The following process is called **Gaussian elimination**.

1. Write the augmented matrix of the SLE.
2. Use elementary row operations to reduce the augmented matrix to row echelon form.
3. Using back substitution, solve the equivalent system that corresponds to the row-reduced matrix.

Example

Solve the system

$$\begin{cases} 2x_1 - x_2 + 5x_3 = -2 \\ x_1 - 2x_2 + 4x_3 = -7 \\ \quad 3x_2 - 2x_3 = 9 \end{cases}.$$

Gauss-Jordan Elimination

Definition

A matrix is in **reduced row echelon form** *if it satisfies the following properties:*

1. *It is in row echelon form.*
2. *The leading entry in each non-zero row is a 1 (called a* **leading** *1).*
3. *Each column containing a leading 1 has zeros everywhere else.*

The entire process is called **Gauss-Jordan elimination**.

Example

The following matrix is in reduced row echelon form:

$$\begin{bmatrix} 0 & 1 & 3 & 0 & 0 & -1 & 4 & 0 & 6 \\ 0 & 0 & 0 & 1 & 0 & 3 & -2 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 2 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example

Solve the system by Gauss-Jordan elimination.

$$\begin{cases} w - 2x + 3y + 2z = 1 \\ 2w - 4x + 7y + 2z = 4 \\ -3w + 6x - 7y - 10z = 1 \end{cases}.$$

The Minus-1 Trick

We introduce a practical trick for reading out the solutions $\mathbf{x}$ of a homogeneous SLE $A\mathbf{x} = \mathbf{0}$, where $A \in \mathbb{R}^{k \times n}, \mathbf{x} \in \mathbb{R}^n$. We assume that A is in **REF without any rows that just contain zeros**, i.e.,

$$A = \begin{bmatrix} 0 & \cdots & 0 & 1 & * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots & * \\ \vdots & & \vdots & 0 & 0 & \cdots & 0 & 1 & * & \cdots & * & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & 0 & * & \ddots & * & 0 & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & * & \ddots & * & 0 & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & * & \cdots & * & 1 & * & \cdots & \vdots \end{bmatrix}$$

We extend this matrix to an $n \times n$-matrix $\tilde{A}$ by adding $n - k$ rows of the form

$$[0 \ \cdots \ 0 \ -1 \ 0 \ \cdots \ 0]$$

so that the diagonal of the augmented matrix $\tilde{A}$ contains either 1 or -1.

The columns of $\tilde{A}$, which contain the -1 as pivots form a basis of the solution space of $A\mathbf{x} = \mathbf{0}$.

Example

Find the solutions of $A\mathbf{x} = \mathbf{0}$ *for*

$$A = \begin{bmatrix} 0 & 1 & 3 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The Gauss-Jordan Method for Computing the Inverse

Fundamental Theorem of Invertible Matrices

Let A be an $n \times n$ invertible matrix.

1. A is invertible.
2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every vector $\mathbf{b} \in \mathbb{R}^n$.
3. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
4. The reduced row echelon form of A is I_n.
5. A is a product of elementary matrices.

Theorem

Let A be a square matrix. If a sequence of elementary row operations reduces A to I, then the same sequence of elementary row operations transforms I into A^{-1}.

If A is row equivalent to I, then elementary row operations will yield

$$[\ A \mid I\ ] \to [\ I \mid A^{-1}\ ].$$

If A cannot be reduced to I, then the Fundamental Theorem guarantees us that A is not invertible.

Note that A^{-1} is the solution of the equation $AX = I$. If $\mathbf{x}_1, \dots, \mathbf{x}_n$ are the columns of matrix X, then

$$AX = I_n \Leftrightarrow A\mathbf{x}_1 = \mathbf{e}_1, \dots, A\mathbf{x}_n = \mathbf{e}_n,$$

and the augmented matrices for these systems $\left[\begin{array}{c|c} A & \mathbf{e}_1 \end{array}\right], \dots, \left[\begin{array}{c|c} A & \mathbf{e}_n \end{array}\right]$ can be combined as

$$\left[\begin{array}{c|c} A & \mathbf{e}_1\mathbf{e}_2\dots\mathbf{e}_n \end{array}\right] = \left[\begin{array}{c|c} A & I_n \end{array}\right]$$

Example

Find the inverse of

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 5 \\ -3 & 2 & 2 \end{bmatrix}$$

if it exists.

Example

Find the inverse of

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 5 \\ 3 & 1 & 7 \end{bmatrix}$$

if it exists.