Modular Forms, and Elliptic curves and Modular curves

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Part I
The Modularity Theorem

The modularity theorem was first partially proven by Andrew Wiles in 1995 (before known as the Taniyama-Shimura conjecture) and led to a proof of Fermat's Last theorem (1637). The proof was then completed in 2001, showing a strong connexion between two kind of objects : modular forms and elliptic curves. In this part we will introduce some notions in order to understand the statement of the modularity theorem :

Theorem (Modularity Theorem) All elliptic curves over $\mathbb{Q}$ are modular.

1 Riemann surfaces

A Riemann surface is a connected one-dimensional complex manifold, in this section we will specify what it means.

Definition 1.1 A n-manifold $\mathcal{M}$ is a Hausdorff topological space locally homeomorphic to $\mathbb{R}^n$. This means that for every point $p \in \mathcal{M}$ there exists an open neighborhood U of p and an homeomorphism $\varphi : U \to \mathbb{R}^n$ to an open subset of $\mathbb{R}^n$. (U, φ) is called a chart.

Definition 1.2 Given two charts (U, φ) and (V, ψ) of an n-manifold $\mathcal{M}$ with $U \cap V \neq \emptyset$ the map $\varphi \circ \psi^{-1} : \psi(U \cap V) \subset \mathbb{R}^n \to \varphi(U \cap V) \subset \mathbb{R}^n$ is called a transition map.

We want to extend theses notions to the complex numbers therfore we can replace $\mathbb{R}^n$ by $\mathbb{C}^n$ in the two previous definitions to define a complex n-manifold. A real n-manifold is said to be $\mathcal{C}^k$ if it's transition maps are all $\mathcal{C}^k$ similarly we will consider the complex 1-manifolds whose transition maps are not only smooth but also holomorphic, wich is a much more stronger constraint.

Definition 1.3 A Riemann surface is a connected complex 1-manifold whose transitions maps are holomorphic.

Definition 1.4 A map $f : \mathcal{M}_1 \to \mathcal{M}_2$ between two Riemann surfaces is said to be holomorphic if for each charts $(U, \varphi), (V, \psi)$ of $\mathcal{M}_1, \mathcal{M}_2$ respectively, the complex function $\psi \circ f \circ \varphi^{-1}$ is holomorphic over $\varphi(U \cap f^{-1}(V))$.

2 Lattices and complex tori

Definition 2.1 A **lattice** is a subset of $\mathbb{C}$ of the form $\omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ with $\{\omega_1, \omega_2\}$ a $\mathbb{R}$-basis of $\mathbb{C}$.

This is equivalent to $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$. We can suppose that $\frac{\omega_1}{\omega_2} \in \mathcal{H}$ for convenience.

Definition 2.2 We say that two lattices Λ and Λ' are isogenuous if there exists an $m \in \mathbb{C}$ such that $m\Lambda = \Lambda'$.

Proposition 2.3 Every lattice $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ is isogenuous to $\mathbb{Z} \oplus \tau\mathbb{Z}$ for some $\tau \in \mathcal{H}$.

Proof : $\omega_1 \neq 0$ so we can consider the isogenuous lattice $(\frac{1}{\omega_1})\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$ with $\tau = \frac{\omega_2}{\omega_1}$, if $\tau \notin \mathcal{H}$ then $-\tau \in \mathcal{H}$ and $\mathbb{Z} \oplus -\tau\mathbb{Z} = \mathbb{Z} \oplus \tau\mathbb{Z}$.

Proposition 2.4 Two lattices $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$ and $\Lambda' = \mathbb{Z} \oplus \tau'\mathbb{Z}$ are isogenuous if and only if $\tau' = \gamma \cdot \tau$ for some $\gamma \in SL_2(\mathbb{Z})$.

Definition 2.5 The complex tori associated to a lattice Λ is the abelian quotient group $\mathbb{C}/\Lambda$.

Proposition 2.6 The quotient topology over $\mathbb{C}/\Lambda$ makes it a Riemann surface.

Definition 2.7 An **isogeny** of complex tori $\varphi : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ is a holomorphic group isomorphism. $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$ are then said to be isogenuous.

Theorem 2.8 Two complex tori $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$ are isogenuous if and only if Λ and Λ' are isogenuous.

Remark: The map $\Psi : \Lambda \mapsto \mathbb{C}/\Lambda$ is a bijection from the set of lattices to the set of complex tori that preserves isogengy. Therefore, the categories of lattices up to isogeny and complex tori up to isogengy are equivalent.

3 Complex tori and elliptic curves

Definition 3.1 Given a sub-field $\mathbb{K}$ of $\mathbb{C}$, an **elliptic curve** over $\mathbb{K}$ is the set of points $(x, y) \in \mathbb{K}^2$ such that

$$y^2 = 4x^3 - ax - b$$

for some $(a, b) \in \mathbb{K}^2$ such that $a^3 - 27b^2 \neq 0$.

Remark: The condition $a^3 - 27b^2 \neq 0$ correspond to the curve being smooth.

Definition 3.2 An isogeny of moduar curves is a map between two modular curves such that $\phi(x, y) = (R_1(x, y), R_2(x, y))$ with R_1 and R_2 two rational fuctions and that is also a group morphism.

Proposition 3.3 If there is an isogeny ϕ from $E1$ to $E2$ then there is an isogeny $\psi : E_1 \to E_2$ called the dual isogeny, and we say that $E1$ and $E2$ are isogenuous. This defines an equivalence relation.

Remark: It can be shown that every curve that is the zero set of a polynomial of the form :

$$y^2z - (ax^3 + bx^2z + cxz^2 + dz^3)$$

is isogenuous (if we adapt the previous definition) to a curve of the form

$$y^2z - (x^3 + bxz^2 + cz^3)$$

whose solutions of the form (x,y,1) satisfies the equation :

$$y^2 = x^3 + bx + c$$

Definition 3.4 Given a complex latice Λ, the Weierstrass function $\wp$ is defined by :

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda, \omega \neq 0} \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2}$$

Proposition 3.5 $\wp$ and $\wp'$ are Λ-periodic.

Proposition 3.6 The Weierstrass function satisfies the differential equation :

$$\wp'^2 = 4\wp^3 - g_2(\Lambda)\wp - g_3(\Lambda)$$

Theorem 3.7 The map :

$$z + \Lambda \to (\wp(z), \wp'(z))$$

is a bijection from nonzero points of $\mathbb{C}/\Lambda$ to the elliptic curve associated to the equation

$$y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda)$$

Definition-Proposition 3.8 (Group law) Given an elliptic curve E, $\mathbb{C}/\Lambda$ its associated complex tori and two points $U, V \in E$. There exists two points $u, v \in \mathbb{C}/\Lambda$ such that $U = (\wp(u), \wp'(u))$ and $V = (\wp(v), \wp'(v))$. Let's consider the line of $\mathbb{C}^2$ going through U and V if $U \neq V$ and the tangent line to E at U if $U = V$. This gives us an line equation of the form

$$ax + by + c = 0$$

Lets consider the function f over $\mathbb{C}/\Lambda$ such that:

$$f(z) = a\wp(z) + b\wp'(z) + c$$

We observe that f has zeros at points u, v. If $b \neq 0$ then f has exactly one other zero w and it is such that $u + v + w + \Lambda = 0 + \Lambda$. If $b = 0$ then $u + v + \Lambda = 0 + \Lambda$ then let $w = 0$ and again $u + v + w + \Lambda = 0 + \Lambda$. Lets define a point O "at infinity" to make notations coherent we denote $(\wp(0), \wp'(0)) := O$. Thus we define the group law on $E \cup \{O\}$ by :

$$U + V := (\wp(-u - v), \wp'(-u - v))$$

This makes the map : $\begin{cases} \mathbb{C}/\Lambda & \longrightarrow & E \cup \{O\} \\ z + \Lambda & \longmapsto & (\wp(z), \wp'(z)) \end{cases}$ a group isomorphism.

Theorem 3.9 Let E_Λ and $E_{\Lambda'}$ be some elliptic curves associated to the lattices Λ and Λ' respectively, then the following statements are equivalent :

i) E_Λ and $E_{\Lambda'}$ are isogenuous
ii) $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$ are isogenuous.
iii) Λ and Λ' are isogenuous.

This theorem reduces the problem of parametrization of elliptic curves up to isogeny to the much simpler problem of parametrization of lattices up to isogeny.

4 Modular curves

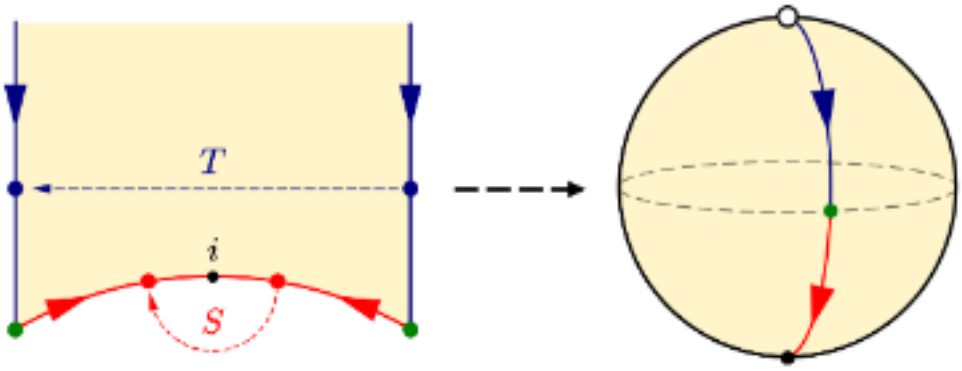
In this section we are looking for a moduli spaces (i.e a space that parametrize) of latices up to isogeny or equivlalently of elliptic curvesup to isogengy.

Definition 4.1 Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$, we define the modular curve associated to Γ by $Y(\Gamma) := \{\Gamma\tau, \tau \in \mathcal{H}\}$, i.e the set of orbits of the action of Γ on $\mathcal{H}$.

Definition 4.2 The fundamental domain $\mathcal{D} \subset \mathcal{H}$ of the modular form $Y(\Gamma)$ is a region that contains exactly one point from each orbit of Γ action except some possible duplication on the edge.

Proposition The region $\mathcal{D} = \{\tau \in \mathcal{H} | \tau| \geq 1 and |Re(\tau)| \leq 1/2\}$ is a fundamental domain for $Y(1)$.

After identification of duplicate points on the edge we see that $Y(1) \simeq SL_2(\mathbb{Z})\backslash\mathcal{D}$ *is a punctured sphere.*



Proposition $Y(1)$ can be made a Riemann surface and compactified into $X(1)$ by adding a point at infinity.

Proposition Similarly for any congruece subgroup Γ, $Y(\Gamma)$ can be made a Riemann surface and compactified into $X(\Gamma)$ by adding a finite number of points. We will denote $X(\Gamma_0(N))$ as $X_0(N)$ for any integer N.

Definition A complex elliptic curve E is said to be **modular** if there exists an integer N such that there is a surjection holomorphic map φ from the modular curve $X_0(N)$ to E as Riemann surfaces. φ is the called a modular parametrization of E.

TODO congruence subgroup, morphisms between Riemann surfaces, ex ref [1] G_2 $g_2(\Lambda)$ Eisenstein series genus...

References

[1] PETER ZHOU. *THE MODULARITY THEOREM*. URL: `https://math.uchicago.edu/~may/REU2023/REUPapers/Zhou,Peter.pdf`.