Basic Mathematics, Fall 2020

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Example

A total of n balls are sequentially and randomly chosen, without replacement, from an urn containing r red and b blue balls $(n \leq r + b)$. Given that k of the n balls are blue, what is the conditional probability that the first ball chosen is blue?

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An ordinary deck of 52 playing cards is randomly divided into 4 piles of 13 cards each. Compute the probability that each pile has exactly 1 ace.

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An ordinary deck of 52 playing cards is randomly divided into 4 piles of 13 cards each. Compute the probability that each pile has exactly 1 ace.
Hint: Denote

$$\text{null} \begin{aligned} E_1 &= \{\textit{the ace of spades is in any one of the piles}\} \\ E_2 &= \{\textit{ the ace of spades and the ace of hearts} \\ &\quad \textit{are in different piles}\} \\ E_3 &= \{\textit{the aces of spades, hearts, and diamonds} \\ &\quad \textit{are all in different piles}\} \\ E_4 &= \{\textit{all 4 aces are in different piles}\} \end{aligned}$$

Natural (contextual) Definition of Independence

Events A and B of the same experiment are called independent if the occurrence of one of them is absolutely uninfluenced by the occurrence of second event. The events will be called dependent if they are not independent.

Formal (mathematical) Definition of Independence

Events A and B of the same experiment are called independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B),$$

The events will be called dependent if they are not independent.

Properties of independent events

Let A, B, C be some events of an experiment.

1. If $\mathbb{P}(A) > 0$ then A and B are independent iff $\mathbb{P}(B|A) = \mathbb{P}(B)$.
2. If A and B are independent events, where $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$, then A and B be cannot be mutually exclusive.
3. If A and B are independent events, then so are $\overline{A}$ and B, also A and $\overline{B}$, and also $\overline{A}$ and $\overline{B}$.
4. Let B and C are mutually exclusive events. If A and B are independent, and A and C are also independent, then A and $B \cup C$ are independent, too.

Example

Pick at random a card from a deck of 52 cards. Then we return that card into the deck, and again pick at random a card. What is the probability that the first card will be an ace and the second card will be a number < 7*?*

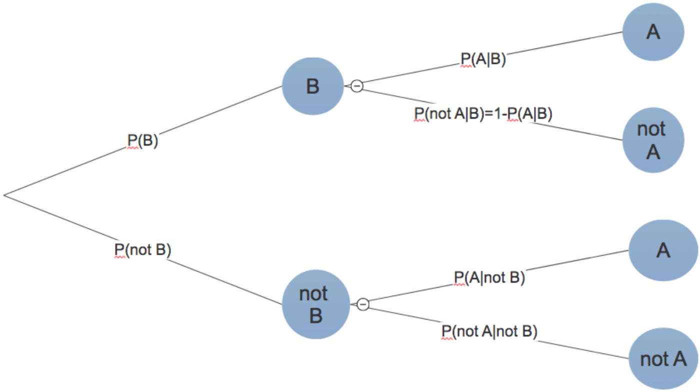
Example

Experiment - 2 hunters shoot at a duck simultaneously. The hunters are equally good and hit the target with probability 0.7.
Event A - First hunter hits the target
Event B - Second hunter hits the target
What is the probability that the duck will survive (both hunters failed)?

Total Probability Formula, TPF

Assume A is some event, and B is another event of the same experiment such that $\mathbb{P}(B) > 0$. Then the following formula is valid:

$$\mathbb{P}(A) = \mathbb{P}(B)\mathbb{P}(A|B) + \mathbb{P}(\overline{B})\mathbb{P}(A|\overline{B}).$$



Example

6 green balls and 4 red balls (identical apart from color) are placed in a box. A child is asked to select a ball at random. Then the second child is asked to draw another ball from the box. What is the probability that the second child chooses a red ball?

General Total Probability Formula

Let $B_1, B_2, \dots, B_n$ be some events of the same experiment such that they are pairwise disjoint, they contain any outcome that belongs to event A, i.e. $A \subset \cup_{k=1}^{n} B_k$. Then

$$\mathbb{P}(A) = \sum_{k=1}^{n} \mathbb{P}(B_k)\mathbb{P}(A|B_k).$$

Example

Assume we are rolling a die. If 1 is shown, we are tossing once a coin. If 2 is shown on the die, we are tossing a coin 2 times, and so on, if 6 is shown on the die top face, we are tossing a coin 6 times. What is the probability that we will have at least one head in this experiment?

Bayes' Formula (Bayes' Rule)

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(B)\mathbb{P}(A|B)}{\mathbb{P}(A)}.$$

Probability, Statistics and Machine Learning experts give the following names to the terms in the Bayes Rule:

- $\mathbb{P}(B)$ - Prior Probability;
- $\mathbb{P}(A|B)$ - Likelihood;
- $\mathbb{P}(B|A)$ - Posterior Probability;
- $\mathbb{P}(A)$ - Evidence.

$\mathbb{P}(B)$ measures the probability of B without any side information, before knowing that some other event happened. This is our Prior Probability.

The probability $\mathbb{P}(B|A)$ is the update of our previous measure under the information that A happened, we update the probability after knowing that A happened - this is our Posterior Probability.

Example

In some country, the risk to develop a lung cancer is 0.1%*. We know that* 20% *of total population are smokers, and the chance to develop a lung cancer for smokers is* 0.4%*.*
Problem 1. (by TPF): What is the probability that a randomly chosen non-smoker will develop a lung cancer?
Problem 2. (by Bayes' Formula): Assume we choose a person randomly, and he has a lung cancer. What is the probability that he is a Smoker?