

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINATION

27 April 2022 (am)

Subject CM2 - Financial Engineering and Loss Reserving Core Principles

Paper A

Time allowed: Three hours and twenty minutes

<p>In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator.</p>
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If you encounter any issues during the examination please contact the Assessment Team on
T. 0044 (0) 1865 268 873.

- 1** A fund earns an annual rate of return i_t , with the rate of return in any year being independent of the rate in any other year. The distribution of $\log(1 + i_t)$ is normal with parameters μ and σ^2 .

The mean of i_t is 5% and the standard deviation is 3%.

- (i) Calculate μ and σ^2 . [3]
- (ii) Calculate the probability that the fund return for any year is between 1% and 3%. [3]
- (iii) Comment on your answer to part (ii). [1]

A sum of £10,000 is invested into the fund.

- (iv) Calculate the probability that the accumulated value of the fund at the end of 3 years is less than £11,000. [2]
- [Total 9]

- 2 (i) Define the term ‘loss ratio’ as used in the Bornhuetter–Ferguson method for estimating outstanding claim amounts. [1]

The run-off triangle below shows cumulative claims incurred on a portfolio of insurance policies.

<i>Accident year</i>	<i>Development year</i>		
	<i>0</i>	<i>1</i>	<i>2</i>
2017	864	1,011	1,072
2018	798	915	
2019	820		

Annual premiums written for accident year 2019 were 1,520 and the ultimate loss ratio is assumed to be 92.5%. Claims can be assumed to be fully run off by the end of development year 2.

- (ii) Calculate the total claims arising from accidents in 2019, using the Bornhuetter–Ferguson method. [5]

1 year later, an unexpected event has resulted in higher claims than expected. The run-off triangle is now as shown below.

<i>Accident year</i>	<i>Development year</i>		
	<i>0</i>	<i>1</i>	<i>2</i>
2018	798	915	1,320
2019	820	1,412	
2020	1,016		

- (iii) Calculate the revised total claims arising from accidents in 2019, using the Bornhuetter–Ferguson method. [3]
- (iv) Discuss the implications of your answer to part (iii) for the insurance company. [3]

[Total 12]

3 Consider a share, S_t , and a derivative on the share with a value at time t of $f(t, S_t)$.

(i) Define, in your own words, what is represented by each of the following Greeks for this derivative:

(a) Delta

(b) Gamma

(c) Vega.

[3]

Consider another share, A_t , which pays no dividends. The continuously compounded risk-free rate is r . Let K be the fair price at time 0 of a forward contract on A_t maturing at time T .

(ii) State the formula for K . [1]

Under the risk-neutral measure, Q , the share is expected to grow at the risk-free rate.

(iii) Demonstrate that the expected present value of the forward contract at time t ($0 \leq t \leq T$) under the measure Q is $A_t - e^{-r(T-t)} K$. [2]

(iv) Calculate Delta, Gamma and Vega for the forward contract. [2]

(v) Comment on how the Greeks for the forward contract in part (iv) compare to the same Greeks for the underlying share. [2]

(vi) Discuss whether it would be appropriate to use a forward contract to Delta hedge a European call option on the share. [3]

[Total 13]

- 4 Suppose that under the unique equivalent martingale measure, Q , for a term structure model, the Stochastic Differential Equation satisfied by the instantaneous interest rate r is:

$$dr_t = \alpha(\mu - r_t)dt + \sigma dZ_t$$

where $\alpha > 0$, μ and σ are fixed parameters and under Q , Z is a standard Brownian Motion.

The process X is defined by:

$$X_t = r_t b(T - t) + \int_0^t r_s ds$$

where the function b is given by $b(s) = \frac{(1 - e^{-\alpha s})}{\alpha}$.

The function f is given by $f(x, t) = \exp(a(T - t) - x)$, where a is a differentiable function.

- (i) Apply Ito's formula to $f(X_t, t)$. [6]

[Hint: You may use, without proof, the fact that $dX_t = A_t dt + B_t dZ_t$ where $A_t = \alpha \mu b(T - t)$, and $B_t = \sigma b(T - t)$.]

- (ii) Find a differential equation that the function a must satisfy, in order for $f(X_t, t)$ to be a martingale. [2]

- (iii) Determine an additional condition on a that is necessary for a bond with unit payoff at time T to have a price given by the formula:

$$B(t, T) = f(X_t, t) \exp\left(\int_0^t r_s ds\right) \quad [5]$$

[Total 13]

- 5 An insurance company holds a large amount of capital and wishes to distribute some to policyholders using one of two possible options.

Option A

A sum of £500 will be invested for each policyholder in a fund in which the expected annual effective rate of return is 3.5% and the standard deviation of annual returns is 2%. The annual rates of return are independent and $(1 + i_t)$ is log-normally distributed with parameters μ and σ^2 , where i_t is the rate of return in year t . The policyholder will receive the accumulated investment at the end of 15 years.

Option B

A sum of £500 will be invested for each policyholder for 10 years at a fixed rate of return of 4% p.a. effective. After 10 years, the accumulated sum will be invested for a further 5 years at the prevailing 5-year spot rate. This spot rate follows the probability density function shown below:

<i>Spot rate (% p.a.)</i>	<i>Probability</i>
0.5	0.15
1.0	0.25
4.5	0.40
7.0	0.20

The policyholder will receive the accumulated investment at the end of the 15 years.

- (i) Demonstrate that $\mu = 0.0342$ and $\sigma = 0.0193$. [4]
 - (ii) Calculate the expected value and standard deviation at the end of year 15 of:
 - (a) Option A.
 - (b) Option B.[12]
 - (iii) Determine, for each of Options A and B, the probability that a policyholder's accumulated investment at the end of the 15 years will be less than £775. [5]
 - (iv) Compare the relative risk of the two options. [2]
- [Total 23]

- 6 Consider a share with price S_0 at time $t = 0$. The share will pay out a dividend of X at time $t = 1$ and again at time $t = 2$. The continuously compounded risk-free rate is r per unit of time.

Assume that the dividend payments are reinvested at the risk-free rate.

- (i) Demonstrate that the fair price of a forward contract on S_t maturing at time $T > 2$ is $K = (S_0 - I)e^{rT}$, where I is the present value of the two dividends. [4]

A share is worth \$100 at time $t = 0$. It will pay a dividend of \$5 at time $t = 1$ and again at time $t = 2$. The continuously compounded risk-free rate is 5% per unit of time.

- (ii) Calculate the fair price of a forward contract on the share maturing at time $T = 3$. [2]

An investor takes a long position in the forward contract in part (ii) at time $t = 0$. Immediately afterwards, the proposed dividends on the underlying share are cancelled.

- (iii) Discuss the implications of this for the investor. [2]
[Total 8]

- 7 An investor makes decisions based on the utility function $U(w) = w - 6w^2$, where w is the investor's wealth in millions of dollars (\$m).

- (i) Demonstrate that the investor has both increasing absolute and relative risk aversion. [3]

The investor has \$50,000 to invest over a 1-year period and has no other wealth. They have three options:

- A Invest in a risk-free account. There will be no change in the value of the investment over 1 year.
B Invest in an asset that will give a 60% return over 1 year with probability 0.2, a 20% return with probability 0.7 and a -40% return with probability 0.1.
C Invest in an asset that will give a 30% return with probability 0.5 and a 20% return with probability 0.5.

The investor makes no allowance for discounting when making investment decisions. The investor must invest the whole \$50,000 in a single option.

- (ii) Determine which option the investor should choose to maximise their expected utility at the end of the year. [5]
(iii) Comment on why the investor could not use $U(w)$ to choose from the above options if their initial wealth was \$65,000. [2]
[Total 10]

- 8** Consider a company funded entirely by debt and equity. The total value of the company's assets is \$40 million. It has debt with a current outstanding amount of \$20 million, with a continuously compounded interest rate of 8% p.a. and maturity in 7 years. Interest is added to the outstanding debt to be paid at maturity.

The volatility of the company's total assets is 10% p.a. The continuously compounded risk-free rate of interest is 2% p.a.

- (i) Calculate the outstanding value of the debt, with interest, at maturity. [1]
- (ii) Calculate the current value of the company's equity using the Merton model. [5]
- (iii) Calculate the implied probability that the company will have sufficient assets to repay the debt at maturity. [1]
- (iv) Demonstrate that the implied value of the company's debt is \$29.7 million to the nearest \$0.1 million (where the total value of the company is the value of the equity plus the value of the debt). [1]
- (v) Explain why your answers to parts (i) and (iv) are different. [2]

An analyst wishes to set up a two-state credit model for the company. The two states will be denoted 'solvent' and 'default' with a constant transition intensity, λ , from solvent to default.

- (vi) Calculate the value of λ that gives the same probability of default at maturity of the debt as calculated in part (iii). [2]

[Total 12]

END OF PAPER