Basic Mathematics, Fall 2020

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Bayes' Formula (Bayes' Rule)

If $\mathbb{P}(A) > 0$ and $0 < \mathbb{P}(B) < 1$ then

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(B)\mathbb{P}(A|B)}{\mathbb{P}(B)\mathbb{P}(A|B) + \mathbb{P}(\overline{B})\mathbb{P}(A|\overline{B})}.$$

Bayes' Formula: Umbrella

If it is raining in the morning there is a 90% chance that I will bring my umbrella. If it is not raining in the morning there is only a 20% chance of me taking my umbrella. On any given morning the probability of rain is 0.1. If you see me with an umbrella, what is the probability that it was raining that morning?

Draw a corresponding tree diagram to make the solution intuitive.

Bayes' General Formula

Assume we have an event A and mutually exclusive events (hypotheses) $B_1, B_2, \ldots, B_n$ such that $\mathbb{P}(A) > 0, \mathbb{P}(B_k) > 0, k = 1, 2, \ldots, n$, and $A \subset \cup_{k=1}^{n} B_k$. Then, for any $i \in \{1, 2, \ldots, n\}$, the following formula takes place:

$$\mathbb{P}(B_i|A) = \frac{\mathbb{P}(B_i)\mathbb{P}(A|B_i)}{\sum_{k=1}^{n} \mathbb{P}(B_k)\mathbb{P}(A|B_k)}.$$

Asian Suppliers

A computer manufacturer receives hard disks from three different supplier countries. Malaysia supplies 40% of the disks, Bangladesh supplies 25%, while Laos supplies the rest. Disks from Malaysia have a defective rate of 4%, those from Bangladesh 3%, while Laos have a 5% rate.

- A disk is checked at random. What is the probability that it is defective?
- A disk is checked and found defective. What is the probability that it was supplied from Laos?

Medical Testing

Assume a person decides to take a medical test for some disease. The probability that the test gives a correct answer is 98%. This means that if a person has that disease, then the test gives positive result in 98% cases. And if a person did not have that disease, then the test gives negative result in 98% cases. It is known that only 0.1% of total population has that disease. Now, a person receives the test results, and the test shows a positive result. What is the probability that the person actually has that disease?

Have a look at videos
https://www.youtube.com/watch?v=BrK7X_XlGB8
https://www.youtube.com/watch?v=R13BD8qKeTg

Definition

Let Ω be a Sample Space of an experiment. Any function

$$X : \Omega \to \mathbb{R}$$

is called a **Random Variable.**

Discrete Random Variables

A random variable X is called **discrete** if the set of the values of X,

$$Range(X) = \{X(w) : w \in \Omega\}$$

is either finite, or countably infinite.

Probability Mass Function

For a discrete random variable X, we define the **probability mass function (PMF)** $p_X(x)$ of X by

$$p_X(x) = \mathbb{P}(\{w \in \Omega : X(w) = x\}) =: \mathbb{P}(X = x).$$

If $Range(X) = \{x_1, x_2, \ldots, x_k, \ldots\}$ then we usually denote

$$p_k = p_X(x_k) = \mathbb{P}(X = x_k).$$

PMF presented in a tabular form.

X	x_1	x_2	...	x_n	$\ldots$
$\mathbb{P}(X = x)$	p_1	p_2	...	p_n	$\ldots$

Clearly $\sum_n p_n = 1$.

CDF of a Random Variable

Let Ω be a sample space of an experiment and $X : \Omega \to \mathbb{R}$ be a random variable . Then the function $F_X : \mathbb{R} \to [0; 1]$ defined by the formula

$$F_X(x) = \mathbb{P}(X \leq x)$$

is called the **cumulative distribution function (CDF)** of the r.v. X.

CDF Properties

For any r.v. X, its CDF, F_X , satisfies the following properties:

- $0 \leq F_X(x) \leq 1$, for any $x \in \mathbb{R}$.
- F_X is an increasing function on $\mathbb{R}$.
- $F_X(-\infty) = 0$ and $F_X(+\infty) = 1$.
- F_X is a right-continuous function, i.e. $F_X(x_0+) = F_X(x_0)$ for any $x_0 \in \mathbb{R}$.

Example (The RV is X - the number of tosses until the first tails comes up)

The PMF:

X	1	2	...	n	$\dots$
$\mathbb{P}(X = x)$	$1/2$	$1/4$	...	$1/2^n$	$\dots$

The CDF formula of X:

$$F_X(x) = \begin{cases} 0, & \text{if } x < 1, \\ 1/2, & \text{if } 1 \leq x < 2, \\ 3/4, & \text{if } 2 \leq x < 3, \\ \dots, & \dots \\ (2^k - 1)/2^k, & \text{if } k \leq x < k+1, \\ \dots, & \dots \end{cases}$$

Definition (Trials)

A fixed number of repetitions of the same experiment can be thought of as one compound experiment. In this case, the individual repetitions are called **trials.**

Definition (Independent Trials)

If the outcomes of any group of the trials do not affect the probabilities of the outcomes of any other trial, then the trials are called **independent trials.**

Bernoulli Trials

Suppose there are only two possible outcomes in the sample space of each trial classified as "Success" and "Failure". Such trials are called **Bernoulli trials.** If moreover, the probability p of Success remains the same for each trial throughout the entire experiment, then such trials are called **repeated** Bernoulli trials.

Binomial Experiment

Let n be some fixed positive integer. An experiment consisting of n repeated Bernoulli trials is called a **Binomial Experiment.**

Examples of Binomial experiments

- Tossing a fair coin 10 times:
  Success = Heads, Failure = Tails
  p = 0.5
- Rolling a fair die 5 times:
  Success = opened number is even,
  Failure = the other numbers
  $p = \frac{3}{6} = \frac{1}{2}$.

Binomial Probabilities

Suppose a Binomial Experiment has n trials. Let X_n be the number of successes in a sequence of n trials. Then for $0 \leq k \leq n$,

$$p_{X_n}(k) = \mathbb{P}(X_n = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

We denote this fact as $X_n \sim B(n, p)$.

Example

We are tossing a fair coin 100 times. The success is "Heads".
1) What is the probability of being successful exactly 5 times?
2) What is the probability that we will be successful at least once?

Example

An examine consists of 16 multiple-choice questions. Each question provides 4 answers, where only 1 is correct. A student, who is absolutely unfamiliar with the subject, chooses 16 answers randomly.
1) What is the probability that the student will answer correctly at least once?
2) The exam is passed if at least 3 questions are answered correctly. What is the probability that the student above will pass the exam?

Example

A communication system consists of 5 components, each of which will, independently, function with probability p. The total system will be able to operate effectively if at least one-half of its components function. Find the probability that a 5-component system will be effective.