ASDS Statistics, YSU, Fall 2020

Lecture 07

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Contents

Last Lecture Recap

Numerical Summaries

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For 1D Datasets, we will consider the following Summaries:

- Summaries (Statistics) about the Center, Mean, Location

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- Summaries (Statistics) about the Center, Mean, Location
- Summaries (Statistics) about the Spread, Variability

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$$x_{(1)} \leq x_{(2)} \leq ... \leq x_{(n)}.$$

In particular,

$$x_{(1)} = \min\{x_1, x_2, ..., x_n\} \qquad \text{and} \qquad x_{(n)} = \max\{x_1, x_2, ..., x_n\}.$$

Example

Example: Let x be the Dataset

$$-2, 1, 3, 2, 2, 1, 1$$

Find the 4-th and 5-th Order Statistics.

Statistical Measures for the Central Tendency/Location

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Here we want to answer to the questions: what are the typical values of our Dataset, where is our Data located at?

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Note: Sometimes this property is a plus, not a drawback! Say, if we want to have an estimator which is sensitive to outliers.

Example

Some examples:

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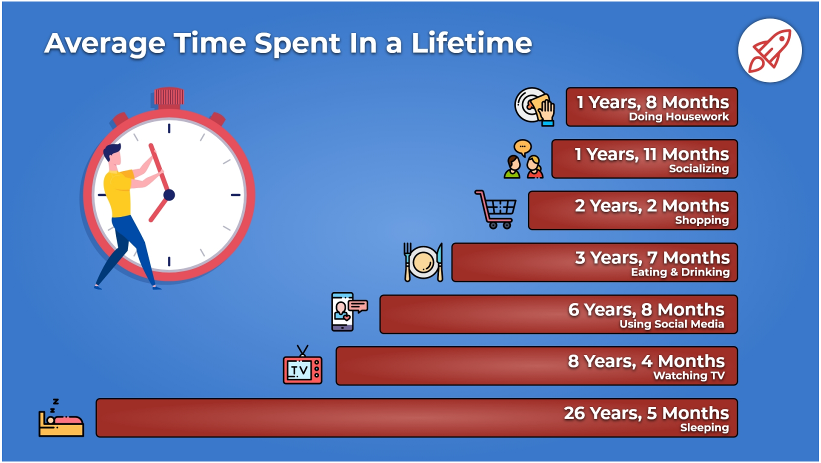
- On average, the number of times a person checks his/her phone is 58, see here
- The average daily time spent on mobile phone is 3h 15min, see here
- The average daily time spent on social media is 144min, see here or here

Example

The average time spent during the lifetime, from this webpage:



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Can we say here that the elements of our Dataset are 115.7143 plus-minus something? Not exactly.

Well, 115.7143 is not describing well our Dataset. This number gives us a wrong information about the elements of the Dataset.

Trimmed Sample Mean

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- **The Trimmed (Truncated) Sample Mean:** First we take a real number $r \in (0, 0.5)$ (or, in percents, from 0 to 50%). We will drop the *lowest r percent and largest r percent* of our data, and then we will calculate the Sample Mean of the rest.

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So we take r (ratio, fraction of points to be deleted from the both ends), we calculate $p = [r \cdot n]$. Then we sort our x in the acsending order, delete first p and last p values from this sorted array, and calculate the mean of the remaining Dataset.

Trimmed Sample Mean

Mathematically,

$$\text{trimmed sample mean}(x) = \bar{x}_{trimmed} =$$

$$= \frac{x_{(p+1)} + x_{(p+2)} + ... + x_{(n-p-1)} + x_{(n-p)}}{n-2p} = \frac{\sum\limits_{k=p+1}^{n-p} x_{(k)}}{n-2p}.$$

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Idea of Trimming: Reduce the influence of outliers. This *Statistics* for the Central Tendency, Center, is more *robust* to outliers, extremes, than the ordinary mean.

Examples

Example: Scores for the Figure Skating Competition is calculated using the Trimmed Mean, see, e.g., Wiki.

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Example: *LIBOR* is calculated using a 22% Trimmed Mean, see, e.g., Wiki

Example: People are calculating Trimmed CPI (Consumer Price Index), see here

Example

```r
x <- c(1, 10, 20, 30, 4, 50)
mean(x)
```

```
## [1] 19.16667
```

```r
mean(x, trim = 0.4)
```

```
## [1] 15
```

Winsorized Sample Mean

- **Winsorized Sample Mean:** Again, to reduce the influence of outliers, one can calculate the *Winsorized Sample Mean*. Here we again take $r \in (0, 0.5)$, take $p = [n \cdot r]$, and calculate

$$\text{winsorized sample mean}(x) =$$

$$\frac{x_{(p+1)} + \ldots + x_{(p+1)} + x_{(p+2)} + x_{(p+3)} + \ldots + x_{(n-p-1)} + x_{(n-p)} + \ldots + x_{(n-p)}}{n}$$

$$= \frac{(p+1) \cdot x_{(p+1)} + \sum_{k=p+2}^{n-p-1} x_{(k)} + (p+1) \cdot x_{(n-p)}}{n}.$$

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The weight of data x_k is then $\frac{w_k}{\sum_{i=1}^{n} w_i}$.

Example: CPI (Consumer Price Index) is percentage change of a weighted average market basket of consumer goods and services purchased by households , see Wiki

Example

```r
x <- c(-1,2,3,2,3,1,4,5, 10)
w <- c(0,1.2,1,1,5,3,2,3, 1)
weighted.mean(x, w)
```

```
## [1] 3.395349
```

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w <- c(0,1.2,1,1,5,3,2,3, 1)
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We can check:

```r
sum(x*w)/sum(w)
```

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The rigorous definition is: let $x : x_1, x_2, ..., x_n$ be our dataset.

- If n is **odd**, then we define

$$median(x) = x_{\left(\frac{n+1}{2}\right)};$$

- If n is **even**,

$$median(x) = \frac{1}{2} \cdot \left(x_{\left(\frac{n}{2}\right)} + x_{\left(\frac{n}{2}+1\right)} \right).$$

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- If n is even: then, in the sorted list, we have 2 elements at the center. We take the average of these two elements.

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```
x <- c(1,3,2, 4,2,3,2,2,1)
mean(x)
```

```
## [1] 2.222222
```

```
median(x)
```

```
## [1] 2
```

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mean(x)
```

```
## [1] 2.222222
```

```
median(x)
```

```
## [1] 2
```

Now, let's add an outlier:

```
x <- c(x, 1000)
mean(x)
```

```
## [1] 102
```

```
median(x)
```

```
## [1] 2
```

Important Property of the Median

- Half of the Datapoints are to the left of the Median, and half of the Datapoints are to the right

Example: Give OTB

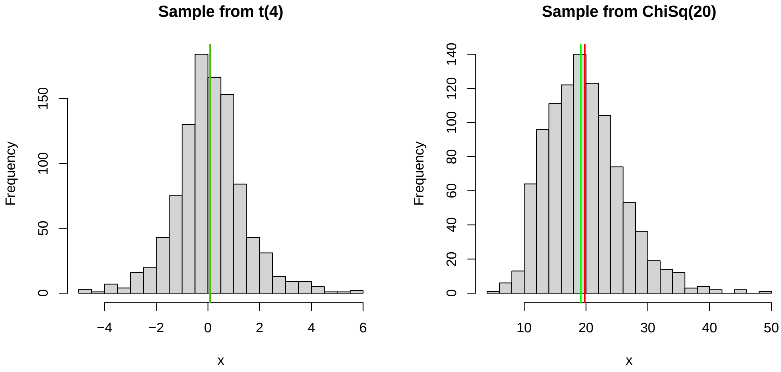
Mean and the Median

- If the Dataset is Symmetric, then the Mean and the Median of that Dataset coincide[1].

[1] Try to define the Symmetry of a Dataset and prove the above statement.

Mean and the Median

- If the Dataset is Symmetric, then the Mean and the Median of that Dataset coincide[1].
- If the Dataset is Skewed, then the Mean and the Median can be very different (Mean is in Red, and the Median is in Green):



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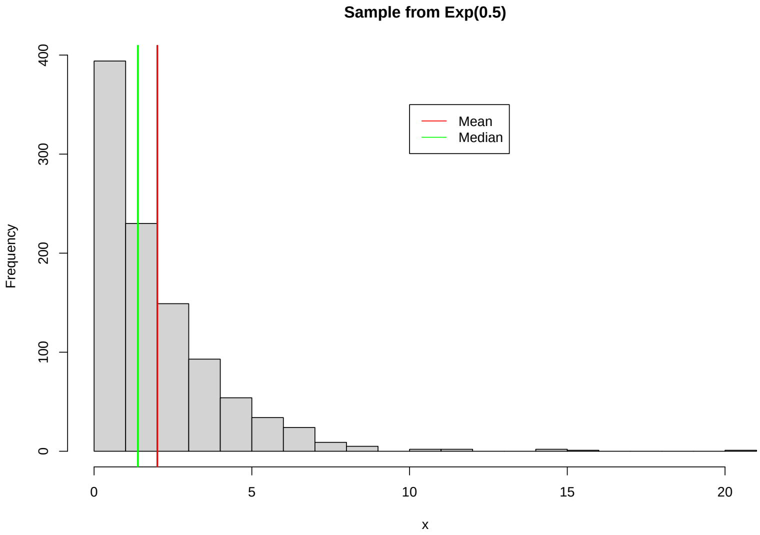

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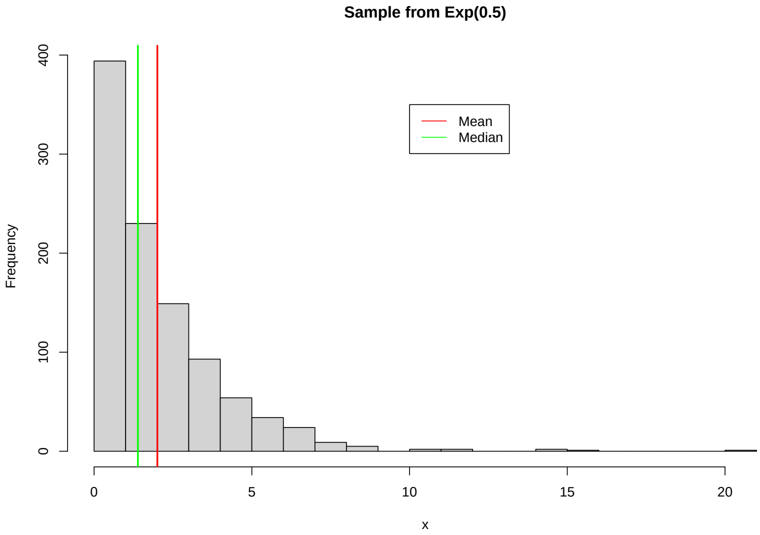

Example: (another one) See, e.g., the Distribution of Wealth article in Wikipedia.

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$$x : 0, -1, 2, 0, 0, 2, 3, 2, 1, 2$$

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Example: The Sample Mode of the following Dataset:

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Remark: Sometimes, one considers also *local Modes* (local maximums of the Frequency Table) and call them just Modes. Just like in Calculus: when saying *extremum*, we think about a *Local*