Basic Mathematics, Fall 2020

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Example

Consider $\mathbb{R}^3$ *with the inner product*

$$\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{x}^T \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix} \mathbf{y}$$

Furthermore, we define $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ *as the standard/canonical basis in* $\mathbb{R}^3$.
1. Determine the orthogonal projection $\pi_U(\mathbf{e}_1)$ *of* $\mathbf{e}_1$ *onto* $U = \mathrm{span}[\mathbf{e}_2, \mathbf{e}_3]$ *w.r.t. the the inner product defined above.*
2. Compute the distance $d(\mathbf{e}_1, U)$.

Remark. Recall that $\boldsymbol{\lambda} = (B^T B)^{-1} B^T \mathbf{x}$ in the case of dot product. Note that (i, j) entry of $Q := B^T B$ is

$$q_{ij} = \mathbf{b}_i^T \mathbf{b}_j = \langle \mathbf{b}_i, \mathbf{b}_j \rangle.$$

Similarly the ith entry of the vector $\mathbf{v} = B^T \mathbf{x}$ is

$$v_i = \mathbf{b}_i^T \mathbf{x} = \langle \mathbf{b}_i, \mathbf{x} \rangle.$$

In the case of general inner product $\langle \cdot, \cdot \rangle$, repeating the steps by which we obtained the formula for $\boldsymbol{\lambda}$ in the case of inner product, we get

$$\boldsymbol{\lambda} = Q^{-1}\mathbf{v}, \text{ where } Q = (\langle \mathbf{b}_i, \mathbf{b}_j \rangle)_{ij=1}^{m}, \ \mathbf{v} = [\langle \mathbf{b}_1, \mathbf{x} \rangle \ \langle \mathbf{b}_2, \mathbf{x} \rangle \ldots \langle \mathbf{b}_m, \mathbf{x} \rangle]^T$$

or equivalently

$$\boldsymbol{\lambda} = \begin{bmatrix} \langle \mathbf{b}_1, \mathbf{b}_1 \rangle & \langle \mathbf{b}_1, \mathbf{b}_2 \rangle & \ldots & \langle \mathbf{b}_1, \mathbf{b}_m \rangle \\ \langle \mathbf{b}_2, \mathbf{b}_1 \rangle & \langle \mathbf{b}_2, \mathbf{b}_2 \rangle & \ldots & \langle \mathbf{b}_2, \mathbf{b}_m \rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle \mathbf{b}_m, \mathbf{b}_1 \rangle & \langle \mathbf{b}_m, \mathbf{b}_2 \rangle & \ldots & \langle \mathbf{b}_m, \mathbf{b}_m \rangle \end{bmatrix}^{-1} \begin{bmatrix} \langle \mathbf{b}_1, \mathbf{x} \rangle \\ \langle \mathbf{b}_2, \mathbf{x} \rangle \\ \vdots \\ \langle \mathbf{b}_m, \mathbf{x} \rangle \end{bmatrix}.$$

In the general case the projection is given by the same formula $\pi_U(\mathbf{x}) = \sum_{i=1}^{m} \lambda_i \mathbf{b}_i = B\boldsymbol{\lambda}$, where $B = [\mathbf{b}_1, \ldots, \mathbf{b}_m] \in \mathbb{R}^{n \times m}$.

Determinant and Trace

Determinants as Measures of Volume

It turns out that the determinant $\det A$ is the signed volume of an n-dimensional parallelepiped formed by columns of a matrix A.

Example

The volume of the rectangle formed by the vectors $[x,\ 0]^T$ *and* $[0,\ y]^T$ *is equal to*

$$\left|\det \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right| = |xy|$$

Example

The volume V *of the parallelogram given by the vectors* $\mathbf{a}_1 = [1\ 2\ 3]^T$, $\mathbf{a}_2 = [0\ 2\ 2]^T$ *and* $\mathbf{a}_3 = [-1\ 0\ 1]^T$ *is equal to*

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$$V = \left|\det\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 0 \\ 3 & 2 & 1 \end{bmatrix}\right| = 4.$$

Theorem (Laplace expansion)

Consider a matrix $A \in \mathbb{R}^{n \times n}$. Then, for all $j = 1, ..., n$:

1. Expansion along column j

$$\det(A) = \sum_{k=1}^{n} (-1)^{k+j} a_{kj} \det(A_{k,j}).$$

2. Expansion along row j

$$\det(A) = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} \det(A_{j,k}).$$

Here $A_{k,j} \in \mathbb{R}^{(n-1)\times(n-1)}$ is the submatrix of A that we obtain when delete row k and column j.
Recall that $\det(A_{k,j})$ is called a minor and $(-1)^{k+j} \det(A_{k,j})$ a cofactor.

Theorem

A square matrix A is invertible if and only if $\det(A) \neq 0$.

Example

Compute the determinant of

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 0 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

using the Laplace expansion. Check whether it is invertible.

Properties of the determinant

- $\det(AB) = \det A \cdot \det B$
- $\det(A^T) = \det A$
- If A is regular, then $\det A^{-1} = \dfrac{1}{\det A}$
- If $A \sim B$ (i.e. $\exists\ Q$ s.t. $A = QBQ^{-1}$), then $\det A = \det B$
- All transformation matrices A_T of a linear mapping $T : V \to V$ have the same determinant.

Theorem

Let $A = [\mathbf{a}_1\mathbf{a}_2 \dots \mathbf{a}_n]$ *be a square matrix.*

a. *if* $\mathbf{a}_i = 0$ *then* $\det A = 0$.

b. $\det[\mathbf{a}_1 \dots \mathbf{a}_i \dots \mathbf{a}_j \dots \mathbf{a}_n] = -\det[\mathbf{a}_1 \dots \mathbf{a}_j \dots \mathbf{a}_i \dots \mathbf{a}_n]$.

c. *If* $\mathbf{a}_i = \mathbf{a}_j$, *then* $\det A = 0$.

d. $\det[\mathbf{a}_1 \dots k\mathbf{a}_i \dots \mathbf{a}_n] = k \det A$.

e.
$\det[\mathbf{a}_1 \dots \mathbf{a}_i + \mathbf{b}_i \dots \mathbf{a}_n] = \det[\mathbf{a}_1 \dots \mathbf{a}_i \dots \mathbf{a}_n] + \det[\mathbf{a}_1 \dots \mathbf{b}_i \dots \mathbf{a}_n]$.

f. $\det[\mathbf{a}_1 \dots \mathbf{a}_i + k\mathbf{a}_j \dots \mathbf{a}_n] = \det A$.