

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINATION

September 2025

Subject CS2 – Risk Modelling and Survival Analysis Core Principles

Paper A

Time allowed: Three hours and twenty minutes

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator.

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1 An insurance company sells policies that cover accidental damage of mobile phones. The policy duration is 1 year with the premium payable in advance. At the end of the policy year, when the policy is due for renewal, the insurance company will make one of the following decisions:

- offer renewal at the standard premium rate
- offer renewal at a higher premium rate
- decline to offer renewal.

Three quarters of policies on the standard premium rate are renewed on the same rate. Of the remainder, the number offered renewal on higher premiums is four times the number declined. Half of the policies on the higher premium rate are renewed on the same, higher premium rate. Of the remainder, a return to the standard premium rate and decline of cover are equally likely.

None of the policies declined renewal are offered cover on any rates in the future. It is assumed that all offers of renewal are accepted.

- (i) Write down the transition matrix needed to model policy renewal terms using a Markov chain. [2]
- (ii) Calculate the probability that a policyholder who begins with cover on the standard premium rate is declined cover at their second renewal. [2]

A policyholder owns a mobile phone for 4 years.

- (iii) Derive the expected number of premiums paid. [5]
- [Total 9]

2 An insurance company has for many years modelled claims for flood damage from its property insurance policies using a gamma distribution. Because of climate change the company believes it may need to change to a distribution with a thicker tail. The company has started considering the Pareto distribution with two parameters (α and λ) and a density function given by $f(x) = \frac{\alpha\lambda^\alpha}{(\lambda+x)^{\alpha+1}}$.

- (i) Explain how it can be shown quantitatively that the Pareto distribution has a thicker tail than the gamma through:
- (a) the moments of the distributions. [3]
- (b) the limiting density ratio. [4]

In each case you should detail what calculations are necessary and how they evidence a thicker tail.

- (ii) Demonstrate that the Pareto distribution with $\alpha = 1$ has a thicker tail than that with $\alpha = 2$ using the limiting density ratio. [3]
- [Total 10]

- 3** Consider the following AR(1) time series model:

$$X_t = 0.9 X_{t-1} + e_t$$

where e_t is a white noise process with variance equal to 1.

- (i) Derive expressions for the autocovariance and Autocorrelation Functions (ACF) for this model. [5]
- (ii) Calculate the variance of the sample mean $(X_1 + X_2 + X_3)/3$ using the autocovariance function in part (i). [4]

[Total 9]

- 4** A university in a certain country has recently completed a mortality study of dogs and has obtained the results summarised in the table below. Crude rates $\hat{\mu}_x$ were calculated using a Poisson model for ages last, $x = 0, 1, \dots, 9$ years. Standard table rates μ_x^s are based on a study by the National Veterinary Institute many years ago.

Age, x	Crude rates, $\hat{\mu}_x$	Exposure (E_x^c) in years	Standard table rates, μ_x^s
0	0.0265500	9,653	0.0270
1	0.0107002	7,991	0.0107
2	0.0092844	9,421	0.0096
3	0.0102735	8,747	0.0105
4	0.0120844	8,536	m
5	0.0184969	8,064	0.0160
6	0.0217086	7,482	0.0210
7	0.0310028	7,102	0.0297
8	0.0453510	6,480	0.0440
9	0.0678304	5,664	0.0639

The team graduated $\hat{\mu}_x$ by reference to the standard table using the function $\mu_x^o = \mu_x^s + k$ where μ_x^o are the graduated rates. They found k to be equal to 0.00068952 using the exposure-weighted least squares method.

- (i) Estimate m , the standard table rate at age $x = 4$, by first deriving an expression for k . [9]

The graduated rates are being checked for goodness-of-fit.

- (ii) Calculate standardised deviations for each age $x = 0, 1, \dots, 9$. [3]
- (iii) State, with reasons for each of the following tests of the standardised deviations in part (ii), if that test is required:
- signs test
 - grouping of signs test.

[Note: You are not required to perform these tests.]

[3]

[Total 15]

- 5** In a ‘roll-of-dice’ game y_n represents the number shown after the dice is thrown. y_n can take values from 1 to 6. X_n is defined as the maximum of $\{y_n, X_{n-1}\}$. The state space of X_n is {1 to 6}. The probability of throwing any particular number at any time is $\frac{1}{6}$.

- (i) Explain why X_n can be modelled using a Markov chain with the transition matrix as follows:

$$\begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{5}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ & & & & & 1 \end{pmatrix}$$

[4]

- (ii) State the initial distribution for X_n . [1]
- (iii) Calculate the probability that $X_4 = 4$ if $X_1 = 3$. [4]

The value of X_n now equals 5.

- (iv) Explain what the expected number of throws is for which the value of X remains at 5 before moving to 6. [3]

[Hint: You may find it useful to consider this as an application of the geometric distribution.]

[Total 12]

- 6** An actuary is asked to model a set of 150 time series observations: x_1, x_2, \dots, x_{150} . The following numerical summaries are obtained:

$$\bar{x} = \frac{1}{150} \sum_{i=1}^{150} x_i = 3.47$$

$$\sum_{i=1}^{150} (x_i - \bar{x})^2 = 517.5$$

$$\sum_{i=2}^{150} (x_i - \bar{x})(x_{i-1} - \bar{x}) = 362.25$$

$$\sum_{i=3}^{150} (x_i - \bar{x})(x_{i-2} - \bar{x}) = -103.5$$

- (i) Derive the estimate for $\hat{\gamma}_0$ using these results as well as the sample ACF for the first two lags: $\hat{\rho}_1$ and $\hat{\rho}_2$. [3]

The actuary considers two different time series models:

$$\begin{aligned} A \quad x_t &= a_1 x_{t-1} + e_t \\ B \quad x_t &= a_1 x_{t-1} + a_2 x_{t-2} + e_t \end{aligned}$$

where e_t is assumed to follow a Normal distribution with zero mean and some variance σ^2 to be estimated. The actuary uses the Yule–Walker equations and the results from part (i) to estimate the parameters.

- (ii) Estimate the parameters a_1 and σ^2 in model A. [3]
- (iii) Estimate the parameters a_1, a_2 and σ^2 in model B. [6]
- (iv) Outline the main methods used in diagnostics checking for time series models of this type. [3]
- [Total 15]

- 7 Each year a football association organises a tournament in which each team plays one game per week for all 52 weeks of the year. Tournament scoring is straightforward: a win earns 1 point, a loss deducts 1 point and a draw awards 0 points. All teams start the season with zero points. At the end of the year, the team with the highest total is crowned the champion of the year.

Consider a team called Piment Noir, and let S_t denote the accumulated number of points by this team at the end of week t .

- (i) Write down the state space of the process S_t . [1]

A new year's tournament is about to begin. Historic data shows that Piment Noir has a probability α of winning each game, where $0 < \alpha < 1$ and they never draw any games.

- (ii) Determine the expected number of points Piment Noir will accumulate over the next year. [2]

Consider the period from week t_1 to week t_2 and let $S_{t1} = N_1$ and $S_{t2} = N_2$.

- (iii) Derive expressions for the number of games won and the number of games lost in the period assuming that no games are drawn. [2]

- (iv) Derive expressions for the m -step transition probabilities for the process S_t . [4]

The fans of Piment Noir are unhappy with the team's results, and they want the team to appoint Smith as the coach. Historic data shows that Smith's teams have never lost a match. The probability that the team with Smith as coach draw their match in week t is $\frac{1}{2^t}$.

- (v) Calculate the expected number of points accumulated by Piment Noir if they hire Smith for the next year. [2]

- (vi) Comment on your result in part (v). [2]
[Total 13]

- 8** A data analyst is seeking to fit multiple linear regression models to a training data set that contains weekly observations over a year with data for many variables. After using a stepwise selection process, the analyst finds a suitable model with four explanatory variables, X_1, X_2, X_3 and X_4 , for the response variable Y .

- (i) Explain carefully the conditions under which this four-variable model would have been preferred to a five-variable model with the additional explanatory variable X_5 if selection had been using:
- (a) the Akaike information criterion. [2]
- (b) the Bayesian information criterion. [2]

After presenting their results, the analyst is asked to revise their model so that a maximum of three explanatory variables are used to help non-statisticians understand the interpretation of the results. There are four models fitted, each using three variables, and these are summarised in the table below that shows the value of the β regression coefficients and the Residual Sum of Squares (RSS) for each model.

<i>Excluded variable</i>	β_1	β_2	β_3	β_4	<i>Model RSS</i>
X_1	–	–2.48	1.02	1.19	1,040
X_2	1.73	–	1.02	0.99	736
X_3	1.70	–1.53	–	1.01	848
X_4	2.03	–1.99	1.11	–	1,008

The analyst decides to select the three-variable model by using a penalised regression method seeking to minimise:

$$\text{Model RSS} + \lambda g(\beta_1, \beta_2, \beta_3, \beta_4)$$

where $\lambda \geq 0$ is a regularisation parameter and $g(\beta_1, \beta_2, \beta_3, \beta_4)$ is the penalty function.

- (ii) Determine the best three-variable model using:
- (a) ridge regression with $\lambda = 0.1$, [5]
- (b) lasso regression with $\lambda = 10$, [5]
- showing all of your working.
- (iii) Describe the conditions under which the analyst would select a small value for the regularisation parameter, λ . [3]
- [Total 17]

END OF PAPER