Matrix Representation of Linear Mappings

Consider a vector space V with an (unordered) basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$. Let $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ be an ordered basis, and $\mathbf{B} = [\mathbf{b_1}, \ldots, \mathbf{b_n}]$ be a matrix whose columns are the vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n$.

Definition

For any $\mathbf{x} \in V$ there is a unique representation

$$\mathbf{x} = \alpha_1 \mathbf{b}_1 + \ldots + \alpha_n \mathbf{b}_n$$

coordinates of $\mathbf{x}$ with respect to B. Then $\alpha_1, \ldots, \alpha_n$ are the coordinates of $\mathbf{x}$ with respect to B, and the vector

$$[\mathbf{x}]_B = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathbb{R}^n$$

is the **coordinate vector** *of $\mathbf{x}$ with respect to the ordered basis B.*

Remark

For an n-dimensional vector space V and an ordered basis B of V, the linear mapping $T : \mathbb{R}^n \to V$, with $T(e_i) = b_i$ is an isomorphism, where $(e_1, \ldots, e_n)$ is the standard basis of $\mathbb{R}^n$.

Definition

Let V, W be vector spaces with corresponding bases $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ and $C = (\mathbf{c}_1, \ldots, \mathbf{c}_m)$. Moreover, let $T : V \to W$ be a linear mapping with

$$T(\mathbf{b}_j) = \alpha_{1j}\mathbf{c}_1 + \ldots + \alpha_{mj}\mathbf{c}_m = \sum_{i=1}^{m} \alpha_{ij}\mathbf{c}_i.$$

Then, we call the $m \times n$-matrix A_T whose elements are given by

$$A_T(i, j) = \alpha_{ij}$$

the **transformation matrix** *of T (with respect to the ordered bases B of V and C of W).*

Note that

$$A_T = [[T(\mathbf{b}_1)]_C \dots [T(\mathbf{b}_n)]_C]$$

and

$$[T(\mathbf{x})]_C = A_T[\mathbf{x}]_B$$

Example

Write the transformation matrix of the linear mapping (homomorphism) $T : V \to W$ w.r.t. ordered bases $B = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ of V and $C = (\mathbf{c}_1, \dots, \mathbf{c}_5)$ of W satisfying

$$T(\mathbf{b}_1) = \mathbf{c}_1 - \mathbf{c}_3 + 3\mathbf{c}_5$$
$$T(\mathbf{b}_2) = \mathbf{c}_2 + 2\mathbf{c}_4 + 5\mathbf{c}_5$$
$$T(\mathbf{b}_3) = \mathbf{c}_1 - 3\mathbf{c}_2 + \mathbf{c}_3$$

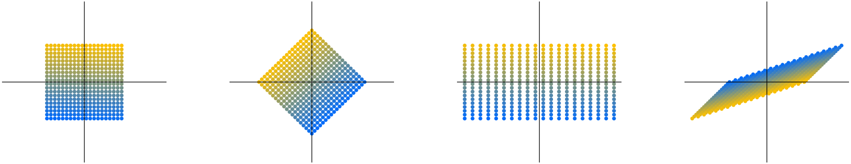
Using the transformation matrix A_T find the coordinate vector $[T(\mathbf{x})]_C$, given that $[\mathbf{x}]_B = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

Example

Let's consider three linear transformations of a set of vectors in $\mathbb{R}^2$ *with the transformation matrices*

$$A_1 = \begin{bmatrix} \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4}) \\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_3 = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ 1 & -1 \end{bmatrix}$$

How are the vectors in figure a) transformed when applying each of the matrices $A_i,\ i = 1, 2, 3$.



(a) Original data. (b) Rotation by 45°. (c) Stretch along the horizontal axis. (d) General linear mapping.

Let $T : V \to W$ be a transformation,

$$B = (\mathbf{b}_1, \ldots, \mathbf{b}_n),\ \tilde{B} = (\tilde{\mathbf{b}}_1, \ldots, \tilde{\mathbf{b}}_n) \text{ are bases of } V$$

and

$$C = (\mathbf{c}_1, \ldots, \mathbf{c}_m),\ \tilde{C} = (\tilde{\mathbf{c}}_1, \ldots, \tilde{\mathbf{c}}_m) \text{ are bases of } W.$$

Moreover, $A_T \in \mathbb{R}^{m \times n}$ is the transformation matrix of the linear mapping $T : V \to W$ with respect to the bases B and C, and $\tilde{A}_T \in \mathbb{R}^{m \times n}$ is the corresponding transformation mapping with respect to $\tilde{B}$ and $\tilde{C}$.

Question

How can we transform A_T into $\tilde{A}_T$?

Example

Consider a transformation matrix

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

with respect to the canonical basis in $\mathbb{R}^2$. *If we define a new basis*

$$B = (\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix})$$

we obtain a diagonal transformation matrix

$$\tilde{A} = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$$

with respect to B, *which is easier to work with than* A.

Theorem

For a linear mapping $T : V \to W$, ordered bases

$$B = (\mathbf{b}_1, \dots, \mathbf{b}_n), \quad \tilde{B} = (\tilde{\mathbf{b}}_1, \dots, \tilde{\mathbf{b}}_n)$$

of V and

$$C = (\mathbf{c}_1, \dots, \mathbf{c}_m), \quad \tilde{C} = (\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_m)$$

of W, and a transformation matrix A_T of T with respect to B and C, the corresponding transformation matrix $\tilde{A}_T$ with respect to the bases $\tilde{B}$ and $\tilde{C}$ is given as

$$\tilde{A}_T = L^{-1} A_T S$$

Here, $S \in \mathbb{R}^{n \times n}$ $(L \in \mathbb{R}^{m \times m})$ is the transformation matrix of id_V (id_W) that maps coordinates with respect to $\tilde{B}(\tilde{C})$ onto coordinates with respect to $B(C)$.

Definition

Matrices $A, \tilde{A} \in \mathbb{R}^{m\times n}$ *are* **equivalent** *if there exist regular matrices* $P \in \mathbb{R}^{n\times n}$ *and* $Q \in \mathbb{R}^{m\times m}$ *such that*

$$\tilde{A} = Q^{-1}AP$$

Definition

Matrices $A, \tilde{A} \in \mathbb{R}^{n\times n}$ *are* **similar** *if there exists a regular matrix* $Q \in \mathbb{R}^{n\times n}$ *such that*

$$\tilde{A} = Q^{-1}AQ.$$

Remark

Similar matrices are always equivalent. However, equivalent matrices are not necessarily similar.

Remark

Clearly for linear mappings $T : U \to V$ and $S : V \to W$ the mapping $S \circ T : U \to W$ is also linear. Moreover transformation matrix $A_{S \circ T}$ is given by $A_{S \circ T} = A_S A_T$.

- *A_T is the transformation matrix of a linear mapping $T_{CB} : U \to V$ with respect to the bases B, C.*
- *$\tilde{A}_T$ is the transformation matrix of the linear mapping $T_{\tilde{C}\tilde{B}} : U \to V$ with respect to the bases $\tilde{B}, \tilde{C}$.*
- *P is the transformation matrix of identity mapping $S_{B\tilde{B}} : U \to U$ (automorphism) that represents $\tilde{B}$ in terms of B.*
- *Q is the transformation matrix of identity mapping $S_{C\tilde{C}} : V \to V$ that represents $\tilde{C}$ in terms of C. Then*

$$\tilde{A}_T = Q^{-1} A_T P$$

is equivalent to $T_{\tilde{C}\tilde{B}} = S^{-1}_{C\tilde{C}} \circ T_{CB} \circ S_{B\tilde{B}}$.

Example

Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ *be a linear mapping with transformation matrix*

$$A_T = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & -1 & 2 & 2 \\ 0 & 3 & 2 & 1 \end{bmatrix}$$

with respect to standard bases B, C *of* $\mathbb{R}^4, \mathbb{R}^3$ *correspondingly, i.e.*

$$B = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right), \quad C = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

Find the transformation matrix $\tilde{A}_T$ *w.r.t. the bases*

$$\tilde{B} = \left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right), \quad \tilde{C} = \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$$