Note Title 21-Nov-20

Fisher Information

- FI for Bernoulli(p)

Take $X \sim \text{Bernoulli}(p)$

X	0	1
$\mathbb{P}$	$1-p$	p

$p \in (0,1)$

- PMF is: $f(x|p) = p^x \cdot (1-p)^{1-x}$, $x \in \{0,1\}$

- Plug X into f :

$$f(X|p) = p^X (1-p)^{1-X}$$

· Calc. the log:

$$\ln f(x|p) = x\ln p + (1-x)\ln(1-p)$$

· Calc.

$$\frac{\partial}{\partial p}\ln f(x|p) = \frac{x}{p} - \frac{1-x}{1-p}$$

$$\frac{\partial^2}{\partial p^2}\ln f(x|p) = -\frac{x}{p^2} - \frac{1-x}{(1-p)^2}$$

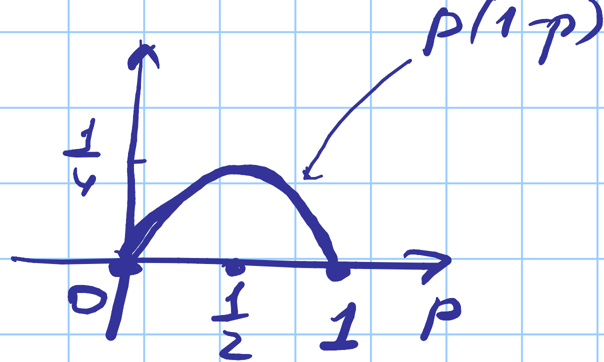
· Calc. FI:

$$I(p) = -\mathbb{E}\left(\frac{\partial^2}{\partial p^2}\ln f(x|p)\right) =$$

$$= \mathbb{E}\left(\frac{x}{p^2} + \frac{1-x}{(1-p)^2}\right) = \frac{\mathbb{E}(x)}{p^2} + \frac{1-\mathbb{E}(x)}{(1-p)^2} \overset{x\sim Bernoulli(p)}{=}$$

$$= \frac{1}{p} + \frac{1}{1-p} = \frac{1}{p(1-p)}$$

$$\boxed{I(p) = \frac{1}{p(1-p)}}$$



X	0	1
$\mathbb{P}$	$1-p$	p

$p=0$

X	0
$\mathbb{P}$	1

← max info

$p=1$

X	1
$\mathbb{P}$	1

X	0	1
$\mathbb{P}$	$2/3$	$1/3$

some info
(0 is more possible)

X	0	1
$\mathbb{P}$	$\frac{1}{2}$	$\frac{1}{2}$

← smallest info

- FI for $Exp(\lambda)$

$$X \sim Exp(\lambda) \qquad X \geq 0 \quad \text{w. Prob } 1$$

- PDF

$$f(x/\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$f(X|\lambda) = \lambda e^{-\lambda \cdot X}$$

$$\ln f(X|\lambda) = -\lambda \cdot X + \ln \lambda$$

$$\frac{\partial}{\partial \lambda} \ln f(X|\lambda) = -X + \frac{1}{\lambda}$$

$$\frac{\partial^2}{\partial \lambda^2} \ln f(X|\lambda) = -\frac{1}{\lambda^2}$$

$$\underline{\underline{I(\lambda)}} = -\mathbb{E}\left(\frac{\partial^2}{\partial\lambda^2}\ln f(x|\lambda)\right) = \mathbb{E}\left(\frac{1}{\lambda^2}\right) = \underline{\underline{\frac{1}{\lambda^2}}}$$

• FI for μ, σ^2 in $N(\mu, \sigma^2)$

$$X \sim N(\mu, \sigma^2)$$

PDF $$f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Denote $\sigma^2 = \theta$

$$f(x|\mu,\theta) = \frac{1}{\sqrt{2\pi\theta}} \cdot e^{-\frac{(x-\mu)^2}{2\theta}}$$

$$f(X|\mu,\theta) = \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{(X-\mu)^2}{2\theta}}$$

$$\ln f(X|\mu,\theta) = -\frac{(X-\mu)^2}{2\theta} - \frac{1}{2} \ln \overline{2\pi\theta}$$

· FI for μ

$$\frac{\partial^2}{\partial \mu^2} \ln f(x|\mu,\theta) = \left(\frac{x-\mu}{\theta} \right)'_{\mu} = -\frac{1}{\theta}$$

$$I(\mu) = -E\left(\frac{\partial^2}{\partial \mu^2} \ln f(x|\mu,\theta)\right) = E\left(\frac{1}{\theta}\right) = \frac{1}{\theta} = \underline{\underline{\frac{1}{\sigma^2}}}$$

· FI for θ

$$\frac{\partial^2}{\partial \theta^2} \ln f(x|\mu,\theta) = \left(\frac{(x-\mu)^2}{2\theta^2} - \frac{1}{2\theta} \right)'_{\theta} = -\frac{(x-\mu)^2}{\theta^3} + \frac{1}{2\theta^2}$$

$$\Rightarrow I(\theta) = -\mathbb{E}\left(\frac{\partial^2}{\partial\theta^2}\ln f(X|\mu,\theta)\right) =$$

$$= \frac{\mathbb{E}(x-\mu)^2}{\theta^3} - \frac{1}{2\theta^2} \underline{\underline{X \sim N(\mu,\sigma^2)}} \frac{\theta}{\theta^3} - \frac{1}{2\theta^2} =$$

$$= \frac{1}{2\theta^2} = \underline{\underline{\frac{1}{2\sigma^4}}}$$

- Efficiency of

$$\hat{p} = \overline{X}$$

$\sim$ Bernoulli(p).

a) $\hat{p}$ is Unbiased $\mathbb{E}(\hat{p}) = \mathbb{E}(X_1) = p \quad \forall p$

b) $\mathrm{Var}(\hat{p}) \stackrel{?}{=} \dfrac{1}{n \cdot I(p)}$

For Bernoulli(p), $I(p) = \dfrac{1}{p(1-p)}$

$$\frac{1}{n I(p)} = \frac{p(1-p)}{n}$$

$$Var(\hat{p}) = Var(\overline{X}) = Var\left(\frac{X_1 + X_2 + \dots + X_n}{n}\right) =$$

$$= \frac{Var(X_1)}{n} \overset{X_1 \sim Bernoulli(p)}{=\!=} \frac{p(1-p)}{n} = \frac{1}{nI(p)}$$

$\Rightarrow$ $\hat{p} = \overline{X}$ is Efficient

$$\boxed{\hat{p} = \overline{X}}$$

- Efficacy of $\hat{\lambda} = \bar{X}$ in Pois(λ) Model.

$X_1, \dots, X_n \sim \text{Pois}(\lambda)$

a) $\hat{\lambda}$ is Unbiased: $\underline{\mathbb{E}(\hat{\lambda})} = \mathbb{E}(\bar{X}) = \mathbb{E}(X_1) = \underline{\lambda}$

b) $\text{Var}(\hat{\lambda}) \stackrel{?}{=} \dfrac{1}{n \cdot I(\lambda)}$

$$\text{Var}(\hat{\lambda}) = \text{Var}(\bar{X}) = \frac{\text{Var}(X_1)}{n} = \underline{\underline{\frac{\lambda}{n}}}$$

<u>FI</u> calc.

PMF: $f(x|\lambda) = e^{-\lambda} \cdot \dfrac{\lambda^x}{x!}$, $x = 0, 1, 2, \dots$

$$X \sim Pois(\lambda)$$

$$f(x|\lambda) = e^{-\lambda} \cdot \frac{\lambda^x}{x!}$$

$$\ln f(x|\lambda) = -\lambda + x \cdot \ln \lambda - \ln(x!)$$

$$\frac{\partial^2}{\partial \lambda^2} \ln f(x|\lambda) = -\frac{x}{\lambda^2}$$

$$I(\lambda) = -\mathbb{E}\left(\frac{\partial^2}{\partial \lambda^2} \ln f(x|\lambda)\right) = \frac{\overset{=\lambda}{\mathbb{E}(X)}}{\lambda^2} = \frac{1}{\lambda}$$

$$\frac{1}{n \cdot I(\lambda)} = \frac{\lambda}{n} = Var(\hat{\lambda})$$

$\Rightarrow$ $\hat{\lambda} = \bar{X}$ is Efficient Est. for λ Pois(λ)