ASDS Statistics, YSU, Fall 2020

Lecture 26

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Contents

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Example: Consider the Model $Unif[\theta, \theta + 1]$.

Note: Sometimes it is necessary to find it numerically, say, using the NR Method

Examples

Example: It is not possible to find the MLE Estimator in a closed form for θ in the one-Parametric Cauchy Distribution $Cauchy(\theta)$ Model. Here, the PDF of $X \sim Cauchy(\theta)$ is given by

$$f(x|\theta) = \frac{1}{\pi(1 + (x - \theta)^2)}, \qquad x \in \mathbb{R},$$

and $\theta \in \mathbb{R}$ is called the *location parameter*.

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  or, put in other way,

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Properties of the MLE, Cont'd

Let us give an interpretation to this last property:

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So, MLE is **Consistent** and **Asymptotocally Efficient**. And this is why, for large Sample Size n, MLE is the Top 1 Choice, is (almost) unbeatable.

Properties of the MLE, Cont'd

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but, instead of $\mathcal{I}(\theta)$ we have $\mathcal{I}\left(\hat{\theta}_n^{MLE}\right)$.

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Note: We will use this later, to construct an (approximate) Confidence Interval for θ and for testing Hypotheses about θ.

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Example Find the MLE for σ in $\mathcal{N}(\mu, \sigma^2)$ Model.

Solution: OTB

Some topics to consider by yourself

- Multivariate Normal and MLE for MVNormal
- Kullback-Leibler Divergence and its relation to MLE
- MLE for the Mixture Model, EM Algorithm
- Bayesian Estimation: MAP and Bayes Estimator

Confidence Intervals

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i.e., we will (almost) **never** be correct in our guess. Sad news!

Prelude No. 2

But the good news is that even when we cannot exactly find the True value of our Parameter using $\hat{\theta}$, if $\hat{\theta}$ possesses some good properties, we believe that the Estimate obtained is a good approximation/Estimate for θ^*.

[1] Recall the $\widehat{SE}$, the Estimated Standard Error reporting story.

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And we can ask questions about how small is the *error* or how much sure are we in our Estimate (for the Unknown Parameter), and how large n needs to be to have a good estimate.

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Here we want to develop the theory of Confidence Intervals, which will contain answers to these questions.

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- which contains the True Parameter value with high Probability;
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Let us state this in Mathematical terms. We will consider here only 1D case, i.e., we will assume $\theta \in \Theta \subset \mathbb{R}$.

Random Intervals and CI

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Example: Assume $X \sim Pois(2.3)$. Then

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Example: Let $X_1, X_2, ..., X_n$ are IID r.v.s. Then

$$(\overline{X} - 0.1,\ \overline{X} + 0.1)$$

is a Random Interval.

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Problem: Using the Random Sample, construct an interval containing the Unknown Parameter value with Probability not less than $1 - \alpha$.

The usual values of the confidence level are $90\%, 95\%, 99\%$, so the usual values of α are $0.1, 0.05$ and 0.01.

CI

Definition: Assume $0 < \alpha < 1$, and let $L = L(x_1, ..., x_n, \alpha)$, $U = U(x_1, ..., x_n, \alpha)$ be two functions with $L(x_1, ..., x_n, \alpha) \leq U(x_1, ..., x_n, \alpha)$ for all $(x_1, ..., x_n, \alpha)$.

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$$(L, U) = \Big(L(X_1, ..., X_n, \alpha)\ ,\quad U(X_1, ..., X_n, \alpha)\Big)$$

is called a **confidence interval (or confidence interval estimator) for θ of confidence level** $1 - \alpha$, if for any $\theta \in \Theta$,

$$\mathbb{P}(L < \theta < U) \geq 1 - \alpha.$$

CI

In the case we have a realization/observation of $X_1, ..., X_n$, say, $x_1, ..., x_n$, then the interval

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Going back to our CI, CI of the confidence level $1-\alpha$ is a Random Interval that contains θ in more than $(1-\alpha)\cdot 100\%$ of cases.

CI, Interpretation

Note: It is important to understand, that in the CI definition

$$\mathbb{P}(L < \theta < U) \geq 1 - \alpha$$

θ is not our r.v., θ is our unknown constant Parameter, so we do not read this as "with high Probability, θ is in (L, U)".

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So, if we will have/generate different observations, we will have different Intervals[2] (L, U), and we want to have that most of the time that interval contains our unknown Parameter value.

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CI, **R** Simulation

Example: Consider an example: our Model is $Exp(\lambda)$, and we have an observation from it. Let us take a Random Sample for the general case: $X_1, X_2, ..., X_n$ from $Exp(\lambda)$.

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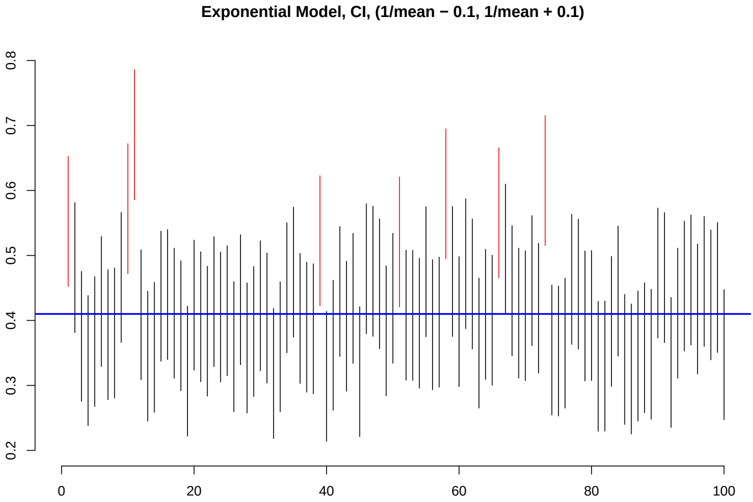
Now, let us take as CI

$$\left(\frac{1}{\overline{X}} - 0.1, \frac{1}{\overline{X}} + 0.1\right)$$

and do some simulations:

CI, **R** Simulation



CI, **R** Simulation, Code

```r
#CI Idea, Exponential Model
lambda <- 0.41
conf.level <- 0.95; a = 1 - conf.level
sample.size <- 50; no.of.intervals <- 100
epsilon <- 0.1

plot.new()
plot.window(xlim = c(0,no.of.intervals), ylim = c(0.2,0.8))
axis(1); axis(2)
title("Exponential Model, CI, (1/mean - 0.1, 1/mean + 0.1)")
for(i in 1:no.of.intervals){
  x <- rexp(sample.size, rate = lambda)
  lo <- 1/mean(x) - epsilon; up <- 1/mean(x) + epsilon
  if(lo > lambda || up < lambda){
    segments(c(i), c(lo), c(i), c(up), col = "red")
  }
  else{
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abline(h = lambda, lwd = 2, col = "blue")
```

Methods to obtain Confidence Intervals

We will consider several methods to construct CIs:

- Chebyshev Inequality Based;
- Pivotal Quantity Based

CI for the Mean, Variance is given, Cheby Method

Example: Assume $X_1, X_2, ..., X_n$ are Independent r.v. with the same Mean $\mathbb{E}(X_k) = \mu$ and the same Variance $Var(X_k) = \sigma^2$.

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By the Chebyshev inequality method, we can obtain that the interval

$$\left(\overline{X} - \frac{\sigma}{\sqrt{n \cdot \alpha}},\ \overline{X} + \frac{\sigma}{\sqrt{n \cdot \alpha}}\right)$$

is a CI for μ of Confidence Level $1 - \alpha$.

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Note: Here

$$\frac{\sigma}{\sqrt{n \cdot \alpha}}$$

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Some Notes

Two notes about the obtained CI - in fact, these notes will work also for other cases too:

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The CI length obtained above is

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Note: If we increase the Confidence Level, i.e., if we decrease α, then the length of CI increases. This is intuitive too: if we want to be more sure where our unknown Parameter is lying, we will get a larger interval.

CI for the Proportion, Cheby Method

Example: Now, let us construct a CI of CLevel $1 - \alpha$ for p in the *Bernoulli*(p) Model.

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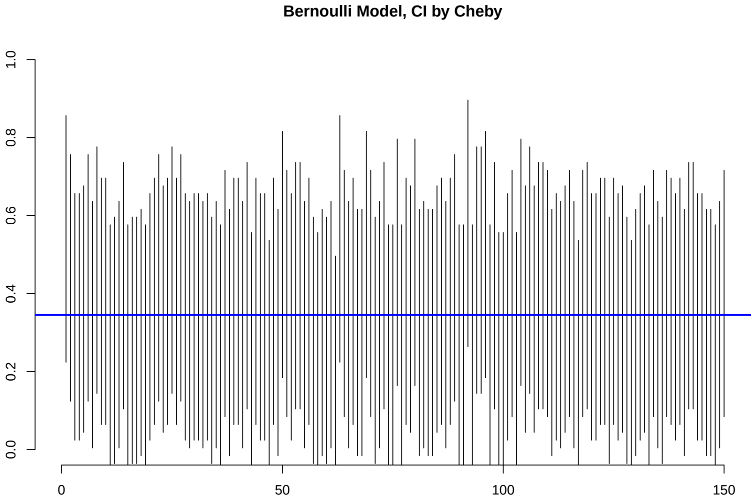
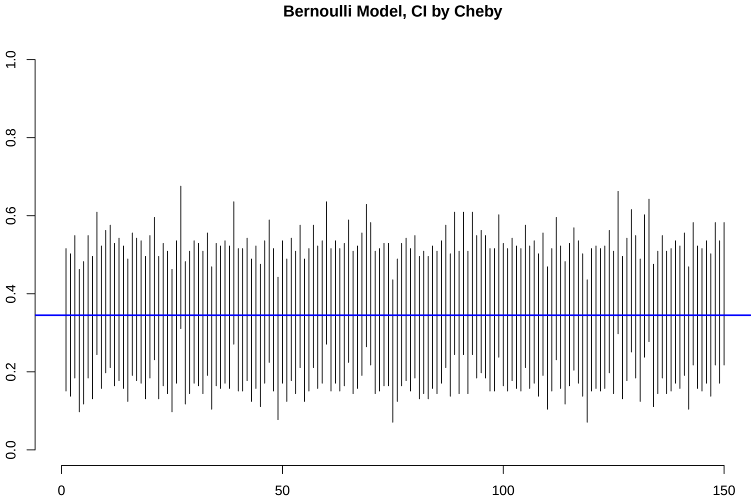
CI for Bernoulli, **R** Simulation



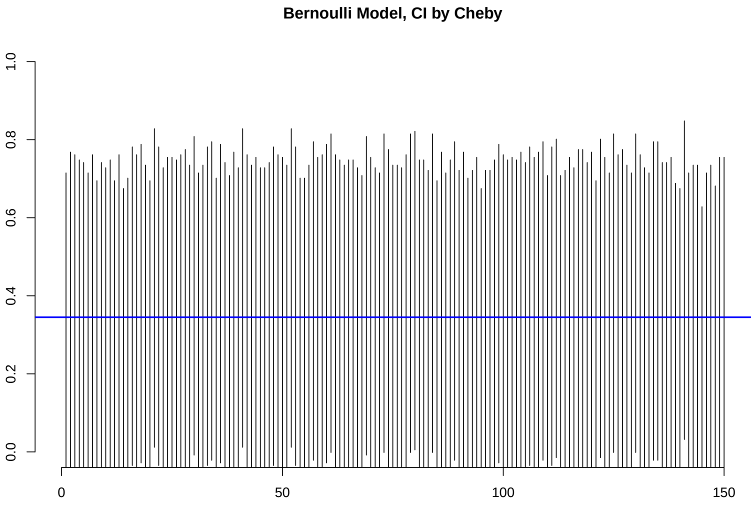
Sample Size $= 50$, $CL = 95\%$

CI for Bernoulli, **R** Simulation



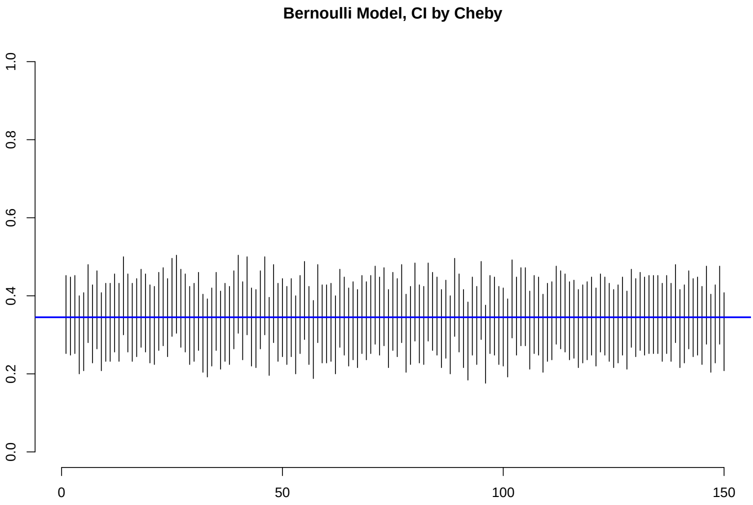
Sample Size $= 150$, $CL = 95\%$

CI for Bernoulli, **R** Simulation



Sample Size $= 150$, $CL = 99\%$

CI for Bernoulli, **R** Simulation



Sample Size $= 250$, $CL = 90\%$

CI, **R** Simulation, Code

```r
#CI Idea, Bernoulli Model
p <- 0.345
conf.level <- 0.9; a = 1 - conf.level
sample.size <- 250; no.of.intervals <- 150
ME <- 1/(2*sqrt(sample.size*a)) #Margin of Error
plot.new()
plot.window(xlim = c(0,no.of.intervals), ylim = c(0,1))
axis(1); axis(2)
title("Bernoulli Model, CI by Cheby")
for(i in 1:no.of.intervals){
  x <- rbinom(sample.size, size = 1, prob = p)
  lo <- mean(x) - ME
  up <- mean(x) + ME
  if(lo > p || up < p){
    segments(c(i), c(lo), c(i), c(up), col = "red")
  }
  else{
    segments(c(i), c(lo), c(i), c(up))
  }
}
abline(h = p, lwd = 2, col = "blue")
```

Examples

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Solution: OTB

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Example: Continuing the above Example: now assume we want to find that Proportion within the Error Margin 0.1, with the CL 95%. At least, how many persons at AUA we need to ask?

Solution: OTB