation

For every linear map $T : R^n \to R^m$ one can find orthonormal bases of R^n and R^m such that T maps the i-th basis vector of R^n to a non-negative multiple of the i-th basis vector of R^m, and sends the left-over basis vectors to zero. With respect to these bases, the map T is therefore represented by a diagonal matrix with non-negative real diagonal entries.

V 1. How to compute the SVD in **the case** $\mathrm{rk}(A) = m \leq n$ **?**

Step 1: Compute the symmetrized matrix $A^T A$ (recall $A \in \mathbb{R}^{m \times n}$).
Step 2: Compute the eigenvalue decomposition of $A^T A = PDP^T$.
From here we obtain

$$V = P, \text{ and } \Sigma^T \Sigma = D,$$

The eigenvalues of $A^T A$ are the squared singular values of Σ.
Step 3. Compute U using the formula

$$\mathbf{u}_i = \frac{1}{\sigma_i} A \mathbf{v}_i, \quad i = 1, \ldots, m (m = \mathrm{rk}(A))$$

V 2. How to compute the SVD in **the case** $\mathrm{rk}(A) = n \leq m$**?**

Step 1: Compute the symmetrized matrix AA^T.
Step 2: Compute the eigenvalue decomposition of $AA^T = QD_1Q^T$. From here we obtain $U = Q$, and $\Sigma\Sigma^T = D_1$,
Step 3. Compute V using the formula

$$\mathbf{v}_i = \frac{1}{\sigma_i} A^T \mathbf{u}_i, \quad i = 1, \ldots, n (n = \mathrm{rk}(A))$$

V 3. How to compute the SVD in **general case** ($\mathrm{rk}(A) = r \leq \min(m, n)$)?

Step 1: Compute the symmetrized matrix AA^T.
Step 2: Compute the eigenvalue decomposition of $AA^T = QD_1Q^T$. From here we obtain $U = Q$, and $\Sigma\Sigma^T = D_1$,
Step 3. Compute V using the formula

$$\mathbf{v}_i = \frac{1}{\sigma_i} A^T \mathbf{u}_i, \quad \text{for } i = 1, \ldots, r$$

and choose $\mathbf{v}_{r+1}, \mathbf{v}_{r+2}, \ldots, \mathbf{v}_n$ so that they form an orthonormal basis of the nullspace of A

Example

Find the SVD of the matrix $A = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

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Answer

$$A = U\Sigma V^T = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Remark

If we were asked to find the SVD of the transpose of the initial matrix, i.e. $A^T = \begin{pmatrix} 0 & 1 \\ 2 & 0 \\ 0 & 0 \end{pmatrix}$*, then we would use the 2nd version of "How to compute the SVD?", as in that case* $\mathrm{rk}(A) = 2 =$*number of columns. So we would first find the* **three** *left singular vectors (new* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$*), then using those by* $\mathbf{v}_i = \frac{1}{\sigma_i} A\mathbf{u}_i$*, we would find the* **two** *right singular vectors (new* $\mathbf{v}_1, \mathbf{v}_2$*).*

Example

Find the SVD of the matrix $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{pmatrix}$

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Answer

$$A = U\Sigma V^T =$$

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{6} & \frac{2}{3} \\ -\frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{6} & -\frac{2}{3} \\ 0 & -\frac{2\sqrt{2}}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Remark

If we were asked to find the SVD of the transpose of the initial matrix, i.e. $A^T = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$*, then we would use the 1st version of "How to compute the SVD?", as in that case* $\mathrm{rk}(A) = 2 =$*number of rows. So we would first find the* **three** *right singular vectors (new* $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$*), then using those by* $\mathbf{u}_i = \frac{1}{\sigma_i} A\mathbf{v}_i$*, we would find the* **two** *left singular vectors (new* $\mathbf{u}_1, \mathbf{u}_2$*).*