Optimization

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Barrier Methods

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & x \in \Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^n$.
Barrier Methods are procedures when constrained optimization problems are approximated by "unconstrained optimization" problem. The approximation is accomplished by adding to the objective function a term that favors points interior to the feasible set over those on the boundary.

Here we assume that the feasible set is robust, i.e., each feasible point can be approached by a sequence of interior points.

Definition

A function $B(x)$ defined on the interior of Ω is called a barrier function for the constrained minimization problem above, if it satisfies the following conditions

1. B is continuous on the interior of Ω ($\text{int}(\Omega)$),
2. $B(x) \geq 0$, $\forall x \in \text{int}(\Omega)$,
3. $B(x) \to \infty$ as x approaches the boundary of Ω.

Assume $\{c_k\}$ is a strictly increasing sequence of positive numbers such that $\lim_{k\to\infty} c_k = \infty$. Now we consider the following problem

$$\begin{aligned} \text{minimize} \quad & f(x) + \frac{1}{c_k} B(x) \\ \text{subject to} \quad & \text{int}(\Omega). \end{aligned}$$

In fact we have a constrained optimization problem but we can apply the methods for unconstrained optimization problems. Take an initial approximation from $\text{int}(\Omega)$, use some method to calculate next approximation and it will be in $\text{int}(\Omega)$ (if carefully implemented) since $B(x) \to \infty$ as x approaches the boundary.

As equality constraints usually don't give robust sets we don't consider problems with equality constraints.

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0, \quad i = 1, \dots p. \end{aligned}$$

We assume that the objective function and constraint functions are continuous.

If $\text{int}(\Omega) = \{x : g_i(x) < 0, \forall i \in \overline{1, p}\}$, then the barrier function can be given by the following formula

$$B(x) = -\sum_{i=1}^{p} \frac{1}{g_i(x)}, \quad x \in \text{int}(\Omega).$$

Example

Consider the following constrained minimization problem

$$\begin{aligned} \text{minimize} \quad & f(x) = -\frac{1}{x^2+1} \\ \text{subject to} \quad & x \leq 2, \\ & x \geq -1. \end{aligned}$$

$$r(x, c) = -\frac{1}{x^2+1} - \frac{1}{c} \cdot \frac{1}{x-2} - \frac{1}{c} \cdot \frac{1}{-x-1}, \quad x \in (-1, 2)$$

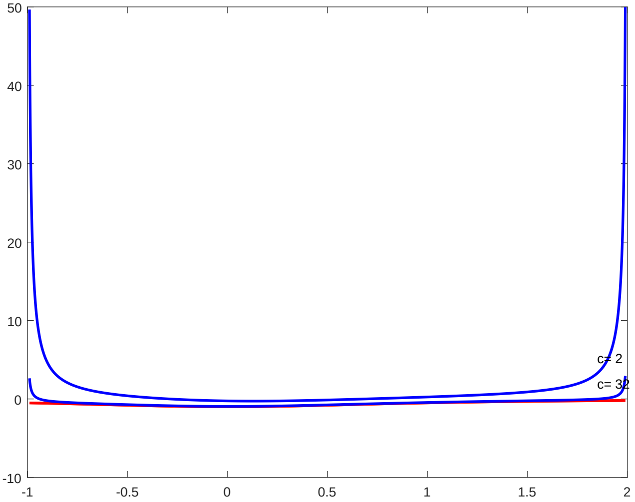

Figure: The red line is the graph of $f(x)$ and blue lines are the graphs of $r(x, c)$ for different parameters.

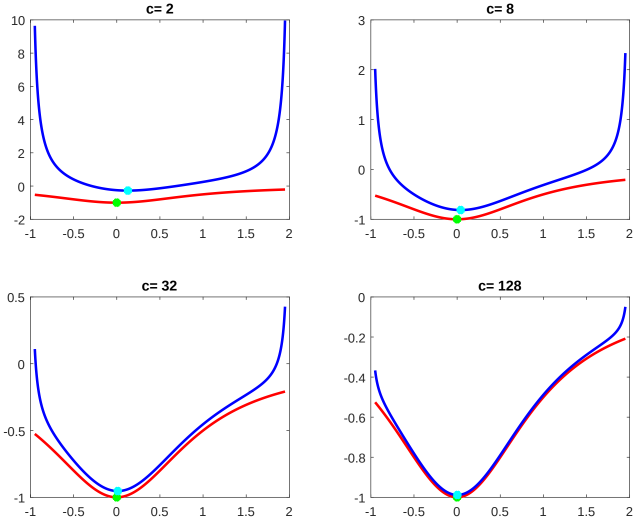

Figure: The green point is the solution of constrained minimization problem and the blue one is the minimizer of $r(x, c)$.

Linear Programming

A linear program is an optimization problem of the form

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0, \end{aligned}$$

where $c \in \mathbb{R}^n$, A is $m \times n$ matrix and $b \in \mathbb{R}^m$. The vector inequality $x \geq 0$ means that each component of x is nonnegative.

Two-Dimensional Linear Programs

Example

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0, \end{aligned}$$

where $x = [x_1, x_2]^T$ and $c = [1, 5]^T$, $A = \begin{bmatrix} 5 & 6 \\ 3 & 2 \end{bmatrix}$, $b = [30, 12]^T$.

To solve the problem geometrically we start with drawing the feasible set

$$\text{FS} = \Big\{(x_1, x_2)^T : 5x_1 + 6x_2 \leq 30,\ 3x_1 + 2x_2 \leq 12,\ x_1 \geq 0, x_2 \geq 0.\Big\}$$

Then, we draw several level curves of objective function

$$f(x_1, x_2) = x_1 + 5x_2$$

to see how large we can make f while satisfying the constraints.

We see that the maximizer is $(0, 5)^T$.

If as objective function we take

- $f(x_1, x_2) = 5x_1 + 4x_2$ as maximizer we get $(1.5, 3.75)^T$;
- $f(x_1, x_2) = 5x_1 + 6x_2$ as maximizers we get $(0, 5)^T$ and $(1.5, 3.75)^T$ and all points on the line segment between thees two points;
- $f(x_1, x_2) = 2x_1 + x_2$ as maximizer we get $(4, 0)^T$;
- $f(x_1, x_2) = -2x_1 + x_2$ as maximizer we get $(0, 5)^T$.

Example

Solve the following Linear Programming Problem (LPP) graphically

$$
\begin{aligned}
\text{minimize} \quad & -x_1 - 3x_2 \\
\text{subject to} \quad & x_1 + x_2 \geq 5 \\
& 4x_2 - 3x_1 \geq -4 \\
& x_2 - 6x_1 \leq 6 \\
& x_1 \geq 0,\ x_2 \geq 0.
\end{aligned}
$$

As we see it is possible that LPP doesn't posses a solution. It is possible when the feasible set is unbounded.

If we change the objective function for $f(x_1, x_2) = 2x_1 + x_2$ than our problem has a solution and the solution is $(0, 5)^T$.

Geometric View of Linear Programs

Recall that a set $\Omega \in \mathbb{R}^n$ is a convex set if $\forall x, y \in \Omega$ and $\forall \in [0, 1]$ the point $\alpha x + (1 - \alpha)y \in \Omega$.

Proposition. Let A be $m \times n$ matrix, $b \in \mathbb{R}^m$. The set of points $\Omega = \{x : Ax = b, x \geq 0\}$ is a convex set.

Definition

A point x in a convex set Ω is said to be an extreme point or corner point of Ω if there are no $x_1, x_2 \in \Omega$, $x_1 \neq x_2$ and $\alpha \in (0, 1)$ such that $x = \alpha x_1 + (1 - \alpha)x_2$.

Proposition. We need to look for the solution of LPP (if exists) among extreme points of the feasible set.

Proposition. If the feasible set is bounded, then LPP has a solution and, moreover, one of the extreme points will be a solution.