Basic Mathematics, Fall 2020

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Definition of the derivative

Assume f is defined in some neighbourhood of $x_0 \in \mathbb{R}$.

Definition

We will say that f is differentiable at x_0, or f has a derivative at x_0, if the following limit

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists and is finite. We will denote

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

and refer the number $f'(x_0)$ as the derivative of f at x_0.

Example

Find the derivative of x^3 using the definition of the derivative.

The list of elementary derivatives

1. $c' = 0;$
2. $(x^\alpha)' = \alpha x^{\alpha-1};$
3. $(e^x)' = e^x$, and in general $(a^x)' = a^x \ln a;$
4. $(\ln x)' = \dfrac{1}{x},$
5. $(\sin x)' = \cos x;$
6. $(\cos x)' = -\sin x;$

Differentiation Rules

If the functions f and g are differentiable, then

Sum Rule: $(f(x) + g(x))' = f'(x) + g'(x)$

Product Rule: $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

Quotient Rule: $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$

Chain Rule: $(g(f(x))' = (g \circ f)'(x) = g'(f(x))f'(x)$

Example

Compute the derivative of the functions $f(x) = e^{x^2}$, $g(x) = x\ln(5x)$, $2f(x) + 3g(x)$ *and* $\dfrac{f(x)}{g(x)}$.

Example

Let X *be a continuous random variable with probability density* f_X, *distribution* F_X *and let* $Y = X^2$. *Find the distribution* F_Y *in terms of* F_X *and the probability density function* f_Y *in terms of* f_X *(for* $y \geq 0$*).*

Definition

The **Taylor polynomial** *of degree* n *of* $f : \mathbb{R} \to \mathbb{R}$ *at* x_0 *is defined as*

$$T_n(x) := \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

where $f^{(k)}(x_0)$ *is the* k*th derivative of* f *at* x_0 *(which we assume exists) and* $\frac{f^{(k)}(x_0)}{k!}$ *are the coefficients of the polynomial.*

Definition

For a smooth function $f \in C^{\infty}, f : \mathbb{R} \to \mathbb{R}$ *the* **Taylor series** *of* f *at* x_0 *is defined as*

$$T_{\infty}(x) := \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

Remark

In general, a Taylor polynomial of degree n is an approximation of a function, which does not need to be a polynomial. The Taylor polynomial is similar to f in a neighborhood around x_0.

Example

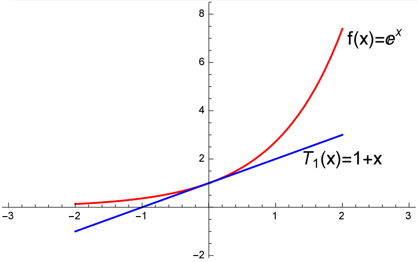
Find the Taylor polynomial $T_5(x)$ of $f(x) = x^3$ at point $x_0 = 2$. Verify that $T_5(x) = f(x)$.

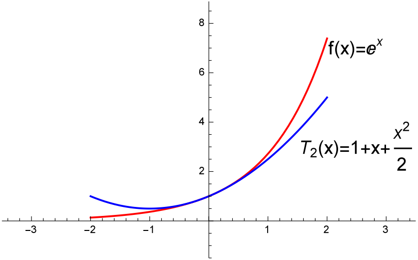
Remark

Taylor polynomial of degree n is an exact representation of a polynomial f of degree $k \leq n$.

Example

Find the Taylor series of the function $f(x) = e^x$ *at* $x_0 = 0$.





Taylor's theorem

If $|f^{(n+1)}(x)| \leq M$ for $x \in (x_0 - h, x_0 + h)$, then

$$f(x) = T_n(x) + r_n(x), \text{ where } |r_n(x)| \leq \frac{M}{n!}|x - x_0|^{n+1},$$

for $x \in (x_0 - h, x_0 + h)$.

To approximate the derivative $f'(x_0)$ one can use

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0 - h)}{2h},$$

which usually yields a better approximation than

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0)}{h}.$$

The list of elementary integrals

Proposition

1. $\int 0\mathrm{d}x = C;$
2. $\int x^{\alpha}\mathrm{d}x = \frac{x^{\alpha+1}}{\alpha+1} + C,$ *if* $\alpha \neq -1;$
3. $\int \frac{\mathrm{d}x}{x} = \ln|x| + C;$
4. $\int e^{x}\mathrm{d}x = e^{x} + C;$
5. $\int \sin x\mathrm{d}x = -\cos x + C;$
6. $\int \cos x\mathrm{d}x = \sin x + C;$
7. $\int \frac{\mathrm{d}x}{1+x^2} = \operatorname{arctg} x + C;$

Example

Compute $E[X]$ *if* X *has a density function given by*

$$f(x) = \begin{cases} \frac{3}{x^2}, & x > 3 \\ 0, & x \leq 3 \end{cases}.$$

Example

For some constant c, *the random variable* X *has the probability density function* $f(x) = \begin{cases} cx^5, & 0 < x < 3 \\ 0, & otherwise \end{cases}$.
Find the constant c, $E[X]$ *and* $Var(X)$.

Integration by parts

Integration by parts for indefinite integrals

If f and g both have antiderivatives on (a, b), then

$$\int f(x)g'(x)\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\mathrm{d}x,$$

or

$$\int f(x)\mathrm{d}g(x) = f(x)g(x) - \int g(x)\mathrm{d}f(x).$$

Example

Evaluate $\int x\sin x\mathrm{d}x$.

Integration by parts for definite integrals

If f and g both are continuously differentiable on $[a, b]$, then

$$\int_a^b f(x)g'(x)\mathrm{d}x = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\mathrm{d}x.$$

Example

Evaluate $\int_0^1 x^2 e^x \mathrm{d}x$.

Example

Let X *be an exponential random variable with parameter* λ.
Calculate $E[X]$ *and* $Var(X)$.
Recall that r.v. X *is said to be* **exponential** *r.v. if its probability density function is given, for some* $\lambda > 0$, *by*

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

Example

a) *Prove that the gamma function* $\Gamma(\alpha)$, *defined as*

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy$$

satisfies the relation

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1).$$

b) *Let* X *be a gamma r.v. with parameters* α *and* λ. *Calculate* $E[X]$ *and* $Var(X)$.
Recall that a r.v. is said to have a gamma distribution with parameters (α, λ), *if its density function is given by*

$$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$