ASDS Statistics, YSU, Fall 2020

Lecture 11

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Contents

Last Lecture Recap

- Give the definition and an interpretation of the Sample α-Quantile.

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- Give the definition and an interpretation of the Sample α-Quantile.
- Give the definition and an interpretation of the Theoretical α-Quantile.

Examples

Example: Find the 70% quantile of the Distribution with the PDF

$$f(x) = \begin{cases} 3x^2, & x \in [0, 1] \\ 0, & \textit{otherwise} \end{cases}$$

Solution: OTB

Theoretical Quantiles, again

Now, if q_α is the α-quantile of some Distribution, and X is a r.v. from that Distribution, then

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Note: Here we are taking inequalities, and not, say, $\mathbb{P}(X \leq q_\alpha) = \alpha$, since, in the Discrete r.v. case, we can have no q_α with exact equality. Say, if $X \sim \mathit{Bernoulli}(0.2)$, and $\alpha = 0.4$, then no q_α exists with $\mathbb{P}(X \leq q_\alpha) = \alpha$.

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Note: If $\alpha = 0.5$, we call $q_\alpha = q_{0.5}$ to be the **Median of the Distribution**.

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Note: If $\alpha = 0.5$, we call $q_\alpha = q_{0.5}$ to be the **Median of the Distribution**. So if we consider a Continuous r.v. and draw the PDF of that r.v., then the Median is the (leftmost) point dividing the area under the PDF curve into 50%-50% portions.

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Say, we will take $\alpha \in (0,1)$ and find two points $a, b \in \mathbb{R}$ such that for $X \sim \mathcal{N}(0,1)$

$$\mathbb{P}(X \leq a) = \mathbb{P}(X \geq b) = \frac{\alpha}{2}.$$

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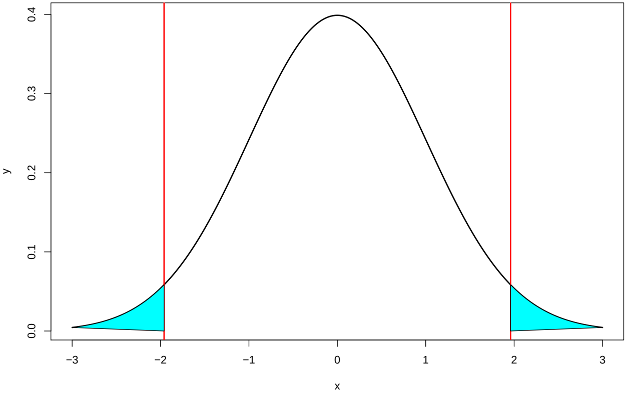
The idea is to find a symmetric (in fact, the smallest length) interval $[a, b]$ such that for a Standard Normal r.v. X, the chances of $X \notin [a, b]$ are small, are exactly α.

Graphically

```
alpha <- 0.05; z.alpha <- qnorm(alpha/2, mean = 0, sd = 1)
x <- seq(-3,3, by = 0.01)
y <- dnorm(x, mean = 0, sd = 1)
plot(x,y, type = "l", xlim = c(-3,3), lwd = 2)
abline(v = z.alpha, lwd = 2, col = "red")
abline(v = -z.alpha, lwd = 2, col = "red")
polygon(c(x[x<=z.alpha], z.alpha),c(y[x<=z.alpha],0),col="cyan")
polygon(c(x[x>=-z.alpha], -z.alpha),c(y[x>=-z.alpha],0),col="cyan")
```



Theoretical Quantiles, again

Then, it is easy to see, if $\alpha \in (0, 0.5)$ because of the symmetry, that $b = -a$, and

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Note: Please be careful when using Normal Tables. Usually, there is a picture above the table, on which you can find the explanation of the process. Just search "Normal tables" in Google Images.

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- two given Datasets (possibly, of different sizes) are from the same Distribution;
- a given Dataset comes from a given Distribution;
- given two theoretical Distributions, check if one of them is a shifted-scaled version of the other one, or check if one has *fatter tails* than the other one

Q-Q Plots, Data vs Data

Now, assume we have two Datasets, not necessarily of the same size:

$$x:\ x_1, x_2, ..., x_n \qquad \text{and} \qquad y:\ y_1, y_2, ..., y_m$$

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Q-Q Plot helps to answer to this question visually. To draw the Q-Q Plot for Datasets, we take some levels of quantiles, say, for some n,

$$\alpha = \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}$$

and then draw the points (q_α^x, q_α^y).

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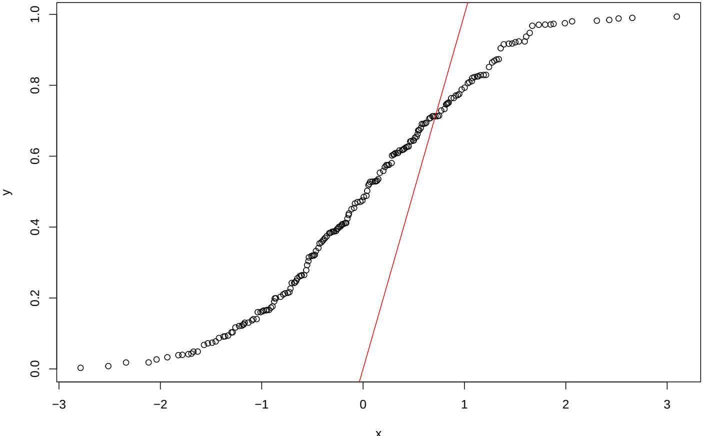
and then draw the points (q_α^x, q_α^y).

Idea: If x and y are coming from the same Distribution, then the Quantiles of x and y need to be approximately the same, $q_\alpha^x \approx q_\alpha^y$, so geometrically, the points (q_α^x, q_α^y) need to be close to the bisector line.

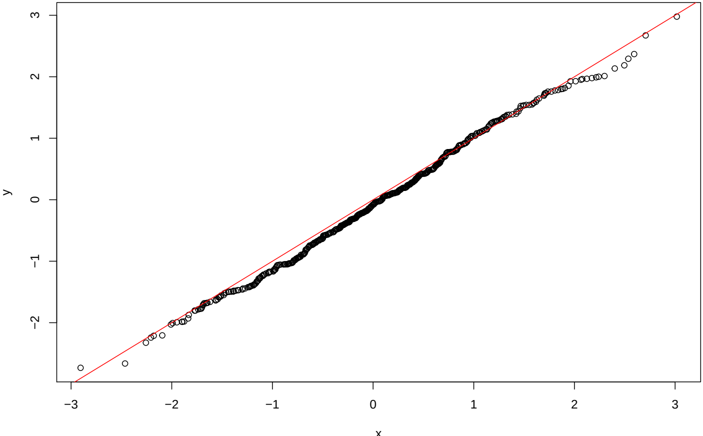
Example, Q-Q Plots, Data vs Data

```
x <- rnorm(1000)
y <- runif(200)
qqplot(x,y)
abline(0,1, col="red")
```



Example, Q-Q Plots, Data vs Data

```
x <- rnorm(1000)
y <- rnorm(500)
qqplot(x,y)
abline(0,1, col="red")
```



Example, Q-Q Plot by Hands, Data vs Data

Example: Assume

$$x: \ -1, 2, 1, 2, 3, 2, 1 \qquad y: \ 0, 3, 4, 1, 1, 1, 1, 2$$

Draw the Q-Q Plot for x and y.

Q-Q Plots, Data vs Theoretical Distribution

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Example: Say, is the following Dataset

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##  [1] -0.063  0.224  0.441  0.300  0.357  0.301  0.182 -0
## [11]  0.974 -0.322  0.288 -0.946  0.425  0.854 -0.820 -0
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from a Normal Distribution?

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from a Normal Distribution?

To answer this question, we again take some levels of quantiles, say, for some n,

$$\alpha = \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}$$

and then draw the points (q_α^F, q_α^x), where q_α^F is the α-quantile of the Theoretical Distribution, and q_α^x is the α-quantile of x.

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Idea: If x is from the Distribution given by F, then we need to have $q_\alpha^F \approx q_\alpha^x$, so, graphically, the point will be close to the bisector.

Normal Q-Q Plot

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Another **R** command is `qqline` which adds a line passing (by default) through the first and third Quartiles,

$$(q^F_{0.25}, q^x_{0.25}) \qquad and \qquad (q^F_{0.75}, q^x_{0.75}).$$

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Some Experiments

Here are some experiments with `qqnorm`

```
x <- runif(100,-3,3)
qqnorm(x)
qqline(x)
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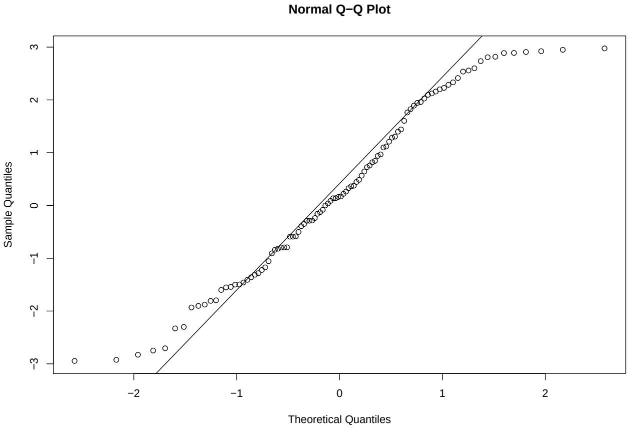

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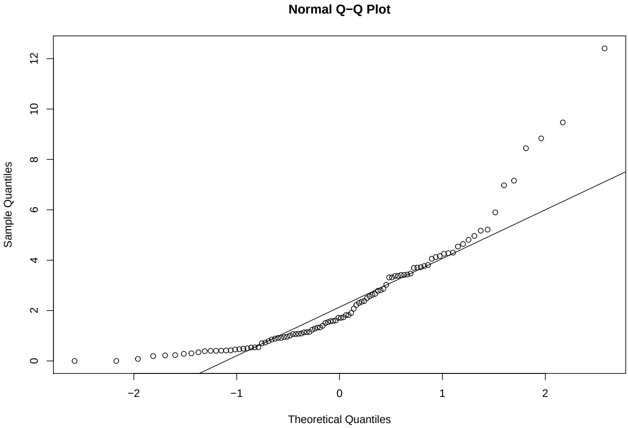
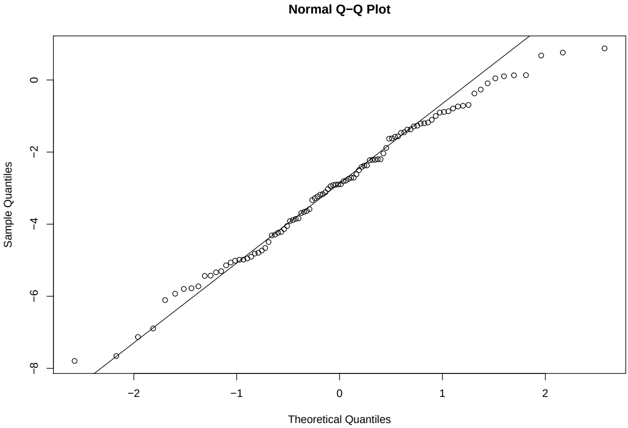
```
x <- rexp(100,0.4)
qqnorm(x)
qqline(x)
```



Some Experiments

Here are some experiments with `qqnorm`

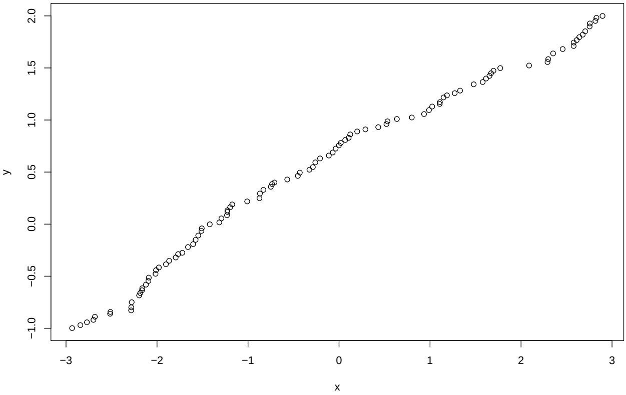
```
x <- rnorm(100, mean = -3, sd = 2)
qqnorm(x)
qqline(x)
```



Some Experiments

Now, assume we want to see if our Dataset x is from $Unif[-1, 2]$:

```
x <- runif(100,-3,3)
y <- runif(1000,-1,2)
qqplot(x,y)
```



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It is important, that, using `qqnorm`, we can check if our Dataset comes from a Normal Distribution, *with some mean and variance*.

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But, for the Normal Distribution, we can use the fact that all Normal Distributions can be obtained from the Standard Normal, by scaling and shifting.

[2] Can you state rigorously and prove this?

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But, for the Normal Distribution, we can use the fact that all Normal Distributions can be obtained from the Standard Normal, by scaling and shifting. This means that the Quantiles of any Normal Distribution can be obtained by a linear transform from the Standard Normal Quantiles[2].

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So if, say, x is a sample from $\mathcal{N}(2, 3^2)$, then

- when doing a Q-Q Plot of x vs $\mathcal{N}(2, 3^2)$, the Quantiles will be

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- when doing a Q-Q Plot of x vs $\mathcal{N}(2, 3^2)$, the Quantiles will be on the bisector;
- when doing a Q-Q Plot of x vs $\mathcal{N}(0, 1)$, the Quantiles will be on some line (can you find the line equation?);

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So if `qqnorm` shows that the quantiles are close to a line, that means that the Dataset is possibly from a Normal Distribution.

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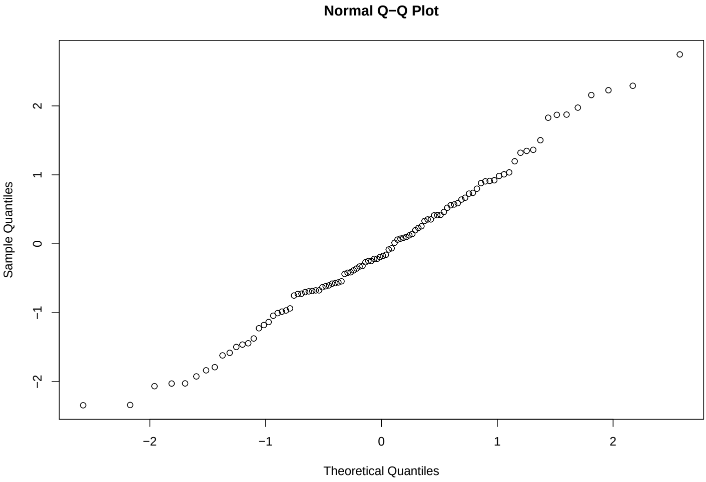
And if `qqnorm` shows that the quantiles are close to the bisector, that means that the Dataset is possibly from the Standard Normal Distribution.

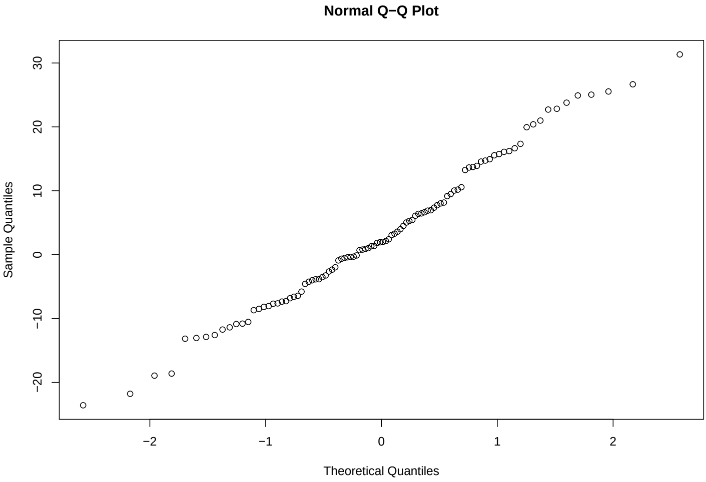
Some Experiments

```
x <- rnorm(100, mean=0, sd=1)
qqnorm(x)
```



Some Experiments

```
x <- rnorm(100, mean=2, sd=12)
qqnorm(x)
```



Important Note, v2

The above important note works also for the Uniform Distribution. This is again because all Uniform Distributions are the scaled-translated versions of the Standard Uniform $Unif\,[0, 1]$.

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So if you will compare your Dataset with $Unif[0,1]$, and Q-Q Plot will show that the Quantiles are close to a line, that means that probably your Dataset is from a Uniform Distribution, with some parameters.