Basic Mathematics , Fall 2020

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October 20, 2020

Singular Value Decomposition (SVD)

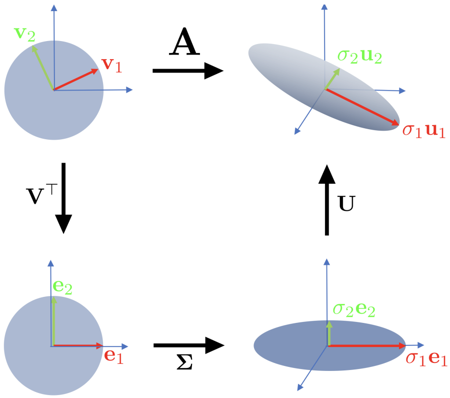
Theorem

Let $A \in \mathbb{R}^{m \times n}$ *be a rectangular matrix of rank* r, *with* $r \in [0, \min(m, n)]$. *The Singular Value Decomposition or SVD of* A *is a decomposition of* A *of the form*

$$A = U \Sigma V^T$$

where $U \in \mathbb{R}^{m \times m}$ *is an orthogonal matrix of column vectors* $\mathbf{u}_i$, *and* $V \in \mathbb{R}^{n \times n}$ *is an orthogonal matrix of column vectors* $\mathbf{v}_j$ *and* Σ *is an* $m \times n$ *matrix with* $\Sigma_{ii} = \sigma_i > 0$ *and* $\Sigma_{ij} = 0, \ i \neq j$. *The SVD is always possible for any matrix* A.

The σ_i are called the singular values, and by convention the singular values are ordered, i.e., $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r \geq 0$
$\mathbf{u}_i$ are called the left-singular vectors and $\mathbf{v}_j$ are called the right-singular vectors.


$\mathbf{v}_2$
$\mathbf{v}_1$
$\mathbf{A}$
$\sigma_2 \mathbf{u}_2$
$\sigma_1 \mathbf{u}_1$
$\mathbf{V}^\top$
$\mathbf{U}$
$\mathbf{e}_2$
$\mathbf{e}_1$
$\boldsymbol{\Sigma}$
$\sigma_2 \mathbf{e}_2$
$\sigma_1 \mathbf{e}_1$

Σ has a diagonal submatrix that contains the singular values and needs additional zero vectors that increase the dimension.

if $m > n$

$$\Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & \sigma_n \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix}$$

if $m < n$

$$\Sigma = \begin{bmatrix} \sigma_1 & \dots & 0 & 0 & \dots & 0 \\ 0 & \ddots & 0 & 0 & \dots & 0 \\ 0 & \dots & \sigma_n & 0 & \dots & 0 \end{bmatrix}$$

The singular value matrix Σ must be of the same size as A

Example

$$A=\begin{pmatrix}-1 & \frac{1}{\sqrt{2}} & 1\\ -1 & -\frac{1}{\sqrt{2}} & 1\end{pmatrix}=U\Sigma V^T$$

$$=\begin{pmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{pmatrix}\begin{pmatrix}2 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}-\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2}\\ 0 & -1 & 0\\ -\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2}\end{pmatrix}$$

Example

$$A=\begin{pmatrix}-\frac{3\sqrt{3}}{4} & -\frac{3}{4}\\ \frac{1}{2} & -\frac{\sqrt{3}}{2}\\ -\frac{9}{4} & -\frac{3\sqrt{3}}{4}\end{pmatrix}=U\Sigma V^T$$

$$=\begin{pmatrix}-\frac{1}{2} & 0 & \frac{\sqrt{3}}{2}\\ 0 & -1 & 0\\ -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}3 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}$$

Remark

The eigenvalue decomposition of a symmetric matrix

$$S = S^T = PDP^T$$

is a special case of the SVD

$$S = U\Sigma V^T,$$

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where $U = V = P$ *and* $\Sigma = D$.

Let $r = \mathrm{rk}(A)$.

Remark

- *The columns of U (m by m) are eigenvectors of AA^T,*
- *the columns of V (n by n) are eigenvectors of A^TA.*
- *The r singular values on the diagonal of Σ (m by n) are the square roots of the nonzero eigenvalues of both AA^T and A^TA.*

Remark

U and V give orthonormal bases for all four fundamental subspaces:

- *first r columns of U: column space of A*
- *last $m-r$ columns of U: nullspace of A^T*
- *first r columns of V: row space of A*
- *last $n-r$ columns of V: nullspace of A*

Remark

When A multiplies a column $\mathbf{v}_i$ of V, it produces σ_i times a column of U.

$$AV = U\Sigma \Leftrightarrow A\mathbf{v}_i = \sigma_i \mathbf{u}_i$$

Geometric interpretation

For every linear map $T : R^n \to R^m$ one can find orthonormal bases of R^n and R^m such that T maps the i-th basis vector of R^n to a non-negative multiple of the i-th basis vector of R^m, and sends the left-over basis vectors to zero. With respect to these bases, the map T is therefore represented by a diagonal matrix with non-negative real diagonal entries.

V 1. How to compute the SVD in **the case** $\mathrm{rk}(A) = m \leq n$?

Step 1: Compute the symmetrized matrix $A^T A$ (recall $A \in \mathbb{R}^{m \times n}$).
Step 2: Compute the eigenvalue decomposition of $A^T A = PDP^T$.
From here we obtain

$$V = P, \text{ and } \Sigma^T \Sigma = D,$$

The eigenvalues of $A^T A$ are the squared singular values of Σ.
Step 3. Compute U using the formula

$$\mathbf{u}_i = \frac{1}{\sigma_i} A \mathbf{v}_i, \quad i = 1, \ldots, m(m = \mathrm{rk}(A))$$

V 2. How to compute the SVD in **the case** $\mathrm{rk}(A) = n \leq m$?

Step 1: Compute the symmetrized matrix AA^T.
Step 2: Compute the eigenvalue decomposition of $AA^T = QD_1Q^T$. From here we obtain $U = Q$, and $\Sigma\Sigma^T = D_1$,
Step 3. Compute V using the formula

$$\mathbf{v}_i = \frac{1}{\sigma_i} A^T \mathbf{u}_i, \quad i = 1, \ldots, n(n = \mathrm{rk}(A))$$

Example

Find the SVD of the matrix $A = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

Solution. Since $\mathrm{rk}(A) = 2 =$number of rows, we will compute $A^T A$ (so we can use the version 1 of "How to compute SVD?").

Note that $A^T A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Hence the characteristic polynomial is $\det \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 4-\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} = -(1-\lambda)(4-\lambda)\lambda.$

So the eigenvalues of $A^T A$ are $\lambda_1 = 4, \lambda_2 = 1$ and $\lambda_3 = 0$.

Thus the singular values are $\sigma_1 = 2, \sigma_2 = 1$ and $\Sigma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

Let us find the eigenvalue decomposition of A^TA. As

$$A^TA - \lambda_1 I_3 = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence $\mathbf{x}_1 = [0\ \ -1\ 0]^T$ and $\mathbf{v}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = [0\ \ -1\ 0]^T$.

Similarly $A^TA - \lambda_2 I_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Hence $\mathbf{x}_2 = [-1\ 0\ ]^T$ and $\mathbf{v}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = [-1\ 0\ 0]^T$.

Analogously $A^TA - \lambda_3 I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

So $\mathbf{x}_3 = [0\ 0\ \ -1]^T$ and $\mathbf{v}_3 = \frac{\mathbf{x}_3}{\|\mathbf{x}_3\|} = [0\ 0\ \ -1]^T$.

By the formula $\mathbf{u}_i = \frac{1}{\sigma_i} A\mathbf{v}_i, \quad i = 1, 2$ (as $2 = \mathrm{rk}(A)$) we have

$\mathbf{u}_1 = \frac{1}{2} A\mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and $\mathbf{u}_2 = \frac{1}{1} A\mathbf{v}_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

Finally we get

$$A = U\Sigma V^T =$$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Remark

If we were asked to find the SVD of the transpose of the initial matrix, i.e. $A^T = \begin{pmatrix} 0 & 1 \\ 2 & 0 \\ 0 & 0 \end{pmatrix}$*, then we would use the 2nd version of "How to compute the SVD?", as in that case* $\mathrm{rk}(A) = 2 =$*number of columns. So we would first find the* **three** *left singular vectors (new* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$*), then using those by* $\mathbf{v}_i = \frac{1}{\sigma_i} A\mathbf{u}_i$*, we would find the* **two** *right singular vectors (new* $\mathbf{v}_1, \mathbf{v}_2$*).*

Example

Find the SVD of the matrix $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{pmatrix}$

Solution. Since $\mathrm{rk}(A) = 2 =$number of columns, we will compute AA^T (so we can use the 2nd version of "How to compute SVD?"). In fact the non-zero eigenvalues of A^TA and AA^T are the same. Hence instead of finding the eigenvalues of AA^T we can find A^TA.

Note that $A^TA = \begin{pmatrix} 17 & 8 \\ 8 & 17 \end{pmatrix}$. Hence the characteristic polynomial is $\det \begin{pmatrix} 17-\lambda & 8 \\ 8 & 17-\lambda \end{pmatrix} = (25-\lambda)(9-\lambda)$. So the eigenvalues of A^TA are $\lambda_1 = 25, \lambda_2 = 9$. Thus the eigenvalues of $AA^T \in \mathbb{R}^{3\times 3}$ are $\lambda_1 = 25, \lambda_2 = 9$ and $\lambda_3 = 0$.

So singular values are $\sigma_1 = 5, \sigma = 3$. and $\Sigma = \begin{pmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{pmatrix}$.

Let us find the eigenvalue decomposition of AA^T. As

$$AA^T - \lambda_1 I_3 = \begin{pmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{pmatrix} - 25I_3 = \begin{pmatrix} -12 & 12 & 2 \\ 12 & -12 & -2 \\ 2 & -2 & -17 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & -1 & -\frac{1}{6} \\ 0 & 0 & 0 \\ 1 & -1 & -\frac{17}{2} \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -\frac{1}{6} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence $\mathbf{x}_1 = [-1 \ \ -1 \ 0]^T$ and $\mathbf{u}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = [-\frac{1}{\sqrt{2}} \ \ -\frac{1}{\sqrt{2}} \ 0]^T$

Similarly

$$AA^T - \lambda_2 I_3 = \begin{pmatrix} 4 & 12 & 2 \\ 12 & 4 & -2 \\ 2 & -2 & -1 \end{pmatrix} \to \begin{pmatrix} 2 & -2 & -1 \\ 0 & 16 & 4 \\ 0 & 16 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence $\mathbf{x}_2 = [-\frac{1}{4} \ \frac{1}{4} \ \ -1]^T$ and $\mathbf{u}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = [-\frac{\sqrt{2}}{6} \ \frac{\sqrt{2}}{6} \ \ -\frac{2\sqrt{2}}{3}]^T$.

Analogously

$$AA^T - \lambda_3 I_3 = \begin{pmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{pmatrix} \to \begin{pmatrix} 2 & -2 & 8 \\ 0 & 25 & -50 \\ 0 & 25 & -50 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}.$$

So $\mathbf{x}_3 = [2\ -2\ -1]^T$ and $\mathbf{u}_3 = \frac{\mathbf{x}_3}{\|\mathbf{x}_3\|} = [\frac{2}{3}\ -\frac{2}{3}\ -\frac{1}{3}]^T$.

By the formula $\mathbf{v}_i = \frac{1}{\sigma_i} A^T \mathbf{u}_i, \quad i = 1, 2$ (as $2 = \mathrm{rk}(A)$) we have

$$\mathbf{v}_1 = \frac{1}{5} A^T \mathbf{u}_1 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \text{ and } \mathbf{v}_2 = \frac{1}{3} A^T \mathbf{u}_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Finally we get

$$A = U \Sigma V^T =$$

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{6} & \frac{2}{3} \\ -\frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{6} & -\frac{2}{3} \\ 0 & -\frac{2\sqrt{2}}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Remark

If we were asked to find the SVD of the transpose of the initial matrix, i.e. $A^T = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$*, then we would use the 1st version of "How to compute the SVD?", as in that case* $\mathrm{rk}(A) = 2 =$*number of rows. So we would first find the* **three** *right singular vectors (new* $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$*), then using those by* $\mathbf{u}_i = \frac{1}{\sigma_i} A\mathbf{v}_i$*, we would find the* **two** *left singular vectors (new* $\mathbf{u}_1, \mathbf{u}_2$*).*