

# **INSTITUTE AND FACULTY OF ACTUARIES**

## **EXAMINATION**

19 September 2022 (am)

### **Subject CS2 – Risk Modelling and Survival Analysis Core Principles**

#### **Paper A**

Time allowed: Three hours and twenty minutes

<p>In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator.</p>
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If you encounter any issues during the examination please contact the Assessment Team on T. 0044 (0) 1865 268 873.

- 1** A leading smartphone manufacturer has developed fast chargers for its mobile phones. Currently, the manufacturer is testing the life of batteries using such chargers. ‘Battery life’ is defined as ‘the number of times the battery can be charged using such chargers’. It has conducted many tests in its laboratories and has found the following:

- 70% of the batteries survived 400 charges.
- 50% of the batteries survived 810 charges.

Calculate, with suitable workings, the number of batteries that would survive 1,000 charges if battery life follows a Weibull distribution. [7]

- 2** Consider the time series process,  $X_t$ , given by:

$$X_t = aX_{t-1} + \frac{1}{2}X_{t-2} + e_t + be_{t-1}$$

where  $e_t$  is a sequence of independent and identically distributed  $N(0, \sigma^2)$  random variables.

Determine the values of the parameters  $a$  and  $b$  such that  $X_t$  is:

(i) stationary. [5]

(ii) invertible. [2]

(iii)  $I(1)$ . [2]

[Total 9]

- 3** Consider the following stochastic process:

$$X_t = \sum_{i=0}^{\infty} 0.5^i e_{t-i}$$

where  $e_j$  is a sequence of independent and identically distributed random variables with a mean of zero and variance  $\sigma^2$ , for  $j = 0, \pm 1, \pm 2, \dots$

(i) Determine whether  $X_t$  is stationary and satisfies the Markov property. [4]

(ii) Determine whether your conclusions from part (i) also apply to the process  $Y_t = X_t - 0.3X_{t-1}$ . [6]

[Total 10]

- 4 When considering rates of mortality, actuaries generally use the initial rate of mortality,  $q_x$ , whereas demographers often use the central rate of mortality,  $m_x$ . In order to estimate  $m_x$  for each age  $x$ , a demographer collects the following data:

<i>Age at last birthday</i>	<i>Observed number of deaths</i>	<i>Central exposed to risk in years</i>
60	34	4,267
61	36	4,037
62	42	4,269
63	47	4,299
64	46	3,929

- (i) Explain the difference between the two definitions of rate of mortality, stating which rate has the higher value at a given age,  $x$ . [2]
- (ii) Estimate  $m_x$  for  $x = 60, 61, 62, 63$  and  $64$  years using the data above. [2]
- (iii) A pension scheme starts paying pensions on a member's 65th birthday. Calculate the probability that the pension scheme member aged 60 exact will start receiving a pension, using the estimates in part (ii) above and carefully explaining all the steps and assumptions needed. [4]

[Total 8]

- 5 An insurance company has graduated the experience of one block of its life business using the following quadratic-Gompertz formula:

$$\mu_x = \exp(a_0 + a_1x + a_2x^2)$$

where  $a_0$ ,  $a_1$  and  $a_2$  are constants and  $x$  is age in years.

The data used for this graduation exercise, together with the graduated rates, are shown in the table below. All data have been collated between 1 January 2018 and 31 December 2018 inclusive on an ‘age nearest birthday’ basis.

<i>Age</i>	<i>Exposed to risk (years)</i>	<i>Death counts</i>	<i>Graduated rates</i>
60	11,362	80	0.0069
61	11,086	85	0.0077
62	10,816	100	0.0085
63	10,530	105	0.0095
64	10,301	108	0.0105
65	10,233	117	0.0115
66	9,970	110	0.0126
67	9,708	131	0.0138
68	9,667	145	0.015
69	9,807	140	0.0163

- (i) Perform a Chi-square test to assess the overall goodness of fit of this graduation, stating the null and alternative hypotheses. [5]

The company would like to extend the graduation to age 70. The death count and exposed to risk at age 70 are 150 deaths and 10,000 years, respectively.

Let  $\hat{\mu}_{70}$  be the graduated rate at age 70, and let the null and alternative hypotheses be the same as in part (i), except that the age range is now from 60 to 70.

- (ii) Determine, using the data in the table above, the condition that  $\hat{\mu}_{70}$  would need to fulfil in order for there to be insufficient evidence, at the 5% significance level, to reject the null hypothesis under the cumulative deviations test. [6]

[Total 11]

6 Consider the zero-mean MA(1) process,  $X_t$ , given by:

$$X_t = e_t + be_{t-1}$$

where  $e_t$  is a sequence of independent and identically distributed  $N(0, \sigma^2)$  random variables and  $|b| \leq 1$ .

A sample  $x_t$ ,  $t = 1, 2, \dots, T$ , is taken, which is believed to be from  $X_t$ , and it is assumed that  $e_0 = 0$ .

(i) Demonstrate that the log-likelihood function is given by:

$$L(b, \sigma^2) = \text{constant} - \frac{1}{2} T \log(\sigma^2) - \frac{1}{2\sigma^2} \sum_{t=1}^T \left( x_t + \sum_{i=1}^{t-1} (-b)^i x_{t-i} \right)^2.$$

[6]

You are given that the maximum likelihood estimator of  $b$ ,  $\hat{b}$ , obtained by setting the partial derivatives of  $L$  to zero satisfies, under appropriate assumptions, the following:

$$\frac{\hat{b}}{1 + \hat{b}^2} = r_1,$$

where  $r_1$  is the sample autocorrelation of  $x_t$  at lag 1.

(ii) Determine  $\hat{b}$ , considering separately the cases:

$$-\frac{1}{2} \leq r_1 < 0, r_1 = 0 \text{ and } 0 < r_1 \leq \frac{1}{2}.$$

[4]

(iii) Explain what conclusion should be drawn if  $|r_1| > \frac{1}{2}$ . [3]

[Total 13]

- 7** An insurer writes three different classes of insurance business: X, Y and Z. The classes have the following total annual claims distributions:

$$X \sim \text{Exp}(0.08)$$

$$Y \sim \text{Normal}(10, 2^2)$$

$$Z \sim \text{Normal}(20, 3^2)$$

The insurer models the level of dependency between the classes' total claim amounts using a Clayton ( $\alpha = 2$ ) copula.

- (i) Calculate the probability, using the Clayton (2) copula, that both  $X < 3$  and  $Y < 8$ . [3]
- (ii) Calculate the probability, using the Clayton (2) copula, that all of the following occur:  $X < 3$ ,  $Y < 8$  and  $Z < 20$ . [3]
- (iii) Calculate the probability, using the Clayton (2) copula, that both  $X > 10$  and  $Y > 12$ . [3]

A student has noted that copulas are a useful modelling tool as 'they allow us to model different degrees of dependency'.

- (iv) Comment on this statement. [3]
- [Total 12]

- 8** A combined 1-year life and health insurance policy pays £10,000 on death within the year and £15,000 on diagnosis of critical illness within the same term. Once the critical illness benefit is paid, no further benefits are payable. On 1st January 2021, the insurance company had 887 policies in force for people aged 56 at last birthday, of which 849 were still active at the end of the year. During 2021, the insurance company paid £115,000 in claims to policyholders aged 56 at last birthday, of which £75,000 were for critical illness.

- (i) Draw a diagram of a three-state model that could be used to evaluate premiums for this policy, fully labelling all states and transition intensities. [3]
- (ii) Beginning with the likelihood function for the model from part (i) above, derive and then solve equations for the maximum likelihood estimates of the transition intensities, stating any assumptions you make. [5]
- (iii) Estimate the expected cost of critical illness benefit for a new policyholder aged 56 exact. [3]
- (iv) Explain why the insurance company should be cautious about using the estimate from part (iii) above in premium calculations. [4]

[Total 15]

- 9 An airline company is concerned about potential financial losses due to currency movements in various countries. An analyst has been hired to model daily currency movements in these economies. The objective is to understand the likelihood of extreme currency movements that may have major financial consequences for the airline.

The analyst has suggested fitting several types of standard statistical distributions to the entire datasets, using maximum likelihood estimation, including Normal, Gamma, Pareto and lognormal. Estimates can then be made regarding probabilities of extreme currency movements based on the chosen model.

- (i) Discuss the problems with this approach. [4]

The analyst proceeds to use the Generalised Pareto Distribution (GPD) to assess only extreme daily losses. For this analysis, currency losses were treated as positive, and gains were treated as negative.

A threshold was set at daily losses of +2.2%, resulting in a data subset of 150 'extreme' values from the 3,000 original data points to be used in the extreme value analysis. The following results were obtained from the subsequent analysis:

- Beta = 0.009785.
- Gamma = 5.417 (shape).

- (ii) Calculate  $G(0.02)$  using the above parameter values, where  $G$  is the distribution function of the GPD, and write down its meaning. [3]

- (iii) Estimate the probability that the currency losses tomorrow exceed 5%, based on this GPD model. [3]

A Normal distribution was also fitted to the data. The best fitting Normal distribution was determined to be  $N(0, 0.014^2)$ . However, currency returns are generally considered to exhibit leptokurtic behaviour.

- (iv) Calculate the probability that daily losses exceed 5% using this Normal distribution, and comment on the results. [3]

- (v) Discuss further investigations the analyst should proceed with, including any problems with using extreme value theory and/or the GPD in this case. [2]

[Total 15]

**END OF PAPER**



Institute  
and Faculty  
of Actuaries

# EXAMINERS' REPORT

**CS2A – Risk Modelling and Survival  
Analysis**

**Core Principles**

**Paper A**

September 2022



## Introduction

The Examiners' Report is written by the Chief Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

Sarah Hutchinson  
Chair of the Board of Examiners  
December 2022

## **A. General comments on the *aims of this subject and how it is marked***

The aim of the Risk Modelling and Survival Analysis Core Principles subject is to provide a grounding in mathematical and statistical modelling techniques that are of particular relevance to actuarial work, including stochastic processes and survival models.

Some of the questions in this paper admit alternative solutions from those presented in this report, or different ways in which the provided answer can be determined. All mathematically correct and valid alternative solutions received credit as appropriate.

In cases where the same error was carried forward to later parts of the answer, candidates were given full credit for the later parts.

In higher order skills questions, where comments were required, well-reasoned comments that differed from those provided in the solutions also received credit as appropriate.

Candidates are advised to take careful note of all instructions that are provided with the exam in order to maximise their performance in future CS2A examinations. The instructions applicable to this diet can be found at the beginning of the solutions contained within this document.

## **B. Comments on *candidate performance in this diet of the examination.***

The syllabus and Core Reading for Risk Modelling and Survival Analysis Core Principles covers multiple statistical techniques and modelling approaches. Candidates' ability to evidence a strong grasp of these techniques and approaches was quite uneven across the syllabus areas with Time Series and Stochastic Process questions generally answered less well than Survival Analysis and Loss Distribution questions. Candidates are reminded that before attempting the examination they need to be thoroughly prepared across the whole syllabus and be ready to use the mathematical and modelling approaches to a range of scenarios.

The least well answered question on this paper was Question 6, on the likelihood function for a moving average process. It was particularly disappointing that a number of candidates chose not to make use of the structure for an answer given in the wording of the question itself.

To demonstrate marginal competence in this subject, candidates need to be able to evidence understanding of the statistical and modelling techniques described in the syllabus and expanded upon in the Core Reading, to apply these core techniques to a range of problems, to set out workings that demonstrate a clear structure for problem-solving and to be able to offer some commentary on the answers derived. This requires time in study to understand the statistical principles and methodologies and practice in answering questions across the syllabus.

Application skills questions were generally not answered well. With computer based examinations, these questions have become more important. Candidates should recognise

that these are generally the questions which differentiate those candidates with a good grasp and understanding of the subject.

The comments that follow the questions in the marking schedule below concentrate on areas where candidates could have improved their performance. Candidates approaching the subject for the first time are advised to concentrate their revision in these areas.

### C. Pass Mark

The Pass Mark for this exam was 55.  
978 presented themselves and 195 passed.

## Solutions for Subject CS2A – September 2022

### Q1

Let 'X' denote Life of the battery

Under, a Weibull distribution

$$P(X > x) = \text{Exp}(-cx^\gamma), \gamma > 0 \text{ \& } c > 0 \quad [1/2]$$

Finding the parameters of Weibull distribution

As per Golden book, Weibull distribution has 2 parameters; namely c & gamma

$$P(X > 400) = 0.70 \quad [1/2]$$

$$\text{Exp}(-c \cdot 400^\gamma) = 0.70 \quad [1/2]$$

$$c \cdot 400^\gamma = \text{Ln}(1/0.70) \quad \text{-- eq 1} \quad [1/2]$$

$$P(X > 810) = 0.50$$

$$\text{Exp}(-c \cdot 810^\gamma) = 0.50 \quad [1/2]$$

$$c \cdot 810^\gamma = \text{Ln}(1/0.50) \quad \text{-- eq 2} \quad [1/2]$$

Eq 2 divided by eq 1 gives

$$1.94336 = 810^\gamma / 400^\gamma \quad [1/2]$$

$$\text{Ln}(1.94336) = \gamma \cdot \ln(810/400)$$

$$\gamma = 0.9417 \quad [1]$$

Substituting gamma in eq 1 or eq 2 gives

$$c = \text{Ln}(1/0.70) / 400^{0.9417} \quad [1/2]$$

$$c = 0.001265 \quad [1/2]$$

Therefore

$$P(X > 1000) = \text{Exp}(-0.001265 \cdot 1000^{0.9417}) \quad [1]$$

$$= 0.42944 \quad [1/2]$$

*This question was well answered. The question is a straightforward application of the Weibull distribution.*

### Q2

(i)

$X_t$  is stationary if and only if the roots of the characteristic polynomial

$$1 - a * \lambda - \frac{1}{2} * \lambda^2$$

are both greater than 1 in magnitude [1]

For  $\lambda = 1$  to be a root,  $a = \frac{1}{2}$  [½]

For  $\lambda = -1$  to be a root,  $a = -\frac{1}{2}$  [½]

If  $a = 0$ , then the characteristic polynomial reduces to  $1 - \frac{1}{2} * \lambda^2$  [½]

This has roots  $\lambda = \sqrt{2}$  and  $-\sqrt{2}$ , which are greater than 1 in magnitude [½]

Stationarity therefore holds for  $a = 0$  [1]

Overall, stationarity holds if and only if  $\text{abs}(a) < \frac{1}{2}$  [1]

(ii)

$X_t$  is invertible if and only if the value of  $\lambda$  satisfying

$$1 + b * \lambda = 0$$

is greater than 1 in magnitude, [1]

i.e. if and only if  $-1 / b$  is greater than 1 in magnitude [½]

Hence invertibility holds if and only if  $\text{abs}(b) < 1$  [½]

(iii)

The condition for  $X_t$  to be  $I(1)$  is that the characteristic polynomial should have a root equal to 1

[1]

This is the case if and only if  $a = \frac{1}{2}$  [1]

**[Total 9]**

*Part (i) was well answered by the majority of candidates. Rather than arguing the values of  $\alpha$  from  $\lambda = \pm 1$ , it is also possible to solve the inequality algebraically.*

*Part (ii) was also well answered.*

*Part (iii) was not well answered with many candidates omitting the condition required.*

### Q3

(i)

$X_t$  is stationary since it is an AR(1) process. In particular

$$E(X_t) = E\left(\sum_{i=0}^{\infty} 0.5^i E(e_{t-i})\right) = 0 \quad [½]$$

$$\begin{aligned} \text{Cov}(X_t, X_{t-s}) &= E\left(\sum_{i=0}^{\infty} 0.5^i E(e_{t-i})\right) E\left(\sum_{j=0}^{\infty} 0.5^j E(e_{t-s-j})\right) \\ &= 0.5^s \sigma^2 / (1 - 0.5^2) \end{aligned} \quad [1½]$$

As  $X_t = 0.5 X_{t-1} + e_t$ , the distribution of  $X_t$  depends on  $X_{t-1}$  and only the information at time  $t-1$  and not any other r.v. before that point [1]

Hence we have the Markov property [1]

(ii)

The stationarity of  $Y_t$  is implied from that of  $X_t$  as

$$E(Y_t) = E(X_t) - 0.3 E(X_{t-1}) = 0 \quad [1]$$

and

$$\text{Cov}(Y_t, Y_{t-s}) = \text{Cov}(X_t - 0.3 X_{t-1}, X_{t-s} - 0.3 X_{t-s-1}) =$$

$$\text{Cov}(X_t, X_{t-s}) - 0.3 \text{Cov}(X_t, X_{t-s-1}) - 0.3 \text{Cov}(X_{t-1}, X_{t-s}) + 0.09$$

$$\text{Cov}(X_{t-1}, X_{t-s-1})$$

All these four components do not depend on  $t$  due to the stationarity of  $X_t$  [1]

For the Markov property  $Y_t$  however this is not the case:

$$\text{As } X_t = 0.5 X_{t-1} + e_t, \quad [1]$$

$$Y_t = 0.5 X_{t-1} + e_t - 0.3 X_{t-1} = e_t + 0.2 X_{t-1} = e_t + 0.2 * 0.5 X_{t-2} + 0.2 e_{t-1} \quad [1]$$

$$\text{Substituting } X_{t-2} = 1/0.2 * (Y_{t-1} - e_{t-1})$$

$$Y_t = 0.5 Y_{t-1} + e_t - 0.3 e_{t-1} \quad [1]$$

In this form one can see that the prediction for  $Y_t$  depends not only on  $Y_{t-1}$  but also on the information contained in  $e_{t-1}$  [1]

Hence the Markov property is NOT satisfied [1]

[Marks available 7, maximum 6]

**[Total 10]**

*Overall, this question was not well answered.*

*For both parts this was often because candidates rushed to a conclusion about whether the Markov property was satisfied rather than setting out structured reasoning. Similarly, in both parts a number of candidates stated that Stationarity held without any reasoning.*

*In part (i) this reasoning could have either been derived from the Covariance term or by observing that the 0.5 coefficient means that the variance is infinite.*

#### Q4

(i)

$q_x$  is the probability someone age  $x$  dies before age  $x+1$  [½]

$m_x$  is the probability of death in that same year per person year lived [½]

$m_x$  is higher than  $q_x$  [½]

unless  $q_x = 0$  [½]

(ii)

$m_x$  is estimated as Deaths / Central exposed to risk [1]

$x$	$m_x$
60	0.007968127
61	0.008917513
62	0.00983837
63	0.010932775
64	0.011707814

[1]

(iii)

if the force of mortality is assumed constant in each individual year of age  $\mu_x$  [½]  
 then  $m_x = \mu_x$  for each  $x$  [½]  
 with constant force of mortality we can use the Exponential model within a year of age [1]  
 and then chain together five exponential survival probabilities [½]  
 so  ${}_5p_{60} = \exp(-0.007968127) \cdot \exp(-0.008917513) \cdot \exp(-0.00983837) \cdot \exp(-0.010932775)$   
 $\cdot \exp(-0.011707814)$  [1]  
 $= 0.951834$  [½]

**[Total 8]**

*Part (i) was generally well answered and is straightforward.*

*Part (ii) was well answered.*

*Part (iii) was generally well answered. Candidates that did less well in this part often tried to calculate the survival probability direct from the central rates of mortality rather than recognise that a constant force of mortality assumption allows use of the exponential model.*

**Q5**

(i)

Null hypothesis: the graduated rates are the true rates underlying the observed data [½]

Alternative hypothesis: The graduated rates are NOT the true rates underlying the observed data [½]

i.  $z_x = (\text{Expected deaths} - \text{Observed deaths}) / \sqrt{\text{Expected deaths}}$  [½]

ii.

Age, $x$	Expected deaths	$z_x$	$z_x^2$
60	78.4	0.1810	0.0327
61	85.4	-0.0392	0.0015
62	91.9	0.8410	0.7073
63	100.0	0.4964	0.2464
64	108.2	-0.0154	0.0002
65	117.7	-0.0626	0.0039
66	125.6	-1.3938	1.9427
67	134.0	-0.2566	0.0659
68	145.0	-0.0004	0.0000
69	159.9	-1.5703	2.4659

[1]

$$X = \sum_x z_x^2$$

The test statistics is

Under the null hypothesis,  $X$  has a Chi2 distribution with  $m$  degrees of freedom, where  $m$  is the number of age groups less one for each parameter fitted, so in this case  $m = 10 - 3$

[½]

The number of degrees of freedom is therefore 7

[½]

The critical value of the Chi2 distribution with 7 degrees of freedom at the 5% level is 14.067

[½]

The observed value of  $X$  is 5.47

[½]

This is lower than the critical value; there is not enough evidence to reject the null hypothesis at 5% level

[½]

(ii)

Under the null hypothesis, the test statistic

$$Y = \frac{\sum \text{Observed deaths} - \text{Expected deaths}}{\sqrt{\sum \text{Expected deaths}}} \sim \mathcal{N}(0,1)$$

[1]

$$Y = \frac{-25.023 + 150 - 10000\hat{\mu}_{70}}{\sqrt{1146.023 + 10000\hat{\mu}_{70}}} = \frac{124.977 - 10000\hat{\mu}_{70}}{\sqrt{1146.023 + 10000\hat{\mu}_{70}}}$$

[1]

We are looking for  $Y$  such that  $-1.96 < Y < 1.96$

[1]

That is

$$\frac{(124.977 - 10000\hat{\mu}_{70})^2}{1146.023 + 10000\hat{\mu}_{70}} < 1.96^2$$

[½]

$$(124.977 - 10000\hat{\mu}_{70})^2 < 1.96^2 \times (1146.023 + 10000\hat{\mu}_{70})$$

$$10000^2 \mu_{70}^2 - 2537956 \mu_{70} + 11216.69 < 0$$

[1]

The roots are 0.0057 and 0.0197

[1]

**[Total 11]**

*Part (i) was very well answered. It is a straightforward application of the Chi-squared test. Candidates are reminded when answering this type of question to set out clearly their hypotheses, the calculation of the test statistic, the reasoning for the number of degrees of freedom, and the conclusion as it relates to the null hypothesis.*

Part (ii) was also well answered. This is more challenging than part (i) but candidates who structured their answer carefully gained the majority of marks available. There are a number of different ways to calculate  $z_{70}$  here, all of which were credited. However some candidates then failed to relate this  $z_{70}$  calculation back to the Cumulative Deviations Test statistic.

**Q6**

(i)

We have  $e_1 = x_1 - b * e_0$  [½]

Since  $e_0 = 0$  by assumption [½]

$e_1 = x_1$  [½]

Suppose we have shown that  $e_t = x_t + \text{Sum}(1,t-1)[(-b)^i x_{t-i}]$  [½]

Then  $e_{t+1} = x_{t+1} - b * e_t$ , [½]

i.e.  $e_{t+1} = x_{t+1} - b * (x_t + \text{Sum}(1,t-1)[(-b)^i x_{t-i}])$ , [½]

i.e.  $e_{t+1} = x_{t+1} + \text{Sum}(1,t)[(-b)^i x_{t-i}]$  [1]

Hence  $e_t = x_t + \text{Sum}(1,t-1)[(-b)^i x_{t-i}]$  for  $t = 1, 2, \dots, T$

The likelihood function is given by

$\text{constant} * \text{Product}(1,T)[1 / \sqrt{\sigma^2} * \exp(-\frac{1}{2} * (e_t / \sigma)^2)]$  [1]

The log-likelihood function is given by

$\text{constant} + \text{Sum}(1,T)[- \frac{1}{2} * \log(\sigma^2) - \frac{1}{2} * (e_t / \sigma)^2]$ , [½]

which reduces to the required expression on substituting for  $e_t$  [½]

(ii)

If  $r_1 = 0$ , then  $bhat = 0$  [½]

Otherwise,  $bhat$  satisfies the quadratic equation  $r_1 * bhat^2 - bhat + r_1 = 0$  [½]

The roots of this quadratic are  $(1 - \sqrt{1 - 4 * r_1^2}) / (2 * r_1)$  [1]

We require the root less than or equal to 1 in magnitude [1]

Thus for  $r_1 > 0$ ,  $bhat = (1 - \sqrt{1 - 4 * r_1^2}) / (2 * r_1)$  [½]

For  $r_1 < 0$ ,  $bhat = (1 + \sqrt{1 - 4 * r_1^2}) / (2 * r_1)$  [½]

(iii)

When  $\text{abs}(r_1) > \frac{1}{2}$ , there is no real value of  $\beta$  equating the theoretical

autocorrelation at lag 1,  $\beta / (1 + \beta^2)$ , to  $r_1$  [1]

Consideration should be given to collecting more data, [½]

as it is possible that the observed value of  $r_1$  has been distorted by sampling error [½]

Otherwise, it should be concluded that the data come from a different type of model

[1]

**[Total 13]**

This question as a whole was very poorly answered. Candidates are reminded of the importance of being able to apply the Core Reading material in this part of the syllabus to unfamiliar situations.



*In part (i) the most common omission was a structured approach that begins with the relationship between  $e_{t+1}$  and  $e_t$  in terms of  $x$ , and then moves to a likelihood function and then its log.*

*In part (ii) which was now well answered, many candidates failed to take heed of the instruction in the question to consider three different conditions for  $r$  separately. As well as the methodology shown above it is also possible to derive the roots algebraically. In part (iii) many candidates simply stated that the equation from part (ii) has no real solution rather than demonstrating why and then engaging with the possible reasons for this.*

**Q7**

(i)

Using  $(u^{-\alpha} + v^{-\alpha} - 1)^{-1/\alpha}$ 

$$F_X(3) = 1 - \exp(-0.08 * 3) \quad [1/2]$$

$$= 0.2134, \quad [1/2]$$

$$F_Y(8) = P(Z < -1) \text{ where } Z \sim N(0, 1) \quad [1/2]$$

$$= 0.1587 \quad [1/2]$$

$$C(0.2134; 0.1587) = (u^{-\alpha} + v^{-\alpha} - 1)^{-1/\alpha} \quad [1/2]$$

$$= (0.2134^{-2} + 0.1587^{-2} - 1)^{-1/2} = 0.1284 \quad [1/2]$$

(ii)

Using  $(u^{-\alpha} + v^{-\alpha} + w^{-\alpha} - 1)^{-1/\alpha}$ 

$$F_Z(20) = P(Z < 0) \text{ where } Z \sim N(0, 1) \quad [1/2]$$

$$= 0.5 \quad [1/2]$$

$$C(0.2134; 0.1587; 0.5) = (u^{-\alpha} + v^{-\alpha} + w^{-\alpha} - 1)^{-1/\alpha} \quad [1]$$

$$= (0.2134^{-2} + 0.1587^{-2} + 0.5^{-2} - 2)^{-1/2} = 0.1253 \quad [1]$$

(iii)

$$F_X(10) = 1 - \exp(-0.08 * 10) = 0.5507 \quad [1/2]$$

$$F_Y(12) = P(Z < 1) \text{ where } Z \sim N(0, 1) \quad [1/2]$$

$$= 0.8413 \quad [1/2]$$

$$C(0.5507; 0.8413) = (u^{-\alpha} + v^{-\alpha} - 1)^{-1/\alpha} \quad [1/2]$$

$$= (0.5507^{-2} + 0.8413^{-2} - 1)^{-1/2} = 0.5191 \quad [1/2]$$

Then using the survival copula, the required probability is

$$1 - F_X(10) - F_Y(12) + C(0.5507; 0.8413) \quad [1]$$

$$= 1 - 0.5507 - 0.8413 + 0.5192 = 0.1271 \quad [1/2]$$

(iv)

It is true that part of the benefit of copulas is that they allow us to model different degrees of dependency [1]

However, copulas also exhibit different patterns of dependency [1]

As well as different degrees of dependency in the body of the distributions, copulas exhibit different degrees of dependency in the tails [1]

Copulas allow the dependency structure between random variables to be modelled separately from the marginal distributions [1]

[Marks available 4, maximum 3]

**[Total 12]**

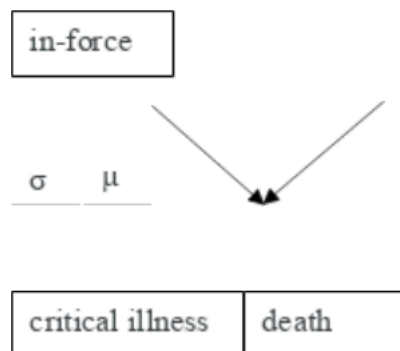
*This question was well answered. In all four parts credit was given either for stating the copula algebraically or for being able to evidence the correct copula from the numbers used in calculating the probabilities required.*

*In part (ii) it is possible to take the answer from part (i) and insert that into a two-part copula rather than derive the three-part copula directly.*

## Q8

(i)

Three-state model with transition intensities



[2]

where  $\mu$  is the transition intensity from in-force to death

[½]

$\sigma$  is the transition intensity from in-force to critical illness

[½]

(ii)

There are  $75000/15000 = 5$  critical illness transitions and  $(115000 - 75000)/10000 = 4$  death claims

[½]

The waiting time in the in-force state can be estimated by census as

$$\frac{1}{2} (887 + 849) = 868 \text{ years}$$

[1]

We assume that transition intensities are constant throughout a single year of age

[½]

and are not policy duration dependent

[½]

The likelihood of the model is  $L(\mu, \sigma) = \exp(-(\mu + \sigma)868) \cdot \mu^4 \cdot \sigma^5$

[1]

and the log-likelihood is  $\log L = 4 \log \mu + 5 \log \sigma - 868\mu - 868\sigma$

[½]

The maximum likelihood estimates are found by differentiating with respect to  $\mu$  and  $\sigma$  respectively, setting to zero and solving gives

$$\hat{\mu} = 4/868 = 0.004608$$

[½]

$$\hat{\sigma} = 5/868 = 0.005760$$

[½]

(iii)

We need the probability of transition to critical illness

$$\Pr[\text{remain in in-force state}] = \exp(-0.004608 - 0.00576) = 0.989685$$

[½]

$$\Pr[\text{making a transition}] = 1 - 0.989685 = 0.010315$$

[½]

$\Pr[\text{transition to critical illness}]$  found by ratio of transition intensities

[½]

$$= 0.010315 \times 0.005760 / (0.004608 + 0.005760)$$

$$= 0.005731$$

[½]

so expected cost of claims =  $15000 \times 0.005731 = £85.96$  [1]

(iv)

Should be cautious because:

number of transitions is low and a small change in observations would lead to a large change in transition intensities [1]

the census approximation might not be valid since the number of policies might not have varied linearly over the year [1]

the large fall in policy numbers suggests censoring is important [1]

would need to investigate whether this censoring is informative [1]

the assumption that transition rates are constant over the year of age should be validated [1]

the assumption that transition rates do not depend on duration should be validated [1]

[Marks available 6, maximum 4]

**[Total 15]**

*This question was well answered.*

*In part (i) candidates are reminded of the need to label and define all transition intensities when displaying a two-state or multi-state model.*

*In part (ii) a number of candidates failed to apply the Census method to the calculation of the waiting time.*

*The expected cost in part (iii) requires both a survival probability and a transition intensity whereas some candidates only included the latter.*

## Q9

(i)

There are more data points in the central part of a distribution than in the tails [1]

Therefore, maximum likelihood estimation gives more weight to the central part of the distribution than to the tails [1]

and hence tends to result in estimated distributions that fit the tails more poorly than the central part of the actual distribution [1]

This will result in inaccurate estimates of the probabilities of extreme currency movements [1]

The best-fitting model under maximum likelihood estimation may be inappropriate for other reasons [1]

[Marks available 5, maximum 4]

(ii)

The formula for the GPD CDF is

$$G(x) = 1 - (1 + x / (\text{gamma} * \text{beta})) ^ (-\text{gamma}). \quad [1/2]$$

Hence

$$G(0.02) = 1 - (1 + 0.02 / (5.417 * 0.009785)) ^ (-5.417) \quad [1]$$

$$= 0.823 \quad [1/2]$$

This is the probability that a loss is less than 4.2% given that it is greater than 2.2% [1]

(iii)

The probability that a loss is less than 5% given that it is greater than 2.2% is  $G(0.028)$  [½]  
 $= 0.8995$ . [½]

The probability that a loss is greater than 5% given that it is greater than 2.2% is  
 therefore  $1 - 0.8995 = 0.1005$  [½]

The estimated probability that a loss is greater than 5% is therefore  
 $0.1005 * 150 / 3,000$  [1]  
 $= 0.0050$  [½]

(iv)

The number of standard deviations is  $0.05 / 0.014 = 3.57$  [½]

From the Tables, the required probability is therefore  $1 - 0.99982 = 0.00018$  [½]

Hence the probability of a loss exceeding 5% is significantly lower under the Normal  
 distribution than under the GPD distribution [½]

Given the expectation of leptokurtic behaviour, there is reason to believe that the GPD  
 distribution is more appropriate than the Normal distribution [1]

Using the Normal distribution would therefore be expected to significantly  
 underestimate the risk of extreme currency movements [½]

(v)

The effect of using a different threshold should be investigated [½]

The goodness of fit of the GPD distribution should be investigated [½]

e.g. by using a chi-squared test [½]

The joint behaviour of movements in different currencies should be investigated [½]

The assumption that the distribution of currency movements remains constant over  
 time should be validated [½]

[Marks available 2½, maximum 2]

**[Total 15]**

*This question was not well answered.*

*In part (i) a number of candidates (presumably having read on through the whole question) rushed to discuss the Generalised Pareto Distribution and missed the points about the nature of the data set and the limitations of maximum likelihood in this type of scenario.*

*In parts (ii) and (iii) a number of candidates completed a GPD calculation but then failed to state the probability that was asked for.*

*Part (iv) was generally well answered.*

*In part (v) it is important to relate answers back to the scenario given in the question, in this case currency movement data and also to the assumptions made.*

**[Paper Total 100]**

## END OF EXAMINERS' REPORT



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