ASDS Statistics, YSU, Fall 2020

Lecture 12

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Q-Q Plots, Theoretical vs Theoretical Distribution

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To answer this question, we again take some levels of quantiles, say, for some n,

$$\alpha = \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}$$

and then draw the points (q_α^F, q_α^G), where q_α^F is the α-quantile of the Theoretical Distribution with the CDF F, and q_α^G is the α-quantile of the Theoretical Distribution with the CDF G.

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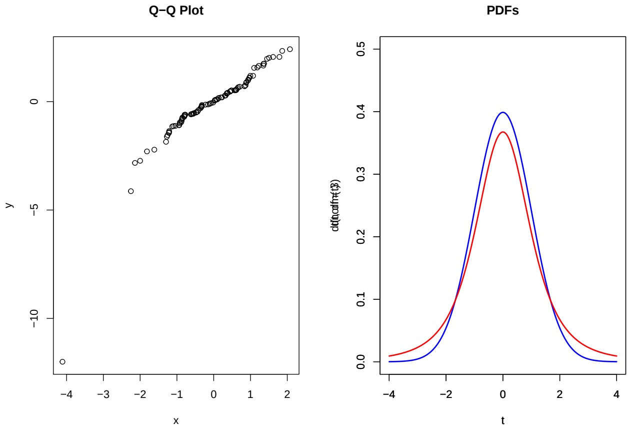
Idea: If G has fatter tails on both sides than F, then we will have graphically some cubic-function graph shape Quantiles.

Some Experiments

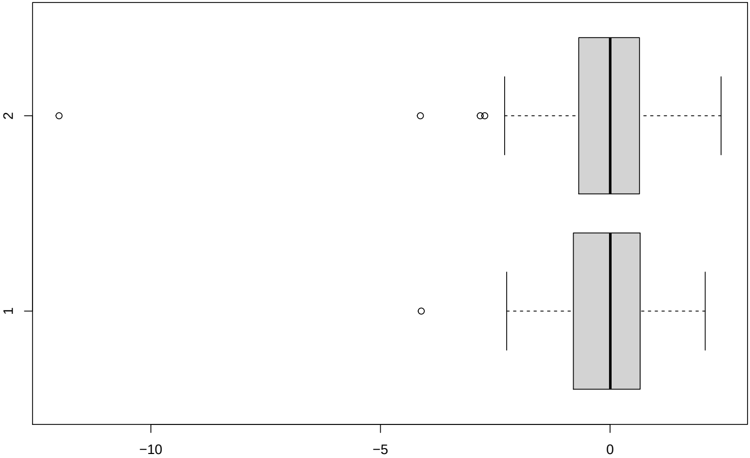
```
par(mfrow = c(1,2))
x <- rnorm(100, mean=0, sd=1); y <- rt(100, df = 3)
qqplot(x,y, main = "Q-Q Plot")
t <- seq(-4,4,0.01)
plot(t, dnorm(t), type = "l", xlim = c(-4,4), ylim = c(0, 0.5), col ="blue", lwd = 2, main = "PDFs")
par(new = TRUE)
plot(t, dt(t, df = 3), type = "l", xlim = c(-4,4), ylim = c(0, 0.5), col ="red", lwd = 2)
```



Some Experiments

The above Datasets, using BoxPlots:

```r
boxplot(x,y, horizontal = T)
```



Some Experiments

```
par(mfrow = c(1,2))
x <- rnorm(100, mean=2, sd=1); y <- rchisq(200, df = 3)
qqplot(x,y, main = "Q-Q Plot")
t <- seq(0,5,0.01)
plot(t, dnorm(t), type = "l", xlim = c(0,5), ylim = c(0, 0.5), col ="blue", lwd = 2,  main = "PDFs")
par(new = TRUE)
plot(t, dchisq(t, df = 3), type = "l", xlim = c(0,5), ylim = c(0, 0.5), col ="red", lwd = 2)
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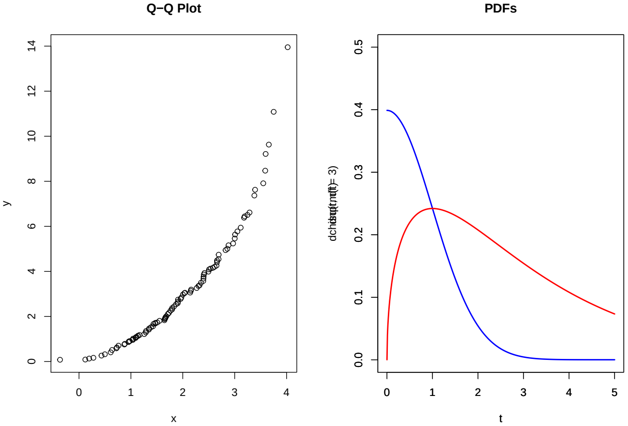
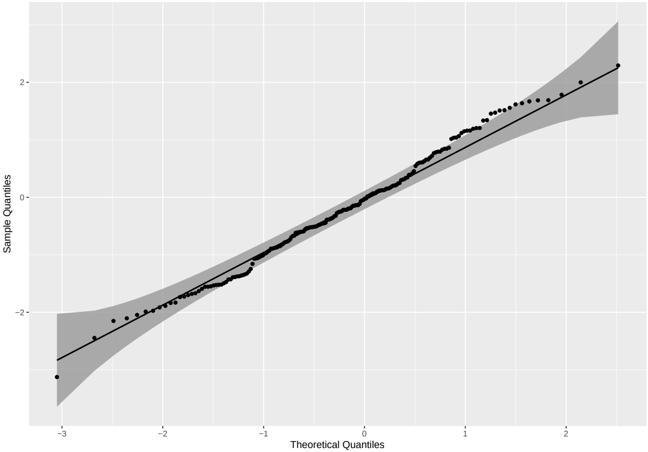


Addition, Q-Q Plot

here you can find some interpretaitons of different shapes of Q-Q Plots: StackExchange Page.

Addition, Q-Q Plot with a Confidence Band

```r
require(qqplotr)
x <- data.frame(variable = rnorm(200))
ggplot(data = x, mapping = aes(sample = variable)) + stat_qq_band() +
  stat_qq_line() + stat_qq_point() + labs(x = "Theoretical Quantiles", y = "Sample Quantiles")
```



Numerical Summaries for Bivariate Data

Sample Covariance and the Correlation Coefficient

Assume now we have a bivariate Dataset

$$(x_1, y_1), \ldots, (x_n, y_n),$$

or just two 1D Datasets of the same size:

$$x: \ x_1, \ldots, x_n \qquad and \qquad y: \ y_1, \ldots, y_n.$$

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Our aim is to see if some linear relationship, association exists between x and y. Of course, the best way is to visualize our Dataset by a ScatterPlot.

Now we want to answer, numerically, how strong/week is the linear relationship between our variables x and y.

Sample Covariance

The **Sample Covariance** of Variables (1D Datasets) x and y is

$$cov(x, y) = s_{xy} = \frac{\sum_{k=1}^{n} (x_k - \overline{x}) \cdot (y_k - \overline{y})}{n}$$

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Note: Recall that for a r.v. X, $Cov(X, X) = Var(X)$.

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Note: Recall that for a r.v. X, $Cov(X, X) = Var(X)$. Here, for Datasets, we have two definitions for the Sample Variance $var(x)$. And we give two definitions of the Sample Covariance, so the property $cov(x, x) = var(x)$ will hold in both cases.

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Remark: For almost all numerical summaries for 1D data, first step was sorting the Dataset to obtain Order Statistics. But please note that for calculating Covariance or Correlation Coefficient (as well as for ScatterPlotting), sorting the Datasets will give incorrect results.

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Remark: For almost all numerical summaries for 1D data, first step was sorting the Dataset to obtain Order Statistics. But please note that for calculating Covariance or Correlation Coefficient (as well as for ScatterPlotting), sorting the Datasets will give incorrect results. This is because we want to find a relationship between x_1 and y_1, x_2 and y_2, ..., not the relationship between the minimal elements of Datasets etc.

Example

Here is the **R** code to calculate the Covariance between the `Speed` and `Dist` variables in the `cars` Dataset:

```
cov(cars$speed, cars$dist)
```

```
## [1] 109.9469
```

Sample Correlation Coefficient

Another measure of the linear relationship between the Variables x and y of Bivariate Dataset is the *Pearson's Correlation Coefficient*:

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If $s_x = 0$ or $s_y = 0$, then we take $cor(x, y) = 0$ by definition.

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If $s_x = 0$ or $s_y = 0$, then we take $cor(x, y) = 0$ by definition.

Note: Please note that we need to calculate the Standard Deviations and Covariance by using the same denominator: either everywhere take n, or take everywhere $n - 1$.

Sample Correlation Coefficient

In both cases, when one calculates Standard Deviations and Covariance by using n simultaneously or $n-1$ simultaneously in the denominator, we will obtain

$$cor(x,y) = \rho_{xy} = \frac{\sum_{k=1}^{n}(x_k - \overline{x}) \cdot (y_k - \overline{y})}{\sqrt{\sum_{k=1}^{n}(x_k - \overline{x})^2 \cdot \sum_{k=1}^{n}(y_k - \overline{y})^2}}$$

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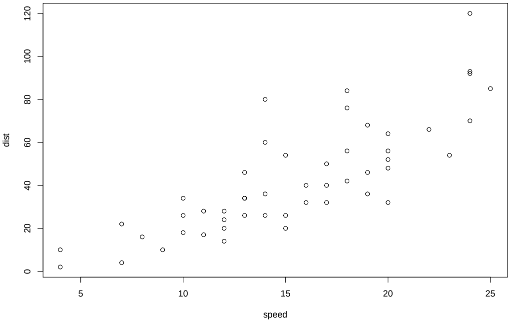
Another formula to calc the correlation coefficient is

$$cor(x,y) = \rho_{xy} = \frac{\displaystyle\sum_{k=1}^{n} x_k y_k - n \cdot \overline{x} \cdot \overline{y}}{\sqrt{\displaystyle\sum_{k=1}^{n} x_k^2 - n \cdot (\overline{x})^2} \cdot \sqrt{\displaystyle\sum_{k=1}^{n} y_k^2 - n \cdot (\overline{y})^2}}.$$

Examples:

Now, the **R** code to calculate the Covariance between the `Speed` and `Dist` variables in the `cars` Dataset:

```
plot(dist~speed, data = cars)
```



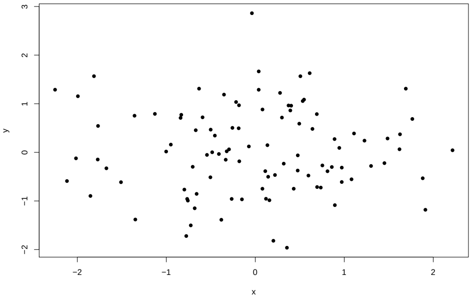

```
cor(cars$speed, cars$dist)
```

```
## [1] 0.8068949
```

Examples:

Some simulations:

```
x <- rnorm(100); y <- rnorm(100);
plot(x,y, pch=16)
```



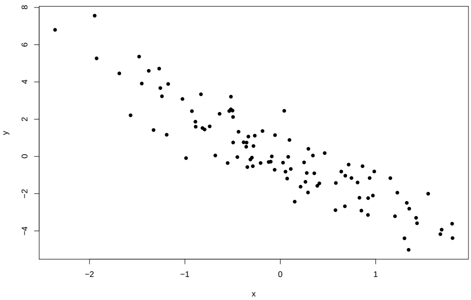

```
c(cor(x,y), cov(x,y))
```

```
## [1] -0.003809632 -0.003336435
```

Examples:

Some simulations:

```
x <- rnorm(100); y <- -2.4*x + rnorm(100);
plot(x,y, pch=16)
```




```
c(cor(x,y), cov(x,y))
```

```
## [1] -0.9072463 -2.1401902
```

Examples:

Let us now use the `state.x77` Dataset from **R**:

```r
head(state.x77)
```

```
##            Population Income Illiteracy Life Exp Murder HS Gr
## Alabama          3615   3624        2.1    69.05   15.1    41
## Alaska            365   6315        1.5    69.31   11.3    66
## Arizona          2212   4530        1.8    70.55    7.8    58
## Arkansas         2110   3378        1.9    70.66   10.1    39
## California      21198   5114        1.1    71.71   10.3    62
## Colorado         2541   4884        0.7    72.06    6.8    63
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```
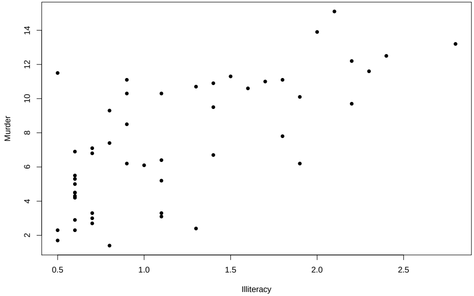

It is not of the DataFrame format, so we change it to DataFrame:

```
state <- as.data.frame(state.x77)
```

Examples:

```
plot(Murder~Illiteracy, data = state, pch=16)
```




```
cor(state$Illiteracy, state$Murder)
```

```
## [1] 0.7029752
```

Examples:

Question: How to generate samples x, y with some given Correlation Coefficient?

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One of the possible methods: take a Matrix

$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix},$$

which is **Positive Definite**, take any 2D vector, say $\mu = [0, 0]^T$, and generate a Sample of size n from the Bivariate Normal Distribution $\mathcal{N}(\mu, \Sigma)$.

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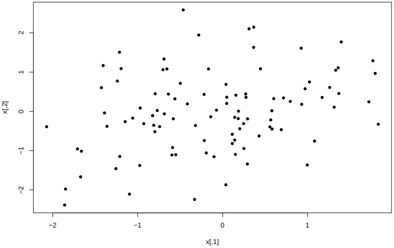
which is **Positive Definite**, take any 2D vector, say $\mu = [0, 0]^T$, and generate a Sample of size n from the Bivariate Normal Distribution $\mathcal{N}(\mu, \Sigma)$.

Then, the $cor(x, y)$ will be approximately ρ (and it will approach ρ as $n \to +\infty$).

Example

```
rho <- 0.35
covmatrix <- matrix(c(1,rho, rho, 1), nrow = 2)
mu <- c(0,0)
x <- mvtnorm::rmvnorm(100, mean = mu, sigma = covmatrix)
plot(x, pch = 16)
```




```
cor(x)
```

```
##              [,1]      [,2]
## [1,] 1.0000000 0.3069956
## [2,] 0.3069956 1.0000000
```

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- For any Dataset x,

$$cov(x, x) = var(x)$$

Properties of the Sample Correlation Coefficient

- $cor(x, y) = cor(y, x)$;

[1] Or $x_i = a \cdot y_i + b$ for any $i = 1, ..., n$ (maybe for another a and b).
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- If $\alpha > 0$ and $\beta \in \mathbb{R}$, then $cor(\alpha \cdot x + \beta, y) = cor(x, y)$

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$$-1 \leq \rho_{xy} \leq 1;$$

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- $\rho_{xy} = -1$ iff there exists a constant $a < 0$ and $b \in \mathbb{R}$ such that[2] $y_i = a \cdot x_i + b$ for any $i = 1, ..., n$.

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