State-space of stochastic system

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1 Introduction

Stochastic systems are ubiquitous in neuroscience, where neural activity and synaptic plasticity dynamics are inherently probabilistic. Therefore, simulations of stochastic systems are essential to understanding and modeling complex neural processes. These simulations enable researchers to explore the effects of noise and variability in the system and to make predictions about the behavior of neural networks under different conditions. For example, simulations of stochastic synapses can reveal the role of synaptic noise in shaping the statistics of neural activity and in modulating information processing. Similarly, simulations of stochastic spiking neurons can provide insights into the coding properties of neurons and the mechanisms underlying spike-timing-dependent plasticity.

Moreover, stochastic simulations can help to bridge the gap between theoretical models and empirical data. By comparing the results of simulations with experimental measurements, researchers can refine and validate their models, and test hypotheses about the underlying mechanisms of neural function. This approach has been particularly fruitful in the study of neural oscillations, where stochastic simulations have revealed the role of noise-induced synchronization in generating coherent oscillatory activity. Thus, simulations of stochastic systems are an essential tool in neuroscience research, providing a means to investigate the complex and dynamic behavior of neural circuits, and to advance our understanding of brain function.
In exercise 4, I evaluated the behavior of a deterministic system by determining the steady states and the effect of several parameters such as weight(w) and initial position. In the current exercise, I will evaluate the stochastic version of the aforementioned system.
Our system is defined as follow:

$$\begin{aligned}
\tau \frac{dE_1}{dt} &= -E_1 + \theta(wE_1 - E_2 - 0.5) + N_1 \\
\tau \frac{dE_2}{dt} &= -E_2 + \theta(E_1 - E_2 - 0.5) + N_2 \\
\tau_n \frac{dN_1}{dt} &= -N_1 + \sigma_n \sqrt{\tau_n} \zeta(t) \\
\tau_n \frac{dN_2}{dt} &= -N_2 + \sigma_n \sqrt{\tau_n} \zeta(t) \\
\theta(x) &= \frac{x^2}{k^2 + x^2}
\end{aligned}$$

Where $\zeta(t)$ is white noise and N_1 and N_2 are independent realizations of Brownian noise (Ornstein-Uhlenbeck process at steady state).

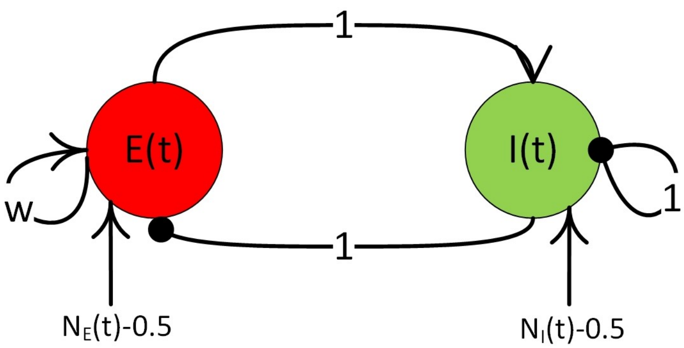

Figure 1: Depiction of the System. In this system, $E(t)$ states for Excitatory population and $I(t)$ for Inhibitory.

As Fig1 depicted, we have two sets of populations that interact with each other. Excitatory population E_1(In Fig1 assigned as $E(t)$) received Inhibitory input from Inhibitory population E_2(In Fig1 assigned as $I(t)$), and both systems received Brownian noise input from the outer world.
In the following sections, I will assess the described system by different approaches.

2 Numerical Simulation and Analysis of Stability

First of all, I simulated the time-evolution of the system by iterative integration:

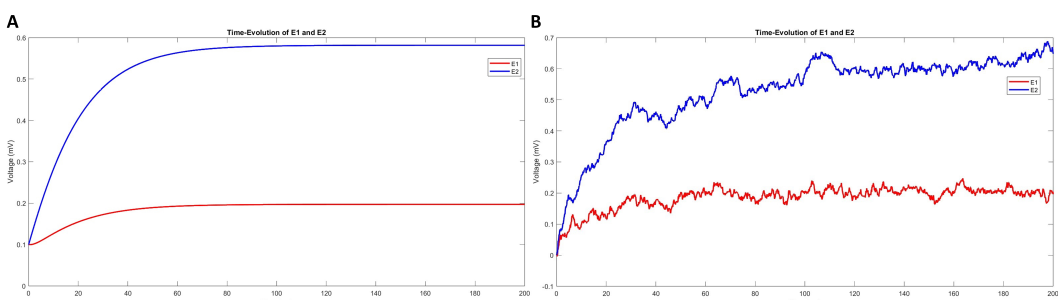

Figure 2: Time-evolution of system. **A.**Deterministic version of the system. **B.**The system with stochastic input.

By selecting $\tau = 10, k = 0.75, w = 3.6, \tau_n = 2, T_end = 20\tau, dt = 0.05\tau_n$, and $\sigma_n = 1.0$, I simulated the system with and without the stochastic input.
The difference between the two versions is obvious in Fig2: The deterministic

version shows the smooth lines in contrast to the noise lines in the second version. Starting at point (0.1, 0.1), both populations evolved to their steady state and remains there (even though because of noisy input we have a fluctuation at that point in the second case).
In the next step, I Visualize the system in the state space (E_1, E_2), starting either at (0.1,0.1) or at (1.2,0.8) for 20 trajectories:

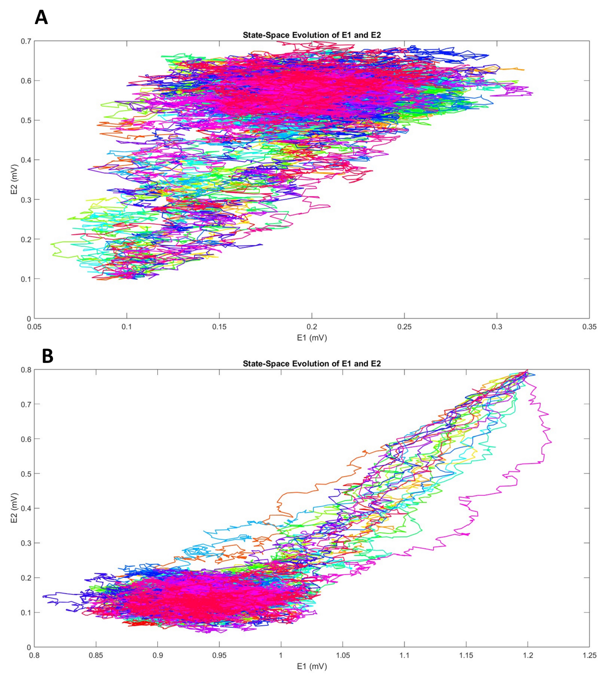

Figure 3: Time-evolution of system in the state-space. **A.**Evolution of system through time, starting from (0.1,0.1). **B.**Evolution of system through time, starting from (1.2,0.8).

In Fig3 you could see starting from different points, the system eventually reaches the stable fixed point with deviation. Stable points for **A** and **B** are different. In order to understand this behavior better, I added the isoclines of the deterministic version of the system to plot:

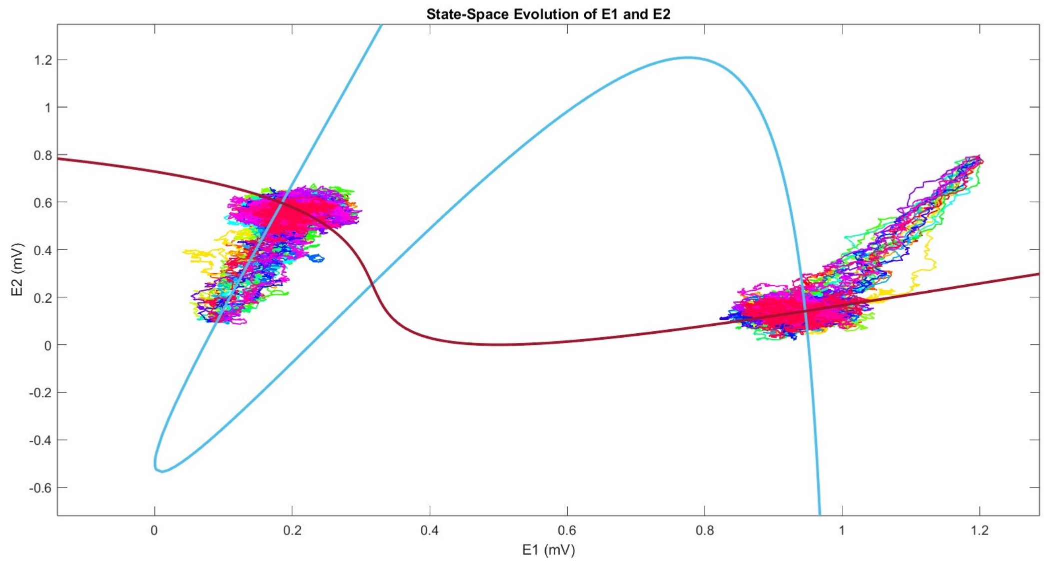

Figure 4: Time-evolution of system in the state-space + isocline of deterministic version of the system.

Based on Fig4, We could conclude that the stochastic version of the system follows the general behavior of the deterministic version. Based on the initial point the system reaches the stable critical point. A more general understanding would be possible by taking advantage of the trajectories of the system:

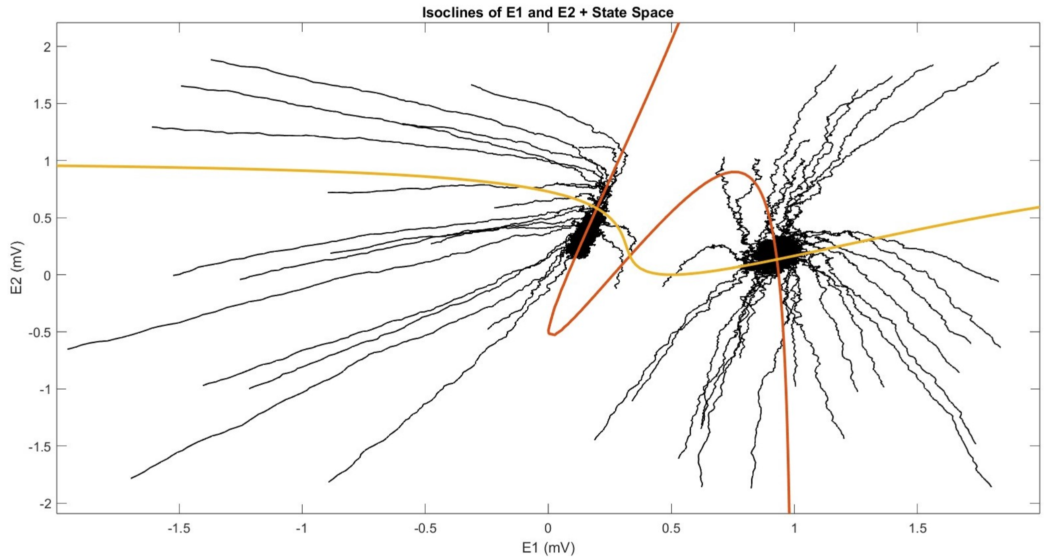

Figure 5: Trajectory of system, plus isoclines. Based on the initial points, trajectories converge to asymptotically stable intersections.

Additional Information from Fig5, is the behavior of the system near the second intersection of isoclines. As you could see the trajectories starting at areas near that point converge to the neighboring fixed point: It is an asymptotically unstable point.

3 First-passage time

First-passage-time(FPT) is a statistical concept used to measure the time taken for a stochastic process to cross a certain threshold or level for the first time. The first-passage-time is an important concept in evaluating the stable state of a system. By calculating the FPT, it is possible to determine whether a system is stable or not. If the FPT is short, it means that the system is likely to reach a stable state quickly. On the other hand, if the FPT is long, it indicates that the system may be unstable or takes a longer time to reach a stable state. Therefore, the FPT is an essential tool for evaluating the stability of a system and determining its behavior over time.
I compute 100 trajectories in the high state (0.9, 0.1) and determine their FPT to the low state, By using the Euler-Maruyama method to numerically integrate the system of equations. I also used the hitting time algorithm to compute the FPT to the low state for each trajectory. Here is the cumulative distribution of FPT from high to low ($\sigma = 2$):

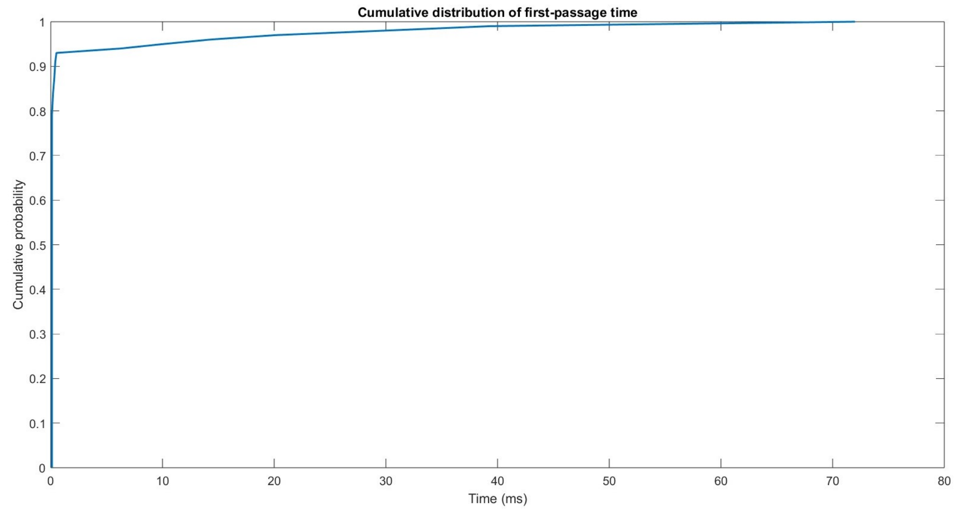

Figure 6: Cumulative distribution of first-passage-time from high to low ($\sigma = 2$)

Fig6 shows the cumulative distribution of FPT from the high state (0.9, 0.1) to the low state ($E_2 <= 0.1$). The FPT is defined as the time at which E2 first crosses the threshold value of 0.1.

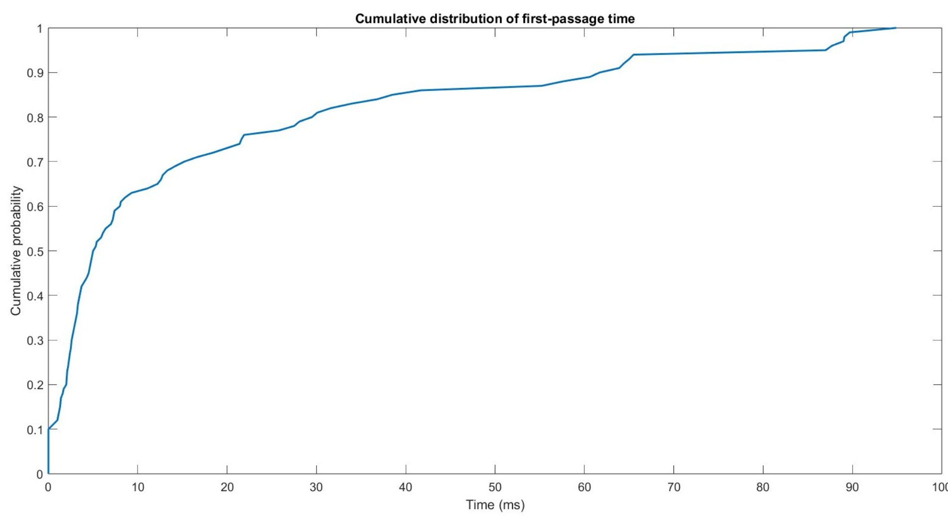

Figure 7: Cumulative distribution of first-passage-time from low to high($\sigma = 2$)

Fig7 shows the cumulative distribution of first-passage time from the low state (0.2, 0.6) to the high state ($E_2 >= 0.9$). The FPT is defined as the time

at which E2 first crosses the threshold value of 0.9.
If the FPT is short, it indicates that the system is less stable and more likely to transition between the two states. On the other hand, if the FPT is long, it indicates that the system is more stable and less likely to transition. In Fig6, you could see higher probability density in the short FPT region, which indicates that the high state is less stable and more likely to transition to the low state. On the other hand, that's not the case for Fig7, implying that the transition from the low state to high state is less probable compared to the high state.

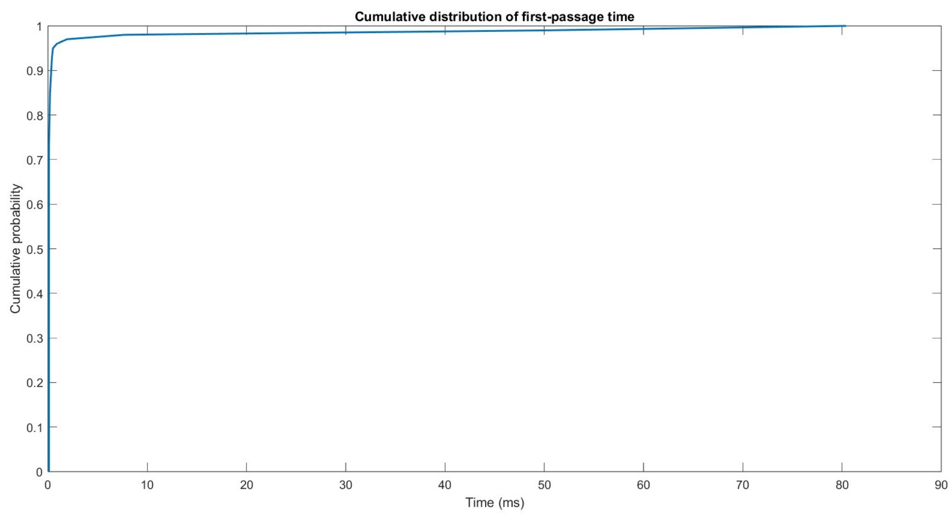

Figure 8: Cumulative distribution of first-passage-time from high to low ($\sigma = 1$ and $T_{end} = 50\tau$)

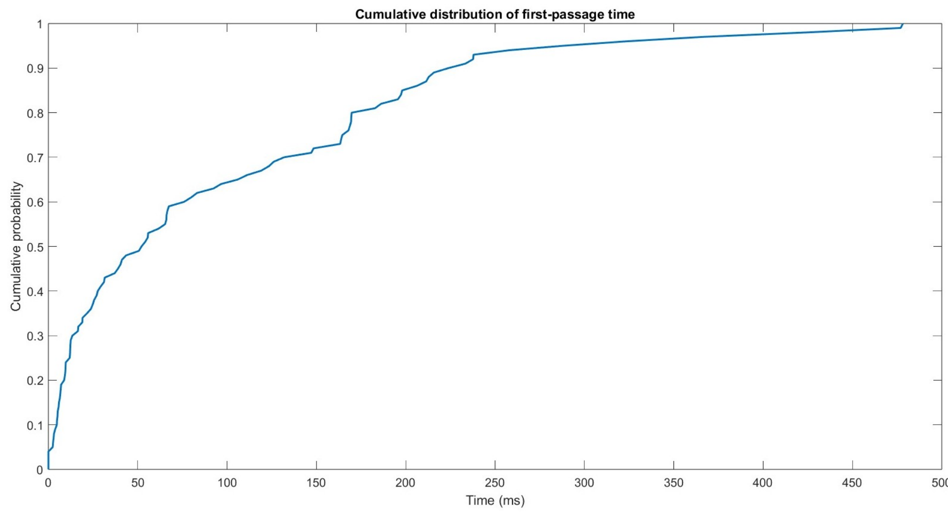

Figure 9: Cumulative distribution of first-passage-time from low to high($\sigma = 1$ and $T_{end} = 50\tau$)

By decreasing $\sigma_n = 1$ we could see the high state shows more tendency for transition but the low state behave completely opposite.

4 Spectral analysis

Spectral analysis can be used to determine the stability of a system by analyzing the frequency response of the system. The frequency response of a system is the output of the system when a sinusoidal input is applied at different frequencies. The frequency response can be represented as a transfer function, which is the ratio of the output to the input in the frequency domain.
To choose a simulation duration (T_{end}) and a noise level (σ) such that at least 95% of all trajectories remain in the initial state for the duration of the stimulation, I performed a parameter sweep by simulating the system with different combinations of T_{end} and σ values and counting the fraction of trajectories that remain in the initial state:

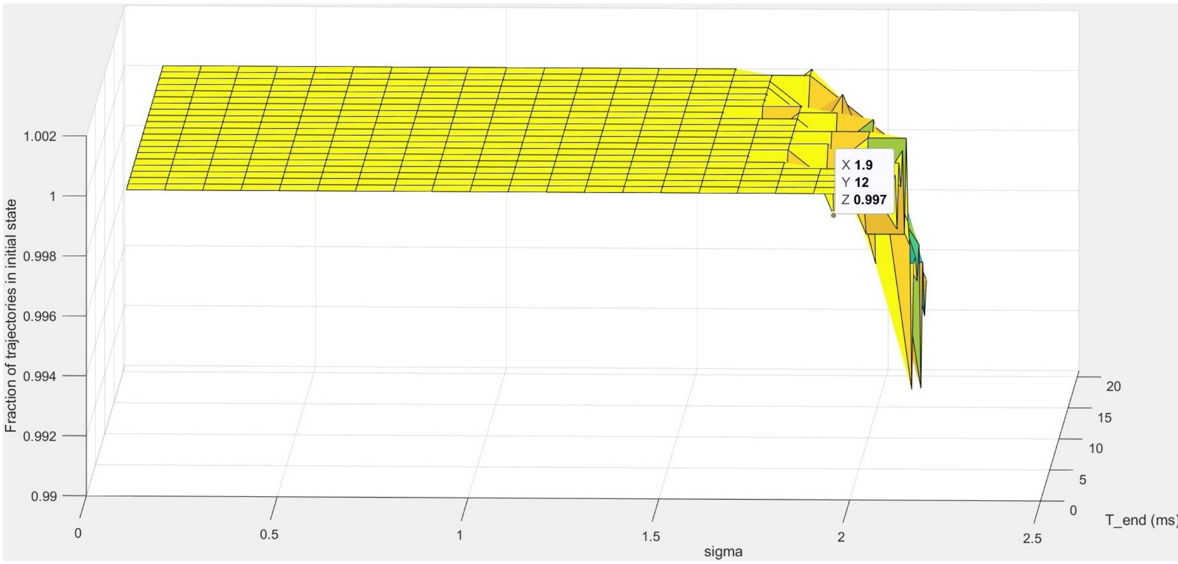

Figure 10: 3D plot of the fraction of trajectories that remains in the the initial state for the duration of the stimulation. On the selected point $X \rightarrow \sigma$, $Y \rightarrow T_{end}$, and $Z \rightarrow Fraction$

Fig10 shows a 3D plot of the fraction of trajectories that remain in the initial state as a function of T_{end} and sigma. We can visually inspect the plot to choose a suitable combination of T_{end} and σ that results in at least 95% of trajectories remaining in the initial state. I selected $T_{end} = 12$ and $\sigma = 1.9$.
In the next step, I computed the average spectrum of $E_1(t)$ and $E_2(t)$ over 100 realizations, we can use the Fourier transform of the time series:

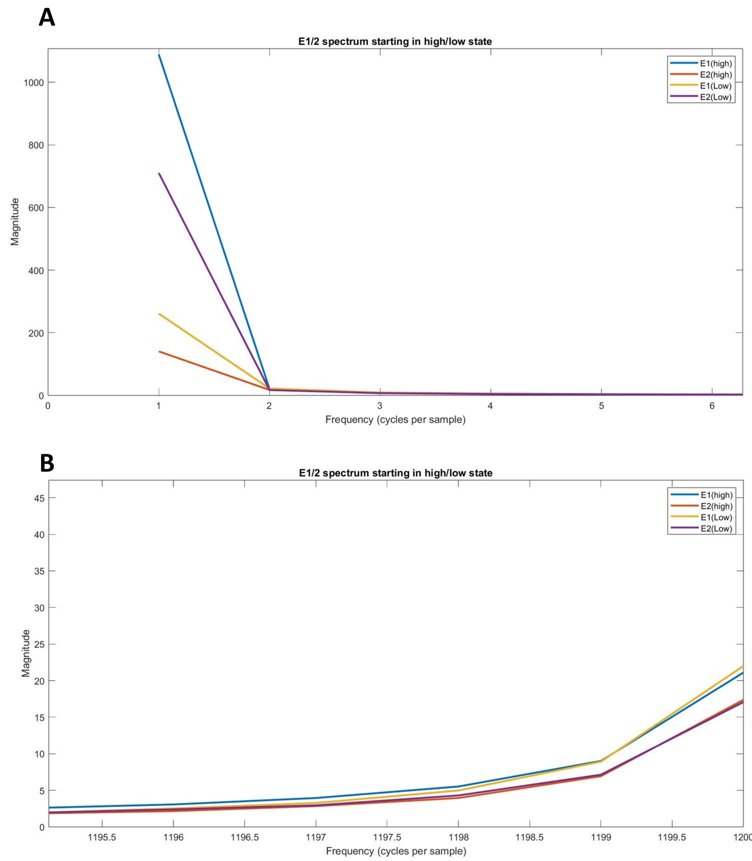

Figure 11: Spectrum of Populations starting in low or high states. **A.**Low-frequency range. **B.**High-frequency range

The spectrum of a signal shows how the power of the signal is distributed across different frequencies. In this case, the spectra of E_1 and E_2 represent the power spectrum of the neural activity in those two populations, while the spectra of N_1 and N_2 represent the power spectrum of the Brownian noise in the system.

The fact that the spectra of E_1 and E_2 have peaks at certain frequencies can suggest the presence of oscillations in the neural activity. The specific frequencies at which these peaks occur can provide information about the underlying mechanisms that give rise to these oscillations.
As The plot in Fig11 depicted, both Excitatory and Inhibitory populations have more power in lower and higher frequencies. To understand these results better I used "Differential spectrum":

$$D(f) = |P(f_{High})| - |P(f_{Low})|$$

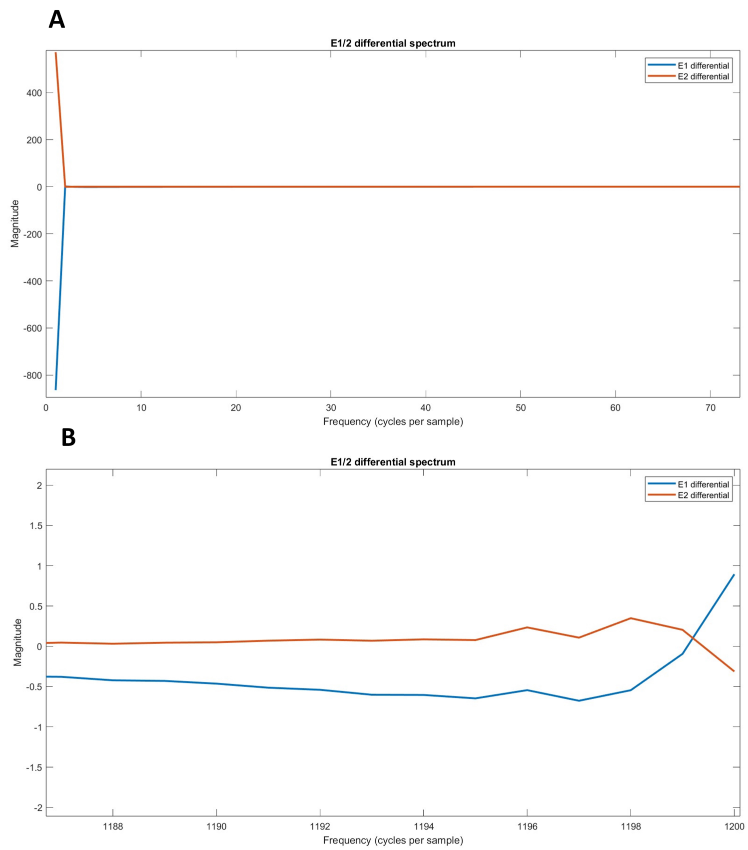

Figure 12: Differential spectrum of Populations starting in low or high states. **A.**Low-frequency range. **B.**High-frequency range

The differential spectrum between two states can give information about the stability of the system. If the differential spectrum shows peaks or significant differences in power between the two states at particular frequencies, this could indicate that the system is more likely to transition between the two states at those frequencies.
For example, if the differential spectrum shows higher power at a certain frequency when starting in the low state compared to the high state, this could suggest that the low state is less stable and more likely to transition to the high state at that frequency. Similarly, if the differential spectrum shows higher power at a certain frequency when starting in the high state compared to the low state, this could suggest that the high state is less stable and more likely to transition to the low state at that frequency.
Here, we could see that at a lower frequency, E_1 is more stable although that's not the case for E_2. This pattern repeats for almost all of the frequencies even

though with very tiny differences.

On The Very Last step, I compared the Theoretical and simulated values of the power spectrum of Brownian noise $N(t)$. The comparison between the theoretical and simulated power spectrum of Brownian noise can give you an indication of how well the theoretical model fits the actual data. In general, if the actual power spectrum closely follows the theoretical power spectrum, it suggests that the underlying process generating the data is consistent with the assumptions made in the theoretical model.
If there are significant deviations between the theoretical and actual power spectrum, it may indicate that the model is not an accurate representation of the underlying process or that there are other factors influencing the observed data that are not accounted for in the model.
The Theoretical spectrum of Brownian noise calculates as follows:

$$S(w) = \frac{|\widetilde{N}(t)|}{T} = \frac{\tau_n \sigma_n^2}{1 + \tau_n^2 w^2}$$

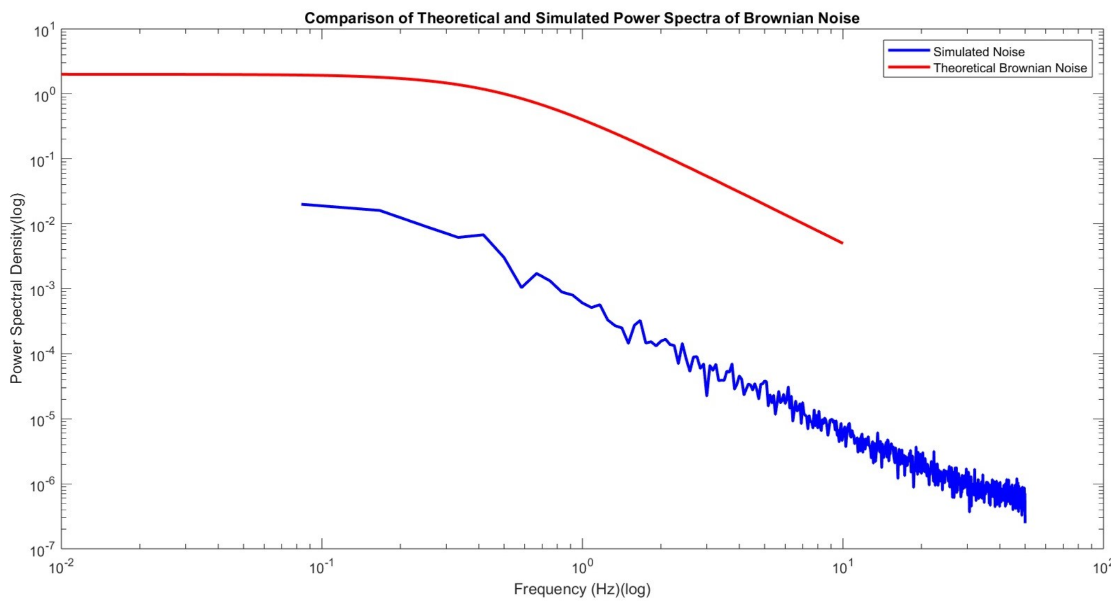

Figure 13: Comparison of power spectrum Theoretical and simulated Brownian noise (Ornstein-Uhlenbeck process at steady-state)

The power spectrum of the simulated data starts at a lower frequency and has a lower power density compared to the theoretical power spectrum.
One possible explanation for the difference in power density between the simu-

lated and theoretical power spectra is that the simulated data may have additional noise or measurement errors that are not accounted for in the theoretical model. It is also possible that the underlying process generating the simulated data is different from the process assumed in the theoretical model.
To further investigate the difference between the simulated and theoretical power spectra, it may be useful to examine other statistical measures and characteristics of the data, such as the autocorrelation function, histogram, or probability distribution. These measures can provide additional information about the underlying process and help to identify potential sources of variation or error in the data.