Basic Mathematics, Fall 2020

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Expected Value

The expectation, or the expected value, of discrete r.v. X, denoted by $\mathbb{E}[X]$, is defined by

$$\mathbb{E}[X] = \sum_{x:p(x)>0} xp(x)$$

Proposition

If X is a discrete random variable that takes on one of the values $x_i, i \geq 1$, with respective probabilities $p(x_i)$, then,

$$\mathbb{E}[X] = \sum_n x_n p(x_n).$$

moreover, for any real-valued function g,

$$\mathbb{E}[g(X)] = \sum_n g(x_n) p(x_n).$$

Variance

If X is a RV with mean μ, then the **variance** of X, denoted by $Var(X)$, is defined by

$$Var(X) = \mathbb{E}[(X - \mu)^2].$$

It turns out that

$$Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

Example

Calculate $\mathbb{E}[X], Var(X)$ *if* X *represents the outcome when a fair die is rolled.*

Properties of Binomial Random Variables

If X is a binomial RV with parameters n and p, then

$$\mathbb{E}[X] = np$$
$$Var(X) = np(1-p)$$

Definition

A random variable X that takes on one of the values $0, 1, 2, \dots$ is said to be a Poisson random variable with parameter $\lambda > 0$ if the PMF of X is representable as

$$\mathbb{P}(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, 2, \dots.$$

The fact that X is a Poisson RV with parameter λ is denoted as $X \sim Po(\lambda)$ or $X \sim Poiss(\lambda)$.

The Poisson random variable may be used as an approximation for a binomial random variable with parameters (n, p) when n is large and p is small enough so that $n \cdot p$ is of moderate size.

$$\text{If } X \sim B(n, P), \text{ then } P(X = k) \approx \frac{\lambda^k}{k!} e^{-\lambda}, \text{ where } \lambda = np.$$

The following random variables can be considered as Poisson RVs because of their closeness to the binomial.

1. The number of typos on a page of a book.
2. The number of people in a community who survive to age 100.
3. The number of telephone calls per hour received by an office.
4. The number of car accidents per year in front of YSU.
5. The number of broken bottles in a delivery.

Example

Assume that the number of university professor in Armenia is 400 and the probability of reaching age 100 is 0.005. Then let's assume that there are only n $=$ 400 professors. Find the probability that at least 3 professors will reach age 100.

Example

Let X be a Poisson random variable with parameter λ. Prove that

$$\mathbb{E}(X) = \lambda, \quad Var(X) = \lambda.$$

Geometric Distribution

Let X be the number of independent Bernoulli trials $B(1, p), 0 < p < 1$ needed to obtain a successful outcome.We will call such X **Geometric** random variable. Its PMF is

$$P(X = k) = (1 - p)^{k-1} p, \quad k = 1, 2, \ldots .$$

The fact that X is a geometric random variable is denoted as $X \sim Geo(p)$.

Example

Let X be a geometric random variable with parameter p. Prove that

$$\mathbb{E}(X) = \frac{1}{p}.$$