ASDS Statistics, YSU, Fall 2020

Lecture 24

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Contents

Methods to find (good) Estimators

The Problem

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Problem: The Problem is to find/construct a good Estimator for θ, using our Random Sample.

The Method of Moments

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Note: Note that, in general, the Theoretical Moments of $\mathcal{F}_\theta$ are functions of θ.

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Note: The Empirical Moment is independent of the Parameter θ, it is just a Statistics, it is a function of $X_1, X_2, ..., X_n$.

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$$0, 1, 1, 2, 1, 0, 0, 1, 1$$

from the following Model:

X	0	1	2
$\mathbb{P}(X = x)$	$\frac{\theta}{10}$	$\frac{\theta}{5}$	$1 - \frac{3\theta}{10}$,

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Example

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Example: Find the MoM Estimator for (a, b) in the $Unif[a, b]$ Model.

Solution: OTB

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Example: Let us do an experiment in **R**, concerning the last example:

```r
a <- 2.5; b <- 3.24
x <- runif(10, min = a, max = b)
x.bar <- mean(x)
z <- sqrt(3* (mean(x^2) - x.bar^2))
a.hat.MoM<- x.bar - z
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c(a.hat.MoM, b.hat.MoM)
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Of course, we can just take $\hat{a} = X_{(1)}$ and $\hat{b} = X_{(n)}$:

```r
c(min(x), max(x))
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```
## [1] 2.515333 3.020296
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Notes

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Note: Sometimes we are interested not in the Estimate of the Parameter itself, but in the Estimate for some function of that Parameter. Say, we want to Estimate $h(\theta)$, where θ is our unknown Parameter. Then one of the approaches is the Plug-in Principle: find an Estimator $\hat{\theta}$ for θ, say, using the MoM, and then plug that in h, to obtain $h(\hat{\theta})$ as an Estimator for $h(\theta)$.

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Now, by the WLLN, $\overline{X}_n \xrightarrow{\mathbb{P}} \mathbb{E}(X_1) = e(\theta)$, so

$$\hat{\theta}_n = e^{-1}(\overline{X}_n) \xrightarrow{\mathbb{P}} e^{-1}(e(\theta)) = \theta.$$