ASDS Statistics, YSU, Fall 2020

Lecture 17

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Convergence a.s.

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Definition: We will say that $X_n \to X$ **almost sure**, and we will write $X_n \to X$ a.s. or $X_n \xrightarrow{a.s.} X$, if

$$\mathbb{P}\Big(\omega \in \Omega : \lim_{n \to +\infty} X_n(\omega) = X(\omega)\Big) = 1,$$

or, for short,

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Equivalently, we can write

$$X_n \xrightarrow{a.s.} X \qquad \text{iff} \qquad \mathbb{P}\Big(X_n \not\to X\Big) = 0.$$

Convergence in Probability

Definition: We will say that $X_n \to X$ **in Probability**, and we will write $X_n \xrightarrow{\mathbb{P}} X$, if

$$\text{for any } \varepsilon > 0, \ \mathbb{P}\Big(|X_n - X| \geq \varepsilon\Big) \to 0, \qquad \text{when} \quad n \to \infty.$$

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$$X_n \xrightarrow{\mathbb{P}} X \qquad \text{iff} \qquad \mathbb{P}\Big(|X_n - X| < \varepsilon\Big) \to 1 \text{ for any } \varepsilon > 0.$$

Convergence in the Mean Square Sence

Definition: We will say that $X_n \to X$ **in the Quadratic Mean Sense or in L^2 (or in the Mean Square Sense)**, and we will write $X_n \xrightarrow{L^2} X$ or $X_n \xrightarrow{qm} X$, if

$$MSE(X_n, X) = \mathbb{E}\Big((X_n - X)^2\Big) \to 0, \qquad \text{when} \quad n \to \infty.$$

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$$MSE(X_n, X) = \mathbb{E}\Big((X_n - X)^2\Big) \to 0, \qquad \text{when} \quad n \to \infty.$$

Here $MSE(X_n, X)$ is the *Mean Square Error* (of the approximation of X by X_n).

Convergence in Distributions

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$F_{X_n}(x) \to F_X(x) \quad \text{as} \quad n \to \infty$ at any point of continuity x of $F_X(x)$.

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Remark: This is equivalent to saying that for (almost) any subsets $A \subset \mathbb{R}$

$$\mathbb{P}(X_n \in A) \to \mathbb{P}(X \in A).$$

Convergence in Distributions

Remark on the notation: Usually, in the case of the Convergence in Distribution, we write the Distribution as the limit, e.g., we write

$$X_n \xrightarrow{D} \mathcal{N}(0,1)$$

instead of writing $X_n \xrightarrow{D} X$, $X \in \mathcal{N}(0,1)$.

Cauchy Principle for a.e, $\mathbb{P}$ and L^2 Convergence

Now, for checking the convergence of a sequence of r.v. X_n, we can use the following Theorem (Cauchy Principle):

Theorem:

- If $X_n - X_m \to 0$ a.e. when $m, n \to +\infty$, then there exists a r.v. X such that $X_n \to X$ a.e.;

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- If for any $\varepsilon > 0$, $\mathbb{P}(|X_n - X_m| \geq \varepsilon) \to 0$ when $m, n \to +\infty$, then there exists a r.v. X such that $X_n \xrightarrow{\mathbb{P}} X$;
- If $\mathbb{E}\Big((X_n - X_m)^2\Big) \to 0$ when $m, n \to +\infty$, then there exists a r.v. X such that $X_n \xrightarrow{L^2} X$.

Example

Example: We have a sequence of infinitely many (independent) tosses of a fair coin, and let X_n be the result of the n-th trial ($Head = 1$, $Tail = 0$). So the Distribution of X_n is

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- Is X_n convergent in the sense of Distributions ?
- Is X_n convergent in the Probability sense ?
- Is X_n convergent in the MS sense ?
- Is X_n convergent in the a.s. sense?

Example

Example: Assume X_n is a Discrete r.v. with the following PMF, defined on the same Probability Space:

X_n	$3 + \frac{1}{n^2}$	n
$\mathbb{P}(X_n = x)$	$1 - \dfrac{1}{n}$	$\dfrac{1}{n}$.

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Which of the followings are true (use only the definitions):

- $X_n \xrightarrow{\mathbb{P}} 3$;
- $X_n \xrightarrow{qm} 3$;
- $X_n \xrightarrow{D} 3$?