ASDS Statistics, YSU, Fall 2020

Lecture 30

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Contents

t-Test Example

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In mathematical terms,

$$\mathcal{H}_0 : \mu = 132.4 \qquad \textit{vs} \qquad \mathcal{H}_1 : \mu \neq 132.4$$

t-Test Example

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$$\mathcal{H}_0 : \mu = 132.4 \qquad \textit{vs} \qquad \mathcal{H}_1 : \mu \neq 132.4$$

Now let's test, at 95%, this Hypo in **R**:

t-Test Example

```r
library(Rlab)
data <- etruscan
x <- data$width[data$group == "ancient"]

t.test(x, mu = 132.4)
```

```
##
##  One Sample t-test
##
## data:  x
## t = 17.46, df = 83, p-value < 2.2e-16
## alternative hypothesis: true mean is not equal to 132.4
## 95 percent confidence interval:
##  142.4781 145.0695
## sample estimates:
## mean of x
##  143.7738
```

Test for the Normal Variance, μ is known

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Large Sample Hypothesis Testing

aka

Asymptotic Testing

Asymptotic Test for the Mean of General Distribution

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$$\mathbb{P}(\text{Type I Error}) = \mathbb{P}(\text{Reject } \mathcal{H}_0 \mid \mathcal{H}_0 \text{ is True}) \to \alpha, \quad \textit{as} \quad n \to +\infty$$

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Test Statistics: $W = \dfrac{\hat{\theta}^{MLE} - \theta_0}{1/\sqrt{n \cdot \mathcal{I}(\theta_0)}}$ or $W = \dfrac{\hat{\theta}^{MLE} - \theta_0}{1/\sqrt{n \cdot \mathcal{I}\left(\hat{\theta}^{MLE}\right)}}$

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Note: When doing Tests, say, with `t.test`, **R** is calculating the p-Value, and sometimes also the CI. So, to decide whether to Reject Null or Not, using **R**, you can use the 2nd and 3rd Methods.

p-Values

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based on the Test Statistics TS. Assume we already have Observations, and we calculate the value of TS, let us denote that by TS_{obs} (this is just a number). We know that, for a given Significance Level α, we will Reject $\mathcal{H}_0$, iff TS_{obs} will be in the RR.

Now, assume the Distribution of TS, our Test Statistics, **under** $\mathcal{H}_0$, is given like this (I am drawing for Z- or t-Statistics, for Two Tailed Test, the other cases can be considered in a similar way):

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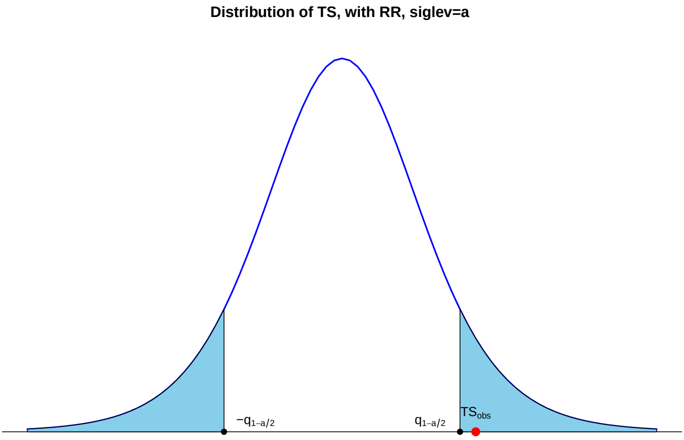

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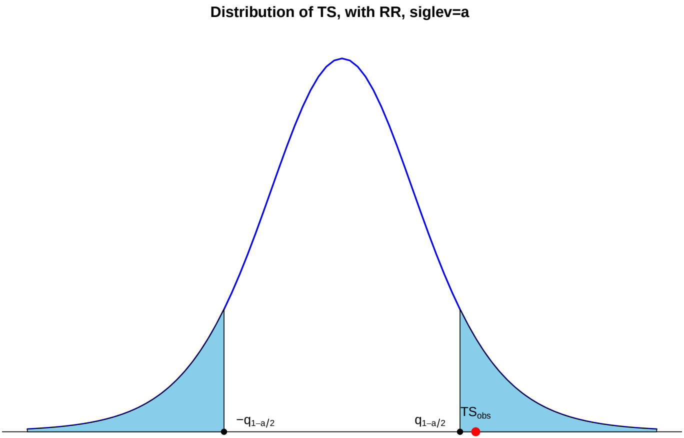

We Reject $\mathcal{H}_0$ at the level a

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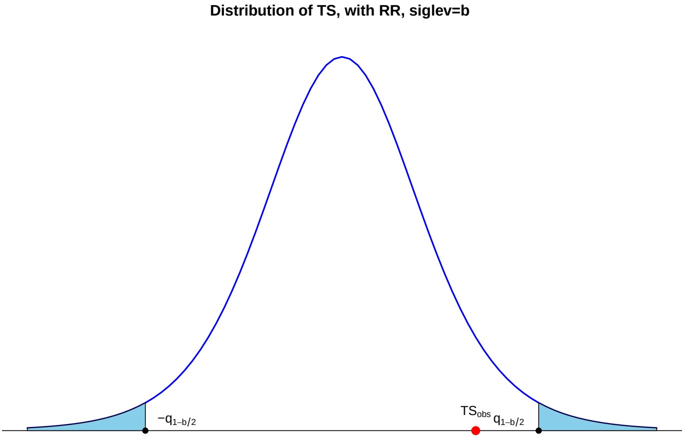

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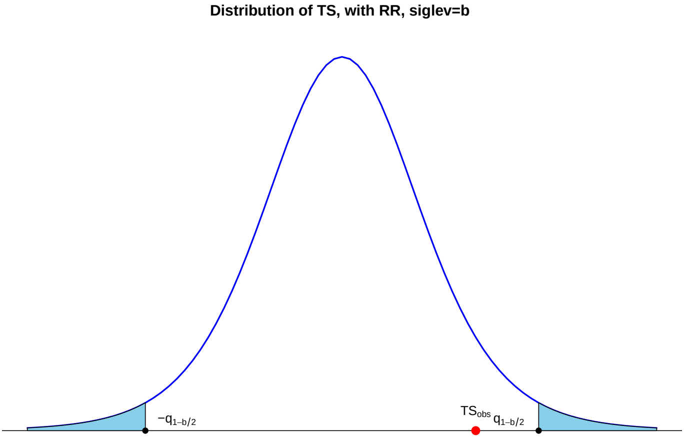

We Do Not Reject $\mathcal{H}_0$ at the level b

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Note: Give here the real line with picture, MP!

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- We Fail to Reject $\mathcal{H}_0$ for any $\alpha < \alpha^*$

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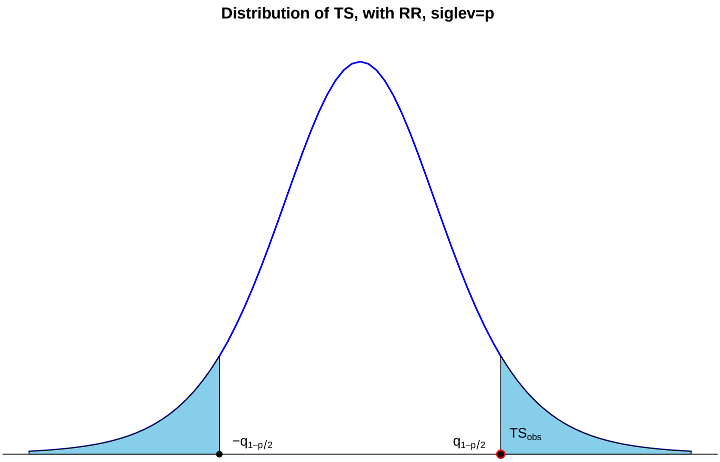
Now, we denote $p = \alpha^*$ and call it the *p*-**Value of the Test:**

$$p\text{-Value} = p = \inf\{\alpha : \text{ we Reject } \mathcal{H}_0 \text{ at level } \alpha\}.$$

p-Values



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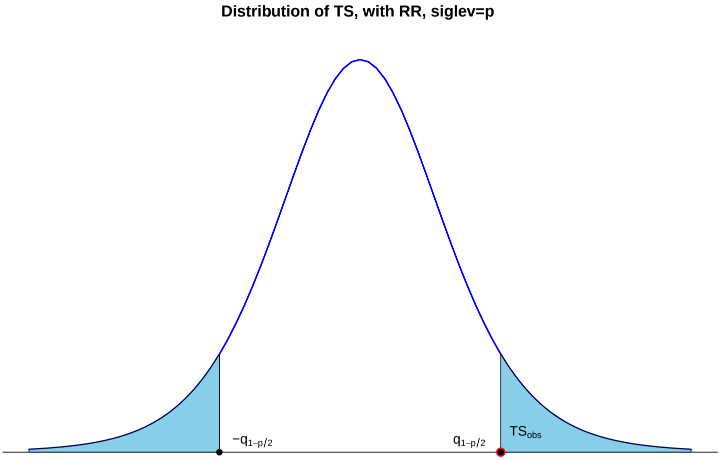

p-Value, the inf value of α at which we Reject $\mathcal{H}_0$

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It is clear from the Figure above that

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To Remember:

- If p-Value$< \alpha$, then we Reject $\mathcal{H}_0$
- If p-Value$\geq \alpha$, then we Fail to Reject $\mathcal{H}_0$

R Code for the Graphics

```
df <- 8;
x <- seq(-4,4,0.1); y <- dt(x, df = df)
plot.new()
plot.window(xlim = c(-4, 4), ylim = c(-0.05,0.4))
plot(x,y, type="l",col="blue",lwd=2,xaxt="n",yaxt="n",
     bty="n",xlab="",ylab="")
abline(h=0)
title("Distribution of TS, with RR, siglev=a ")
qpoint <- 1.5; tspoint <- 1.7
cord.x <- c(qpoint,seq(qpoint,4,0.01),4)
cord.y <- c(0,dt(seq(qpoint,4,0.01), df=df),0)
polygon(cord.x,cord.y,col='skyblue')
points(c(qpoint), c(0), pch=20, cex=1.4)
text(c(qpoint-0.38),c(0.01),labels=expression("q"[1-a/2]))
cord.x1 <- c(-4,seq(-4,-qpoint,0.01),-qpoint)
cord.y1 <- c(0,dt(seq(-4,-qpoint,0.01), df=df),0)
polygon(cord.x1,cord.y1,col='skyblue')
points(c(-qpoint), c(0), pch=20, cex=1.4)
text(c(-qpoint+0.4),c(0.01),labels=expression("-q"[1-a/2]))
points(c(tspoint), c(0), col="red", pch=19, cex=1.4)
text(c(tspoint), c(0.02), labels = expression("TS"[obs]))
```

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$$\mathcal{H}_0: \ \mu = 1.2 \qquad \textit{vs} \qquad \mathcal{H}_1: \ \mu \neq 1.2.$$

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and the value of p-Value is ⌣̈

```
1-(pnorm(1.72,mean=0,sd=1)-pnorm(-1.72,mean=0,sd=1))
```

```
## [1] 0.08543244
```

Goodness-of-Fit Tests

Intro to GoF Tests

Here, we have a DataSet $x_1, x_2, ..., x_n$, and a Statistical Model, and we want to see how good our Model is fitting the Data.

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First we consider Pearson's χ^2-Test: a famous GoF Test for the Multinomial Distribution.

Goodness-of-Fit Tests: Pearson's χ^2 Test

Model: Here we assume that the result of an Experiment can be one of the $A_1, ..., A_m$ (different classes), with Probabilities

$$p_1 = \mathbb{P}(A_1), ..., p_m = \mathbb{P}(A_m).$$

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Significance Level: $\alpha \in (0, 1)$;

Goodness-of-Fit Tests: Pearson's χ^2 Test

Test Statistics:

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Here usually one constructs the following χ^2-Table:

	A_1	A_2	...	A_m
Observed Freq., O_k	X_1	X_2	...	X_m
Expected Freq., E_k	$n \cdot p_1$	$n \cdot p_2$	...	$n \cdot p_m$

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Rejection Region: $\chi^2 > \chi^2_{m-1,1-\alpha}$

Example

Example: I am claiming that, for my Stat courses, the percentage of A-grade students is 15%, of B-grade students is 25%, of C-grades are 20%, for D I have 15%, and all others are Failing the course.

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$$\#A = 27,\ \#B = 22,\ \#C = 10,\ \#D = 10,\ \#F = 12.$$

Is this Data supporting my claim?

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Solution: We have $n = 27 + 22 + 10 + 10 + 12 = 81$. Next, we make the Table:

	A	B	C	D	E
Obs. Frq., O_k	27	22	10	10	12
Exp. Frq., E_k	$81 \cdot 0.15$	$81 \cdot 0.25$	$81 \cdot 0.2$	$81 \cdot 0.15$	$81 \cdot 0.25$

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Exp. Frq., E_k	$81 \cdot 0.15$	$81 \cdot 0.25$	$81 \cdot 0.2$	$81 \cdot 0.15$	$81 \cdot 0.25$

Now, we can calculate the TS:

$$\chi^2 = \sum_{k=1}^{5} \frac{(O_k - E_k)^2}{E_k} = \frac{(27 - 81 \cdot 0.15)^2}{81 \cdot 0.15} + \ldots + \frac{(12 - 81 \cdot 0.15)^2}{81 \cdot 0.15}$$

Example, Cont'd

The rest is in **R**:

```
obsd <-  c(27, 22, 10, 10, 12)
expd <- 81* c(0.15, 0.25, 0.2, 0.15, 0.25)
xi2 <- sum((obsd-expd)^2/expd)
xi2
```

```
## [1] 24.41564
```

```
q <- qchisq(1-0.05, df = length(obsd)-1)
q
```

```
## [1] 9.487729
```

```
xi2 > q
```

```
## [1] TRUE
```

Example, Cont'd

```r
obsd <-  c(27, 22, 10, 10, 12)
p <-  c(0.15, 0.25, 0.2, 0.15, 0.25)
chisq.test(obsd, p = p)
```

```
##
##  Chi-squared test for given probabilities
##
## data:  obsd
## X-squared = 24.416, df = 4, p-value = 6.592e-05
```

Kolmogorov-Smirnov Test

```
x <- rnorm(50, mean = 3, sd = 1)
ks.test(x, y = "pnorm", mean = 0, sd = 1)
```

```
##
##  One-sample Kolmogorov-Smirnov test
##
## data:  x
## D = 0.87562, p-value = 8.882e-16
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## data:  x
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## alternative hypothesis: two-sided
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```r
x <- rexp(50, rate = 3.1)
ks.test(x, y = "pnorm", mean = 0, sd = 1)
```

```
##
##  One-sample Kolmogorov-Smirnov test
##
## data:  x
## D = 0.50228, p-value = 3.382e-12
## alternative hypothesis: two-sided
```

Example

```r
x <- runif(40)
y <- rexp(30)
ks.test(x,y)
```

```
##
##  Two-sample Kolmogorov-Smirnov test
##
## data:  x and y
## D = 0.31667, p-value = 0.05272
## alternative hypothesis: two-sided
```

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- **F-Test of equality of variances:** Given 2 Samples `x, y` from Normal Distributions, it tests if the variances are equal.

  usage: `var.test(x,y)`

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- **Runs Test:** given a vector `x` performs a test if that vector is random

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- **One Sample Proportions Test:** Given a Bernoulli Experiment, it tests if the probability of success is the given value

  usage: `prop.test(no.of.successes, no.of.trials, p)` or `binom.test(no.of.successes, no.of.trials, p)`

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- **Location Tests:** See, for example, Here