Optimization

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The Rate Convergence of Numerical Sequence

Definition

A sequence x_n exhibits **linear** convergence to a limit x if there is a constant C in the interval (0 , 1) and an integer N such that

$$|x_{n+1} - x| \leq C|x_n - x|, \quad \forall n \geq N.$$

Example

$$x_n = \frac{1}{2^n}.$$

Definition

A sequence x_n exhibits **superlinear** convergence to a limit x if there is a sequence β_n, which converges to 0, and an integer N such that

$$|x_{n+1} - x| \leq \beta_n |x_n - x|, \quad \forall n \geq N.$$

Example

$$x_n = \frac{n}{2^{n^2}} + 1.$$

Definition

We will say that $\alpha \geq 1$ is the rate of convergence of sequence x_n if α is the largest number for which there exist a constant $C > 0$ (if $\alpha = 1$ then $0 < C < 1$) and an integer N such that

$$|x_{n+1} - x| \leq C|x_n - x|^\alpha, \quad \forall n \geq N.$$

Example

Find the limit and the rate of convergence to that limit for the following sequences:

a. $x_n = \frac{1}{2^{2^n}}$;

b. $x_n = \frac{1}{4^{3^n}+n}$;

One Dimensional Search Methods

Here we consider the minimization of univariate function $f : [a, b] \to \mathbb{R}$.

In an iterative algorithm we start with an initial candidate solution x_0 and generate sequence of points $x_1, x_2, \dots$. Each x_{k+1} iteration depends on previous points $x_0, x_1, \dots, x_k$. The algorithm may also use the values of f or f' or even f'' at some points:

- Golden section method (uses only f);
- Fibonacci method (uses only f);
- Bisection method (uses only f');
- Newton's method (uses f' and f'');
- Secant method (uses only f');.

Definition

The function $f : [a, b] \to \mathbb{R}$ is called a unimodal function on interval $[a, b]$, if f has only one local minimizer on $[a, b]$.

Proposition

If $f : [a, b] \to \mathbb{R}$ is continuous and unimodal on $[a, b]$, then f is strictly decreasing up to the minimum point x^* and increasing thereafter.

Let's assume that f is not strictly decreasing on $[a, x_2]$, that is, there exist $a \leq x_1 < x_2 \leq x^*$ such that $f(x_1) \leq f(x_2)$. As $f \in \mathbb{C}[a, x_2]$, then it attains its minimum and let the minimizer be x^{**}. We can assume that x^{**} is different from x_2. x^{**} is a local minimizer of f and it contradicts the unimodality of f.

Example

Check if the function f is unimodal on the given interval, if

a. $f(x) = \sin(x)$, $x \in [\pi/2, 2\pi]$;

b. $f(x) = \sin(x)$, $x \in [0, 2\pi]$;

c. $f(x) = \frac{x^5}{5} - x^3$, $x \in [-5, 2]$.

Golden Section Search

Assume $f : [a, b] \to \mathbb{R}$ is unimodal and continuous on interval $[a, b]$. Let x^* be the minimum point of f over $[a, b]$.

Let's denote

$$[a_0, b_0] = [a, b].$$

$$A = a_0 + \gamma(b_0 - a_0),$$

$$B = b_0 - \gamma(b_0 - a_0),$$

where $\gamma \in \left(0, \frac{1}{2}\right)$.

We define the new interval in the following way

$$[a_1, b_1] = \begin{cases} [a_0, B], & \text{if } f(A) < f(B), \\ [A, b_0], & \text{if } f(A) \geq f(B). \end{cases}$$

$$x^* \in [a_1, b_1]$$

$$b_1 - a_1 = (1 - \gamma)(b_0 - a_0).$$

The first approximation will be

$$x_1 = \frac{a_1 + b_1}{2}.$$

n-th step

$$[a_n, b_n] = \begin{cases} [a_{n-1}, B], & \text{if } f(A) < f(B), \\ [A, b_{n-1}], & \text{if } f(A) \geq f(B). \end{cases}$$

$$x^* \in [a_n, b_n]$$

$$b_n - a_n = (1 - \gamma)(b_{n-1} - a_{n-1})$$

The *n*-th approximation will be

$$x_n = \frac{a_n + b_n}{2}.$$

Theorem

If f is a unimodal and continuous function on $[a, b]$ and $\gamma \in \left(0, \frac{1}{2}\right)$, then the golden section search approximation x_n converges to x^* and we have

$$|x_n - x^*| \leq 0.5(1-\gamma)^n(b-a).$$