ASDS Statistics, YSU, Fall 2020

Lecture 19

Michael Poghosyan

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Contents

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and to calculate the limit of this sequence $\overline{X}_1, \overline{X}_2, ..., \overline{X}_n, ...$, we will use our famous Limit Theorems: LLN and CLT.

Limit Theorems

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- X_n-s are independent. Say, in particular,

$$Var(X_1+X_2+...+X_n)=Var(X_1)+Var(X_2)+...+Var(X_n)=n\cdot Var(X_1).$$

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The interpretation of $\mathbb{E}(\overline{X}_n) = \mathbb{E}(X_1)$ and $Var(\overline{X}_n) = \frac{Var(X_1)}{n}$:the values of $\overline{X}_n$ are centered at $\mathbb{E}(X_1)$ and are becoming more and more concentrated around that number as n increases.

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The Weak LLN

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The Weak Law of Large Numbers, WLLN:

If $X_1, X_2, ..., X_n$ are IID, with finite $\mathbb{E}(X_1)$ and Variance $Var(X_1)$, then

$$\overline{X}_n = \frac{X_1 + X_2 + ... + X_n}{n} \xrightarrow{\mathbb{P}} \mathbb{E}(X_1), \qquad n \to +\infty,$$

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i.e., for any $\varepsilon > 0$,

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Note: This means that for any $\varepsilon > 0$, the chances that $\overline{X}_n$ is far from $\mathbb{E}(X_1)$ more than ε, is very small, if n is large.

The Strong LLN

The Strong Law of Large Numbers, SLLN, Kolmogorov

If $X_1, X_2, \ldots, X_n$ are IID, with finite $\mathbb{E}(|X_1|)$, then

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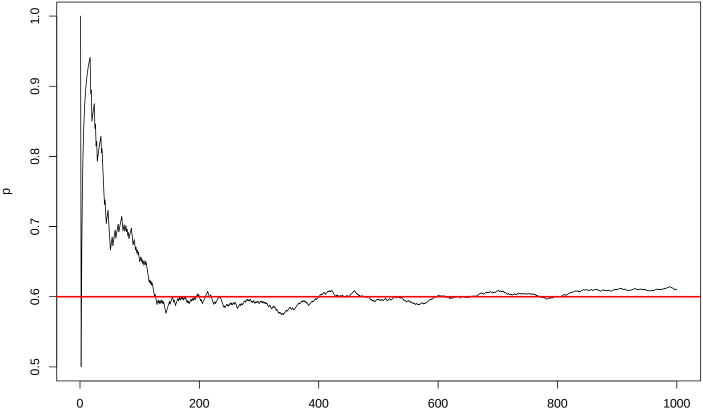
that is,

$$\mathbb{P}\left(\lim_{n\to+\infty} \frac{X_1 + X_2 + ... + X_n}{n} = \mathbb{E}(X_1)\right) = 1.$$

Visualization of the LLN

```
set.seed(111); n <- 1000; expect <- 0.6
X <- rbinom(n, 1, expect)
S <- cumsum(X); p <- S/(1:n)
plot(p, type = "l")
abline(expect,0, col = "red", lwd = 2)
```



Supplements, LLN

Sometimes we are required to calculate limits of the form:

$$\lim_{n \to +\infty} \frac{g(X_1) + g(X_2) + \ldots + g(X_n)}{n}$$

in the *Probability* or *a.s.* sense, for some nice function g.

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To give the general idea of the CLT, let us use the following transform: for a r.v. X, let us denote

$$Z = Standardize(X) = \frac{X - \mathbb{E}(X)}{\sqrt{Var(X)}} = \frac{X - \mathbb{E}(X)}{SD(X)},$$

the Standardization (normalization, scaling) of X.

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the Standardization (normalization, scaling) of X. Clearly,

$$\mathbb{E}(Z) = 0 \qquad \text{and} \qquad Var(Z) = 1.$$

Basic Idea of the CLT

The basic idea of the CLT is the following: if we have a sequence of IID r.v. X_n, and we consider their sum S_n or their average $\overline{X}_n$, then

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Easy and beautiful, isn't it?

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Two forms of CLT

Of course, these two forms of the CLT are the same: we have

$$Standardize(S_n) = \frac{S_n - \mathbb{E}(S_n)}{\sqrt{Var(S_n)}} = \frac{S_n - n \cdot \mu}{\sqrt{n} \cdot \sigma}$$

and

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Now,

$$\frac{S_n - n \cdot \mu}{\sqrt{n} \cdot \sigma} = \frac{n \cdot (\frac{S_n}{n} - \mu)}{\sqrt{n} \cdot \sigma} = \frac{\frac{S_n}{n} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}},$$

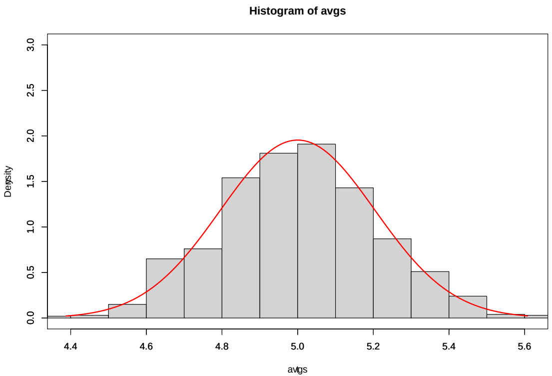
so

$$Standardize(S_n) = Standardize(\overline{X}_n).$$

Hence, the above two versions of CLT are the same, just one is in terms of S_n, the other one is in terms of $\overline{X}_n$.

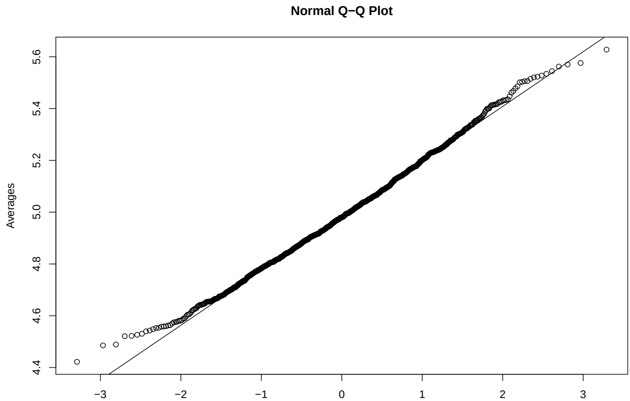
CLT, Visually

```
n <- 600 # Sample Size
m <- 1000 # no of Samples
rate <- 0.2
x <- rexp(n*m, rate = rate)
theo.mean <- 1/rate #theoretical mean
theo.sd <- 1/rate #theoretical SD
m <- matrix(x, ncol = m); d <- data.frame(m)
avgs <- sapply(d, mean)
a = theo.mean-3*theo.sd/sqrt(n); b = theo.mean+3*theo.sd/sqrt(n)
hist(avgs, freq = F, xlim = c(a, b), ylim=c(0,3))
par(new = T)
t <- seq(a,b, 0.01)
y <- dnorm(t, mean = theo.mean, sd = theo.sd/sqrt(n))
plot(t,y, type = "l", col="red", lwd = 2, xlim = c(a,b), ylim=c(0,3))
```



CLT, Visually, v2

```
n <- 600 # Sample Size
m <- 1000 # no of Samples
rate <- 0.2
x <- rexp(n*m, rate = rate)
m <- matrix(x, ncol = m); d <- data.frame(m)
avgs <- sapply(d, mean)
qqnorm(avgs, ylab = "Averages"); qqline(avgs)
```



CLT, Roughly

In a non-rigorous way, we can write, for large n (here $\approx$ means approximately distributed as):

$$\frac{S_n - n \cdot \mu}{\sqrt{n} \cdot \sigma} \approx \mathcal{N}(0,1) \qquad \textit{and} \qquad \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \approx \mathcal{N}(0,1).$$

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or

$$S_n \approx \mathcal{N}(n\mu, n\sigma^2) \qquad \textit{and} \qquad \overline{X}_n \approx \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right).$$

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Let us summarize:

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- If X_k-s are independent, have the Mean $\mathbb{E}(X_k) = \mu$ and $Var(X_K) = \sigma^2$, and **are Normally Distributed**, i.e., $X_k \sim \mathcal{N}(\mu, \sigma^2)$, then

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so we know the **exact Distributions** of S_n and $\overline{X}_n$.

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- If X_k-s are independent, have the Mean $\mathbb{E}(X_k) = \mu$ and $Var(X_K) = \sigma^2$, and **from any Distribution** (but the same Distribution), then

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Let us summarize:

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- If X_k-s are independent, have the Mean $\mathbb{E}(X_k) = \mu$ and $Var(X_K) = \sigma^2$, and **from any Distribution** (but the same Distribution), then

  $$S_n \approx \mathcal{N}(n\mu, n\sigma^2) \qquad \text{and} \qquad \overline{X}_n \approx \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right);$$

  and we know the **asymptotic Distributions** (approximate Distributions for large n) of S_n and $\overline{X}_n$.

CLT, Berry-Eseen Inequality

Now, quickly about the convergence rate of CLT:

Theorem(18+, Berry-Esseen): Assume X_k are IID r.v.s with finite $\mathbb{E}(X_1) = \mu$, $Var(X_1) = \sigma^2$ and $\mathbb{E}(|X_1|^3)$. Then, for any $n \in \mathbb{N}$,

$$\sup_{x \in \mathbb{R}} |\mathbb{P}(Z_n \leq x) - \Phi(x)| \leq \frac{\mathbb{E}(|X_1 - \mu|^3)}{\sigma^3 \cdot \sqrt{n}},$$

where

$$Z_n = Standardize(S_n) = Standardize(\overline{X}_n),$$

and $\Phi(x)$ is the CDF of $\mathcal{N}(0, 1)$.