MLE

Note Title 28-Nov-20

- MLE for Bernoulli(p)

<u>PMF</u> is $f(x|p) = p^{x} \cdot (1-p)^{1-x}$, $x \in \{0,1\}$

$X_1, \dots, X_n \sim \text{Bernoulli}(p)$

- Likelihood funct.

$$\mathcal{L}(p) = f(X_1|p) \cdot f(X_2|p) \cdots f(X_n|p) =$$

$$= p^{X_1} \cdot (1-p)^{1-X_1} \cdot p^{X_2} (1-p)^{1-X_2} \cdots p^{X_n} (1-p)^{1-X_n}$$

$S_n = X_1 + \dots + X_n$

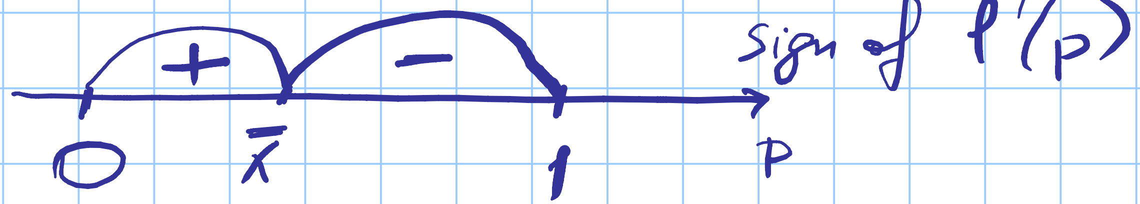
$$= p^{S_n} \cdot (1-p)^{n-S_n}$$

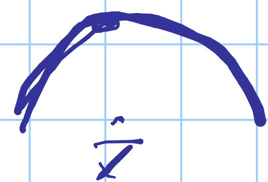
$$\ell(p) = \ln \mathcal{L}(p) = S_n \ln p + (n-S_n)\ln(1-p), \quad p \in (0,1)$$

$$\ell'(p) = \frac{S_n}{p} - \frac{n-S_n}{1-p} = \frac{S_n - np}{p(1-p)} = \frac{n(\bar{X}-p)}{p(1-p)}$$

$$\ell'(p) = 0 \iff \frac{S_n}{p} = \frac{n-S_n}{1-p} \qquad S_n - \cancel{pS_n} = np - \cancel{pS_n}$$

$$p = \frac{S_n}{n} = \frac{X_1 + \dots + X_n}{n} = \bar{X}$$





$p = \bar{X}$ is a glob max point of $\ell(p)$

$$\Rightarrow \boxed{\hat{p}^{MLE} = \overline{X}} = \hat{p}^{MoM}$$

• MLE for $Exp(\lambda)$, $\lambda \in (0, +\infty)$

PDF is $f(x|\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

$X_1, \dots, X_n \sim Exp(\lambda)$

$$\mathcal{L}(\lambda) = f(X_1|\lambda) \cdot f(X_2|\lambda) \cdots f(X_n|\lambda) =$$

$$= \lambda \cdot e^{-\lambda X_1} \cdot \lambda e^{-\lambda X_2} \cdots \lambda \cdot e^{-\lambda X_n} =$$

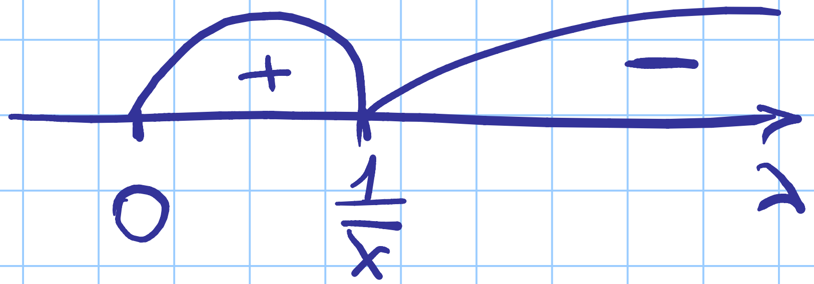
$$= \lambda^n \cdot e^{-\lambda \cdot S_n}$$

$$\ell(\lambda) = \ln \mathcal{L}(\lambda) = n \ln \lambda - \lambda \cdot S_n$$

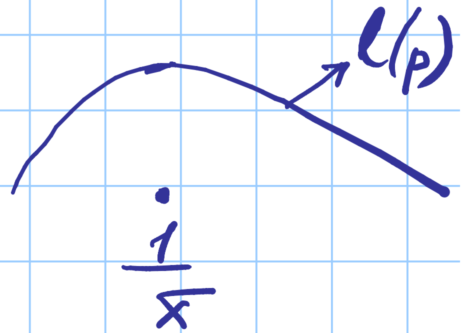
$$l'(\lambda) = \frac{n}{\lambda} - S_n$$

$$l'(\lambda) = 0 \iff \frac{n}{\lambda} = S_n$$

$$\lambda = \frac{n}{S_n} = \frac{1}{\bar{x}}$$



sign of $l'(\lambda)$



$$\boxed{\hat{\lambda}^{MLE} = \frac{1}{\bar{x}}} = \hat{\lambda}^{MOM}$$

- MLE for $\mathcal{U}nif[0,\theta]$

PDF is $f(x|\theta) = \begin{cases} \frac{1}{\theta}, & \text{if } 0 \le x \le \theta \\ 0, & \text{if } x \notin [0,\theta] \end{cases}$

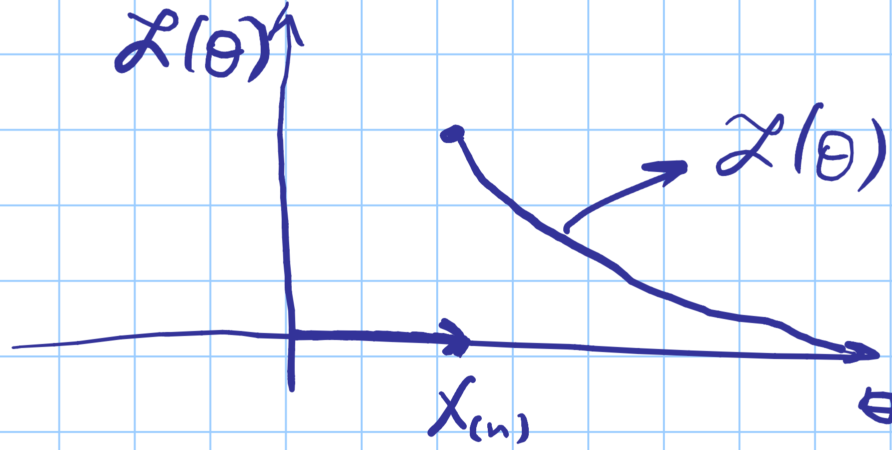
$$X_1, \dots, X_n \sim \mathcal{U}nif[0,\theta], \quad \theta \in (0,+\infty)$$

$$\mathcal{L}(\theta) = \underbrace{f(X_1|\theta)}_{\overset{\shortparallel}{\frac{1}{\theta}}} \cdot \underbrace{f(X_2|\theta)}_{\overset{\shortparallel}{\frac{1}{\theta}}} \cdots \underbrace{f(X_n|\theta)}_{\overset{\shortparallel}{\frac{1}{\theta}}}$$

$$f(X_1|\theta) = \begin{cases} \frac{1}{\theta}, & X_1 \le \theta \\ 0, & X_1 > \theta \end{cases}$$

$$\mathcal{L}(\theta) = \begin{cases} \frac{1}{\theta^n}, & \theta \geq X_1, \theta \geq X_2, \dots, \theta \geq X_n \\ 0, & \text{otherwise} \end{cases} =$$

$$= \begin{cases} \frac{1}{\theta^n}, & \theta \geq X_{(n)} \\ 0, & \text{otherwise} \end{cases}$$



$$\boxed{\hat{\theta}^{MLE} = X_{(n)}}$$

$$\hat{\theta}^{MoM} = 2 \cdot \overline{X}$$

$$X_{(n)} = \max\{X_1, X_2, \dots, X_n\} \qquad X_i \sim \mathcal{U}nif[0,\theta]$$

$$\mathbb{E}(X_{(n)}) \overset{<}{\neq} \theta$$

$\frac{n+1}{n} X_{(n)}$ is Unbiased

· MLE for $N(\mu, \sigma^2)$

PDF: $f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $\underline{\underline{\sigma^2 = \theta}}$

$$= \frac{1}{\sqrt{2\pi\theta}} \cdot e^{-\frac{(x-\mu)^2}{2\theta}}$$

$X_1, \dots, X_n \sim N(\mu, \theta)$

$$\mathcal{L}(\mu, \theta) = f(X_1|\mu,\theta) \cdot \ldots \cdot f(X_n|\mu,\theta) =$$

$$= \left(\frac{1}{2\pi\theta}\right)^{n/2} \cdot e^{-\frac{1}{2\theta} \cdot \sum_{k=1}^{n} (X_k - \mu)^2}$$

$$\ell(\mu,\theta) = -\frac{n}{2}\ln(2\pi\theta) - \frac{1}{2\theta}\sum_{k=1}^{n}(X_k-\mu)^2 \ , \quad \begin{array}{l}\mu\in\mathbb{R}\\ \theta>0\end{array}$$

$$\begin{cases}\dfrac{\partial \ell}{\partial \mu}=0\\[2ex] \dfrac{\partial \ell}{\partial \theta}=0\end{cases} \Leftrightarrow \begin{cases}\dfrac{1}{2\theta}\displaystyle\sum_{k=1}^{n}(X_k-\mu)=0\\[2ex] -\dfrac{n}{2\theta}+\dfrac{1}{2\theta^2}\displaystyle\sum_{k=1}^{n}(X_k-\mu)^2=0\end{cases}$$

$$\Leftrightarrow \begin{cases}\displaystyle\sum_{k=1}^{n}X_k - n\mu = 0\\[2ex] \dfrac{1}{\theta}\displaystyle\sum_{1}^{n}(X_k-\overline{X})^2 = n\end{cases} \qquad \begin{array}{l}\mu=\overline{X}\\[2ex] \theta = \dfrac{\sum_{k=1}^{n}(X_k-\overline{X})^2}{n}\end{array}$$

$$\hat{\mu}^{MLE} = \overline{X} \qquad \hat{\theta}^{MLE} = \widehat{\sigma^2}^{MLE} = \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n}$$

$$\begin{array}{ccc} \hat{\mu}^{MLE} & & \widehat{\sigma^2}^{MLE} \\ \parallel & & \parallel \\ \hat{\mu}^{MoM} & & \widehat{\sigma^2}^{MoM} \end{array}$$

· MLE for Discr Model

X	0	1	2
$\mathbb{P}(X=x)$	$\frac{\theta}{10}$	$\frac{\theta}{5}$	$1-\frac{3\theta}{10}$

$\theta \in \left[0, \frac{10}{3}\right]$

PMF

$$f(x|\theta) = \begin{cases} \frac{\theta}{10}, & x=0 \\ \frac{\theta}{5}, & x=1 \\ 1-\frac{3\theta}{10}, & x=2 \end{cases}$$

$X_1, X_2, \dots, X_n \sim F_\theta$

n_0 n_1

$$\mathcal{L}(\theta) = f(x_1|\theta)\cdot\ldots\cdot f(x_n|\theta) = \left(\frac{\theta}{10}\right)^{\#X_k=0}\cdot\left(\frac{\theta}{5}\right)^{\#X_k=1}\cdot$$

$$\cdot\left(1-\frac{3\theta}{10}\right)^{\overbrace{\#X_k=2}^{n_2}} = \left(\frac{\theta}{10}\right)^{n_0}\cdot\left(\frac{\theta}{5}\right)^{n_1}\cdot\left(1-\frac{3\theta}{10}\right)^{n_2}$$

$$\ln(\theta) = n_0 \ln\frac{\theta}{10} + n_1\cdot\ln\frac{\theta}{5} + n_2\cdot\ln\left(1-\frac{3\theta}{10}\right)$$

$$\underline{l'(\theta) = 0} \quad \ldots$$