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ORP 5001

Introduction to Linear Optimization

Trajectory Optimization for RVD with Rotating Target through Single Waypoint Placement

Submitted to

Munevver Subasi, PhD.

Associate Professor

Department of Mathematical Sciences

Submitted by

Armando Rolins, B.S.

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Linearized Dynamic Equations for Relative Motion in Orbit

Most spacecraft orbital maneuvers and trajectories are performed based on an inertial frame of reference fixed on a large body's center of mass (i.e. Earth's center of mass for Earth bound orbits). However this coordinate frame is not ideal for two objects within close proximity to each other to perform RVD.

Close proximity operations between two objects in orbit are based on a target, or reference frame between object A to object B. This coordinate frame is rotating and noninertial, which means that Newton's second law cannot be used directly to model the dynamics of relative motion. The

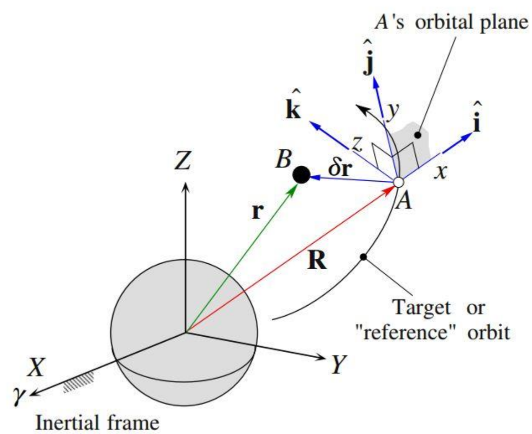

Figure 1 Relative motion reference frame [1]

coordinate frame for relative motion is shown in Figure 1

The coordinate frame is centered on one of the objects, object A (the target) and describes the state of object B (the chaser) to the origin. If **R** is defined as the inertial position of the target, then the position of the chaser is described by

$$\boldsymbol{r} = \boldsymbol{R} + \delta\boldsymbol{r} \tag{1}$$

Where $\delta\boldsymbol{r}$ is the relative position between the chaser and the target. Because this coordinate frame is defined in close proximity between target and chaser, $\frac{\delta r}{R} \ll 1$ where $\delta r = \|\delta\boldsymbol{r}\|$ and $R = \|\boldsymbol{R}\|$. The motion of the chaser relative to geocentric inertial frame can be expressed by

$$\ddot{\boldsymbol{r}} = -\mu\frac{\boldsymbol{r}}{r^3} \tag{2}$$

Where $r = \|\boldsymbol{r}\|$. It is possible to express the motion of the chaser relative to the target by inserting Equation (1) into Equation (2) and linearizing the result, neglecting terms with higher order than 1;

$$\delta\ddot{\boldsymbol{r}} = -\frac{\mu}{R^3}\left[\delta\boldsymbol{r} - \frac{3}{R^2}(\boldsymbol{R}\cdot\delta\boldsymbol{r})\boldsymbol{R}\right] \tag{3}$$

The only resulting unknown in this equation is $\delta\boldsymbol{r}$ which is only a first order term.

The vector components for $\delta\boldsymbol{r}$ in a commoving frame can be expressed as

$$\delta\boldsymbol{r} = \delta x\hat{\boldsymbol{\imath}} + \delta y\hat{\boldsymbol{\jmath}} + \delta z\hat{\boldsymbol{k}} \tag{4}$$

So the equation of motion of the chaser relative to the target in the commoving frame can be expressed by substituting Equation (4) into Equation (3), and in its simplest form yields

$$\delta\ddot{\boldsymbol{r}} = -\frac{\mu}{R^3}\left[-2\delta x\hat{\boldsymbol{\imath}} + \delta y\hat{\boldsymbol{\jmath}} + \delta z\hat{\boldsymbol{k}}\right] \tag{5}$$

Through a sequence of substitutions found in [1, pp. 379,380], it is possible to arrive at three scalar differential equations

$$\delta\ddot{x} - \left(\frac{2\mu}{R^3} + \frac{h^2}{R^4}\right)\delta x + \frac{2(\boldsymbol{V}\cdot\boldsymbol{R})h}{R^4}\delta y - \frac{2h}{R^2}d\dot{y} = 0 \tag{6a}$$

$$\delta\ddot{y} + \left(\frac{\mu}{R^3} - \frac{h^2}{R^4}\right)\delta y - \frac{2(\boldsymbol{V}\cdot\boldsymbol{R})h}{R^4}\delta x + \frac{2h}{R^2}d\dot{x} = 0 \tag{7b}$$

$$\delta\ddot{z} + \frac{\mu}{R^3}\delta z = 0 \tag{8c}$$

If the reference orbit – that of the target – is taken to be circular, $\boldsymbol{V}\cdot\boldsymbol{R} = 0$, $h = \sqrt{\mu R}$ and the angular velocity $\omega = \sqrt{\frac{\mu}{R^3}}$. So Equations (6a),(7b),(8c) can be expressed in a simplified form as

$$\delta\ddot{x} - 3\omega^2\delta x - 2\omega\delta\dot{y} = 0 \tag{7(a)}$$

$$\delta\ddot{y} + 2\omega\delta\dot{x} = 0 \tag{7(b)}$$

$$\delta\ddot{z} + \omega^2\delta z = 0 \tag{7(c)}$$

These three equations are collectively called the Clohessy-Wiltshire-Hill equations, for which an analytical solution exists and model the dynamics of RVD used in this research.

Derivation of Optimal Feedback Guidance Law

Given the dynamics of the environment of the problem in question it is necessary to define the control laws that should be used for the chaser to follow a given trajectory to the target. The control laws will dictate the control input in the form of acceleration $a(t)$ or equivalently ΔV based on the relative position and velocity of the chaser to the target. Because the control input is directly related to propellant expenditure, it is also taken as the cost function the control law will try minimize. The optimal control law will therefore define a series of control inputs as the chaser moves towards the target and thus will define the trajectory which requires least propellant expenditure.

Optimal Control Law for Simple Rendezvous

The optimal control law for a simple rendezvous problem is discussed by Bryson & Ho in [2, pp. 154,155], it uses a simple linear system with a quadratic criteria as such;

$$\dot{\boldsymbol{r}} = \boldsymbol{v}$$

$$\dot{\boldsymbol{v}} = \boldsymbol{a}(t)$$

$$\boldsymbol{J} = \frac{1}{2}\left[[\boldsymbol{v} \quad \boldsymbol{r}] \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{r} \end{bmatrix} \right]_{t=t_f} + \frac{1}{2}\int_t^{t_f} \boldsymbol{a}^T \boldsymbol{a}\, dt$$

Where J is the objective function, C_1, C_2 are constants and t_f is the final time. If the acceleration is defined in terms of lagrange multipliers, then

$$\boldsymbol{a}(t) = -\Lambda_r(t)\boldsymbol{r}(t) - \Lambda_v(t)\boldsymbol{v}(t) \quad 8(a)$$

$$\Lambda_r = \frac{\frac{1}{C_1}\left(t_f - t\right) + \frac{1}{2}(t_f - t)^2}{D(t_f - t)} \quad 8(b)$$

$$\Lambda_v = \frac{\frac{1}{C_2} + \frac{1}{C_1}(t_f - t)^2 + \frac{1}{3}(t_f - t)^3}{D(t_f - t)} \quad 8(c)$$

Where

$$D\left(t_f - t\right) = \left[\frac{1}{C_1} + \left(t_f - t\right)\right]\left[\frac{1}{C_2} + \frac{1}{3}\left(t_f - t\right)^3\right] - \frac{1}{4}(t_f - t)^4 \quad 9$$

In this form, the acceleration term puts into light some properties of the problem by varying the constants C_1 and C_2.

Case 1: If C_1 is unconstrained and $C_2 \rightarrow \infty$

$$\Lambda_r = 0 \tag{10(a)}$$

$$\Lambda_v = \frac{1}{\frac{1}{C_1} + (t_f - t)} \tag{10(b)}$$

$$\boldsymbol{a}(t) = -\frac{\boldsymbol{v}(t)}{\left[\frac{1}{C_1} + (t_f - t)\right]} \tag{10(c)}$$

Then $r(t_f)$ is uncontrolled, and this is called "velocity to be gained/lost" control

Case 2: If $C_1 \rightarrow 0$ and $C_2 \rightarrow \infty$

$$\Lambda_r = \frac{(t_f - t)}{\frac{1}{C_2} + \frac{1}{3}(t_f - t)^3} \tag{11(a)}$$

$$\Lambda_v = \frac{(t_f - t)^2}{\frac{1}{C_2} + \frac{1}{3}(t_f - t)^3} \tag{11(b)}$$

$$\boldsymbol{a}(t) = -3\left[\frac{\boldsymbol{v}(t)}{(t_f - t)} + \frac{\boldsymbol{r}(t)}{(t_f - t)^2}\right] \tag{11(c)}$$

Then $r(t_f) \rightarrow 0$. If V is defined as the velocity along the line of sight, and we let the line of sight angle

$$\lambda = \frac{\boldsymbol{r}(t)}{\boldsymbol{V}(t_f - t)} \tag{12}$$

So that the line of sight defines the angle between the current chaser velocity vector and the relative position vector to the target. Then

$$\boldsymbol{a}(t) = -3\boldsymbol{V}\dot{\lambda} \tag{13}$$

Which is also known as Proportional Navigation Guidance (PNG). This is usually used for missiles which require a target intercept, which means reducing relative distance to the target to 0 but having no constraint on final velocity.

For a rendezvous intercept however, the final time boundary is constrained both with respect to relative position as well as relative velocity, since it is desirable to make the intercept as soft as possible to avoid mechanical damage to the docking port. This leads to the last case of the problem.

Case 3: If $C_1 \to 0$ and $C_2 \to 0$

Then the lagrange multipliers remain the same as in Case 2, but

$$\boldsymbol{a}(t) = -\frac{6\boldsymbol{r}(t)}{\left(t_f - t\right)^2} - \frac{4\boldsymbol{v}(t)}{\left(t_f - t\right)} \tag{14}$$

Which is the guidance law used for rendezvous where the desired final state is set to 0.

Constrained Terminal-Velocity Guidance (CTVG)

In a similar way, it is possible to consider the same problem with a different cost function. This is outlined in [3] as such

$$\dot{\boldsymbol{r}} = \boldsymbol{v} \tag{15(a)}$$

$$\dot{\boldsymbol{v}} = \boldsymbol{g} + \boldsymbol{a}(t) \tag{15(b)}$$

$$\boldsymbol{J} = \frac{1}{2}\int_{t}^{t_f} \boldsymbol{a}^T \boldsymbol{a}\, dt \tag{15(c)}$$

Subject to the following boundary conditions;

$$\boldsymbol{r}(t_0) = \boldsymbol{r_0}, \qquad \boldsymbol{r}\left(t_f\right) = \boldsymbol{r}_f \tag{16(a)}$$

$$\boldsymbol{v}(t_0) = \boldsymbol{v}_0, \qquad \boldsymbol{v}\left(t_f\right) = \boldsymbol{v}_f \tag{16(b)}$$

The Hamiltonian function is then defined as

$$H = \frac{1}{2}\boldsymbol{a^T a} + \boldsymbol{p}_r{}^T \boldsymbol{v} + \boldsymbol{p}_v{}^T(\boldsymbol{g} + \boldsymbol{a}) \tag{17}$$

where costate vector $\boldsymbol{p_r}$ is associated with position and $\boldsymbol{p_v}$ is associated with velocity. As mentioned in the previous section, the close proximity rendezvous problem is one in which gravitational variation with time and position can be considered negligible, and therefore $\boldsymbol{g}$ here is assumed to be a constant. This gives the costate equations and optimal control as:

$$\dot{\boldsymbol{p}}_r = -\frac{\delta H}{\delta \boldsymbol{r}} = 0 \tag{18(a)}$$

$$\dot{\boldsymbol{p}}_v = -\frac{\delta H}{\delta \boldsymbol{v}} = -\boldsymbol{p}_r \tag{18(b)}$$

$$\frac{\delta H}{\delta \boldsymbol{a}} = 0 \Rightarrow \boldsymbol{a} = -\boldsymbol{p}_v \tag{18(c)}$$

If we define the $t_{go} = \left(t_f - t\right)$ then the solutions to the costate equations can be expressed as

$$\boldsymbol{p}_r = \boldsymbol{p}_r\left(t_f\right), \qquad \boldsymbol{p_v} = t_{go}\boldsymbol{p}_r\left(t_f\right) + \boldsymbol{p}_v(t_f) \tag{19(a,b)}$$

So the optimal control law can be expressed in terms of the Hamiltonian as

$$\boldsymbol{a}(t) = -t_{go}\boldsymbol{p}_r(t_f) + \boldsymbol{p}_v(t_f) \quad 20$$

So the states of the system are

$$\boldsymbol{v} = \frac{{t_{go}}^2}{2}\boldsymbol{p}_r(t_f) + t_{go}\boldsymbol{p}_v(t_f) - t_{go}\boldsymbol{g} + \boldsymbol{v_f} \quad 21(a)$$

$$\boldsymbol{r} = -\frac{{t_{go}}^3}{6}\boldsymbol{p}_r(t_f) - \frac{{t_{go}}^2}{2}\left(\boldsymbol{p}_v(t_f) + \boldsymbol{g}\right) - t_{go}\boldsymbol{v_f} + \boldsymbol{r_f} \quad 21(b)$$

So the costate vectors can be written as

$$\boldsymbol{p}_r(t_f) = \frac{6(\boldsymbol{v} + \boldsymbol{v_f})}{{t_{go}}^2} + \frac{12(\boldsymbol{r} - \boldsymbol{r_f})}{{t_{go}}^3} \quad 22(a)$$

$$\boldsymbol{p}_v(t_f) = -\frac{2(\boldsymbol{v} + 2\boldsymbol{v_f})}{t_{go}} - \frac{2(\boldsymbol{r} - \boldsymbol{r_f})}{{t_{go}}^2} + \boldsymbol{g} \quad 22(b)$$

So the CTVG can be described by the desired state as

$$\boldsymbol{a}(t) = \frac{6[\boldsymbol{r_f} - (\boldsymbol{r} + t_{go}\boldsymbol{v})]}{{t_{go}}^2} - \frac{2(\boldsymbol{v_f} - \boldsymbol{v})}{t_{go}} - \boldsymbol{g} \quad 23(a)$$

Or

$$\boldsymbol{a}(t) = \frac{6[\boldsymbol{r_f} - (\boldsymbol{r} + t_{go}\boldsymbol{v})]}{{t_{go}}^2} + \frac{4(\boldsymbol{v_f} - \boldsymbol{v})}{t_{go}} - \boldsymbol{g} \quad 23(b)$$

For the soft rendezvous constraint where $\boldsymbol{v_f} = 0$, then the CTGV has the form

$$\boldsymbol{a}(t) = -\frac{6(\boldsymbol{r_f} - \boldsymbol{r})}{{t_{go}}^2} - \frac{4\boldsymbol{v}}{t_{go}} - \boldsymbol{g} \quad 24$$

Which is equivalent to equation mentioned previously. If the terms ZEM and ZEV are defined as $\boldsymbol{ZEM} = (\boldsymbol{r} - \boldsymbol{r_f})$, and $\boldsymbol{ZEV} = (\boldsymbol{v} - \boldsymbol{v_f})$, then

$$\boldsymbol{a}(t) = \frac{6\boldsymbol{ZEM}}{{t_{go}}^2} - \frac{2\boldsymbol{ZEV}}{t_{go}} - \boldsymbol{g} \quad 25$$

Which is the guidance law used for the algorithm for RVD with a rotating target.

Numerical Simulations

The first step to investigate the feasibility of any algorithm designed to solve a real world problem is to run a numerical simulation. The parameters of the simulator are based off of the dimensions found in the Florida Institute of Technology ORION lab [4], with the target object placed in the center. The center is also the origin of the coordinate system, which means that for an area of dimensions 6 m x 3.6 m the coordinate grid ranges from -3 m to 3 m on the x-axis and -1.8 to 1.8 on the y-axis.

The target and chaser objects are similarly dimensioned cubes of size 0.8 m x 0.8 m x 0.8 m, the distance from the origin to the docking port on the face of the target cube is therefore 0.4 m. When the target cube is set to rotate, the docking port position traces a circle around the origin which delimits an area inside the target. Any trajectory which penetrates that area therefore causes a collision between the chaser and the target.

The initial position chosen for the chaser to start at is $\boldsymbol{r} = [2.5\ 1.3]^T$ and its velocities are set to be 0 so that it is stationary. The target is set to rotate at $\omega = 1\ rpm$. These parameters are kept constant for every simulation.

Unconstrained ZEM/ZEV Trajectory

The unconstrained trajectory is the simplest form of the problem where the acceleration of the chaser is the acceleration due to the CWH terms added to the input acceleration determined by the ZEM/ZEV Guidance law.

$$\boldsymbol{a}(t) = \boldsymbol{a}_{CWH}(t) + \boldsymbol{a}_{ZEM/ZEV}(t) \qquad (26)$$

When this acceleration is applied to the chaser, its path can be propagated to find the optimal trajectory for a smooth unconstrained rendezvous, the one variable the problem is dependent on is therefore the total time of flight. The objective function can be defined as

$$\boldsymbol{a}(t) = f(t_{flight}) \qquad (27)$$

where the time to go as defined in the ZEM/ZEV law is defined as

$$t_{go} = t_{flight} - t \qquad (28)$$

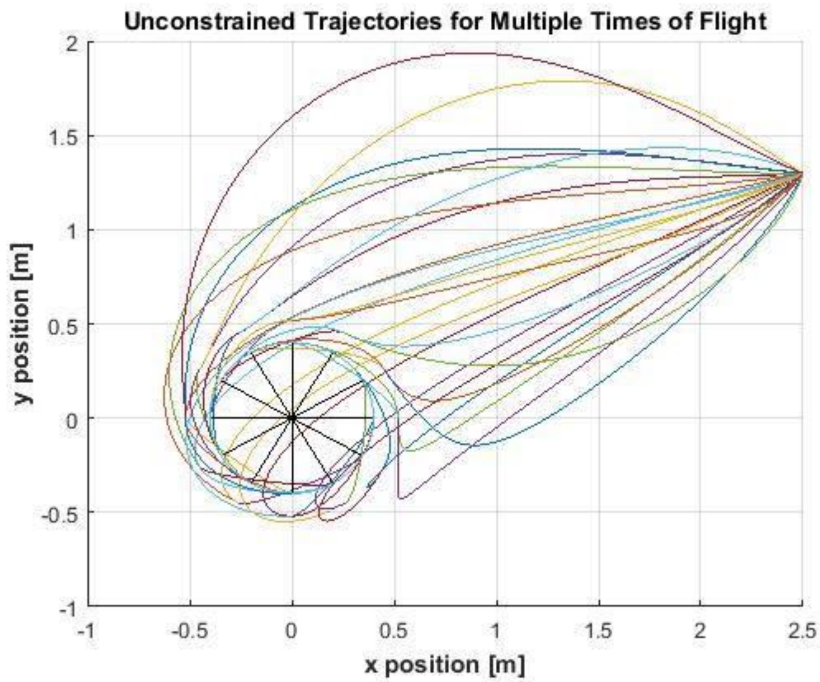

Figure 2 Unconstrained Trajectories from 30 seconds to 150 seconds time of flight

The figure above shows the optimal trajectory and smooth rendezvous with the target for 30 seconds to 150 seconds time of flight. It is readily apparent however that some of these trajectories pass through the inside of the circle which represents the target, this implies a collision between the chaser and the target and therefore cannot be deemed as feasible solutions to the rendezvous problem. The unconstrained ZEM/ZEV guidance law is insufficient to ensure a feasible trajectory, it is necessary to drive solutions away from the sweeping radius of the target docking port.

Waypoint constraint driven ZEM/ZEV Trajectory

One approach to the problem of avoiding trajectories that intersect the target mid-flight is to introduce a waypoint through which the chaser has to fly during its trajectories. The new problem now becomes choosing the waypoint parameters such as to produce trajectories which steer clear of the target. The objective function now can be defined as dependent on more variables

$$\boldsymbol{a}(t) = f(\boldsymbol{r_{waypoint}}, \boldsymbol{v_{waypoint}}, t_{waypoint}, t_{flight}) \quad 29$$

Where $\boldsymbol{r}_{waypoint}$ is a 2 dimensional position of the waypoint and $\boldsymbol{v}_{waypoint}$ is the 2 dimensional velocity at which the target will fly through the waypoint, and $t_{waypoint}$ is the time of flight until the waypoint. To better understand what the solution region looks like, a comparison of the special case when t_{flight} will be held constant at 135 seconds, and $t_{waypoint}$ will be established at 90% of t_{flight}.

Brute Force Search for Global Minimum Solution

One of the ways to search for the solution in this case is to calculate the cost for the trajectory at several possible waypoints throughout the field. The points were linearly spaced out as to have a field defined by 100 x 50 waypoints. At each waypoint, an array of 100 random velocity vectors were assigned and the cost function was calculated, and returned the velocity vector which yielded the lowest acceleration cost for each point.

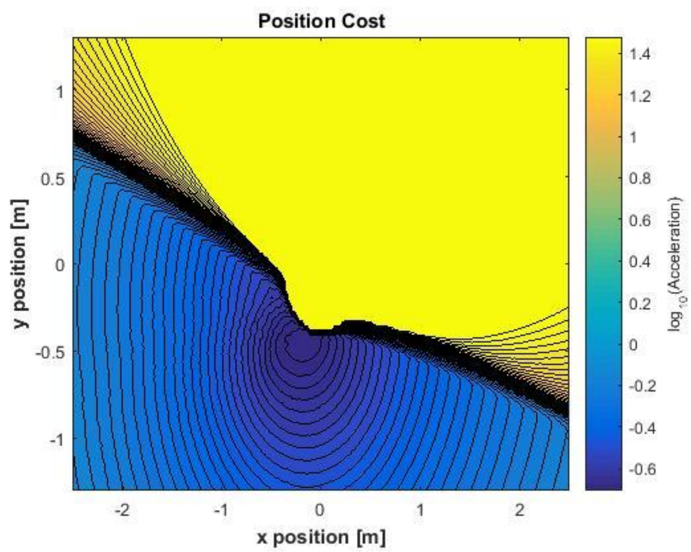

Figure 3 Cost Contour for waypoint position selection

Figure 3 shows how the cost changes depending on the selection of the position of the waypoint. Since the time of flight is fixed at 135 seconds - which results in the docking part being at the lower end of the target - it is clear that the optimal waypoint position is located close to that location since the time of flight to the waypoint is 90% of the time of flight. Any waypoint located within the target was penalized with extra acceleration and points close to the starting point would require a strong control input to catch up to the target in the remaining 10% time.

The actual position of the point which yielded the least cost in this test is

x = -0.113065326633166 m

y = -0.407070707070707 m

As previously mentioned, this waypoint selection is also associated with a velocity vector, the cost for velocity vectors is shown in Figure 4.

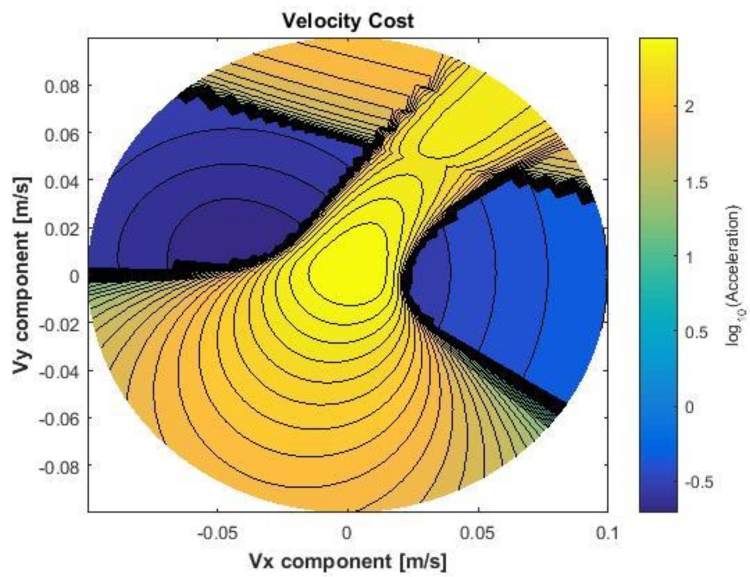

Figure 4 Cost Contour for velocity vector selection

This plot shows some interesting characteristics of this problem. Velocity vectors which point directly into the target attain high acceleration costs since they would require control input corrections to avoid a direct collision as well as enter a smooth intersect trajectory. Velocity vectors which point directly opposite to the target also have high costs since they will also require large control input to direct its motion towards the rendezvous point within the allotted 10% of flight time remaining. The optimal region seems to be a short magnitude velocity vector in the retrograde direction of the target's motion.

The optimal parameters as achieved by this method have a total cost of 0.1957 m/s^2 and are

x = -0.113065326633166 m

y = -0.407070707070707 m

Vx = -0.03828000 m/s

Vy = 0.00669293 m/s

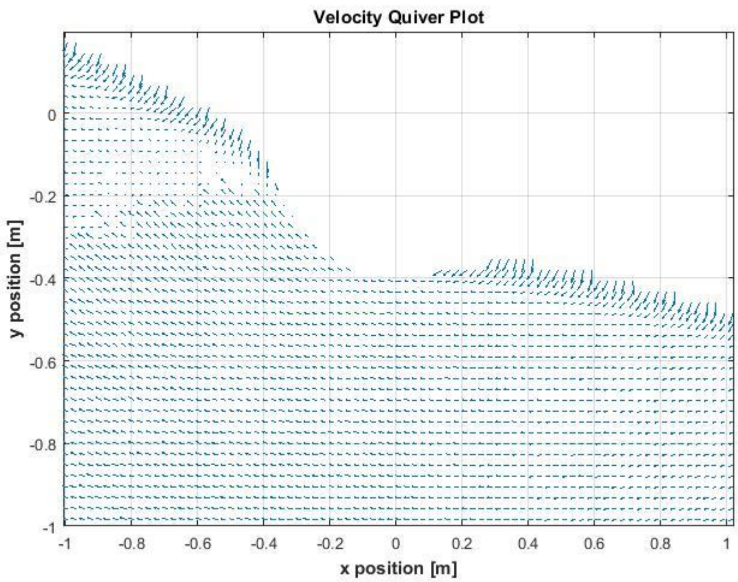

Figure 5 Waypoint Velocities Quiver Plot

Figure 5 shows a quiver plot which confirms the pick of velocity vector for that given waypoint. It shows that most waypoints picked within that region would have an similar optimal velocity vector. Only relatively few points in the top left quadrant seem to yield velocity vectors in significantly different directions.

Line Search for Optimal Solution

The problem with the brute force test is that it is computationally heavy, a Matlab script running to obtain all those plots would run on the order of 40h. The data mentioned previously was acquired through parallel processing in C language.

An alternate way of obtaining a solution is through a Line Search Method [5]. This involves taking an initial guess and updating the guess through a step size α in the descent gradient direction p_k. So the next guess is described by

$$x_{k+1} = x_k + \alpha_k p_k \tag{30}$$

where the index k denotes the step taken in the iterative procedure. The algorithm stops when $\|\nabla f(x_k)\| < tolerance$ where the tolerance for this test was set at 1×10^{-6}. The descent direction in this case was chosen by the steepest descent method, so

$$\boldsymbol{p_k} = -\nabla \boldsymbol{f}(\boldsymbol{x_k}) \tag{31}$$

Finally the step size α_k was chosen to satisfy the Wolfe Conditions

$$\boldsymbol{f}(\boldsymbol{x_k} + \alpha_k \boldsymbol{p_k}) \leq \boldsymbol{f}(\boldsymbol{x_k}) + c_1 \alpha_k \boldsymbol{p_k}^{\boldsymbol{T}} \nabla \boldsymbol{f}(\boldsymbol{x_k}) \tag{32(a)}$$

$$\boldsymbol{p_k}^{\boldsymbol{T}} \nabla \boldsymbol{f}(\boldsymbol{x_k} + \alpha_k \boldsymbol{p_k}) \geq c_2 \boldsymbol{p_k}^{\boldsymbol{T}} \boldsymbol{\nabla} \boldsymbol{f}(\boldsymbol{x_k}) \tag{32(b)}$$

Constants c_1 and c_2 were picked to be 10^{-4} and 0.9 respectively. The initial guess was set to be $x_0 = [-0.5, -0.5, 0, 0.075]$ with $f(x_0) = 0.4355\ m/s^2$.

The line search method improved the solution to $f(x) = 0.23837\ m/s^2$.

The method had trouble dealing with the steep gradients at the collision regions however and crashed quite frequently with many choices of parameters.

Conclusion

The two methods clearly gave different solutions, however these two solutions are reasonably close given computational issues faced with the Line Search algorithm. The plots shown for cost based on position and velocities make it clear that the problem is highly nonlinear and therefore requires some more robust technique for optimization.

Future work will probably involve acquiring solutions through Newton's Method as well as Matlab's optimization functions to have better results for comparison. This research will also involve varying the total time of flight and the time of flight to the waypoint as the 5th and 6th variable in the cost function. This will give a more general picture of how this problem behaves and possibly better techniques to placing this single waypoint can be developed with less computational effort.

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