Modular Forms, and Elliptic curves and Modular curves

Armand Perrin, Samy, Damian

April 15, 2025

Part I
The Modularity Theorem

Theorem (Modularity Theorem) All elliptic curves over $\mathbb{Q}$ are modular.

Definition Given a sub-field $\mathbb{K}$ of $\mathbb{C}$, an **elliptic curve** over $\mathbb{K}$ is the set of points $(x, y) \in \mathbb{K}^2$ such that

$$y^2 = x^3 + ax + b$$

for some $(a, b) \in \mathbb{K}^2$.

Definition A **Riemann surface** is a connected one-dimensional complex manifold.

Proposition We can associate any elliptic curve E with a Riemann surface : the complex tori $\mathbb{C}/\Lambda$, where Λ is the latice associated to E.

Proposition The Riemann surface $Y(\Gamma)$ can be compactified into $X(\Gamma)$ by adding a finite number of points.

Definition A complex elliptic curve E is said to be **Modular** if there exists an integer N such that there is a surjection morphism from the modular curve $X_0(N)$ to E as Riemann surfaces.

TODO morphisms between riemann surfaces, $Y(\Gamma), X(\Gamma), X_0(N)$ ex ref [1]

References

[1] PETER ZHOU. *THE MODULARITY THEOREM*. URL: `https://math.uchicago.edu/~may/REU2023/REUPapers/Zhou,Peter.pdf`.