Optimization

Lusine Poghosyan

YSU

November 15, 2020

Golden Section Search

Assume $f : [a, b] \to \mathbb{R}$ is unimodal and continuous on interval $[a, b]$. Let x^* be the minimum point of f over $[a, b]$.

Let's denote

$$[a_0, b_0] = [a, b].$$

$$A = a_0 + \gamma(b_0 - a_0),$$

$$B = b_0 - \gamma(b_0 - a_0),$$

where $\gamma \in \left(0, \frac{1}{2}\right)$.

We define the new interval in the following way

$$[a_1, b_1] = \begin{cases} [a_0, B], & \text{if } f(A) < f(B), \\ [A, b_0], & \text{if } f(A) \geq f(B). \end{cases}$$

$$x^* \in [a_1, b_1]$$

$$b_1 - a_1 = (1 - \gamma)(b_0 - a_0).$$

The first approximation will be

$$x_1 = \frac{a_1 + b_1}{2}.$$

n-th step

$$[a_n, b_n] = \begin{cases} [a_{n-1}, B], & \text{if } f(A) < f(B), \\ [A, b_{n-1}], & \text{if } f(A) \geq f(B). \end{cases}$$

$$x^* \in [a_n, b_n]$$

$$b_n - a_n = (1 - \gamma)(b_{n-1} - a_{n-1})$$

The n-th approximation will be

$$x_n = \frac{a_n + b_n}{2}.$$

Theorem

If f is a unimodal and continuous function on $[a, b]$ and $\gamma \in \left(0, \frac{1}{2}\right)$, then the golden section search approximation x_n converges to x^* and we have

$$|x_n - x^*| \leq 0.5(1-\gamma)^n(b-a).$$

Example

Calculate the second approximation x_2 of the golden section search method with $\gamma = \frac{1}{3}$ for function $f(x) = \frac{x^3}{3} - 2x$ on the interval $[-3, 0]$.

In general case at each step we calculate the value of *f* at two points: *A* and *B*.

It is possible to find $\gamma \in \left(0, \frac{1}{2}\right)$ such that we evaluate *f* at two points at the first step but at each next step we evaluate *f* at one point.

Let's assume that at the step n $f(A) < f(B)$, which means that our new interval $[a_n, b_n]$ is going to be $[a_{n-1}, B]$.

$$\begin{aligned} a_{n-1} + \gamma(b_{n-1} - a_{n-1}) &= b_n - \gamma(b_n - a_n) \\ &= b_{n-1} - \gamma(2-\gamma)(b_{n-1} - a_{n-1}) \end{aligned}$$

This implies

$$\gamma^2 - 3\gamma + 1 = 0$$

and

$$\gamma = \frac{3-\sqrt{5}}{2} \approx 0.382.$$

$$|x_n - x^*| \leq 0.5 \left(\frac{\sqrt{5}-1}{2}\right)^n (b-a).$$

Stopping conditions

- $|b_n - a_n| < \varepsilon$
- $|x_n - x_{n-1}| < \varepsilon$ or $\frac{|x_n - x_{n-1}|}{|x_{n-1}|} < \varepsilon$, if $x_{n-1} \neq 0$
- $|f(x_n) - f(x_{n-1})| < \varepsilon$ or $\frac{|f(x_n) - f(x_{n-1})|}{|f(x_{n-1})|} < \varepsilon$, if $f\left(x_{n-1}\right) \neq 0$

Bisection Method

Assume $f : [a, b] \to \mathbb{R}$ is a unimodal and continuously differentiable function on $[a, b]$ and $f'(a)f'(b) < 0$.

Let's denote

$$[a_0, b_0] = [a, b]$$

and

$$x_0 = \frac{a_0 + b_0}{2}.$$

If $f'(x_0) = 0$, then we stop here.

$$[a_1, b_1] = \begin{cases} [a_0, x_0], & \text{if} \quad f'(x_0) > 0, \\ [x_0, b_0], & \text{if} \quad f'(x_0) < 0. \end{cases}$$

n-th step

$$[a_n, b_n] = \begin{cases} [a_{n-1}, x_{n-1}], & \text{if} \quad f'(x_{n-1}) > 0, \\ [x_{n-1}, b_{n-1}], & \text{if} \quad f'(x_{n-1}) < 0. \end{cases}$$

$$x_n = \frac{a_n + b_n}{2}.$$

If at n-th step $f'(x_n) = 0$, then we terminate our search.

Stopping conditions for the Bisection Method

- $|x_n - x_{n-1}| < \varepsilon$ or $\frac{|x_n - x_{n-1}|}{|x_{n-1}|} < \varepsilon$, if $x_{n-1} \neq 0$
- $|f'(x_n)| < \varepsilon$
- $|f(x_n) - f(x_{n-1})| < \varepsilon$ or $\frac{|f(x_n) - f(x_{n-1})|}{|f(x_{n-1})|} < \varepsilon$, if $f\left(x_{n-1}\right) \neq 0$

Example

Calculate the second approximation x_2 of the bisection method for the function $f(x) = -\frac{x^3}{3} + 2x$ on the interval $[-4, 0]$.