14

Estimation and Test of Significance

14.1 Introduction

The object of sampling is to study the features of the population on the basis of sample observations. A carefully selected sample is expected to reveal these features, and hence we shall infer about the population from a statistical analysis of the sample. This process is known as *Statistical Inference.*

There are two types of problems. Firstly, we may have no information at all about some characteristics of the population, especially the values of the parameters involved in the distribution, and it is required to obtain estimates of these parameters. This is the problem of *Estimation.* Secondly, some information or hypothetical values of the parameters may be available, and it is required to test how far the hypothesis is tenable in the light of the information provided by the sample. This is the problem of *Test of Hypothesis* or *Test of Significance.*

14.2 Theory of Estimation

Suppose we have a random sample $x_1, x_2, \ldots x_n$ on a variable x, whose distribution in the population involves an unknown parameter θ. It is required to find an estimate of θ on the basis of sample values. The estimation is done in two different ways:—(i) *Point Estimation,* and (ii) *Interval Estimation.* In point estimation, the estimated value is given by a single quantity, which is a function of sample observations (i.e. statistic). This function is called the *'estimator'* of the parameter, and the value of the estimator in a particular sample is called an *'estimate'*. In interval estimation, an interval within which the parameter is expected to lie is given by using two quantities based on sample values. This is known as *Confidence Interval,* and the two quantities which are used to specify the interval, are known as *Confidence Limits.*

14.3 Point Estimation—Criteria for good estimators

Many functions of sample observations may be proposed as estimators of the same parameter. For example, either the mean or median or mode of the sample values may be used to estimate the parameter μ of the Normal distribution with p.d.f.

$$\frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/2\sigma^2}$$

which we shall in future refer to as $N(\mu, \sigma^2)$. Naturally we have to choose one among these estimators on the basis of certain criteria.

According to R.A. Fisher, the criteria for a good estimator are

(i) Unbiasedness,
(ii) Consistency,
(iii) Efficiency,
(iv) Sufficiency.

Unbiasedness—

A statistic t is said to be an *Unbiased Estimator* of a parameter θ, if the expected value of t is θ.

$$E(t) = \theta \tag{14.3.1}$$

Otherwise, the estimator is said to be '*biased*'. The bias of a statistic in estimating θ is given as

$$Bias = E(t) - \theta \tag{14.3.2}$$

Let $x_1, x_2, \ldots x_n$ be a random sample drawn from a population with mean μ and variance σ^2. Then

$$\text{Sample mean } (\bar{x}) = \Sigma x_i / n$$
$$\text{Sample variance } (S^2) = \Sigma(x_i - \bar{x})^2 / n \tag{14.3.3}$$

Theorem 1. The sample mean $\bar{x}$ is an *unbiased* estimator of the population mean μ ; because

$$E(\bar{x}) = \mu \tag{14.3.4}$$

Theorem 2. The sample variance S^2 is a *biased* estimator of the population variance σ^2 ; because

$$E(S^2) = \frac{n-1}{n}\sigma^2 \neq \sigma^2 \tag{14.3.5}$$

Theorem 3. An unbiased estimator of the population variance σ^2 is given by

$$s^2 = \Sigma(x_i - \bar{x})^2/(n-1) \tag{14.3.6}$$

because

$$E(s^2) = \sigma^2 \tag{14.3.7}$$

Note the distinction between S^2 and s^2 in which only the denominators are different. S^2 is the variance of the sample observations, but s^2 is the 'unbiased estimator' of the variance (σ^2) in the population.

Example 14 : 1 Show that the sample mean based on a simple random sample with replacement (srswr) is an unbiased estimator of the population mean. [*C.U., B.A.(Econ) '78*]

Solution : Let $x_1, x_2, \ldots x_n$ be a simple random sample with replacement from a finite population of N members $X_1, X_2, \ldots X_N$ (suppose).

Sample mean $(\bar{x}) = (x_1+x_2+ \ldots +x_n)/n$

Population mean $(\mu) = (X_1+X_2+ \ldots +X_N)/N$

We have to show that $E(\bar{x}) = \mu$

Now, each of the sample members $x_1, x_2, \ldots x_n$ behaves like a random variable, because their values in any particular sample depend on chance. For instance, in srswr any of the population members $X_1, X_2, \ldots X_N$ may appear at the i-th drawing, i.e. x_i is a random variable with the following probability distribution:

x_i :	X_1	X_2	...	X_N	Total
Prob :	$1/N$	$1/N$	...	$1/N$	1

Therefore, $E(x_i) = (1/N)X_1+(1/N).X_2+ \ldots +(1/N).X_N$

$$= (X_1+X_2+ \ldots +X_N)/N$$

$$= \mu$$

Hence,
$$E(\bar{x}) = E\left[\frac{1}{n}(x_1 + x_2 + \ldots + x_n)\right]$$

$$= \frac{1}{n}\left[E(x_1) + E(x_2) + \ldots + E(x_n)\right]$$

$$= (\mu + \mu + \ldots + \mu)/n$$

$$= n\mu/n = \mu$$

This shows that $\bar{x}$ is an unbiased estimator of μ (also see Examples 13 : 21 and 13 : 24).

[**Note :** This result holds in all cases of random sampling, irrespective of whether the sample is drawn 'with replacement' or 'without replacement' from a finite population or from an infinite population.]

Example 14 : 2 If $x_1, x_2, \ldots, x_n$ is a random sample from an infinite population with variance σ^2, and $\bar{x}$ is the sample mean, show that $\sum_{i=1}^{n}(x_i - \bar{x})^2/n$ is a biased estimator of σ^2, but the bias becomes negligible for large n. Give an unbiased estimator of σ^2 here.

[*W.B.H.S. '79, 82*]

Solution : Let μ and σ^2 be the mean and variance of the population. Then $E(x_i) = \mu$, $\text{Var}(x_i) = E(x_i-\mu)^2 = \sigma^2$ for each $i = 1, 2, \ldots n$. The variance of the sample is

$$S^2 = \Sigma(x_i - \bar{x})^2/n$$

We have to show that $E(S^2) \neq \sigma^2$

Now, $S^2 = \Sigma(x_i - \bar{x})^2/n = \Sigma x_i^2/n - \bar{x}^2$

$= \Sigma y_i^2/n - \bar{y}^2$, where $y_i = x_i - \mu$

($\because$ S.D. is unaffected by change of origin)

$= \Sigma(x_i - \mu)^2/n - (\bar{x} - \mu)^2$

$$\therefore \quad E(S^2) = \Sigma E(x_i - \mu)^2/n - E(\bar{x} - \mu)^2$$
$$= \Sigma\sigma^2/n - \text{Var}(\bar{x})$$
$$= \sigma^2 - \sigma^2/n \qquad \text{(see } 13.7.1\text{)}$$
$$= \frac{n-1}{n}\sigma^2 \neq \sigma^2$$

This shows that S^2 is a *biased* estimator of σ^2.

$$\text{Bias} = E(S^2) - \sigma^2$$
$$= \frac{n-1}{n}\sigma^2 - \sigma^2 = \frac{-\sigma^2}{n}$$

Thus for large n, Bias will be negligibly small. If we write

$$s^2 = \Sigma(x_i - \bar{x})^2/(n-1)$$

(note that the divisor is $n-1$ instead of n) we see that

$$s^2 = \frac{n}{n-1} S^2,$$

$$\therefore \quad E(s^2) = \frac{n}{n-1} E(S^2) = \frac{n}{n-1} \cdot \frac{n-1}{n}\sigma^2 = \sigma^2$$

This shows that s^2 is an *unbiased* estimator of σ^2.

Example 14 : 3 The following observations constitute a random sample from an unknown population. Estimate the mean and standard deviation of the population. Also, find the estimate of standard error of sample mean.

14, 19, 17, 20, 25. [*M.B.A.* '79]

Solution : The unbiased estimators of the population mean (μ) and the population variance (σ^2) are $\bar{x} = \Sigma x_i/n$ and $s^2 = \Sigma(x_i - \bar{x})^2/(n-1)$ respectively. Here $n = 5$ and $\bar{x} = \Sigma x_i/n = 95/5 = 19$

$$\Sigma(x_i - \bar{x})^2 = (-5)^2 + 0^2 + (-2)^2 + 1^2 + 6^2 = 66$$
$$\therefore \quad s^2 = \tfrac{66}{4} = 16.5, \quad s = \sqrt{16.5} = 4.06$$

The estimates of μ and σ are 19 and 4.06 respectively.

The standard error of sample mean is S.E. $(\bar{x}) = \sigma/\sqrt{n}$. But as σ is not known, it is estimated by s,

$$\text{Estimate of S.E. } (\bar{x}) = \frac{s}{\sqrt{n}} = \frac{\sqrt{66/4}}{\sqrt{5}} = \sqrt{\frac{66}{20}}$$
$$= \sqrt{3 \cdot 3} = 1 \cdot 82$$

Consistency—

A desirable property of a good estimator is that its accuracy should increase when the sample size becomes larger. That is, the estimator is expected to come closer to the parameter as the size of the sample increases.

A statistic t_n computed from a sample of n observations is said to be a *Consistent Estimator* of a parameter θ, if it converges in probability to θ as n tends to infinity. This means that the larger the sample size (n), the less is the chance that the difference between t_n and θ will exceed any fixed value. In symbols, given any arbitrary small positive quantity ϵ,

$$\underset{n \to \infty}{\text{Lt}} \; P\{ \mid t_n - \theta \mid > \epsilon \} = 0 \tag{14.3.8}$$

If $E(t_n) \to \theta$ and Var $(t_n) \to 0$ as $n \to \infty$, then the statistic t_n will be a 'consistent estimator' of θ.

Consistency is a limiting property. Moreover, several consistent estimators may exist for the same parameter. For example, in sampling from a Normal population $N(\mu, \sigma^2)$, both the sample mean and the sample median are consistent estimators of μ.

Efficiency and Minimum Variance—

In order to make a choire amongst consistent estimators, we have to introduce the idea of 'efficiency'. Of two consistent estimators for the same parameter, the statistic with the smaller sampling variance is said to be *"more efficient"*. Thus if t and t' are both consistent estimators of θ, and

$$\text{Var}\,(t) < \text{Var}\,(t')$$

then t is 'more efficient' than t' in estimating θ; because it is grouped more closely around θ and will on the average deviate less from θ.

If a consistent estimator exists whose sampling variance is less than that of any other consistent estimator, it is said to be "most efficient"; and it provides a standard for the measurement of *'efficiency'* of a statistic. If V_0 be the variance of the most efficient estimator and V be the variance of any other estimator, then the efficiency of the estimator is defined as

$$\text{Efficiency} = \frac{V_0}{V} \tag{14.3.9}$$

Obviously, the measure of efficiency cannot exceed 1. In sampling from a Normal population $N(\mu, \sigma^2)$, both the sample mean and the sample median are consistent estimators of μ, but

$$\text{Var}\,(\bar{x}) = \frac{\sigma^2}{n}, \qquad \text{Var}\,(\textit{median}) = \frac{\pi\sigma^2}{2n}$$

(for large n). Since Var ($\bar{x}$) is smaller than Var (median), mean is more efficient than median in estimating the parameter μ. It can be shown that the sample mean is the most efficient estimator. Hence

$$\text{Efficiency of median} = \frac{\sigma^2/n}{\pi\sigma^2/2n} = \frac{2}{\pi} = 0.64 \text{ approx.}$$

A statistic t which has the minimum variance among all estimators of θ is called the *Minimum Variance* (MV) estimator.

A statistic which is unbiased and has also the minimum variance (i.e. most efficient) is said to be the *Minimum Variance Unbiased Estimator* (MVUE). The variance of MVUE is often given by the R.H.S. of (*14.3.10*).

Theorem 4. In sampling from a Normal population $N(\mu, \sigma^2)$ the sample mean $\bar{x}$ is the Minimum Variance Unbiased Estimator for the parameter μ.

Let $x_1, x_2, \ldots x_n$ be a random sample and

$$T = a_1x_1 + a_2x_2 + \ldots + a_nx_n$$

where $a_1, a_2, \ldots a_n$ are constants. If the linear function T is an unbiased estimator of a parameter and also has the minimum variance, it is said to be the *Best Linear Unbiased Estimator* (BLUE).

Theorem 5. The sample mean $\bar{x}$ is the Best Linear Unbiased Estimator of the population mean μ.

Example 14 : 4 If T_1, T_2, T_3 are independent, unbiased estimates of θ and all have the same variance, which of the following unbiased estimates of θ would you prefer? $(T_1+2T_2+T_3)/4$, $(2T_1+T_2+2T_3)/5$, $(T_1+T_2+T_3)/3$. [*W.B.H.S. '78*]

Solution : The last expression, viz. $(T_1+T_2+T_3)/3$ would be the most preferable ; because it is the *mean* of T_1, T_2, T_3 and hence the Best Linear Unbiased Estimator, *i.e.* has the minimum variance among all linear functions which may be proposed as unbiased estimators of θ.

Theorem 6. (*Cramer-Rao inequality*) If t is an unbiased estimator of a parameter θ, based on a random sample from a continuous population with p.d.f. $f(x, \theta)$, then

$$\text{Var}(t) \geqslant \frac{1}{n\,E\left\{\frac{\partial}{\partial\theta}\log f(x,\theta)\right\}^2} \qquad (14.3.10)$$

An estimator whose sampling variance attains the lower bound given by (*14.3.10*) is called the *Minimum Variance Bound* (MVB) *estimator*. For example, the mean ($\bar{x}$) of a random sample from a normal population is a MVB estimator of μ.

Sufficiency—

A statistic is said to be a *'sufficient estimator'* of a parameter θ, if it contains all information in the sample about θ. If a statistic t exists such that the joint distribution of the sample is expressible as the product of two factors, one of which is the sampling distribution of t and contains θ, but the other factor is independent of θ, then t will be a sufficient estimator of θ.

Thus, if $x_1, x_2, \ldots x_n$ is a random sample from a population whose p.m.f. or p.d.f. is $f(x, \theta)$, and t is a sufficient estimator of θ then we can write

$$f(x_1, \theta).\ f(x_2, \theta).\ \ldots\ f(x_n, \theta) = g(t, \theta).\ h(x_1, x_2, \ldots x_n)$$

where $g(t, \theta)$ is the sampling distribution of t and contains θ, but $h(x_1, x_2, \ldots x_n)$ is independent of θ. Since the parameter θ occurring in the joint distribution of all the sample observations can be contained in the distribution of the statistic t, it is said that t alone can provide all 'information' about θ and is therefore "sufficient" for θ.

Sufficient estimators are the most desirable kind of estimators, but unfortunately they exist in only relatively few cases. If a sufficient estimator exists, it can be found by the Method of Maximum Likelihood.

Theorem 7. In random sampling from a Normal population $N(\mu, \sigma^2)$, the sample mean $\bar{x}$ is a sufficient estimator of μ.

14.4 Methods of Point Estimation

(1) Method of Maximum Likelihood

This is a convenient method for finding an estimator which satisfies most of the criteria discussed earlier. Let $x_1, x_2, \ldots x_n$ be a random sample from a population with p.m.f. (for discrete case) or p.d.f. (for continuous case) $f(x, \theta)$, where θ is the parameter. Then the joint distribution of the sample observations, viz.

$$L = f(x_1, \theta).\ f(x_2, \theta).\ \ldots\ f(x_n, \theta) \qquad (14.4.1)$$

is called the *Likelihood Function* of the sample.

The *Method of Maximum Likelihood* consists in choosing as an estimator of θ that statistic, which when substituted for θ, maximises the likelihood function L. Such a statistic is called a *maximum likelihood estimator* (*m.l.e.*). We shall denote the *m.l.e.* of θ by the symbol θ_0.

Since log L is maximum when L is maximum, in practice the m.l.e. of θ is obtained by maximising log L. This is achieved by differentiating log L partially with respect to θ, and using the two relations

$$\left[\frac{\partial}{\partial\theta}\log L\right]_{\theta=\theta_0}=0, \qquad \left[\frac{\partial^2}{\partial\theta^2}\log L\right]_{\theta=\theta_0}<0 \qquad (14.4.2)$$

Properties of maximum likelihood estimator (m.l.e.)—

(1) The m.l.e. is consistent, most efficient, and also sufficient, provided a sufficient estimator exists.

(2) The m.l.e. is not necessarily unbiased. But when the m.l.e. is biased, by a slight modification, it can be converted into an unbiased estimator.

(3) The m.l.e. tends to be distributed normally for large samples.

(4) The m.l.e. is invariant under functional transformations. This means that if T is an m.l.e. of θ, and $g(\theta)$ is a function of θ, then $g(T)$ is the m.l.e. of $g(\theta)$.

Example 14 : 5 On the basis of a random sample find the maximum likelihood estimator of the parameter of a Poisson distribution.

Solution : The Poisson distribution with parameter m has p.m.f.

$$f(x, m) = \frac{e^{-m}.m^x}{x\,!} \qquad (x = 0, 1, 2, \ldots \infty)$$

The likelihood function of the sample observations is

$$L = f(x_1, m).f(x_2, m). \ldots f(x_n, m)$$

$$\therefore \quad \log L = \log f(x_1, m) + \log f(x_2, m) + \ldots + \log f(x_n, m)$$

$$= \sum_{i=1}^{n} \log f(x_i, m)$$

$$= \Sigma\,[-m + x_i\,(\log m) - \log x_i\,!\,]$$

$$= -\,nm + (\log m)\,\Sigma x_i - \Sigma \log (x_i\,!\,)$$

Taking partial derivative of log L with respect to the parameter m,

$$\frac{\partial \log L}{\partial m} = -\,n + \frac{\Sigma x_i}{m} = -\,n + \frac{n\bar{x}}{m}$$

Now replacing m by m_0 and equating the result to zero,

$$\left[\frac{\partial \log L}{\partial m}\right]_{m=m_0} = -\,n + \frac{n\bar{x}}{m_0} = 0$$

Solving, we get $\quad m_0 = \bar{x}$. Again,

$$\left[\frac{\partial^2 \log L}{\partial m^2}\right]_{m=m_0} = -\,\frac{n\bar{x}}{m_0{}^2} = -\,\frac{n\bar{x}}{\bar{x}^2} = -\,\frac{n}{\bar{x}} \text{ which is negative.}$$

This shows that log L is maximum at $m = m_0 = \bar{x}$. That is, the m.l.e. of m is $m_0 = \bar{x}$, the sample mean.

Example 14 : 6 Find the maximum likelihood estimator of the variance σ^2 of a Normal population $N(\mu, \sigma^2)$, when the parameter μ is known. Show that this estimator is unbiased.

Solution : The p.d.f. of Normal distribution is

$$f(x, \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}; \quad (-\infty < x < +\infty)$$

The logarithm (to the base e) of the likelihood function L is

$$\log L = \sum_{i=1}^{n} \log f(x_i, \mu, \sigma^2)$$

$$= \Sigma\left[-\log\sigma - \frac{1}{2}\log(2\pi) - \frac{(x_i-\mu)^2}{2\sigma^2}\right]$$

$$= -\frac{n}{2}\log\sigma^2 - \frac{n}{2}\log(2\pi) - \frac{\Sigma(x_i-\mu)^2}{2\sigma^2}$$

Differentiating partially with respect to σ^2,

$$\frac{\partial}{\partial(\sigma^2)}\log L = -\frac{n}{2\sigma^2} + \frac{\Sigma(x_i-\mu)^2}{2(\sigma^2)^2}$$

The m.l.e. of σ^2 is obtained by solving

$$-\frac{n}{2\sigma_0^2} + \frac{\Sigma(x_i-\mu)^2}{2\sigma_0^4} = 0$$

$$\therefore \quad \sigma_0^2 = \sum_{i=1}^{n}(x_i-\mu)^2/n$$

It can be shown that $\left[\frac{\partial^2\log L}{\partial(\sigma^2)^2}\right]_{\sigma^2=\sigma_0^2} = \frac{-n}{2\sigma_0^4}$

which is negative. Thus the maximum likelihood estimator of σ^2 is

$$\sigma_0^2 = \Sigma(x_i-\mu)^2/n, \ (\mu \text{ known})$$

Again, since $x_1, x_2, \ldots x_n$ is a random sample and μ is the population mean, we have $E(x_i-\mu)^2 = \sigma^2$. Therefore,

$$E(\sigma_0^2) = \sum_{i=1}^{n} E(x_i-\mu)^2/n = \Sigma\sigma^2/n = \sigma^2$$

Thus, σ_0^2 is an unbiased estimator of σ^2.

Example 14 : 7 Find the m.l.e. of the parameters μ and σ^2 in random samples from a $N(\mu, \sigma^2)$ population, when both the parameters are unknown.

Solution : As in the preceding example,

$$\log L = -\frac{n}{2}\log\sigma^2 - n\log\sqrt{2\pi} - \frac{\Sigma(x_i-\mu)^2}{2\sigma^2}$$

$$\therefore \quad \left[\frac{\partial\log L}{\partial\mu}\right]_{\mu=\mu_0} = \frac{-1}{2\sigma^2}\Sigma\, 2(x_i-\mu_0)(-1) = 0$$

This gives $\Sigma(x_i - \mu_0) = 0$; i.e., $\mu_0 = \bar{x}$, the sample mean. The m.l.e. of the parameter μ is the sample mean $\bar{x}$. (Note that this estimator is *unbiased*.)

Proceeding as in Example 14 : 6 we have $\sigma_0{}^2 = \Sigma(x_i - \mu)^2/n$. But since the parameter μ is not known, it is replaced by the m.l.e. $\mu_0 = \bar{x}$. The m.l.e. of σ^2 is now

$$\sigma_0{}^2 = \Sigma\,(x_i - \bar{x})^2/n = S^2$$

which is the sample variance. (Note that this estimator is *biased*).

Example 14 : 8 *A* tossed a biased coin 50 times and got head 20 times, while *B* tossed it 90 times and got 40 heads. Find the maximum likelihood estimate of the probability of getting head when the coin is tossed.

Solution : Let P be the unknown probability of obtaining a head. Using binomial distribution,

Probability of 20 heads in 50 tosses $= {}^{50}C_{20}\, P^{20}\, (1-P)^{30}$

Probability of 40 heads in 90 tosses $= {}^{90}C_{40}\, P^{40}\, (1-P)^{50}$

The likelihood function is given by the product of these probabilities :

$$L = {}^{50}C_{20} \,.\, {}^{90}C_{40}\, P^{60}(1-P)^{80}$$

$$\therefore\ \log L = \log\,({}^{50}C_{20} \,.\, {}^{90}C_{40}) + 60 \log P + 80 \log\,(1-P)$$

Hence, $$\frac{\partial \log L}{\partial P} = \frac{60}{P} - \frac{80}{1-P}$$

The maximum likelihood estimate P_0 is therefore obtained by solving

$$\frac{60}{P_0} - \frac{80}{1-P_0} = 0.$$

This gives $P_0 = 60/140 = 3/7$. *Ans.* 3/7

(2) Method of Moments

The *Method of Moments* consists in equating the first few moments of the population with the corresponding moments of the sample, i.e. setting

$$\mu_r{}' = m_r{}' \qquad (14.4.3)$$

where $\mu_r{}' = E(x^r)$ and $m_r{}' = \Sigma x_i{}^r/n$. Since the parameters enter into the population moments, these relations when solved for the parameters give the estimates by the method of moments. Of course, this method is applicable only when the population moments exist. The method is generally applied for fitting theoretical distributions to observed data.

Example 14 : 9 Estimate the parameter p of the binomial distribution by the method of moments (when n is known).

Solution : For the binomial distribution $\mu_1' = E(x) = np$. Also $m_1' = \bar{x}$. Setting $\mu_1' = m_1'$, we have $np = \bar{x}$. Thus

$$p = \bar{x}/n$$

i.e. the estimated value of p is given by the sample mean divided by the parameter n (known).

Example 14 : 10 Find the estimates of μ and σ in the Normal population $N(\mu, \sigma^2)$ by the method of moments.

Solution : Equate the first two moments of the population and the sample, $\mu_1' = m_1'$ and $\mu_2' = m_2'$, i.e. $\mu_2 = m_2$. Thus

$$\mu = \bar{x} \text{ and } \sigma^2 = S^2, \text{ the sample variance,}$$

The parameters μ and σ are estimated by the sample mean $\bar{x}$ and the sample standard deviation S respectively.

14.5 Interval Estimation

In the theory of point estimation, developed earlier, any unknown parameter is estimated by a single quantity. Thus the sample mean ($\bar{x}$) is used to estimate the population mean (μ), and the sample proportion (p) is taken as an estimator of the population proportion (P). A single estimator of this kind, however good it may be, cannot be expected to coincide with the true value of the parameter, and may in some cases differ widely from it. In the theory of interval estimation, it is desired to find an interval which is expected to include the unknown parameter with a specified probability.

Let $x_1, x_2, \ldots x_n$ be a random sample from a population of a known mathematical form which involves an unknown parameter θ. We would try to find two functions t_1 and t_2 based on sample observations such that the probability of θ being included in the interval (t_1, t_2) has a given value, say c.

$$P(t_1 \leqslant \theta \leqslant t_2) = c \tag{14.5.1}$$

Such an interval, when it exists, is called a *Confidence Interval* for θ. The two quantities t_1 and t_2 which serve as the lower and upper limits of the interval are known as *Confidence Limits*. The probability (c) with which the confidence interval will include the true value of the parameter is known as *Confidence Coefficient* of the interval.

The significance of confidence limits is that if many independent random samples are drawn from the same population and the confidence interval is calculated from each sample, then the parameter will actually be included in the intervals in c proportion of cases in the long run. Thus the estimate of the parameter is stated as an interval with a specified degree of confidence.

The calculation of confidence limits is based on the knowledge of sampling distribution of an appropriate statistic. Suppose, we have a random sample of size n from a Normal population $N(\mu, \sigma^2)$, where the variance σ^2 is known. It is required to find 95% confidence limits for the unknown parameter μ. We know that the sample mean $(\bar{x})$ follows normal distribution with mean μ and variance σ^2/n, and so

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

has a standard normal distribution (See *13.8.5*, page 193). Since 95% of the area under the standard normal curve lies between the ordinates at $z = \pm 1.96$ (see *13.8.3*), we have

$$P\left[-1.96 \leqslant \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \leqslant 1.96\right] = 0.95$$

i.e. in 95% of cases the following inequalities hold

$$-1.96 \leqslant \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \leqslant 1.96$$

Separating out μ we get

$$\bar{x} - 1.96\frac{\sigma}{\sqrt{n}} \leqslant \mu \leqslant \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}$$

The interval $\left(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}},\ \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}\right)$ is known as the 95% confidence interval for μ, and the 95% confidence limits are

$$\bar{x} \pm 1.96\frac{\sigma}{\sqrt{n}}$$

Again, 99% of area under the standard normal curve lies between the ordinates at $z = \pm 2.58$, and 99.73% (i.e. almost whole) of the area lies between $z = \pm 3$. Hence proceeding exactly in the same manner, the 99% confidence limits for μ are

$$\bar{x} \pm 2.58\frac{\sigma}{\sqrt{n}}$$

and almost sure limits for μ are

$$\bar{x} \pm 3\frac{\sigma}{\sqrt{n}}$$

In fact, using values from the normal probability integral table (showing areas under standard normal curve), confidence limits corresponding to any specified percentage can be obtained. These are *exact* confidence limits.

In some cases, the population may not be truely a normal distribution, but the sampling distributions of statistics based on large samples are approximately normal. For example, the sample mean $(\bar{x})$ based on a large random sample drawn (with or without replacement) from any population is approximately normally distributed. Similarly, the sample proportion (p) calculated from a large random sample has approximately a normal distribution (see Section 13.6). It is therefore possible to utilise the properties relating to the percentage of area under the standard normal curve (page 193) to find approximate confidence limits for the population mean μ and the population proportion P, provided the sample size n is large.

Approximate Confidence Limits (large samples)
any Distribution

(1) for Mean μ :

$$\begin{aligned} &\text{95\% confidence limits} &&= \bar{x} \pm 1.96 \text{ (S.E. of } \bar{x} \text{)} \\ &\text{99\% confidence limits} &&= \bar{x} \pm 2.58 \text{ (S.E. of } \bar{x} \text{)} \\ &\text{Almost sure limits} &&= \bar{x} \pm 3 \text{ (S.E. of } \bar{x} \text{)} \end{aligned} \qquad (14.5.2)$$

(See Example 14 : 12)

(2) for Proportion P :

$$\begin{aligned} &\text{95\% confidence limits} &&= p \pm 1.96 \text{ (S.E. of } p \text{)} \\ &\text{99\% confidence limits} &&= p \pm 2.58 \text{ (S.E. of } p \text{)} \\ &\text{Almost sure limits} &&= p \pm 3 \text{ (S.E. of } p \text{)} \end{aligned} \qquad (14.5.3)$$

(See Examples 14 : 11 & 14 : 13)

(3) for Difference of Means $(\mu_1 - \mu_2)$:

$$\begin{aligned} &\text{95\% confidence limits} &&= (\bar{x}_1 - \bar{x}_2) \pm 1.96 \text{ (S.E. of } \bar{x}_1 - \bar{x}_2) \\ &\text{99\% confidence limits} &&= (\bar{x}_1 - \bar{x}_2) \pm 2.58 \text{ (S.E. of } \bar{x}_1 - \bar{x}_2) \\ &\text{Almost sure limits} &&= (\bar{x}_1 - \bar{x}_2) \pm 3 \text{ (S.E. of } \bar{x}_1 - \bar{x}_2) \end{aligned}$$

(See Example 14 : 29)

(14.5.4)

(4) for Difference of Proportions $(P_1 - P_2)$:

$$\begin{aligned} &\text{95\% confidence limits} &&= (p_1 - p_2) \pm 1.96 \text{ (S.E. of } p_1 - p_2) \\ &\text{99\% confidence limits} &&= (p_1 - p_2) \pm 2.58 \text{ (S.E. of } p_1 - p_2) \\ &\text{Almost sure limits} &&= (p_1 - p_2) \pm 3 \text{ (S.E. of } p_1 - p_2) \end{aligned}$$

(See Example 14 : 23) (14.5.5)

Note : —(i) The '*probable limits*' (without any reference to the degree of confidence) may be taken to be '*almost sure limits*' in all the above cases.

(ii) The formulae for S.E. involve population parameters. If these parameters are not known, an approximate value of S.E. may be obtained by substituting the statistic for the corresponding parameter.]

Example14 : 11 A sample of 600 screws is taken from a large consignment and 75 are found to be defective. Estimate the percentage of defectives in the consignment and assign limits within which the percentage lies. *[I.C.W.A. (old), June '75]*

Solution : There are 75 defectives in a sample of size $n = 600$. Therefore, the sample proportion of defectives is

$$p = 75/600 = 1/8 = 12.5\%$$

This may be taken as an estimate of the percentage of defectives (P) in the whole consignment ('Point estimation'.)

The 'limits' to the percentage of defectives refer to the confidence limits, which may be given as $p \pm 3$(S.E. of p).

$$\text{S.E. of } p = \sqrt{\frac{PQ}{n}}$$

$$= \sqrt{\frac{pq}{n}} \text{ approximately ; (since the population proportion } P \text{ is not known).}$$

$$= \sqrt{\frac{\frac{1}{8}(1-\frac{1}{8})}{600}} = \frac{1}{80}\sqrt{\frac{7}{6}} = \cdot 0135$$

Thus, the limits for P are

$$\tfrac{1}{8} \pm 3 \times .0135 = .125 \pm .0405$$
$$= .1655 \text{ and } .0845 = 16.55\% \text{ and } 8 \cdot 45\%$$

The limits to the percentage of defectives in the consignment are 8.45% to 16.55% ('Interval estimation'.) *Ans.* 12.5% ; 8.45% to 16.55%

Example 14 : 12 A random sample of 100 ball bearings selected from a shipment of 2000 ball bearings has an average diameter of 0.354 inch with a S.D. = ·048 inch. Find 95% confidence interval for the average diameter of these 2000 ball bearings.

[*C.U., M.Com. '72; I.C.W.A., Dec. '79*]

Solution : Theory : If a random sample of large size n is drawn *without replacement* from a *finite population* of size N, then the 95% confidence limits for the population mean μ are $\bar{x} \pm 1.96$ (S.E. of x), where $\bar{x}$ denotes the sample mean and

$$\text{S.E. of } \bar{x} = \frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}},$$

σ denoting the Standard Deviation (S.D.) of the population. Here,

Sample size (n) = 100,	Population size (N) = 2000
Sample mean ($\bar{x}$) = 0·354,	Sample S.D. (S) = ·048

Since σ is not known, an approximate value of S.E. is obtained on replacing the population S.D. (σ) by the sample S.D. (S).

$$\text{S.E. of } \bar{x} = \frac{S}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}} \text{ approximately}$$

$$= \frac{\cdot 048}{\sqrt{100}}\sqrt{\frac{2000-100}{2000-1}} = .0047$$

The 95% confidence limits for the population mean μ are

$$\bar{x} \pm 1.96 \text{ (S.E. of } \bar{x}) = 0.354 \pm 1.96 \times .0047$$
$$= 0.354 \pm .0092 = 0.3632 \text{ and } 0.3448$$

Thus, the 95% confidence interval is (0.3448 to 0.3632) inch.

Example 14 : 13 A random sample of 100 articles taken from a large batch of articles contains 5 defective articles. (a) Set up 96 per cent confidence limits for the proportion of defective articles in the batch. (b) If the batch contains 2696 articles set up 95% confidence interval for the proportion of defective articles.

[I.C.W.A., June '74]

Solution : (a) The 96% confidence limits for the population proportion (P) are given by $p \pm 2.05$ (S.E. of p), where p is the sample proportion.

$$\text{S.E. of } p = \sqrt{\frac{PQ}{n}}$$

Since the formula involves the unknown population proportion P, an approximate value of S.E. is obtained on replacing the population proportion (P) by the sample proportion (p). Putting $n = 100$ and $p = 5/100 = .05$, ($q = 1-p = .95$)

$$\text{S.E. of } p = \sqrt{\frac{pq}{n}} = \sqrt{\frac{.05 \times .95}{100}} = .022$$

Hence, the 96% confidence limits for P are

$$p \pm 2.05 \text{ (S.E. of } p) = .05 \pm 2.05 \times .022 = .05 \pm .045$$
$$= .05 + .045 \text{ and } .05 - .045$$
$$= .095 \text{ and } .005$$

(b) The 95% confidence limits for proportion (P) are given by $p \pm 1.96$ (S.E. of p). But, when the population is of a finite size N,

$$\text{S.E of } p = \sqrt{\frac{pq}{n}} \sqrt{\frac{N-n}{N-1}} \text{ (approximately)}$$

Here, $n = 100$, $N = 2696$, $p = .05$. Putting these values

$$\text{S.E. of } p = \sqrt{\frac{.05 \times .95}{100}} \sqrt{\frac{2696-100}{2696-1}} = .022\sqrt{\frac{2596}{2695}}$$
$$= .022 \times \sqrt{.963} = .022 \times .98 = .0216 \text{ (approx.)}$$

Hence, the required 95% confidence limits for P are

$$p \pm 1.96 \text{(S.E. of } p) = .05 \pm 1.96 \times .0216 = .092 \text{ and } .008$$

The 95% *confidence interval* for the proportion of defective articles is 008 to .092.

Ans. .005 and .095 ; .008 to .092

Example 14 : 14 10 Life Insurance Policies in a sample of 200 taken out of 50,000 were found to be insured for less than Rs. 5,000. How many policies can be reasonably expected to be insured for less than Rs. 5,000 in the whole lot at 95% Confidence level?

[*C.A., Nov. '80*]

Solution : Let us first find the confidence limits for the 'proportion' of life insurance policies insured for less than Rs. 5,000 in the whole lot. Here,

Population size (N) $= 50{,}000$,
Sample size (n) $= 200$
Sample proportion (p) $= 10/200 = .05$

Using (*13.7.10*),

$$\text{S.E. of } p = \sqrt{\frac{.05 \times .95}{200}} \sqrt{\frac{50{,}000 - 200}{50{,}000 - 1}}$$

$$= \sqrt{\frac{.05 \times .95 \times 49800}{200 \times 50\,000}} \text{ (approx.)}$$

$$= .0154$$

95% confidence limits for the population proportion (P) are

$$p \pm 1.96 \text{ (S.E. of } p) = .05 \pm 1.96 \times .0154$$
$$= .05 \pm .030 = .080 \text{ and } .020$$

This means that out of the lot of 50,000, the 'proportion' of policies insured for less than Rs. 5000 lies between .020 and .080, with probability 95%. Thus the 'number' of such policies lies between $50{,}000 \times .020 = 1000$ and $50{,}000 \times .080 = 4{,}000$.

Ans. Between 1,000 and 4,000.

Exact Confidence Limits (any sample size)
Normal Distribution

In the following cases, it is assumed that samples are drawn at random from Normal populations.

(5) for Mean μ : (s.d. known)

95% confidence limits $= \bar{x} \pm 1.96\,(\sigma/\sqrt{n})$
99% confidence limits $= \bar{x} \pm 2.58\,(\sigma/\sqrt{n})$ (*14.5.6*)

(See Examples 14:15 & 14:37)

(6) for Difference of Means ($\mu_1 - \mu_2$) : (s.d.s known)

95% confidence limits $= (\bar{x}_1 - \bar{x}_2) \pm 1.96 \sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}$
99% confidence limits $= (\bar{x}_1 - \bar{x}_2) \pm 2.58 \sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}$ (*14.5.7*)

(See Example 14:39)

Example 14 : 15 A random sample of size 10 was drawn from a normal population with an unknown mean and a variance of 44.1 (inch)^2. If the observations are (in inches): 65, 71, 80, 76, 78, 82, 68, 72, 65 and 81, obtain the 95% confidence interval for the population mean.

Solution : We are given $n = 10$, $\sigma^2 = 44.1$ and $\Sigma x_i = 738$.

$\therefore$ $\bar{x} = 738/10 = 73.8$. Since the population s.d. σ is known, using formula (*14.5.6*), 95% confidence limits for μ are given by

$$73.8 \pm 1.96 \times \sqrt{44.1} / \sqrt{10} = 73.8 \pm 1.96 \times \sqrt{4.41}$$
$$= 73.8 \pm 1.96 \times 2.1$$
$$= 73.8 \pm 4.1 = 77.9 \text{ and } 69.7.$$

The 95% confidence interval for μ is therefore 69.7 to 77.9 inches.

(7) for Mean μ: (s.d. unknown)

In random samples from a Normal population $N(\mu, \sigma^2)$

$$t = \frac{\bar{x} - \mu}{S/\sqrt{n-1}}$$

follows t distribution with $(n-1)$ degree of freedom, where S is the sample s.d. (see *13.8.15*). If $t_{.025}$ denotes the upper 2.5% point of t distribution with $(n-1)$ d.f., then the 95% confidence interval for μ is obtained from

$$-t_{.025} \leqslant \frac{\bar{x} - \mu}{S/\sqrt{n-1}} \leqslant t_{.025}$$

Hence, for the population mean μ

95% confidence limits $= \bar{x} \pm t_{.025}\,(S/\sqrt{n-1})$ (*14.5.8a*)

99% confidence limits $= \bar{x} \pm t_{.005}\,(S/\sqrt{n-1})$ (*14.5.8b*)

(see Examples 14:16 & 14:40)

(8) for Difference of Means $(\mu_1 - \mu_2)$: (common s.d. unknown)

Assuming that two independent samples are drawn from two Normal populations with means μ_1, μ_2 but a common *unknown* s.d. σ.

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{s\sqrt{1/n_1 + 1/n_1}}$$

follows t distribution with $(n_1 + n_2 - 2)$ d.f. (see *13.8.16*), where

$$s^2 = (n_1 S_1^2 + n_2 S_2^2)/(n_1 + n_2 - 2)$$

Hence, with 95% probability the following inequalities hold

$$-t_{.025} \leqslant \frac{(\bar{x}_1-\bar{x}_2)-(\mu_1-\mu_2)}{s\sqrt{(1/n_1+1/n_2)}} \leqslant t_{.025}$$

from which the 95% confidence limits for $(\mu_1-\mu_2)$ are

$$(\bar{x}_1-\bar{x}_2) \pm t_{.025}.\, s\sqrt{(1/n_1+1/n_2)} \qquad (14.5.9a)$$

Similarly the 99% confidence limits for $(\mu_1-\mu_2)$ are

$$(\bar{x}_1-\bar{x}_2) \pm t_{.005}.\, s\sqrt{(1/n_1+1/n_2)} \qquad (14.5.9b)$$

(See Example 14:43)

Example 14 : 16 A random sample of 10 students of class II was selected from schools in a certain region, and their weights recorded are shown below (in lb.) : 38, 46, 45, 40, 35, 39, 44, 45, 33, 37. Find 95% confidence limits within which the mean weight of all such students in the region is expected to lie. (Given $t_{.025} = 2.262$ for 9 d.f. and 2.228 for 10 d.f.).

Solution : From the given data, $\bar{x} = 402/10 = 40.2$. To calculate the s.d. (S), we take deviations from 40, i.e. $d = x-40$.

$$-2,\ 6,\ 5,\ 0,\ -5,\ -1,\ 4,\ 5,\ -7,\ -3$$

$$\Sigma d = 2,\ \Sigma d^2 = 190$$

$$\therefore \quad S^2 = \Sigma d^2/n - (\Sigma d/n)^2 = 190/10 - (2/10)^2 = 18.96$$

$$S = \sqrt{18.96} = 4.35.$$

Since the population s.d. σ is ***unknown*** the 95% confidence limits for μ are (see formula *14.5.8a*)

$$40{\cdot}2 \pm 2.262 \times 4.35/\sqrt{9} \quad \text{(degrees of freedom = 9)}$$

$$= 40.2 \pm 3.28 \quad = 36.92 \text{ and } 43.48$$

The 95% confidence limits for the mean weight are (in lb.) 36.9 and 43.5.

(9) for Variance σ^2 :

Case I. (mean known)—In random samples from $N(\mu, \sigma^2)$ population, $\Sigma(x_i-\mu)^2/\sigma^2$ follows chi-square distribution with n degrees of freedom (see *13.8.11*, page 196). If $\chi^2_{.975}$ and $\chi^2_{.025}$ denote the lower and the upper 2.5% points of chi-square distribution with n d.f., then with probability 95% we have

$$\chi^2_{.975} \leqslant \Sigma(x_i-\mu)^2/\sigma^2 \leqslant \chi^2_{.025}$$

from which the 95% confidence interval for σ^2 is

$$\Sigma(x_i-\mu)^2/\chi^2_{.025} \leqslant \sigma^2 \leqslant \Sigma(x_i-\mu)^2/\chi^2_{.975} \qquad (14.5.10)$$

Case II. (mean unknown)—In this case (see *13.8.12*) $nS^2/\sigma^2 = \Sigma(x_i-\bar{x})^2/\sigma^2$ follows chi-square distribution with $(n-1)$ degrees of freedom. Using the lower and the upper 2.5% points of chi-square distribution with $(n-1)$ d.f., we have with probability 95% the following inequalities

$$\chi^2_{.975} \leqslant nS^2/\sigma^2 \leqslant \chi^2_{.025}$$

from which the 95% confidence interval for σ^2 can be given as

$$nS^2/\chi^2_{.025} \leqslant \sigma^2 \leqslant nS^2/\chi^2_{.975} \qquad (14.5.11$$

Example 14 : 17 The standard deviation of a random sample of size 12 drawn from a normal population is 5.5. Calculate the 95% confidence limits for the standard deviation (σ) in the population (Given $\chi^2_{.975} = 3.82$ and $\chi^2_{.025} = 21.92$ for 11 degrees of freedom).

Solution : Here $n = 12$ and the sample s.d. $(S) = 5.5$. Substituting the values in formula (*14.5.11*), the 95% confidence interval for σ^2 is

$$\frac{12\times(5.5)^2}{21.92} \leqslant \sigma^2 < \frac{12\times(5.5)^2}{3.82}$$

or, $\quad 16.56 \leqslant \sigma^2 \leqslant 95.03$

i.e. $\quad 4.1 \leqslant \sigma \leqslant 9.7$

The 95% confidence limits for the population s.d. (σ) are 4.1 and 9.7.

Example 14 : 18 A sample of size 8 from a normal population yields as the unbiased estimate of population variance the value 4.4. Obtain the 99% confidence limits for the population variance σ^2 (Given $\chi^2_{.995} = 0.99$ and $\chi^2_{.005} = 20.3$ for 7 d.f.).

Solution : Here $n = 8$, and the unbiased estimate $s^2 = 4.4$. So, $nS^2 = (n-1)s^2 = 7\times4.4 = 30.8$. Hence the 99% confidence limits for σ^2 are obtained from

$$\frac{nS^2}{\chi^2_{.005}} < \sigma^2 \leqslant \frac{nS^2}{\chi^2_{.995}}$$

or, $\quad \dfrac{30.8}{20.3} \leqslant \sigma^2 \leqslant \dfrac{30.8}{0.99}$; i.e. $\quad 1.52 \leqslant \sigma^2 \leqslant 31.1$

(10) for Variance-Ratio σ_1^2/σ_2^2 : (means unknown)

If two independent random samples of sizes n_1 and n_2 are drawn from two Normal populations with unknown means μ_1, μ_2 and variances σ_1^2, σ_2^2 respectively, then $\dfrac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2}$ follows F distribution with degrees of freedom $(n_1-1,\ n_2-1)$. If $F_{.975}$ and $F_{.025}$ denote the lower and the upper 2.5% points of F distribution, we have with probability 95% the following inequalities

$$F_{.975} \leqslant \frac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2} < F_{.025} \qquad (14.5.12)$$

The 95% confidence interval for σ_1^2/σ_2^2 can be obtained from this as

$$\frac{1}{F_{.025}} \cdot \frac{s_1^2}{s_2^2} \leqslant \frac{\sigma_1^2}{\sigma_2^2} \leqslant \frac{1}{F_{.975}} \cdot \frac{s_1^2}{s_2^2} \qquad (14.5.13)$$

where s_1^2 and s_2^2 denote *unbiased estimators* of σ_1^2, σ_2^2 respectively from the two samples. (see Example 14:51)

14.6 Theory of Test of Significance

Statistical Hypothesis :

In many practical problems, statisticians are called upon to make decisions about a statistical population on the basis of sample observations. For example, given a random sample, it may be required to decide whether the population, from which the sample has been obtained, is a normal distribution with mean = 40 and s.d. = 3. In attempting to reach such decisions, it is necessary to make certain assumptions or guesses about the characteristics of population, particularly about the probability distribution or the values of its parameters. Such an assumption or statement about the population is called a *Statistical Hypothesis*. The validity of a hypothesis will be tested by analysing the sample. The procedure which enables us to decide whether a certain hypothesis is true or not, is called *Test of Significance* or *Test of Hypothesis.*

Null Hypothesis and Alternative Hypothesis :

In tests of significance, we start with a certain hypothesis about the population characteristics. This is called *Null Hypothesis*, and denoted by the symbol H_0. For example, the null hypothesis may be that the population mean is 40. We write

$$H_0\ (\mu = 40)$$

Any hypothesis which differs from the null hypothesis is called *Alternative Hypothesis*, and is denoted by the symbol H_1. The null hypothesis is tested against an alternative hypothesis which in the above case, may be either that the population mean is not 40, or that it is greater than 40, or that it is less than 40 ; i.e. any one of

$$H_1(\mu \neq 40), \qquad H_1\ (\mu > 40), \qquad H_1\ (\mu < 40)$$

The sample is then analysed to decide whether to 'reject' or not to reject the null hypothesis H_0. For this purpose, we choose a suitable statistic, called *"Test Statistic"* and find its sampling distribution, assuming that H_0 is really true. The 'observed value' of the statistic in the sample will in general be different from the 'expected value' because of sampling fluctuations. If the difference between them is large, the null hypothesis H_0 is rejected, and we doubt the validity of our assumption. If the difference is not large, H_0 is not rejected, and the difference may be considered to have arisen solely due to fluctuations of sampling. It is therefore necessary to decide how much of difference is tolerable before we are able to conclude that the null hypothesis is acceptable.

Level of significance, and Critical Region :

The decision about rejection or otherwise of the null hypothesis is based on probability considerations. Assuming the null hypothesis to be true, we calculate the *probability of obtaining a difference equal to or greater than the observed difference.* If this probability is found to be small, say less than .05, the conclusion is that the observed value of the statistic is rather unusual and has arisen because the underlying assumption (i.e. null hypothesis) is not true. We say that the observed difference is *significant at 5 per cent level,* and hence the 'null hypothesis is rejected' at 5 per cent *level of significance.* If, however, this probability is not very small, say more than .05, the observed difference cannot be considered to be unusual and is attributed to sampling fluctuations only. The difference is, now, said to be *not significant* at 5 per cent level, and we conclude that 'there is no reason to reject the null hypothesis' at 5 per cent level of significance. It has become customary to use 5% and 1% levels of significance, although other levels, such as 2% or 0.5% may also be used.

Without actually going to calculate this probability, the test of significance may be simplified as follows. From the sampling distribution of the statistic, we find the maximum difference which is exceeded in (say) 5 per cent of cases. If the observed difference is larger than this value, the null hypothesis is rejected. If it is less, there is no reason to reject the null hypothesis.

Suppose, the sampling distribution of the statistic is a normal distribution. Since the area under normal curve outside the ordinates at *mean* $\pm$ *1.96 (s.d.)* is only 5%, the probability that the observed value of the statistic differs from the expected value by 1.96 times the S.E. or more is .05 ; and the probability of a larger difference will be still smaller. If, therefore,

$$z = \frac{(Observed\ value)-(Expected\ value)}{S.E.} \tag{14.6.1}$$

is either greater than 1.96 or less than -1.96 (i.e. numerically greater than 1.96), the null hypothesis H_0 is rejected at 5% level of significance. The set of values $z \geqslant 1.96$ or $z \leqslant -1.96$, i.e.

$$|z| \geqslant 1.96$$

constitutes what is called the *Critical Region* for the test. Similarly since the area outside *mean* $\pm$ 2.58 (*s.d.*) is only 1%, H_0 is rejected at 1% level of significance, if z numerically exceeds 2.58, i.e. the critical region is $|z| \geqslant 2.58$ at 1% level.

Using the sampling distribution of an appropriate test statistic we are thus able to establish the maximum difference at a specified level between the observed and expected values that is consistent with the null hypothesis H_0. The set of values of the test statistic corresponding to this difference which lead to the acceptance of H_0 is called *Region of Acceptance*. Conversely, the set of values of the test statistic leading to the rejection of H_0 is referred to as *Region of Rejection* or "*Critical Region*" of the test. The value of the statistic which lies at the boundary of the regions of acceptance and rejection is called *Critical Value*. When the null hypothesis is true, the probability of observed value of the test statistic falling in the critical region is often called the "*Size of Critical Region*".

Size of Critical Region $\leqslant$ *Level of Significance*

However, for a continuous population, the critical region is so determined that its Size equals the Level of Significance (α).

Two-tailed and One-tailed tests :

Our discussions above were centred round testing the significance of 'difference' between the observed and expected values, i.e. whether the observed value is *significantly different* from (i.e. either larger or smaller than) the expected value, as could arise due to fluctuations of random sampling. In the illustration, the null hypothesis is tested against "*both-sided alternatives*" ($\mu > 40$ or $\mu < 40$), i.e.

$$H_0 \ (\mu = 40) \quad \text{against} \quad H_1 \ (\mu \neq 40)$$

Thus assuming H_0 to be true, we would be looking for large differences on both sides of the expected value, i.e. in "both tails" of the distribution. Such tests are, therefore, called "*two-tailed tests*".

Sometimes we are interested in tests for large differences on one side only i.e. in one 'one tail' of the distribution. For example, whether a new process of manufacture produces bricks with a 'higher' breaking strength, or whether a change in the production technique yields 'lower' percentage of defectives. These are known as "*one-tailed tests*".

For testing the null hypothesis against "*one-sided alternatives (right side)*" ($\mu > 40$), i.e.

$$H_0 \ (\mu = 40) \quad \text{against} \quad H_1 \ (\mu > 40)$$

the calculated value of the statistic z is compared with 1.645, since 5% of the area under the standard normal curve lies to the right of 1.645. If the observed value of z exceeds 1.645, the null hypothesis H_0 is rejected at 5% level of significance. If a 1% level were used, we would replace 1.645 by 2.33. Thus the critical regions for test at 5% and 1% levels are $z \geq 1.645$ and $z \geq 2.33$ respectively.

For testing the null hypothesis against *"one-sided alternatives (left side)"* ($\mu < 40$) i.e.

$$H_0 (\mu = 40) \quad \text{against} \quad H_1 (\mu < 40)$$

the value of z is compared with -1.645 for significance at 5% level, and with -2.33 for significance at 1% level. The critical regions are now $z < -1.645$ and $z < -2.33$ for 5% and 1% levels respectively.

In fact, the sampling distributions of many of the commonly-used statistics can be approximated by normal distributions as the sample size increases, so that these rules are applicable in most cases when the sample size is 'large', say, more than 30.

It is evident that the same null hypothesis may be tested against alternative hypotheses of different types depending on the nature of the problem (see Table 14.1). Correspondingly, the type of test and the critical region associated with each test will also be different (Table 14.2).

TABLE 14.1

Formulation of Null Hypothesis and Alternative Hypothesis

Problem	Null Hypothesis	Alternative Hypothesis
Test whether $\mu = 40$, or		
(1) μ is *different* from 40	$H_0 (\mu = 40)$	$H_1 (\mu \neq 40)$
(2) μ is *more* than 40	$H_0 (\mu = 40)$	$H_1 (\mu > 40)$
(3) μ is *less* than 40	$H_0 (\mu = 40)$	$H_1 (\mu < 40)$

TABLE 14.2

Type of Test and Critical Region

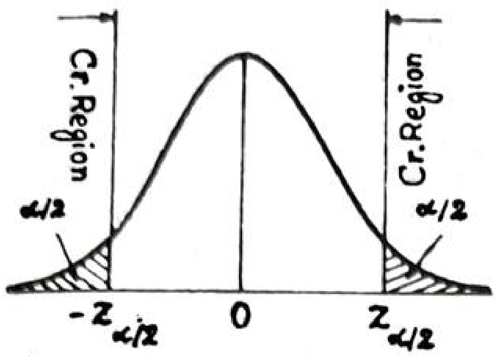
Alternative Hypothesis	Type of Alternative	Type of Test	Critical Region
$H_1 (\mu \neq 40)$	Both-sided	Two-tailed	Both Tails
$H_1 (\mu > 40)$	One-sided	One-tailed	Right Tail
$H_1 (\mu < 40)$	One-sided	One-tailed	Left Tail

Using z-statistic, the critical regions at α level of significance are shown in Fig. 14.1.

FIGURES 14.1

Critical Regions for Two-tailed and One-tailed Tests

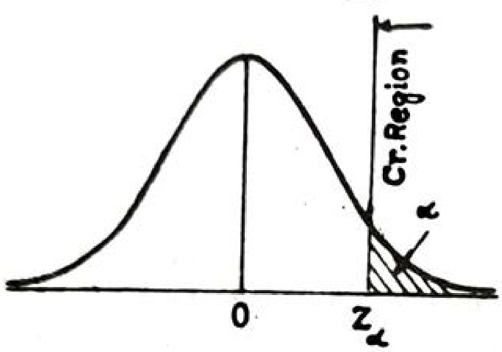
(1) Two-tailed Test



Critical Region:
$z \geq z_{\alpha/2}$ or $z \leq -z_{\alpha/2}$

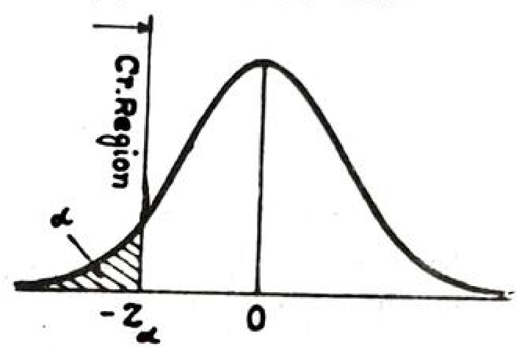
i.e. $|z| \geq z_{\alpha/2}$ (Both Tails)

(2) One-tailed Test



Critical Region:
$z \geq z_{\alpha}$ (Right Tail)

(3) One-tailed Test



Critical Region:
$z \leq -z_{\alpha}$ (Left Tail)

Type I & Type II Errors :

The procedure of testing statistical hypothesis does not guarantee that all decisions are perfectly accurate. At times, the tests may lead to erroneous conclusions. This is so, because the decision is taken on the basis of sample values, which are themselves fluctuating and depend purely on chance.

The errors in statistical decisions are of two types :

(i) **Type I error**—This is the error committed by the test in *rejecting a true null hypothesis.*

(ii) **Type II error**—This is the error committed by the test in *accepting a false null hypothesis.*

Considering the previous illustration for testing whether the population mean is 40, i.e. H_0 ($\mu = 40$), let us imagine that we have a random sample from a population whose mean is really 40. If we apply the test for H_0 ($\mu = 40$), we might find that the value of test statistic lies in the critical region, thereby leading to the conclusion that the population mean is not 40; i.e. the test rejects the null hypothesis although it is true. We have thus committed what is known as *"Type I error"* or *"Error of First Kind"*.

On the other hand, suppose that we have a random sample from a population whose mean is known to be different from 40, say 43. If we apply the test for H_0 ($\mu = 40$), the value of the test statistic may, by chance, lie in the acceptance region, leading to the conclusion that the mean may be 40; i.e. the test does not reject the null hypothesis H_0 ($\mu = 40$), although it is false. This is again another form of incorrect decision, and the error thus committed is known as *"Type II error"* or *"Error of Second Kind"*.

TABLE 14·3—**Errors in Test of Significance** $H_0(\theta=\theta_0)$

True situation	Statistical Decision	
	$\theta=\theta_0$	$\theta\neq\theta_0$
$\theta=\theta_0$	Correct decision	*Type I error*
$\theta\neq\theta_0$	*Type II error*	Correct decision

Using the sampling distribution of the test statistic, we can measure in advance the probabilities of committing the two types of errors. Since the null hypothesis is rejected only when the test statistic falls in the critical region,

Probability of Type I error

= Probability of rejecting H_0 ($\theta = \theta_0$), when it is true
= Probability that the test statistic lies in the critical region, assuming $\theta = \theta_0$.

The probability of Type I error must not exceed the level of significance (α) of the test.

Probability of Type I error $\leqslant$ Level of Significance

The probability of Type II error assumes different values for different values of θ covered by the alternative hypothesis H_1. Since the null hypothesis is accepted only when the observed value of the test statistic lies outside the critical region,

Probability of Type II error (when $\theta = \theta_1$),

= Probability of accepting H_0 ($\theta = \theta_0$), when it is false
= Probability that the test statistic lies in the region of acceptance, assuming $\theta = \theta_1$.

The probability of Type I error is necessary for constructing a test of significance. It is in fact the 'Size of the Critical Region.' The probability of Type II error is used to measure the "power" of the test in detecting falsity of the null hypothesis.

It is desirable that the test procedure be so framed which minimises both the types of error. But this is not possible, because for a given sample size an attempt to reduce one type of error is generally accompanied by an increase in the other type of error. The tests of significance are designed so as to limit the probability of Type I error to a specified value (usually 5% or 1%) and at the same time to minimise the probability of Type II error. Note that when the population has a continuous distribution,

Probability of Type I error
= Level of significance
= Size of critical region

Power of a Test :

The null hypothesis $H_0(\theta = \theta_0)$ is accepted when the observed value of test statistic lies outside the critical region, as determined by the test procedure. Suppose that the true value of θ is not θ_0, but another value θ_1, i.e. a specified alternative hypothesis $H_1(\theta = \theta_1)$ is true. Type II error is committed if H_0 is not rejected, i.e. the test statistic lies outside the critical region. Hence the probability of Type II error is a function of θ_1, because now $\theta=\theta_1$ is assumed to be true.

If $\beta(\theta_1)$ denotes the probability of Type II error, when $\theta = \theta_1$ is true, the complementary probability $1-\beta(\theta_1)$ is called *Power* of the test against the specified alternative $H_1(\theta = \theta_1)$.

Power = 1 − Probability of Type II error
= Probability of rejecting H_0 when H_1 is true

Obviously, we would like a test to be as 'powerful' as possible for all critical regions of the same size. Treated as a function of θ, the expression $P(\theta) = 1-\beta(\theta)$ is called *Power Function* of the test for θ_0 against θ. The curve obtained by plotting $P(\theta)$ against all possible values of θ, is known as *Power Curve.*

Example 14 : 19 The fraction of defective items in a large lot is P. To test the null hypothesis $H_0 : P = 0.2$, one considers the number f of defectives in a sample of 8 items and accepts the hypothesis if $f \leqslant 6$, and rejects the hypothesis otherwise. What is the probability of type I error of this test? What is the probability of type II error corresponding to $P = 0.1$? [*W.B.H.S.*, '79]

Solution : We are going to test whether the fraction (P) of defectives in the lot is 0.2 or not.

Null hypothesis H_0 $(P = 0.2)$. Alternative hypothesis $H_1 (P \neq 0.2)$. The test procedure is as follows :—

(1) Take a random sample of 8 items from the lot.

(2) Count the number of defectives in the sample.

(3) *Accept* H_0, if the number of defectives found in the sample is 6 or less $(f \leqslant 6)$, and conclude that the fraction of defectives in the lot may be 0.2.

Reject H_0, if the number of defectives actually obtained in the sample is 7 or 8. Conclusion: The fraction of defectives in the lot is not 0.2; i.e. H_0 is not tenable.

It may be seen that the number of defectives (f) in the sample is a random variable, which follows Binomial distribution with parameters $n = 8$ and P. The probability of r defectives is ${}^8C_r P^r (1-P)^{8-r}$.

Probability of Type I error

$= $ Probability of rejecting H_0, when H_0 is true.
$= $ Probability of 7 or 8 defectives, when $P = 0.2$
$= {}^8C_7 (0.2)^7 (0.8)^1 + (0.2)^8$
$= 0{\cdot}0000{,}8448$

Probability of Type II error

$= $ Probability of accepting H_0, when a specified H_1 is true.
$= $ Probability of 6 or less defectives, when $P = 0.1$.
$= 1 - $ Probability of 7 or 8 defectives, when $P = 0.1$.
$= 1 - [{}^8C_7 (0.1)^7 (0.9)^1 + (0.1)^8]$
$= 0{\cdot}9999{,}9927$

Definition of Useful Terms :

(1) **Statistical Hypothesis.** Any statement or assertion about a statistical population or the values of its parameters is called a *Statistical Hypothesis.* There are two types of hypotheses—Simple and Composite.

(2) **Simple Hypothesis.** A statistical hypothesis which specifies the population completely (i.e. the probability distribution and all parameters are known) is called a *Simple Hypothesis.*

(3) **Composite Hypothesis.** A statistical hypothesis which does not specify the population completely (i.e. either the form of probability distribution or some parameters remain unknown) is called a *Composite Hypothesis.*

(4) **Test of Hypothesis (or Test of Significance).** A *Test of Hypothesis* is a procedure which specifies a set of "rules for decision" whether to 'accept' or 'reject' the hypothesis under consideration (i.e. null hypothesis).

(5) **Null Hypothesis.** A statistical hypothesis which is set up (i.e. assumed) and whose validity is tested for possible rejection on the basis of sample observations is called a *Null Hypothesis*. It is denoted by H_0 and tested against alternatives. Tests of hypotheses deal with rejection or acceptance of null hypothesis only.

(6) **Alternative Hypothesis.** A statistical hypothesis which differs from the null hypothesis is called an *Alternative Hypothesis*, and is denoted by H_1. The alternative hypothesis is not tested, but its acceptance (rejection) depends on the rejection (acceptance) of the null hypothesis. Alternative hypothesis contradicts the null hypothesis. The choice of an appropriate critical region depends on the type of alternative hypothesis, viz. whether both-sided, one-sided (right/left) or specified alternative.

(7) **Test Statistic.** A function of sample observations (i.e. statistic) whose computed value determines the final decision regarding acceptance or rejection of H_0, is called a *Test Statistic*. The appropriate test statistic has to be chosen very carefully and a knowledge of its sampling distribution under H_0 (i.e. when the null hypothesis is true) is essential in framing the decision rules. If the value of the test statistic falls in the critical region, the null hypothesis is rejected.

(8) **Critical Region.** The set of values of the test statistic which lead to rejection of the null hypothesis is called *Critical Region* of the test. The probability with which a true null hypothesis is rejected by the test is often referred to as *"Size" of the Critical Region*. Geometrically, a sample $x_1, x_2, \ldots x_n$ of size n is looked upon as just a point **x**, called *Sample point*, within the region of all possible samples, called the *Sample Space* (W). The critical region is then defined as a subset (w) of those sample points which lead to the rejection of the null hypothesis.

(9) **Level of Significance.** The maximum probability with which a true null hypothesis is rejected is known as *Level of Significance* of the test, and is denoted by α. In framing decision rules, the level of significance is arbitrarily chosen in advance depending on the consequences of a statistical decision. Customarily, 5% or 1% level of significance is taken, although other levels such as 2% or $\frac{1}{2}$% is also used. The level of significance α is used to indicate the upper limit of the probability of committing Type I error, i.e. the size of the critical region.

(10) **Type I Error (or Error of First Kind).** This is the error committed in rejecting a null hypothesis by the test when it is really true. The critical region is so determined that the probability of Type I error does not exceed the level of significance of the test.

(11) **Type II Error (or Error of Second Kind).** This is the error committed in accepting a null hypothesis by the test when it is really

false. The probability of Type II error depends on the specified value of the alternative hypothesis, and is used in evaluating the efficiency of a test.

(12) **Power.** The probability of rejecting a false null hypothesis is called *Power of the test*. Therefore, Power is the probability of drawing a correct conclusion by the test, when the null hypothesis is false. For a specified value of the parameter consistent with the alternative hypothesis,

$$Power = 1 - Probability\ of\ Type\ II\ error.$$

If the power is graphically plotted against all specified alternatives, the curve obtained is known as *Power Curve*. The power of a test determines the extent to which a test can discriminate between true and false null hypotheses.

Steps in Test of Significance :

(1) *Set up the "Null Hypothesis" H_0 and the "Alternative Hypothesis" H_1* on the basis of the given problem. The null hypothesis usually specifies the values of some parameters involved in the population : $H_0(\theta = \theta_0)$. The alternative hypothesis may be any one of the following types : $H_1(\theta \neq \theta_0)$, $H_1(\theta > \theta_0)$, $H_1(\theta < \theta_0)$. The type of alternative hypothesis determines whether to use a two-tailed or one-tailed test (right or left tail).

(2) *State the appropriate "test statistic" T and also its sampling distribution, when the null hypothesis is true.* In large sample tests the statistic $z = (T - \theta_0)/\text{S.E.}\ (T)$, which approximately follows Standard Normal distribution, is often used. In small sample tests, the population is assumed to be Normal and various test statistics are used which follow Standard Normal, Chi-square, t or F distribution exactly.

(3) *Select the "level of significance" α of the test,* if it is not specified in the given problem. This represents the maximum probability of committing a Type I error, i.e. of making a wrong decision by the test procedure when in fact the null hypothesis is true. Usually, a 5% or 1% level of significance is used (If nothing is mentioned, use 5% level).

(4) *Find the "critical region" of the test at the chosen level of significance.* This represents the set of values of the test statistic which lead to rejection of the null hypothesis. The critical region always appears in one or both tails of the distribution, depending on whether the alternative hypothesis is one-sided or both-sided. The area in the tails (called 'size of the critical region') must be equal to the level of significance α. For a one-tailed test, α appears in

one tail and for a two-tailed test $\alpha/2$ appears in each tail of the distribution. The critical region is

$$
\begin{aligned}
T \geq T_{\alpha/2} \text{ or } T \leq T_{1-\alpha/2} &\quad \text{when } H_1\ (\theta \neq \theta_0) \\
T \geq T_{\alpha} &\quad \text{when } H_1\ (\theta > \theta_0) \\
T \leq T_{1-\alpha} &\quad \text{when } H_1\ (\theta < \theta_0)
\end{aligned}
$$

where T_α is the value of T such that the area to its right is α.

(5) *Compute the value of the test statistic T on the basis of sample data and the null hypothesis.* In large sample tests, if some parameters remain unknown they should be estimated from the sample.

(6) *If the computed value of test statistic T lies in the critical region, "reject H_0"; otherwise "do not reject H_0".* The decision regarding rejection or otherwise of H_0 is made after a comparison of the computed value of T with the critical value (i.e. boundary value of the appropriate critical region).

(7) *Write the conclusion in plain non-technical language.* If H_0 is rejected, the interpretation is: *"the data are not consistent with the assumption that the null hypothesis is true and hence H_0 is not tenable"*. If H_0 is not rejected, *"the data cannot provide any evidence against the null hypothesis and hence H_0 may be accepted to be true"*. The conclusion should preferably be given in the words stated in the problem.

14.7 Large Sample Tests

The following tests of significance are valid only for large sample sizes. These tests are also called "approximate tests", because the sampling distributions used are only approximately true, when the number of observations in the sample is large (say, more than 30). The accuracy however increases with larger sample sizes.

(A) Using Normal Distribution

(1) Test for a specified proportion :

A random sample of size n shows that the proportion of members possessing a certain attribute (e.g. defectives) is p. It is required to test the hypothesis that the proportion P in the population has a specified value P_0.

$$H_0\ (P = P_0)$$

If the sample size n is large, we use the test statistic

$$z = \frac{p - P_0}{(\text{S.E. of } p)} \qquad (14.7.1)$$

which approximately follows standard normal distribution.

For testing significance at 5% level, the rules are as follows:

(i) If the alternative hypothesis is that the population proportion P is 'different' from P_0, reject H_0 when the value of z lies outside the range -1.96 to 1.96.

$$H_1\ (P \neq P_0)\ ; \qquad \text{Critical Region } |z| \geqslant 1.96$$

(ii) If the alternative hypothesis is that the population proportion P is 'greater' than P_0, reject H_0 when the value of z is greater than 1.645.

$$H_1\ (P > P_0)\ ; \qquad \text{Critical Region } z \geqslant 1.645$$

(iii) If the alternative hypothesis is that the population proportion P is 'less' than P_0, reject H_0 when the value of z is less than -1.645.

$$H_1\ (P < P_0); \qquad \text{Critical Region } z \leqslant -1.645$$

Otherwise, do not reject the null hypothesis H_0.

For testing at 1% level, the values 1.96, 1.645 and -1.645, given above, should be replaced by 2.58, 2.33 and -2.33 respectively.

TABLE 14.4— **Rejection Rules for H_0 $(P = P_0)$**

Alternative hypothesis H_1	Critical Region	
	5% level	1% level
$P \neq P_0$	$\lvert z \rvert \geqslant 1.96$	$\lvert z \rvert \geqslant 2.58$
$P > P_0$	$z \geqslant 1.645$	$z \geqslant 2.33$
$P < P_0$	$z \leqslant -1.645$	$z \leqslant -2.33$

Confidence limits for the population proportion P are (see *14.5.3*)

95% confidence limits $= p \pm 1.96$ (S.E. of p)

99% ,, ,, $= p \pm 2.58$ (S.E. of p) *(14.7.2)*

where S.E. of $p = \sqrt{pq/n}$, if H_0 is rejected.

Example 14 : 20 A die was thrown 400 times and 'six' resulted 80 times. Do the data justify the hypothesis of an unbiased die?

[*I.C.W.A. June '77*]

Solution: Let us assume that the die is unbiased, i.e. the null hypothesis is that the probability of obtaining a 'six' with the die is 1/6.

$$H_0\ (P = \tfrac{1}{6})$$

Alternatively the die is not unbiased, i.e. the alternative hypothesis

$$H_1 \ (P \neq \tfrac{1}{6})$$

Since 'six' occurred 80 times out of 400, the *observed value* of the proportion (p) of 'six' is

$$p = 80/400 = 0.2$$

On the assumption that H_0 is true (i.e. the die is unbiased), the expected value of the proportion (P_0) of 'six' is

$$P_0 = \tfrac{1}{6} = 0.167$$

$$(\text{S.E. of } p) = \sqrt{\frac{\frac{1}{6} \times \frac{5}{6}}{400}} = \frac{\sqrt{5}}{120} = .0186$$

Using the proportion of 'six' in the sample, our test statistic is

$$z = \frac{\textit{Observed value} - \textit{Expected value}}{\textit{Standard Error}}$$

$$= \frac{0.2 - 0.167}{.0186} = 1.77$$

When H_0 is true, the statistic z follows Standard Normal distribution. Since the value of z does not fall in the Critical Region (critical region at 5% level is $|z| \geqslant 1.96$), it is not significant at 5% level. We have, therefore, no reason to reject the null hypothesis, and conclude that the die may be unbiased.

Example 14 : 21 In a big city 325 men out of 600 were found to be smokers. Does this information support the conclusion that the majority of men in this city are smokers? (State the hypotheses clearly). [*I.C.W.A., June '82*]

Solution : Null Hypothesis is that the proportion of smokers in the whole city is 50%, i.e. $50/100 = 0.5$.

$$H_0 \ (P = 0.5)$$

We are interested to see if the proportion of smokers is *more than* 50%; i.e. Alternative Hypothesis is

$$H_1 \ (P > 0.5)$$

The proportion of smokers in the sample is 325 out of $n = 600$. Using sample proportion,

$$\text{Observed Value } (p) = 325/600 = 0.542$$

If the null hypothesis H_0 is true,

$$\text{Expected Value } (P_0) = 0.5$$

and $$\text{S.E. of } p = \sqrt{\frac{0.5\,(1 - 0.5)}{600}} = 0.0204$$

The test statistic is

$$z = \frac{\text{Observed value} - \text{Expected value}}{\text{S.E.}} = \frac{0.542 - 0.5}{0.0204} = 2.1$$

Since the alternative hypothesis H_1 $(P > 0.5)$ is one-sided, the Critical Region of the test is one-tailed. At 5% level of significance

Critical Region is $z \geqslant 1.645$

(Note that the tail-area of Standard Normal curve for $z \geqslant 1.645$ is 5%).

The value of test statistic z, viz. 2.1, lies in the critical region, and hence is "significant". We therefore *reject* the null hypothesis at 5% level of significance and conclude that the data support the hypothesis that majority of men in the city are smokers.

Example 14 : 22 A manufacturer claimed that at least 90% of the components which he supplied, conformed to specifications. A random sample of 200 components showed that only 164 were upto the standard. Test his claim at 1% level of significance.

Solution : Null hypothesis is that the proportion of components conforming to specifications is 90%, i.e. 0.90

$$H_0 \ (P = 0.9)$$

The alternative hypothesis is that it is less than 90%, H_1 $(P < 0.9)$
The proportion of articles conforming to specifications, i.e.

$$\text{Observed value} = 164/200 = 0.82$$

Assuming that H_0 is true,

$$\text{Expected value} = 0.9$$

$$\text{Standard Error} = \sqrt{\frac{0.9 \times 0.1}{200}} = .0212$$

The value of the test statistic is

$$z = \frac{0.82 - 0.9}{.0212} = -3.77$$

Since the value of z, viz. -3.77, is less then -2.33 (critical region at 1% level is $z \leqslant -2.33$), the null hypothesis is rejected. The test therefore reveals that the manufacturer's claim is not justified.

[**Note :** (1) Examples 14:20 to 14:22 all relate to "Test for a specified proportion", but the tests are different depending on the type of alternative hypothesis. Example 14 : 20 shows a "two-tailed" test, Ex. 14 : 21 a "one-tailed" (right tail) test and Ex. 14:22 a "one-tailed" (left tail) test.

(2) As far as possible, the Examples below illustrate various types of tests—two tailed tests, followed by the two kinds of one-tailed tests.]

(2) Test for equality of two proportions :

Let p_1 and p_2 be the proportions in two large random samples of sizes n_1 and n_2 drawn respectively from two populations. We are interested to test whether the proportions in the two populations are equal or not, i.e. whether the difference (p_1-p_2) as observed in the samples has arisen only due to fluctuations of sampling.

$$H_0\ (P_1 = P_2) \text{ against } H_1\ (P_1 \neq P_2)$$

If H_0 is true, i.e. the two population proportions are equal, say P, the estimate of P is

$$p = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} \qquad (14.7.3)$$

The expected value of the difference (p_1-p_2) is then 0, and

$$\text{S.E. of } p_1-p_2 = \sqrt{pq\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}, \quad \text{where } p+q = 1$$

If the sample sizes n_1 and n_2 are large, the test statistic

$$z = \frac{p_1-p_2}{\text{S.E.}} \qquad (14.7.4)$$

approximately follows standard normal distribution. The critical regions are

$|z| \geqslant 1.96$ at 5% level of significance
$|z| \geqslant 2.58$ „ 1% „ „ „

Confidence limits for P_1-P_2 are (see *14.5.5*)

95% confidence limits $= (p_1-p_2) \pm 1.96$ (S.E.)
99% „ $= (p_1-p_2) \pm 2.58$ (S.E.) (14.7.5)

where $\text{S.E.} = \sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}$, if H_0 is rejected.

Example 14 : 23 In a large city A, 20 per cent of a random sample of 900 school children had defective eye-sight. In another large city B, 15 per cent of a random sample of 1600 children had the same defect. Is this difference between the two proportions significant? Obtain 95% confidence limits for the difference in the population proportions.

Solution : Null hypothesis is that the population proportions are equal.

$$H_0\ (P_1 = P_2)$$

Alternative hypothesis is that they are not equal, $H_1\ (P_1 \neq P_2)$

$$n_1 = 900 \qquad p_1 = 20\% = 0.20$$
$$n_2 = 1600, \qquad p_2 = 15\% = 0.15$$

Under H_0, the estimate of the equal but unknown proportion P is

$$p = \frac{900 \times 0.20 + 1600 \times 0.15}{900 + 1600} = 0.168$$

$$\text{S.E. of } (p_1 - p_2) = \sqrt{0.168 \times 0.832\,(1/900 + 1/1600)}$$
$$= \sqrt{.0002427} = .0156$$
$$\therefore \quad z = \frac{0.20 - 0.15}{.0156} = 3.21$$

Since the value of z falls in the critical region $|z| \geqslant 2.58$, it is significant at 1% level. We therefore reject the null hypothesis H_0 and conclude that the difference between the two proportions is significant; i.e. P_1 and P_2 are not equal.

[**Note :** This is a *two-tailed* test.]

The 95% confidence limits for the difference P_1-P_2 are

$$(p_1-p_2) \pm 1.96 \text{ (S.E. of } p_1-p_2)$$

where S.E. of (p_1-p_2) is given by (*13.7.4*) at page 177. But since P_1 and P_2 are not known, and the significance test has shown that $P_1 \neq P_2$, they are estimated by the sample proportions $p_1 = 0.20$ and $p_2 = 0.15$ respectively.

$$\text{S.E. of } p_1 - p_2 = \sqrt{P_1Q_1/n_1 + P_2Q_2/n_2}$$
$$= \sqrt{p_1q_1/n_1 + p_2q_2/n_2} \quad \text{approximately}$$
$$= \sqrt{\frac{0.20 \times 0.80}{900} + \frac{0.15 \times 0.85}{1600}} = .016$$

Hence, the 95% confidence limits for P_1-P_2 are

$$(0.20-0.15) \pm 1.96(.016) = .05 \pm .031 = .019 \text{ and } .081$$

Example 14 : 24 A machine produced 20 defective articles in a batch of 400. After overhauling it produced 10 defectives in a batch of 300. Has the machine improved? [*C.A., Nov. '81*]

Solution : Null Hypothesis is that the proportions of defectives before and after overhauling are equal. Alternative Hypothesis is that the proportion of defectives has decreased after overhauling.

$$H_0\ (P_1 = P_2) \quad \text{against} \quad H_1\ (P_1 > P_2)$$

We have

	Sample size	*Sample proportion*
Sample 1 :	$n_1 = 400$	$p_1 = 20/400 = 1/20$
Sample 2 :	$n_2 = 300$	$p_2 = 10/300 = 1/30$

Assuming that the null hypothesis is true, the estimate of the common proportion in the population is

$$p = \frac{20 + 10}{400 + 300} = \frac{3}{70}$$

Taking the difference of proportions $(p_1 - p_2)$,

Observed value $= 1/20 - 1/30 = .0167$

Expected value $= 0$ (the two proportions assumed equal)

$$\text{S.E. of } (p_1 - p_2) = \sqrt{\tfrac{3}{70} \times \tfrac{67}{70} \left(\tfrac{1}{400} + \tfrac{1}{300}\right)} = .0155$$

Using the test statistic (*14.7.4*),

$$z = \frac{\text{Observed value} - \text{Expected value}}{\text{S.E.}} = \frac{.0167}{.0155} = 1.08$$

Here the alternative hypothesis H_1 $(P_1 > P_2)$ is one-sided, and the critical region is therefore given by one tail of the Standard Normal curve. At 5% level, the Critical Region is $z \geqslant 1.645$.

Since the observed value of z, viz. 1.08 does not fall in the critical region (i.e. is not greater than 1.645), it is "not significant". We have therefore *no reason to reject* the null hypothesis and conclude that the two population proportions may be equal; i.e. the machine has not improved after overhauling.

[**Note:** This is a *one-tailed* (right tail) test. The next one shows a *one-tailed* (left tail) test.]

Example 14 : 25 In a certain city 100 men in a sample of 400 were found to be smokers. In another city, the number of smokers was 300 in a random sample of 800. Does this indicate that there is a greater proportion of smokers in the second city than in the first?

Solution : Null hypothesis H_0 $(P_1 = P_2)$

Alternative hypothesis H_1 $(P_1 < P_2)$

$n_1 = 400$, $p_1 = 100/400 = 0.25$

$n_2 = 800$. $p_2 = 300/800 = 0.375$

Under H_0 the estimate of the equal but unknown proportion is

$$p = \frac{100 + 300}{400 + 800} = \frac{1}{3}$$

Expected value of $(p_1 - p_2)$ is 0, and

$$\text{S.E. of } p_1 - p_2 = \sqrt{\tfrac{1}{3} \times \tfrac{2}{3} \left(\tfrac{1}{400} + \tfrac{1}{800}\right)} = .029$$

$$\therefore \quad z = \frac{0.25 - 0.375}{.029} = -4.31$$

Here the alternative hypothesis H_1 $(P_1 < P_2)$ is one sided, and for this one-tailed test the critical regions are

$z \leqslant -1.645$ at 5% level of significance

$z \leqslant -2.33$ „ 1% „ „ „

Since the observed value of z is less than -2.33, it is significant at 1% level. We reject the null hypothesis and conclude that the proportion of smokers is greater in the second city than in the first

[**Note : —Which Level of Significance to use—5% or 1% ?**

(1) The critical region at 1% level is always included in the critical region at 5% level (i.e. the former is a *subset* of the latter). Therefore, if an observed value of the test statistic is found to be 'significant' at 1% level, it is no doubt significant at 5% level also. On the other hand, if the observed value is 'not significant' at 5% level, it is obviously not significant at 1% level.

(2) The following general rules may be followed in the choice of level of significance of a test:—Consider the critical regions both at 5% and 1% levels. If the observed value of the test statistic is

(a) *not significant at 5% level,* mention that only. It is unnecessary to consider significance at 1% level. See Examples 14 : 20, 24, 26, 40, etc.

(b) *significant at 5% level, but not significant at 1% level,* mention only that it is significant at 5% level. Do not mention that the observed value is not significant at 1% level. See Examples 14 : 21 & 42.

(c) *significant at 1% level,* mention that only. It is useless to mention significance at 5% level. See Examples 14:23, 25, 27 to 31, etc.

If the level of significance is mentioned in the problem, there is no choice, but to use it. See Examples 14:22, 32, 33, 36, 39, 41, 43 to 45, etc.

(3) An observed value of the test statistic that is significant at 5% level is said to be simply "significant"; if significant at 1% level, it is said to be "highly significant".]

(3) Test for a specified mean (large sample) :

A random sample of large size n gives a sample mean $\bar{x}$. It is required to test the hypothesis that the population mean μ has a specified value μ_0. H_0 ($\mu = \mu_0$)

For large n, the sampling distribution of sample mean ($\bar{x}$) is approximately normal (see *13.6.1*). Therefore, when H_0 is true, the test statistic

$$z = \frac{\bar{x} - \mu_0}{\text{S.E. of } \bar{x}} \qquad (14.7.6)$$

approximately follows standard normal distribution. The critical region of the test, which depends on the nature of the alternative hypothesis, is given in the table below.

TABLE 14.5—**Rejection Rules for H_0 ($\mu = \mu_0$)**

Alternative hypothesis H_1	Critical Region	
	5% level	1% level
$\mu \neq \mu_0$	$\|z\| \geqslant 1.96$	$\|z\| \geqslant 2.58$
$\mu > \mu_0$	$z \geqslant 1.645$	$z > 2.33$
$\mu < \mu_0$	$z < -1.645$	$z \leqslant -2{\cdot}33$

Confidence limits for the population mean μ are (see *14.5.2*)

$$95\% \text{ confidence limits } = \bar{x} \pm 1.96 \text{ (S.E. of } \bar{x})$$
$$99\% \quad ,, \quad ,, \quad = \bar{x} \pm 2.58 \text{ (S.E. of } \bar{x}) \qquad (14.7.7.)$$

where S.E. of $\bar{x} = \sigma/\sqrt{n}$. If the population s.d. σ is not known, it is replaced by the sample s.d. S.

(4) Test for equality of two means (large samples) :

Suppose we have two *independent* random samples of large sizes n_1 and n_2 from two populations, and the sample means are $\bar{x}_1$ and $\bar{x}_2$ respectively. It is required to test whether the two populations have the same mean.

The standard error of the difference of means $(\bar{x}_1 - \bar{x}_2)$ is

$$\text{S.E.} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \qquad (14.7.8)$$

where σ_1 and σ_2 are the standard deviations in the two populations. In order to test the null hypothesis that the means are equal

$$H_0 \; (\mu_1 = \mu_2)$$

we use the test statistic

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\text{S.E.}} \qquad (14.7.9)$$

which approximately follows standard normal distribution. For testing equality of means against H_1 $(\mu_1 \neq \mu_2)$ the critical regions are :

$$|z| \geqslant 1.96 \text{ at } 5\% \text{ level of significance}$$
$$|z| \geqslant 2.58 \text{ at } 1\% \text{ level of significance}$$

Confidence limits for $\mu_1 - \mu_2$ are (see *14.5.4*)

$$95\% \text{ confidence limits} = (\bar{x}_1 - \bar{x}_2) \pm 1.96(\text{S.E.})$$
$$99\% \quad ,, \quad ,, \quad = (\bar{x}_1 - \bar{x}_2) \pm 2.58(\text{S.E.}) \qquad (14.7.10)$$

where S.E. is given by (*14.7.8*).

Example 14 : 26 A sample of 400 male students is found to have a mean height of 171.38 cms. Can it be reasonably regarded as a sample from a large population with mean height 171.17 cms. and s.d. 3.30 cms. [*I.C.W.A., June '80*]

Solution : We assume that the sample really comes from a large population with mean 171.17 and s.d. 3.30 (cms.). Null Hypothesis is

$$H_0 \; (\mu = 171.17, \; \sigma = 3.30)$$

Alternative Hypothesis is that the sample does not come from such a population.

$$H_1 \; (\mu \neq 171.17)$$

Since the sample size $n = 400$ is large, the sample mean $(\bar{x})$ is approximately normally distributed.

Observed value $(\bar{x}) = 171.38$
Expected value $(\mu_0) = 171.17$
S.E. of $\bar{x} = \sigma/\sqrt{n} = 3.30/\sqrt{400} = 0.165$

When H_0 is true, the z-statistic (*14.7.6*) follows N (0, 1).

$$z = \frac{\text{Observed value} - \text{Expected value}}{\text{S.E.}} = \frac{171.38 - 171.17}{0.165} = 1.27$$

Since the alternative hypothesis is both sided (i.e. μ is either more than or less than 171.17), the critical region of the test is also two-tailed (i.e. given by two tails of Standard Normal curve). At 5% level

Critical Region is $|z| \geqslant 1.96$

The value of test statistic z (viz. 1.27) does not fall in the critical region and hence is "not significant". We have therefore *no reason to reject* the null hypothesis at 5% level of significance and conclude that the sample may be regarded as having arisen from the given population.

Example 14 : 27 An automatic machine was designed to pack exactly 2.0 kg. of Vanaspati. A sample of 100 tins was examined to test the machine. The average weight was found to be 1.94 kg. with Standard Deviation 0.10 kg. Is the machine working properly?

[*C.A., May '68*]

Solution : Given Sample size $(n) = 100$
Sample mean $(\bar{x}) = 1.94$ kg.
Sample s.d. $(S) = 0.10$ kg.

it is required to test the hypothesis that the population mean is 2.0 kg.

Null hypothesis : H_0 ($\mu = 2.0$ kg.)
Alternative hypothesis : H_1 ($\mu \neq 2.0$ kg.)

Since the sample size (n) is large, the sample mean (x) is approximately normally distributed with mean μ and S.E. $= \sigma/\sqrt{n}$. However, since the population s.d. σ is not known, an aproximate value of S.E. is

$$\text{S.E.} = \frac{S}{\sqrt{n}} = \frac{0.10}{\sqrt{100}} = 0.01$$

Therefore, $$z = \frac{1.94 - 2.0}{0.01} = -6$$

Since $|z| = 6$ exceeds 2.58, we reject the null hypothesis at 1% level of significance and conclude that the machine is not functioning properly.

[**Note** :—Examples 14:26 & 27 are *two-tailed tests* for specified mean, the former with σ *known*, and the latter with σ *unknown*. Example 14:28 is a *one-tailed test* (σ *unknown*).]

Example 14 : 28 A machine part was designed to withstand an average pressure of 120 units. A random sample of size 100 from a large batch was tested and it was found that the average pressure which these parts can withstand is 105 units with a standard deviation of 20 units. Test whether the batch meets the specification.

[C.U., M. Com. '72, '74]

Solution : Null hypothesis :—Mean pressure in the population which a machine part can withstand is 120 units. Alternative hypothesis :—Mean pressure is smaller than 120 units. H_0 ($\mu = 120$) against H_1 ($\mu < 120$).

S.E. of $x = S/\sqrt{n}$ approximately, (since σ is not known)
$= 20/\sqrt{100} = 2.$

Here, sample mean is $\bar{x} = 105$. Therefore,

$$z = \frac{105-120}{2} = -7.5$$

Since the value of $z = -7.5$ is smaller than -2.33 (critical region at 1% level for the one-tailed test is $z \leqslant -2.33$), we reject the null hypothesis at 1% level of significance and conclude that the average breaking strength in the population is less than 120 units, i.e. that the batch of machine parts does not meet the specification. (If the value is found to be significant at 1% level, we have stronger grounds for rejection of H_0 than at 5% level).

Example 14 : 29 In a certain factory there are two different processes of manufacturing the same item. The average weight in a sample of 250 items produced from one process is found to be 120 grammes with a S.D. of 12 grammes; the corresponding figures in a sample of 400 items from the other process are 124 and 14. Compute the Standard Error of difference between the two sample means. Is this difference significant? Also find 99% confidence limits for the difference in the average weights of items produced by the two processes.

[I.C.W.A., Dec. '78]

Solution : Null hypothesis is that the average weights of items produced from the two processes are equal. Alternative hypothesis is that the averages are not equal. H_0 ($\mu = \mu_2$) against H_1 ($\mu_1 \neq \mu_2$).

Sample sizes	$n_1 = 250$	$n_2 = 400$
Sample means	$\bar{x}_1 = 120$ gms.	$\bar{x}_2 = 124$ gms.
Sample S.D.s	$S_1 = 12$ gms.	$S_2 = 14$ gms.

Since the population S.D.s are not known, we use the sample S.D.s to calculate the Standard Error of the difference $\bar{x}_1 - \bar{x}_2$) between the two sample means (see *14.7.8*)

$$\text{S.E. of } \bar{x}_1 - \bar{x}_2 = \sqrt{S_1^2/n_1 + S_2^2/n_2}$$
$$= \sqrt{12^2/250 + 14^2/400} = 1.03$$

The observed value of the test statistic is

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\text{S.E.}} = \frac{120 - 124}{1.03} = -\ 3.9$$

Critical region for both sided alternatives H_1 $(\mu_1 \neq \mu_2)$ is $|z| \geqslant 1.96$ at 5% level and $|z| \geqslant 2.58$ at 1% level of significance. Since $|z| =$ 3.9 exceeds 2.58, we reject H_0 at 1% level of significance, and conclude that the difference between two means is significant; i.e. $\mu_1 \neq \mu_2$.

99% confidence limits for $\mu_1 - \mu_2$ are (see *14.7.10*).

$$(\bar{x}_1 - \bar{x}_2) \pm 2.58\ (\text{S.E.}) = (120 - 124) \pm 2.58(1.03)$$
$$= -\ 4 \pm 2.66 = -\ 1.34 \text{ and } -\ 6.66$$

This means that confidence limits for the difference $\mu_2 - \mu_1$ are 1.34 and 6.66 gms.)

Example 14 : 30 The means of two large samples of sizes 1000 and 2000 are 67.5 and 68.0 respectively. Test the equality of means of the two populations each with s.d. 2.5. (No credit if the null and alternative hypotheses are not stated. Assumptions should be stated clearly). *[I.C.W.A., June '82]*

Solution : Null Hypothesis is that the means of the two populations are equal, and Alternative Hypothesis is that they are not equal.

$$H_0\ (\mu_1 = \mu_2) \qquad \text{against} \qquad H_1\ (\mu_1 \neq \mu_2)$$

Since the population sizes are large, the appropriate test statistic is

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\text{S.E. of } (\bar{x}_1 - \bar{x}_2)}$$

which follows Standard Normal distribution, when H_0 is true. Given

$$n_1 = 1000 \qquad n_2 = 2000$$
$$\bar{x}_1 = 67.5 \qquad \bar{x}_2 = 68.0$$

Population s.d. $(\sigma) = 2.5$ (common for both)

Using *(13.7.3a)*,

$$\text{S.E. of } (\bar{x}_1 - \bar{x}_2) = 2.5\sqrt{1/1000 + 1/2000}$$
$$= 2.5\sqrt{\cdot 0015} = 0.097$$

Observed value of $z = (67.5 - 68.0)/0.097 = -5.2$

The alternative hypothesis is both-sided, and so the Critical Region of the test is given by both tails of Standard Normal curve. At 1% level

Critical Region : $|z| \geqslant 2.58$

In the present case, $|z| = |-5.2| = 5.2$ is actually greater than 2.58. So, we *reject* the null hypothesis H_0 at 1% level of significance and conclude that the means of the two populations are *not equal*.

[**Note** :—Examples 14:29 to 31 are all *two-tailed tests* for equality of two means. The two population s.d.s are (i) *unknown* in Ex. 14:29; (ii) *known* and *equal*, but sample s.d.s are not given in Ex. 14:30; and (iii) *known* and *equal*, but the sample s.d.s are also given in Ex. 14:31.]

Example 14 : 31 The mean yield of wheat from a district A was 210 lbs. with S.D. = 10 lbs. per acre from a sample of 100 plots. In another district B, the mean yield was 220 lbs. with S.D. = 12 lbs. from a sample of 150 plots. Assuming that the standard deviation of yield in the entire state was 11 lbs., test whether there is any significant difference between the mean yield of crops in the two districts. [*C.A., May '76*]

Solution : Let μ_1 and μ_2 denote the mean yields of crops in districts A and B respectively. H_0 ($\mu_1 = \mu_2$) against H_1 ($\mu_1 \neq \mu_2$)

Given $n_1 = 100$ $\qquad n_2 = 150$

$\bar{x}_1 = 210$ lbs. $\qquad \bar{x}_2 = 220$ lbs.

$S_1 = 10$ lbs. $\qquad S_2 = 12$ lbs.

Population S.D. (σ) = 11 lbs.

We may assume that the S.D. of yield in the whole state is the S.D. of yields in the two districts. That is, the two populations have the same S.D. (σ) = 11 lbs. Using (*13.7.3a*)

$$\text{S.E. of } \bar{x}_1 - \bar{x}_2 = \sigma \sqrt{1/n_1 + 1/n_2}$$
$$= 11 \sqrt{1/100 + 1/150} = 1.42$$

The observed value of the test statistic is

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\text{S.E.}} = \frac{210 - 220}{1.42} = -7.04$$

Since $|z| = 7.04$ exceeds 2.58, we reject H_0 at 1% level and conclude that there is a significant difference in the mean yields of crops in the two districts.

(B) Using Chi-square (χ^2) Distribution

The chi-square distribution (Page 195) is used in both large sample and small sample tests. It is mainly used in

(1) Test for goodness of fit.
(2) Test for independence of attributes.
(3) Test for a specified standard deviation (Small Sample test).

(1) Test for goodness of fit (Pearsonian χ^2)

This test, devised by Karl Pearson, is used to decide whether the observations are in good agreement with a hypothetical distribution, i.e. whether the sample may be supposed to have arisen from a specified population. The observed frequencies (f_0) of different classes are compared with the expected frequencies (f_e) by the test statistic

$$\chi^2 = \Sigma \left\{ \frac{(f_0 - f_e)^2}{f_e} \right\} \qquad (14.7.11)$$

This is called *"Pearsonian Chi-square"* or *"goodness-of-fit chi-square"* When the null hypothesis, viz.

H_0 (Data are in agreement with the hypothetical population)

is true, the statistic approximately follows chi-square distribution with ($k-1$) degrees of freedom, where k is the number of classes. If the observed value of the statistic exceeds the tabulated value of χ^2 at a given level, the null hypothesis is rejected.

Example 14 : 32 In his experiments on pea-breeding, Mendel obtained the following frequencies of seeds : Round and yellow—315; Wrinkled and yellow—101; Round and green—108; Wrinkled and green—32; Total—556. Theory predicts that the frequencies should be in the proportions 9 : 3 : 3 : 1. Examine the correspondence between theory and observations. (Given that 5% value of χ^2 for 3 d.f. is 7.815).

Solution : On the assumption that the data agree with theory, and that the divergences have arisen only due to sampling fluctuations, the probabilities for the classes should be 9/16, 3/16, 3/16, 1/16 respectively. The expected frequencies are calculated below :

Class		Observed frequency (f_0)	Expected frequency (f_e)
Round and yellow	...	315	$\frac{9}{16} \times 556 = 313$
Wrinkled and yellow	...	101	$\frac{3}{16} \times 556 = 104$
Round and green	...	108	$\frac{3}{16} \times 556 = 104$
Wrinkled and green	...	32	$\frac{1}{16} \times 556 = 35$
		556	556

No : The totals for Observed and Expected freauencies must be equal.

$$\therefore \quad \chi^2 = \frac{(315-313)^2}{313} + \frac{(101-104)^2}{104} + \frac{(108-104)^2}{104} + \frac{(32-35)^2}{35}$$

$$= 4/313 + 9/104 + 16/104 + 9/35$$

$$= .01 + .09 + .15 + .26 = 0.51$$

Since there are 4 classes. Degrees of freedom = (4—1) = 3

We are given that the 5% value of χ^2 for 3 d.f. is 7.815. Since the observed value 0.51 is less than the tabulated value, the null hypothesis cannot be rejected. We conclude that the observations support the theory.

Example 14 : 33 A die was thrown 60 times with the following results :

Face	1	2	3	4	5	6	Total
Frequency	6	10	8	13	11	12	60

Are the data consistent with the hypothesis that the die is unbiased ? (Given $\chi^2_{.01} = 15.09$ for 5 degrees of freedom).

Solution : Null hypothesis is that the die is unbiased. Then the probability of each face is 1/6, and the expected frequency is $60\times\frac{1}{6} = 10$ for each.

Observed frequency (f_o)	6	10	8	13	11	12
Expected frequency (f_e)	10	10	10	10	10	10
$(f_o-f_e)^2$	16	0	4	9	1	4

$$\therefore \quad \chi^2 = \tfrac{16}{10} + \tfrac{0}{10} + \tfrac{4}{10} + \tfrac{9}{10} + \tfrac{1}{10} + \tfrac{4}{10} = 3.4$$

There are 6 classes, Degrees of freedom $= (6-1) = 5$

Since the observed value of χ^2 (viz. 3.4) is less than the tabulated value 15.09 at 1% for 5 degrees of freedom, we cannot reject the null hypothesis at 1% level of significance. The conclusion is that the data are in agreement with the hypothesis of an unbiased die.

Example 14 : 34 5 identical coins are tossed 320 times, and the number of heads appearing each time is recorded. The results are :

Number of heads	0	1	2	3	4	5	Total
Frequency :	14	45	80	112	61	8	320

Would you conclude that the coins are biased ? (Given $\chi^2_{.05} = 11.07$ and $\chi^2_{.01} = 15.09$ for 5 degrees of freedom).

Solution : This is a problem for testing "goodness of fit" of a binomial distribution. The null hypothesis is that the coins are unbiased, i.e. the probability of obtaining head is 1/2 for each coin.

H_0 (Data support Binomial distribution with $P = 1/2$)

The test statistic is $\chi^2 = \Sigma\,\{(f_0 - f_e)^2/f_e\}$.

If H_0 is true, i.e. the coins are assumed to be unbiased, the probability of obtaining r heads in one throw of the set of 5 coins is given by the binomial distribution

$$P(r) = {}^5C_r(1/2)^r(1/2)^{5-r} = {}^5C_r/32$$

So, the expected frequency (f_e) of r heads in 320 tosses is

$$f_e = 320.P(r) = 10\times{}^5C_r$$

These are shown below for different values of r :

Number of heads (r) :	0	1	2	3	4	5	Total
Observed frequency (f_o) :	14	45	80	112	61	8	320
Expected frequency (f_e) :	10	50	100	100	50	10	320

$$\therefore \quad \chi^2 = 4^2/10 + 5^2/50 + 20^2/100 + 12^2/100 + 11^2/50 + 2^2/10$$
$$= 1.6 + 0.5 + 4.0 + 1.44 + 2.42 + 0.4 = 10.36.$$

There are 6 classes; so Degrees of freedom $= 6-1 = 5$

Since the observed value of χ^2 (viz. 10.36) is less than the tabulated value at 5% level (viz. 11.07), we cannot reject H_0 at 5% level of significance. We therefore conclude that the coins may not be biased.

(2) Test for independence of attributes :

[**Note :—(1) Contingency Table**—In Statistics, sometimes we have to deal with "attributes" (see Part I, Section 1.2) or qualitative characters of members which cannot be measured accurately, although the members can be divided into two or more categories with respect to the attributes. Let us consider two attributes A and B, where A is shown in m categories $A_1, A_2, \ldots, A_m$, and B in n categories $B_1, B_2, \ldots, B_n$. The data can be shown in the form of a two-way table with m rows and n columns, as in a 'bivariate' frequency distribution (Tables 9.2 & 9.7). This two-way frequency table for attributes is known as $(m \times n)$ *Contingency Table* (see Example 14:35). The frequency of members belonging to both the categories A_i and B_j simultaneously is shown in the cell at the i-th row and j-th column, and denoted by (A_iB_j).

Similarly, (A_i) and (B_j) denote the frequency of members belonging to categories A_i and B_j respectively, and N the total frequency, as in the table below.

(3 × 4) Contingency Table

		Attribute B				
		B_1	B_2	B_3	B_4	Total
Attribute A	A_1	(A_1B_1)	(A_1B_2)	(A_1B_3)	(A_1B_4)	(A_1)
	A_2	(A_2B_1)	(A_2B_2)	(A_2B_3)	(A_2B_4)	(A_2)
	A_3	(A_3B_1)	(A_3B_2)	(A_3B_3)	(A_3B_4)	(A_3)
Total		(B_1)	(B_2)	(B_3)	(B_4)	N

Often, the members are divided into two categories only in respect of each attribute, according to the presence or absence of attribute A (denoted by A and α) and presence or absence of attribute B (denoted by B and β). See Example 14:36.

(2 × 2) Contingency Table

		Attribute A: A	Attribute A: α	Total
Attribute B	B	$(AB) = a$	$(\alpha B) = b$	$(B) = a+b$
	β	$(A\beta) = c$	$(\alpha\beta) = d$	$(\beta) = c+d$
Total		$(A) = a+c$	$(\alpha) = b+d$	N

(2) **Independence of Attributes**—Two attributes are said to be "independent", if they are unrelated to each other; i.e., the presence or absence of any attribute among the members of the population does in no way influence whether the other attribute will be present. In general, if there are m categories A_i for attribute A and n categories B_j for attribute B, then A and B will be independent if

$$(A_i\, B_j) = \frac{(A_i)\,(B_j)}{N}$$

for all values of i and j. This means that in the population each cell frequency will be equal to the product of the corresponding row- and column-totals divided by the total number of members in the population. In particular, for a (2×2) contingency table,

$$(AB) = \frac{(A)\,(B)}{N}$$

If the attributes are not independent, they are said to be "associated".

(3) **Association of Attributes**—The word "association" is used to indicate the degree of relationship between attributes (the corresponding word for variable is "correlation"). If two attributes are not "independent", they are said to be "associated"; i.e. there is "association" between the two attributes. For a (2×2) table, a measure of association is given by Yule's "Coefficient of Association"

$$Q = \frac{(AB)\,(\alpha\beta) - (A\beta)\,(\alpha B)}{(AB)\,(\alpha\beta) + (A\beta)\,(\alpha B)} = \frac{ad - bc}{ad + bc} \quad]$$

Suppose that the observations are classified simultaneously according to two attributes, and the frequencies (f_0) in the difference categories are shown in a two-way table (known as *Contingency Table*). On the basis of cell frequencies it is required to test whether the two attributes are associated or not. Under the null hypothesis

H_0 (Attributes are independent)

the expected frequency (f_e) of any cell is given by

$$f_e = \frac{\text{(Row total)} \times \text{(Column total)}}{\text{Total frequency}} = \frac{(A_i)\,(B_j)}{N}$$

Then we calculate the statistic

$$\chi^2 = \Sigma\left\{\frac{(f_0 - f_e)^2}{f_e}\right\} \tag{14.7.12}$$

which approximately follows chi-square distribution with

d.f. = (number of rows -1) $\times$ (number of columns -1)

If the calculated value of χ^2 exceeds the tabulated value for the given d.f. and at a specified level, the null hypothesis is rejected, and we conclude that the attributes are not independent, but associated.

Simplified formula for (2×2) table :

Suppose, the contingency table has only 2 rows and 2 columns with the four cell frequencies *a, b, c, d* as shown below:

			Total
	a	b	R_1
	c	d	R_2
Total	C_1	C_2	N

[**Note:** It has become customary to use *a* for the frequency in the *upper left cell* and *d* for that in the *lower right cell; b* and *c* are frequencies in the other diagonal positions.

For simplicity we write R_1, R_2 to denote the row totals, and C_1, C_2 to denote the column totals; i.e.

$$R_1 = a+b,\quad R_2 = c+d,\quad C_1 = a+c,\quad C_2 = b+d$$
$$N = a+b+c+d = R_1+R_2 = C_1+C_2$$

In this case, formula (*14.7.12*) reduces to

$$\chi^2 = \frac{N(ad - bc)^2}{R_1\, R_2\, C_1\, C_2} \tag{14.7.13}$$

with degrees of freedom $= (2-1)\times(2-1) = 1$.

Yates' correction :

For a (2×2) table, there is only one degree of freedom; i.e. only one of the four cell frequencies can be arbitrarily given, if the row and column totals should remain fixed. It is therefore necessary to make a correction to formula (*14.7.13*), so that its approximation to the continuous chi-square distribution can be improved.

If $ad > bc$, reduce a and d by $\frac{1}{2}$, and increase b and c by $\frac{1}{2}$;
If $ad < bc$, increase ,, ,, ,, ,, reduce ,, ,, ,,

This is known as *Yates' Correction for Continuity.*

$$\chi^2 \text{ (corrected)} = \frac{N\{\,|\,ad-bc\,|\,-N/2\,\}^2}{R_1\ R_2\ C_1\ C_2} \tag{14.7.14}$$

It should be noted that Yates' correction can be applied only for 1 d.f.

Example 14 : 35 A random sample of 500 students were classified according to economic condition of their family and also according to merit, as shown below :

Merit	*Economic Condition* Rich	Middle class	Poor	Total
Meritorious	42	137	61	240
Not meritorious	58	113	89	260
Total	100	250	150	500

Test whether the two attributes Merit and Economic Condition are associated or not, (Given, $\chi^2_{.05} = 5.99$ and $\chi^2_{.01} = 9.21$ for 2 d.f.)

Solution : Null hypothesis is that the attributes are independent. The expected frequencies are calculated as follows :

TABLE 14.6—**Calculation of Expected Frequencies**

	Rich	Middle Class	Poor	Total
Meritorious	$\frac{240 \times 100}{500} = 48$	$\frac{240 \times 250}{500} = 120$	$\frac{240 \times 150}{500} = 72$	240
Not meritorious	$\frac{260 \times 100}{500} = 52$	$\frac{260 \times 250}{500} = 130$	$\frac{260 \times 150}{500} = 78$	260
Total	100	250	150	500

$$\chi^2. = \frac{(42-48)^2}{48} + \frac{(137-120)^2}{120} + \frac{(61-72)^2}{72} + \frac{(58-52)^2}{52} + \frac{(113-130)^2}{130} + \frac{(89-78)^2}{78}$$

$$= 0.75 + 2.41 + 1.68 + 0.69 + 2.22 + 1.55 = 9.30$$

Degrees of freedom $= (2-1) \times (3-1) = 1 \times 2 = 2$

For 2 d.f. the tabulated value of χ^2 at 5% level is 5.99 and at 1% level is 9.21. Since the observed value of χ^2 exceeds even the 1% tabulated value, it is highly significant. We reject the null hypothesis and conclude that Merit and Economic Condition are not independent; i.e. they are associated.

Example 14 : 36 In a survey of 200 boys, of which 75 were intelligent, 40 had skilled fathers; while 85 of the unintelligent boys had unskilled fathers. Do these figures support the hypothesis that skilled fathers have intelligent boys? Use χ^2 test. Value of χ^2 for 1 d.f. at 5% level is 3.84. *[C.A., Nov. '82]*

Solution: The data are shown in the following (2×2) table:

Intelligence of son	Skill of father		Total
	Skilled	Unskilled	
Intelligent	**40**	35	**75**
Unintelligent	40	**85**	125
Total	80	120	**200**

[**Note**:—Figures in bold type are only given.]

Null Hypothesis is that the two attributes "Skill of father" and "Intelligence of son" are *independent*, and the Alternative Hypothesis is that they are not independent. On the hypothesis of independence, the test statistic (*14.7.14*) follows χ^2 distribution with 1 d.f.

$$\chi^2 = \frac{200\{\,|\,40 \times 85 - 35 \times 40\,|\, - 100\,\}^2}{80 \times 120 \times 75 \times 125}$$

$$= \frac{200 \times 1900 \times 1900}{80 \times 120 \times 75 \times 125} = \frac{361}{45} = 8.02$$

Since the observed value of the statistic 8.02 is greater than the tabulated value 3.84 (given), it is significant. We therefore reject the null hypothesis at 5% level of significance and conclude that the attributes are *not independent*; i.e. the data support the alternative hypothesis that 'skilled fathers have intelligent boys'.

14.8 Small Sample Tests

These are "exact tests" and are applicable to samples of any size, although they are generally used for small samples; say $n < 30$.

(A) Using Normal Distribution

(1) Test for a specified population mean (s.d. known) :

Consider a random sample of size n from a *normal* population, whose standard deviation σ is known. It is required to test the null hypothesis that the mean of population has a specified value.

$$H_0\ (\mu = \mu_0)$$

The sampling distribution of sample mean ($\bar{x}$) is exactly normal (see theorem 1, page 193) with standard error S.E. $= \sigma/\sqrt{n}$. When the null hypothesis is true, the test statistic.

$$z = \frac{\bar{x} - \mu_0}{\text{S.E.}} = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \qquad (14.8.1)$$

follows standard normal distribution exactly. If the value of z falls in the critical region, shown below, the null hypothesis is rejected.

TABLE 14.7—**Rejection Rules for $H_0(\mu = \mu_0)$**

Alternative hypothesis H_1	Critical Region	
	5% level	1% level
$\mu \neq \mu_0$	$\lvert z \rvert \geqslant 1.96$	$\lvert z \rvert \geqslant 2.58$
$\mu > \mu_0$	$z \geqslant 1.645$	$z \geqslant 2.33$
$\mu < \mu_0$	$z \leqslant -1.645$	$z \leqslant -2.33$

Confidence limits for the population mean μ are (see *14.5.6*):—

$$\begin{aligned} 95\% \text{ confidence limits} &= \bar{x} \pm 1.96\ (\sigma/\sqrt{n}) \\ 99\% \text{ confidence limits} &= \bar{x} \pm 2.58\ (\sigma/\sqrt{n}) \end{aligned} \qquad (14.8.2)$$

(2) Test for equality of two means (s.d.s known) :

Consider two *independent* samples of sizes n_1 and n_2 from two *normal* populations with unknown means, but *known* standard deviations σ_1 and σ_2 respectively. It is required to test the hypothesis that the population means are equal.

$$H_0(\mu_1 = \mu_2)$$

The sampling distribution of the difference of sample means ($\bar{x}_1 - \bar{x}_2$) is exactly normal with standard error

$$\text{S.E.} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \qquad (14.8.3)$$

The test statistic

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}} \tag{14.8.4}$$

follows standard normal distribution exactly. If the value of z falls in the critical region, the null hypothesis is rejected.

TABLE 14.8—**Rejection Rules for $H_0(\mu_1 = \mu_2)$**

Alternative hypothesis H_1	Critical Region	
	5% level	1% level
$\mu_1 \neq \mu_2$	$\lvert z \rvert \geq 1.96$	$\lvert z \rvert \geq 2.58$
$\mu_1 > \mu_2$	$z \geq 1.645$	$z \geq 2.33$
$\mu_1 < \mu_2$	$z \leq -1.645$	$z \leq -2.33$

Confidence limits for the difference of population means $(\mu_1 - \mu_2)$ are (see *14.5.7*):

$$\begin{aligned} 95\% \text{ confidence limits} &= (\bar{x}_1 - \bar{x}_2) \pm 1.96\,(\text{S.E.}) \\ 99\% \text{ confidence limits} &= (\bar{x}_1 - \bar{x}_2) \pm 2.58\,(\text{S.E.}) \end{aligned} \tag{14.8.5}$$

where S.E. is given by (*14.8.3*).

Example 14 : 37 A random sample of size 10 was taken from a normal population, whose variance is known to be 7.056 sq. inches. If the observations are (in inches) 65, 71, 64, 71, 70, 69, 64, 63, 67 and 68, test the hypothesis that the population mean is 69 inches. Also obtain 99% confidence limits for the population mean.

Solution : Null hypothesis is that the population mean is 69 inches, and the alternative hypothesis is that it is not so; i.e. H_0 ($\mu = 69$) against H_1 ($\mu \neq 69$). Population s.d. $(\sigma) = \sqrt{7.056}$, and $n = 10$.

For the given data, sample mean $\bar{x} = 67.2$. The value of test statistic is

$$z = \frac{67.2 - 69}{\sqrt{7.056}/\sqrt{10}} = \frac{-1.8}{\sqrt{.7056}} = -2.14$$

Since $|z| = 2.14$ exceeds 1.96 (the critical region at 5% level is $|z| \geq 1.96$), the null hypothesis is rejected. We conclude that the population mean is not 69 inches.

99% confidence limits for μ are $67.2 \pm 2.58\,(\sqrt{7.056}/\sqrt{10})$

$= 67.2 \pm 2.17 = 65.03$ and 69.37 inches.

Example 14 : 38 It is required to estimate the mean of a normal population using a sample sufficiently large so that the probability will be 0.95 that the sample mean will not differ from the population mean by more than 25% of the population s.d. How large should the sample be?

Solution : We have to find the sample size (n) so that

$$P(|\bar{x}-\mu| \leqslant \sigma/4) = 0.95$$

i.e. $$P\left(\left|\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right| \leqslant \frac{\sigma/4}{\sigma/\sqrt{n}}\right) = 0.95$$

i.e. $$P(|z| \leqslant \sqrt{n}/4) = 0.95 \qquad \ldots \quad \text{(i)}$$

where z is a standard normal variate (see *13.8.5*). But we know that (see *13.8.3*)

$$P(|z| \leqslant 1.96) = 0.95 \qquad \ldots \quad \text{(ii)}$$

Comparing (i) and (ii),

$$\sqrt{n}/4 = 1.96\,; \quad \therefore \; n = (4\times 1.96)^2 = 61.47$$

Since the sample size works out to be larger than 61, the least value of n is 62. *Ans.* 62

Example 14 : 39 Two independent random samples were taken from two normal populations and the following information is given:

	Population I	Population II
Sample size	$n_1 = 10$	$n_2 = 12$
Sample mean	$\bar{x}_1 = 20$	$\bar{x}_2 = 27$
Population s.d.	$\sigma_1 = 8$	$\sigma_2 = 6$

Is it likely that the mean of Population I is smaller than that of Population II? (Use 1% level of significance). Also find 99% confidence limits for the difference of population means.

Solution : Null hypothesis $H_0(\mu_1 = \mu_2)$; i.e. we assume that the means are equal. Alternative hypothesis $H_1(\mu_1 < \mu_2)$. The observed value of z is

$$z = \frac{20-27}{\sqrt{8^2/10 + 6^2/12}} = \frac{-7}{3.07} = -2.28$$

Since the value of z is not less than -2.33 (critical region at 1% is $z < -2.33$), it is not significant at 1% level. We conclude that the mean of Population I may not be smaller than that of Population II.

99% confidence limits for $(\mu_1 - \mu_2)$ are

$$(\bar{x}_1 - \bar{x}_2) \pm 2.58\ (\text{S.E.})$$
$$= (20-27) \pm 2.58 \times 3.07$$
$$= -7 \pm 7.92 = -14.92 \text{ and } 0.92$$

(B) Using t distribution

(1) Test for a specified mean (s.d. unknown) :

For testing the null hypothesis H_0 ($\mu = \mu_0$), if the population standard deviation (σ) is not known, formula (*14.8.1*) cannot be used. However the "unbiased" estimator of the population variance (σ^2) is

$$s^2 = \frac{n}{n-1} S^2$$

where S^2 is the variance of a sample of size n. Thus the estimate of σ is $s = \sqrt{\dfrac{n}{n-1}}\, S$. If this is substituted for σ in formula (*14.8.1*), then the statistic (known as **Student's t**)

$$t = \frac{\bar{x}-\mu_0}{s/\sqrt{n}} = \frac{\bar{x}-\mu_0}{S/\sqrt{n-1}} \tag{14.8.6}$$

follows t distribution (*13.8.15*) with degrees of freedom $(n-1)$. For tests of significance, the tables of t must therefore be used. If $t_{.025}$ denotes the value of t distribution corresponding to the upper tail area 025, then the null hypothesis is rejected, if $|t| \geqslant t_{.025}$.

95% confidence limits for the population mean μ are (see *14.5.8a*)

$$\bar{x} \pm t_{.025}\, s/\sqrt{n} \quad \text{or,} \quad \bar{x} \pm t_{.025}\, S/\sqrt{n-1} \tag{14.8.7}$$

99% confidence limits for μ are

$$\bar{x} \pm t_{.005}\, s/\sqrt{n} \quad \text{or,} \quad \bar{x} \pm t_{.005}\, S/\sqrt{n-1} \tag{14.8.7a}$$

(2) Test for equality of two means (s.d.s unknown) :

Consider two *independent* random samples of sizes n_1 and n_2 from two normal populations with means μ_1 and μ_2. It is required to test the hypothesis that the means are equal.

$$H_0\ (\mu_1 = \mu_2)$$

If the two population *standard deviations are assumed to be equal*, an "unbiased" estimator of the common variance is given by

$$s^2 = \frac{n_1 S_1^2 + n_2 S_2^2}{n_1 + n_2 - 2} \tag{14.8.8}$$

where S_1, S_2 are the sample standard deviations.
If this is substituted for σ_1^2 and σ_2^2 in formula (*14.8.4*) the statistic

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s\sqrt{1/n_1 + 1/n_2}} \tag{14.8.9}$$

(known as **Fisher's t**) follows t distribution (*13.8.16*) with degrees of freedom $(n_1 + n_2 - 2)$.

Confidence limits for $(\mu_1-\mu_2)$ are (see *14.5.9a*):

95% confidence limits $= (\bar{x}_1 - \bar{x}_2) \pm t_{.025} \times s\sqrt{1/n_1 + 1/n_2}$

99% confidence limits $= (\bar{x}_1 - \bar{x}_2) \pm t_{.005} \times s\sqrt{1/n_1 + 1/n_2}$

(*14.8.10*)

Example 14 : 40 A random sample of size 20 from a normal population gives a sample mean of 42 and sample standard deviation of 6. Test the hypothesis that the population mean is 44. State clearly the alternative hypothesis you allow for and the level of significance adopted. [*C.A., Nov. '74*]

Solution : Null hypothesis is that the population mean is 44. H_0 (μ = 44). Alternative hypothesis is that the population mean is different from 44. H_1 ($\mu \neq$ 44), i.e. both-sided alternatives.

Since the population s.d. (σ) is not known, we use t test. Here, n = 20, $\bar{x}$ = 42 and S = 6. Therefore using (*14.8.6*)

$$t = \frac{42-44}{6/\sqrt{19}} = -1.45$$

Degrees of freedom = (20−1) = 19

From the tables of t distribution, we find for 19 d.f. the percentage points of t are respectively $t_{.025}$ = 2.09 and $t_{.005}$ = 2.86. We have to use a two-tailed test, because the alternatives are both-sided. Since $|t|$ = 1.45 is less than the 5% tabulated value corresponding to the two tails (*viz.* 2.09) there is no reason to reject the null hypothesis at 5% level of significance, and we conclude that the population mean may be 44.

[99% confidence limits for μ are $42 \pm 2.86 \times (6/\sqrt{19})$
$= 42 \pm 3.94 = 38.06$ and 45.94]

Example 14 : 41 A fertiliser mixing machine is set to give 12 kg. of nitrate for every quintal bag of fertiliser. Ten 100-kg. bags are examined. The percentages of nitrate are: 11, 14, 13, 12, 13, 12, 13, 14, 11, 12. Is there reason to believe that the machine is defective? Value of t for 9 d.f. is 2.262. [*C.A., Nov. '82*]

Solution : *Assumption*—The 'weight (in kg.) of nitrate in one bag of the fertiliser' follows normal distribution with mean μ and s.d. σ.

Null Hypothesis H_0(μ = 12); Alternative Hypothesis H_1($\mu \neq$ 12). Since σ is not known, the appropriate test statistic is *Student's t*, given by (*14.8.6*). When H_0 is true, this follows t distribution with $(n-1)$ d.f.

From the given data, we calculate the sample mean ($\bar{x}$) and the sample s.d. (S). $\bar{x} = \Sigma x/n = 125/10 = 12.5$

For the s.d., take deviations from 12, i.e. $d = x - 12$.

−1, 2, 1, 0, 1, 0, 1, 2, −1, 0 ; $\Sigma d = 5$, $\Sigma d^2 = 13$

$\therefore \quad S^2 = \Sigma d^2/n - (\Sigma d/n)^2 = 13/10 - (5/10)^2 = 1.05$

Thus, $\quad \bar{x} = 12.5, \quad S = \sqrt{1.05} = 1.02$

Using the second form in (*14.8.6*),

$$t = \frac{12.5 - 12}{1.02/\sqrt{(10-1)}} = \frac{0.5 \times 3}{1.02} = 1.47$$

Since the alternative hypothesis is both-sided, the critical region is given by both tails of the t distribution with 9 d.f. At 5% level,

Critical Region: $|t| \geqslant 2.262$

The observed value of $t = 1.47$ does not fall in the critical region (because it is not more than 2.262) and is therefore *not significant*. So, we cannot reject the null hypothesis and conclude that the population mean is 12 ; i.e. the data do not indicate that the machine is defective.

[**Note:** The previous two examples show the use of *Student's t* statistic in two-tailed tests. In Ex. 14:40, the sample mean and s.d. are given, while in Ex. 14:41, these are obtained from the sample data. Ex. 14:42 shows a one-tailed test (right tail), using *Student's t.*]

Example 14 : 42 A soap manufacturing company was distributing a particular brand of soap through a large number of retail shops. Before a heavy advertisement campaign, the mean sales per week per shop was 140 dozens. After the campaign, a sample of 26 shops was taken and the mean sales was found to be 147 dozens with standard deviation 16. Can you consider the advertisement effective?

[*C.A., Nov. '78*]

Solution: We assume that the sales per week follow normal distribution with mean $\mu = 140$ (σ unknown); and it is required to test whether the average sales after the campaign is larger than 140. H_0 ($\mu = 140$) against one sided alternative H_1($\mu >$ 140).

The test statistic is

$$t = \frac{\bar{x} - \mu_0}{S/\sqrt{n-1}} = \frac{147 - 140}{16/\sqrt{25}} = 2.19$$

From Statistical tables, we find that for 25 degrees of freedom, the value of t corresponding to upper 5% and 1% "tail areas" are namely $t_{.05} = 1.708$ and $t_{.01} = 2.485$. Since the observed value of t, 2.19, is larger than $t_{.05}$, we reject H_0 and conclude that μ is greater than 140; i.e. the advertisement has been effective in increasing the sales.

Example 14 : 43 Two types of batteries are tested for their length of life and the following data are obtained :

	No. of sample	*Mean life in Hours*	*Variance*
Type *A*	9	600	121
Type *B*	8	640	144

Is there a significant difference in the two means ? Value of t for 15 degrees of freedom at 5% level is 2.131. [*C.A., Nov. '82*]

Solution: Assumptions—(1) The two populations are normal distributions with means μ_1, μ_2 and a common s.d. σ. (2) The two samples are randomly drawn and independent.

$$H_0\ (\mu_1 = \mu_2) \quad \text{against} \quad H_1\ (\mu_1 \neq \mu_2)$$

Given

	Population A	*Population B*
Sample size	$n_1 = 9$	$n_2 = 8$
Sample mean	$\bar{x}_1 = 600$	$\bar{x}_2 = 640$
Sample variance	$S_1^2 = 121$	$S_2^2 = 144$

(Compare with Example 14 : 39)

The appropriate test statistic is *Fisher's t* (*14.8.9*) which, under H_0, follows t distribution with (n_1+n_2-2) d.f. An estimate of the common but unknown s.d. (σ) is obtained from (*14.8.8*)

$$s = \sqrt{\frac{9 \times 121 + 8 \times 144}{9+8-2}} = \sqrt{149.4} = 12.2$$

The observed value of the statistic is

$$t = \frac{600-640}{12.2\sqrt{1/9+1/8}} = \frac{-40}{12.2 \times 0.486} = -6.7$$

$$\text{Degrees of freedom} = 9+8-2 = 15$$

Since the alternative hypothesis is both-sided, the test is two-tailed. The critical region is given by both the tails of t distribution. At 5% level

$$\text{Critical Region : } |t| \geqslant 2.131$$

Since the observed value falls in the critical region (because $|t| = |-6.7| = 6.7$ is larger than 2.131), we reject the null hypothesis at 5% level of significance and conclude that there is 'significant difference in the two means'.

[Confidence interval for the difference $(\mu_1-\mu_2)$ is given by (*14.5.9a*)

$$\begin{aligned} 95\% \text{ confidence limits} &= (600-640) \pm 2.131 \times 12.2\sqrt{1/9+1/8} \\ &= -40 \pm 12.63 \\ &= -27.37 \text{ and } -52.63 \end{aligned}$$

Thus, 95% confidence interval for $(\mu_2-\mu_1)$ is 27.37 to 52.63.]

[**Note:** Examples 14:43 & 44 illustrate the use of *Fisher's t* statistic, the former showing a two-tailed test and the latter a one-tailed test (left tail).]

Example 14:44 A group of 5 patients treated with medicine A weigh 42, 39, 48, 60, 41 kg.; a second group of 7 patients from the same Hospital treated with medicine B weigh 38, 42, 56, 64, 68, 69, 62 kg. Do you agree with the claim that medicine B increases the weight significantly? (The value of 't' at 5% level of significance for 10 degrees freedom is 2.2281). [*I.C.W.A., June '79*]

Solution : This is a problem for testing equality of two population means based on independent samples. We assume that the samples are drawn from normal populations having the same standard deviation σ (unknown) and means μ_1 and μ_2 respectively. The null hypothesis is that patients treated with medicines A and B have the same average weight (in the population) and the alternative hypothesis is that medicine B increases the average weight. H_0 ($\mu_1 = \mu_2$) against the alternative H_1 ($\mu_1 < \mu_2$).

The appropriate statistic is Fisher's t *(14.8.9)*. Since the alternative hypothesis is one-sided H_1 ($\mu_1 < \mu_2$), the null hypothesis H_0 will be rejected, if $t \leqslant -t_{.05}$.

From the given data, we obtain the following results :—

	Medicine A	*Medicine B*
Sample size	$n_1 = 5$	$n_2 = 7$
Sample mean	$\bar{x}_1 = 46$	$\bar{x}_2 = 57$
Sample variance	$S_1^2 = 58$	$S_2^2 = 926/7$

As in the previous example, the estimate of σ is

$$s = \sqrt{\frac{5\times 58+7\times(926/7)}{5+7-2}} = \sqrt{121.6} = 11.03$$

$$\therefore \quad t = \frac{46-57}{11.03\sqrt{(1/5+1/7)}} = \frac{-11}{6.46} = -1.70$$

Since the observed value of t, viz. -1.70, is not less than -2.2281, we cannot reject H_0 at 5% level of significance. Thus the claim is not justified.

[*Comment*: The given value 2.2281 corresponds to "both tail areas" equal to 5%, and is not appropriate for this test. The correct value 1.812 corresponding to one tail area 5% for 10 d.f. should have been given and used for comparison].

(3) **Test for equality of two means—Paired t :**

Suppose, we have a random sample of n pairs of observations (x_1, y_1), (x_2, y_2), ..., (x_n, y_n) from a *bivariate normal* population. The values of x and y are now *correlated* and no longer independent. It is required to test whether the means of x and y in the population are equal, i.e. H_0 ($\mu_x = \mu_y$).

Let $d_i = x_i - y_i$ denote the difference (with proper sign) in the values of x and y for the i-th pair ($i = 1, 2, \ldots, n$).

The test statistic is

$$t = \frac{\bar{d}}{S/\sqrt{n-1}} \qquad (14.8.10a)$$

where $\bar{d}$ and S denote respectively the mean and standard deviation of the differences d_i; i.e.

$$\bar{d} = \Sigma d_i/n \qquad (14.8.10b)$$

$$S^2 = \Sigma(d_i - \bar{d})^2/n = \Sigma d_i^2/n - (\Sigma d_i/n)^2$$

When H_0 is true, the statistic (*14.8.10a*), known as *Paired t*, follows t distribution with $(n-1)$ degrees of freedom. The critical region is given by two tails or one tail (right or left tail) of t distribution depending on the alternative hypothesis H_1 ($\mu_x \neq \mu_y$), H_1 ($\mu_x > \mu_y$) or H_1 ($\mu_x < \mu_y$).

Remarks : Paired t-test for H_0 ($\mu_x = \mu_y$) may be viewed as Student's t-test for H_0 ($\mu_d = 0$), where $d = x - y$; because $\mu_d = \mu_x - \mu_y = 0$, when $\mu_x = \mu_y$. Thus, for testing equality of means from correlated pairs (x, y), we take deviations $d = x - y$ and apply Student's t-test for a specified mean 0, using the values of d.

Example 14 : 45 An I.Q. test was administered to 5 persons before and after they were trained. The results are given below :

	I	II	III	IV	V
I. Q. before training	110	120	123	132	125
I. Q. after training	120	118	125	136	121

Test whether there is any change in I.Q. after the training programme. (Given $t_{0\cdot01(4)} = 4.6$) [*C.A., May '80*]

Solution : This is a problem of *Paired t test*. Because, the scores before training (x) and after training (y) are not independent, but the latter are likely to be affected by the former; i.e. x and y are correlated.

Null Hypothesis : H_0 ($\mu_x = \mu_y$); i.e. there is no change in the mean scores before and after the training. *Alternative Hypothesis* : H_1 ($\mu_x \neq \mu_y$). The test statistic is (*14.8.10a*).

Calculations for Paired t

						Total
x :	110	120	123	132	125	Total
y :	120	118	125	136	121	—
$d = x-y$:	-10	2	-2	-4	4	-10
d^2 :	100	4	4	16	16	140

By (*14.8.10b*), $\bar{d} = -10/5 = -2$

$$S^2 = 140/5 - (-10/5)^2 = 28 - 4 = 24$$

The observed value of the test statistic is

$$t = \frac{-2}{\sqrt{24} / \sqrt{5-1}} = \frac{-2}{\sqrt{6}}$$

$$= \frac{-2\sqrt{6}}{6} = \frac{-2.45}{3} = -0.82$$

Critical Region at 1% level of significance is $|t| > 4.6$. Since the observed value $|t| = 0.82$ is less than 4.6, H_0 cannot be rejected and we conclude that there is no significant change in the mean I. Q. after the training programme.

[**Note**: (i) In this book, we have used the notation $t_{01(4)}$ to denote the value of t for 4 d.f. corresponding to *one tail* area (right tail) .01, i.e. 1%. Some authors use the same notation $t_{.01(4)}$ corresponding to the *two tail* areas 1%, i.e. right tail area ½%, as has been done in the above problem. However, whatever be the notation followed, the value 4.6 shown here is the appropriate critical value at 1% level for the two-tailed t-test with 4 d.f.

(ii) Example 14:45 shows a *two-tailed* test, and Example 14:45A a *one-tailed* (right tail) test.]

Example 14:45A A certain stimulus administered to each of 12 patients resulted in the following changes in blood pressure:

5, 2, 8, −1, 3, 0, −2, 1, 5, 0, 4, 6.

Can it be concluded that the stimulus will in general be accompanied by an increase in blood pressure? (Given for 11 d.f., $t_{.05} = 2.2$)

[*I.C.W.A., Dec. '80*]

Solution: Here, the 'changes' $d = x-y$ in blood pressure are given; i.e. x is the (final) blood pressure after administering the stimulus and y is the initial blood pressure. We are required to test whether the mean blood pressure has increased, i.e. μ_x is greater than μ_y.

Null Hypothesis: $H_0\,(\mu_x = \mu_y)$; i.e. mean blood pressure has not changed. *Alternative Hypothesis*: $H_1\,(\mu_x > \mu_y)$; i.e. mean blood pressure *increases* after the stimulus.

When H_0 is true, the statistic *(14.8.10a)* follows t distribution with $(n-1)$ degrees of freedom. For the given data,

$$n = 12, \quad \Sigma d_i = 31, \quad \Sigma d_i^2 = 185$$

$$\therefore \quad \bar{d} = 31/12 = 2.58, \text{ and}$$

$$S^2 = 185/12 - (31/12)^2 = 1259/144. \text{ Hence}$$

$$t = \frac{\bar{d}}{\sqrt{S^2/(n-1)}} = \frac{2.58}{\sqrt{\dfrac{1259}{144 \times 11}}} = \frac{2.58 \times 12}{\sqrt{114.5}}$$

$$= \frac{30.96}{10.7} = 2.89$$

Degrees of freedom $= 12-1 = 11$

Since the alternative hypothesis is one-sided, this is a one-tailed test, and the critical region is given by the right tail.

Critical Region: $t \geqslant t_{.05}$ at 5% level of significance.

Since the calculated value 2.89 is larger than the tabulated value 2.2 (given), it is 'significant'. We therefore reject H_0 and conclude that "the stimulus will in general be accompanied by an increase in blood pressure".

[**Note**: The value of t given in the problem is not correct. This should be $t_{.05} = 1.8$ corresponding to the right tail area of t distribution for 11 d.f. The given value actually corresponds to the two tail areas and is appropriate only for a two-tailed test, i.e. $t_{.025} = 2.2$. However, the 'conclusion' remains unaltered.]

(4) **Test for 'significance' of observed correlation coefficient:** [See (*14.9.3*)]

(C) Using Chi-square (χ^2) Distribution

(1) **Test for a specified value of s.d.:**

Suppose, we have a random sample $x_1, x_2, \ldots, x_n$ from a normal population with mean μ and s.d. σ (both unknown). It is required to test the hypothesis that the population s.d. σ has a specified value σ_0. Null hypothesis is H_0 $(\sigma = \sigma_0)$ against alternative hypothesis H_1 $(\sigma > \sigma_0)$. We use the test statistic

$$\chi^2 = \frac{\Sigma(x_i - \bar{x})^2}{\sigma_0^2} = \frac{nS^2}{\sigma_0^2} \qquad (14.8.11)$$

where S is the s.d. of the sample. When the null hypothesis H_0 is true, the statistic follows chi-square distribution with $(n-1)$ d.f. If the observed value of χ^2 exceeds the tabulated 5% value, we conclude that the population s.d. is not σ_0.

Example 14 : 46 A random sample of size 20 from a normal population gives a sample mean of 42 and a sample standard deviation of 6. Test the hypothesis that the population s.d. is 9. Clearly state the alternative hypothesis you allow for and the level of significance adopted. [*C.A., Nov. '74*]

Solution: Given that Sample size $(n) = 20$. Sample s.d. $(S) = 6$, it is required to test the null hypothesis H_0 $(\sigma = 9)$ against the alternative hypothesis H_1 $(\sigma > 9)$.

$$\chi^2 = \frac{nS^2}{\sigma_0^2} = \frac{20 \times 6^2}{9^2} = 8.89$$

From the tables we find that for 19 d.f., 5% value of $\chi^2 = 30.14$. Since the observed value of χ^2, viz. 8.89, is less than the 5% tabulated value, it is not significant at 5% level. Therefore we cannot reject the null hypothesis, and conclude that the population s.d. may be 9.

[**Note:** In testing for a specified s.d. (σ), the sample mean ($\bar{x}$) is not required; but in testing for a specified mean (μ) the sample s.d. (S) is also necessary (see *14.8.6*).]

Example 14 : 47 Weights (in kg.) of 10 students are given below: 38, 40, 45, 53, 47, 43, 55, 48, 52, 49. Can we say that variance of the distribution of weights of all students from which the above sample of 10 students was drawn, is equal to 20 square kg.? [*C.A., Nov. '79*]

Degrees of freedom	$\chi^2_{\cdot 05}$	$\chi^2_{\cdot 01}$
9	16.92	21.67
10	18.31	23.21

Solution: We assume that the sample comes from a normal population with variance 20 sq kg. (mean unknown).

$$H_0\ (\sigma^2 = 20) \text{ against } H_1\ (\sigma^2 > 20)$$

The appropriate test statistic is χ^2 defined by (*14.8.11*). For the calculation of $\Sigma(x_i - \bar{x})^2$, we take deviations from an arbitrary origin 45: $-7, -5, 0, 8, 2, -2, 10, 3, 7, 4$; $\Sigma d = 20$ and $\Sigma d^2 = 320$

$$\Sigma(x_i - \bar{x})^2 = \Sigma d^2 - (\Sigma d)^2/n = 320 - 20^2/10 = 280$$

$$\chi^2 = \frac{\Sigma(x_i - \bar{x})^2}{\sigma_0^2} = \frac{280}{20} = 14$$

$$\text{d.f.} = 10 - 1 = 9$$

Critical region at 5% level is $\chi^2 > \chi^2_{\cdot 05}$ for 9 d.f. Since the observed value of $\chi^2 = 14$ is less than 16.92, we have no reason to reject H_0 and thus conclude that the population variance may be 20 sq. kg.

[**Note:** 95% confidence limits for σ may be obtained from (see *14.5.11*)

$$\begin{aligned}
& \chi^2_{\cdot 975} && \leqslant 280/\sigma^2 && \leqslant \chi^2_{\cdot 025} \\
or,\ & 270 && \leqslant 280/\sigma^2 && \leqslant 19.02 \\
or,\ & 1/19.02 && \leqslant \sigma^2/280 && \leqslant 1/2.70 \\
or,\ & 280/19.02 && \leqslant \sigma^2 && \leqslant 280/2.70 \\
or,\ & 14.7 && \leqslant \sigma^2 && \leqslant 103.7 \\
\therefore\ & 3.8 && \leqslant \sigma && \leqslant 10.2
\end{aligned}$$
]

(D) Using F distribution

(1) Test for equality of two s.d.s (means unknown):

Given two *independent* random samples of sizes n_1 and n_2 from two normal populations with unknown means, we may be required to test the hypothesis that the population s.d.s are equal, H_0 $(\sigma_1 = \sigma_2)$. The estimates of variances σ_1^2 and σ_2^2 are obtained from the samples (see p. 207)

$$s_1^2 = \left(\frac{n_1}{n_1 - 1}\right) S_1^2 ; \qquad s_2^2 = \left(\frac{n_2}{n_2 - 1}\right) S_2^2$$

where S_1 and S_2 are the standard deviations in the samples. If the null hypothesis is true, the statistic (see *13.8.22*)

$$F = \frac{s_1^2}{s_2^2} \tag{14.8.12}$$

follows F distribution with d.f. (n_1-1, n_2-1). If the observed value of F is larger than the 5% tabulated value, we reject the null hypothesis H_0 and conclude that the standard deviations in the two populations are not equal.

Example 14 : 48 The standard deviations calculated from two random samples of sizes 9 and 13 are 2.1 and 1.8 respectively. May the samples be regarded as drawn from normal populations with the same s.d.? (The 5% value of F from tables with d.f. 8 and 12 is $F_{.05} = 2.85$)

Solution : Here, $n_1 = 9$, $n_2 = 13$, $S_1 = 2.1$ and $S_2 = 1.8$. The "unbiased" estimates of the variances σ_1^2 and σ_2^2 in the two populations are respectively

$$s_1^2 = \frac{9 \times (2.1)^2}{8} = 4.96 , \qquad s_2^2 = \frac{13 \times (1.8)^2}{12} = 3.51$$

We have to test H_0 $(\sigma_1 = \sigma_2)$ against H_1 $(\sigma_1 > \sigma_2)$. The value of the statistic is

$$F = \frac{4.96}{3.51} = 1.41$$

Degrees of freedom are (8, 12).

Since the observed value of F, viz. 1.41, is less than the 5% tabulated value (viz. 2.85) corresponding to d.f. (8, 12), we cannot reject the null hypothesis at 5% level of significance. The conclusion is that the population s.d.s may be equal.

[**Note :** F statistic is the ratio of two independent "unbiased estimators" of the population variance, and not simply the ratio of sample variances.]

Example 14 : 49 Two random samples are drawn from two populations and the following results were obtained:

Sample I :	16	17	18	19	20	21	22	24	26	27		
Sample II :	19	22	23	25	26	28	29	30	31	32	35	36

Find the variances of the two samples and test whether the two populations have the same variance. [*C.A., Nov. '81*]

Solution : For Samples I and II, the means are

$$\bar{x}_1 = 210/10 = 21, \quad \bar{x}_2 = 336/12 = 28$$

TABLE 14.9—**Calculation of Sample Variances**

Sample I (x) :	16	17	18	19	20	21	22	24	26	27			Total
$x-\bar{x}_1$:	−5	−4	−3	−2	−1	0	1	3	5	6			0
$(x-\bar{x}_1)^2$:	25	16	9	4	1	0	1	9	25	36			126
Sample II (x) :	19	22	23	25	26	28	29	30	31	32	35	36	Total
$x-\bar{x}_2$:	−9	−6	−5	−3	−2	0	1	2	3	4	7	8	0
$(x-\bar{x}_2)^2$:	81	36	25	9	4	0	1	4	9	16	49	64	298

$$S_1^2 = \text{Variance of Sample I} = \Sigma(x-\bar{x}_1)^2/n_1 = 126/10 = 12.6$$
$$S_2^2 = \text{Variance of Sample II} = \Sigma(x-\bar{x}_2)^2/n_2 = 298/12 = 24.83$$

For testing the equality of population variances, $H_0(\sigma_1^2 = \sigma_2^2)$, we use F statistic (*14.8.12*).

$$n_1 = 10, \; n_2 = 12, \; S_1^2 = 12.6, \; S_2^2 = 24.83$$

The unbiased estimates of population variances are

$$s_1^2 = 10\times 12.6/9 = 14, \qquad s_2^2 = 12\times 24.83/11 = 27.1$$

Since $s_2^2 > s_1^2$, the test statistic is

$$F = s_2^2/s_1^2 = 27.1/14 = 1.94; \quad \text{d.f. } (11, 9)$$

Alternative Hypothesis $H_1\,(\sigma_2^2 > \sigma_1^2)$. The tabulated value at 5% level is $F_{\cdot 05} = 3.10$ (approx.).

$$\text{Critical Region : } F \geqslant 3.10$$

Since the calculated value $F = 1.94$ is less than 3.10, it is not significant, and we conclude that the two populations may have the same variance.

Example 14 : 50 In a sample of 8 observations, the sum of the squared deviations of items from the mean was 94.5. In another sample of 10 observations, the value was found to be 101.7. Test whether the difference is significant at 5% level. (You are given that at 5% level, critical value of F for $\nu_1 = 7$ and $\nu_2 = 9$ degrees of freedom is 3.29 and for $\nu_1 = 8$ and $\nu_2 = 10$ d.f. its value is 3.07.

[*C.A., May '79*]

Solution : This is a problem of testing for equality of two population variances, based on small samples.

$$H_0\;(\sigma_1^2 = \sigma_2^2) \text{ against } H_1(\sigma_1^2 > \sigma_2^2)$$

The test statistic is F defined by (*14.8.12*). If H_0 is true, the estimates of the common variance from two samples are (see *14.3.6*):

$$s_1^2 = 94.5/(8-1) = 13.5, \qquad s_2^2 = 101.7/(10-1) = 11.3$$
$$F = 13.5/11.3 = 1.19 \quad \text{(Degrees of freedom 7 and 9)}$$

Critical region at 5% level is $F > F_{.05}$. Since the observed value of $F = 1.19$ is less than the critical value 3.29 (for $\nu_1 = 7$, $\nu_2 = 9$), we have no reason to reject H_0, and thus conclude that there is no significant difference between the two variances.

Example 14 : 51 With the data of Example 14:50, obtain the 95% confidence limits for the ratio of variances σ_1^2/σ_2^2. (Given, the upper 2.5% value of F with d.f. (7, 9) is 4.20 and d.f. (9, 7) is 4.82).

Solution : From (*14.5.12*) we have with probability 95%

$$\begin{array}{c}\text{Lower 2.5\% value of } F\\ \text{with d.f. } (7,9)\end{array} < \frac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2} < \begin{array}{c}\text{Upper 2.5\% value of } F\\ \text{with d.f. } (7, 9)\end{array}$$

But, using (*13.8.19*),

Lower 2.5% value of F with d.f. (7, 9)

$= 1 / [$ Upper 2.5% value of F with d.f. (9, 7)]

$= 1 / 4.82$

Putting the values,

$$\frac{1}{4.82} \leqslant \frac{13.5}{11.3}\cdot\frac{\sigma_2^2}{\sigma_1^2} \leqslant 4.20$$

$$\text{or,}\quad \frac{1}{4.82} \leqslant 1.19\,\frac{\sigma_2^2}{\sigma_1^2} < 4.20$$

$$\text{or,}\quad 4.82 > \frac{1}{1.19}\cdot\frac{\sigma_1^2}{\sigma_2^2} > \frac{1}{4.20}$$

$$\text{or,}\quad \frac{1.19}{4.20} < \frac{\sigma_1^2}{\sigma_2^2} < 1.19\times 4.82$$

$$\text{i.e.}\quad 0.28 \leqslant \sigma_1^2/\sigma_2^2 \leqslant 5.74$$

The 95% confidence limits for σ_1^2/σ_2^2 are 0.28 and 5.74, and the confidence interval is 0.28 to 5.74.

Example 14 : 52 The following results were obtained from two independent random samples:

	Sample size	Mean	Standard deviation
Sample I	5	25	2.1
Sample II	6	29	4.0

Test whether the two samples may be regarded as drawn from the same *Normal* population.

Given $t_{.025} = 1.82$ for 9 d.f. and $F_{.05} = 6.26$ for (5, 4) d.f.

Solution : A normal distribution has two parameters, namely mean μ and variance σ^2. In order to see if the samples come from the same normal population, we have to test for

(i) equality of two means H_0 ($\mu_1 = \mu_2$), by t-test,
(ii) equality of two variances H_0 ($\sigma_1^2 = \sigma_2^2$), by F-test.

For (i) we must assume that the population variances are equal, and hence we first apply F-test for (ii) using the test statistic (*14.8.12*).

$$s_1^2 = n_1S_1^2/(n_1-1) = 5\times(2.1)^2/4 = 5.51$$
$$s_2^2 = n_2S_2^2/(n_2-1) = 6\times(4.0)^2/5 = 19.20$$

Since $s_2^2 > s_1^2$, we set up $H_1\ (\sigma_2^2 > \sigma_1^2)$.

$$F = s_2^2/s_1^2 = 19.20/5.51 = 3.48 \text{ ; d.f. } (5, 4)$$

[**Note:** In the computation of F, the larger variance appears in the numerator]

Critical Region : $F \geqslant 6.26$

Since the observed value $F = 3.48$ does not fall in the critical region, it is not significant. We therefore cannot reject the null hypothesis at 5% level and conclude that the population *variances may be equal.*

We can now apply t-test for (i), using *Fisher's t* statistic (*14.8.9*). $H_1(\mu_1 \neq \mu_2)$. An estimate of the unknown common variance in the population is

$$s^2 = \frac{n_1S_1^2 + n_2S_2^2}{n_1 + n_2 - 2} = \frac{5(2.1)^2 + 6(4.0)^2}{5+6-2} = \frac{118.05}{9} = 13.12$$

$$\therefore \quad t = \frac{25-29}{\sqrt{13.12}\ \sqrt{1/5 + 1/6}} = \frac{-4}{2.19} = -1.83 \text{ ; } d.f.\ 9$$

Since the alternative hypothesis is two-sided, the critical region is at the two tails of the distribution. At 5% level, the critical region is $|t| \geqslant t_{.025}$. From Tables $t_{.025} = 2.26$ for 9 d.f.

Critical Region: $|t| > 2.26$

Since $|t| = 1.82$ is not greater than 2.26, the observed value is not significant. We cannot reject H_0 at 5% level and conclude that the population *means may be equal.*

Both the hypotheses $H_0(\mu_1 = \mu_2)$ and $H_0(\sigma_1^2 = \sigma_2^2)$ are found to be tenable. We conclude that the two samples may be regarded as drawn from the same Normal population.

14.9 Tests for Correlation Coefficient.

In sampling from a *bivariate normal* population, the sampling distribution of sample correlation coefficient (r) is far from normal even for large sample size (n). Therefore, tests for correlation coefficient have to be treated separately depending on the largeness of sample size n and whether the population correlation coefficient is $\rho = 0$ or $\rho \neq 0$, i.e. whether the variables are *uncorrelated* or *correlated* in the population.

(1) **Test for a specified correlation coefficient (very large sample):**

When the sample size (n) is considerably large (say more than 500), the sample correlation coefficient (r) may be assumed to follow roughly a normal distribution with mean ρ (i.e. population correlation coefficient) and standard error (S.E. of r) $= (1-\rho^2)/\sqrt{n}$. Hence,

$$z = \frac{r-\rho}{(\text{S.E. of } r)} \tag{14.9.1}$$

has approximately a Standard Normal distribution. This may therefore be used to test the hypothesis $H_0(\rho = \rho_0)$, i.e. whether the sample may be assumed to have arisen from a population with correlation coefficient $\rho = \rho_0$, and also for finding approximate confidence limits for ρ.

$$\begin{aligned} 95\% \text{ confidence limits} &= r \pm 1.96(\text{S.E. of } r) \\ 99\% \text{ confidence limits} &= r \pm 2.58(\text{S.E. of } r) \end{aligned} \tag{14.9.2}$$

where (S.E. of r) $= (1-r^2)/\sqrt{n}$ approx.

Example 14 : 53 A random sample of 900 pairs of observations from a bivariate normal population shows a correlation coefficient of 0.74. Is it likely that the population correlation coefficient is 0.8 ? Also find the 99% confidence limits for the correlation coefficient in the population.

Solution : Null Hypothesis is that the population correlation coefficient is 0.8 and Alternative Hypothesis is that it is not 0.8. $H_0(\rho = 0.8)$ against $H_1(\rho = 0.8)$.

Sample size $n = 900$ is quite large. Hence, assuming approximate normal distribution for the sample correlation coefficient and under H_0,

$$\text{Observed value } (r) = 0.74$$
$$\text{Expected value } (\rho) = 0.8$$
$$\text{S.E. of } r = \frac{1-\rho^2}{\sqrt{n}} = \frac{1-0.64}{\sqrt{900}} = .012$$

The test statistic (*14.9.1*) is

$$z = \frac{r-\rho}{\text{S.E. of } r} = \frac{0.74-0.8}{.012} = -5$$

The alternative hypothesis is both-sided, and hence this is a two-tailed test. Critical Region is $|z| > 1.96$ at 5% level, and $|z| > 2.58$ at 1% level.

Since, here $|z| = 5$ is larger than 2.58, z is "highly significant". We therefore reject H_0 and conclude that the population correlation coefficient is not 0.8.

In order to find the confidence limits, we have

$$\text{S.E. of } r = \frac{1 - r^2}{\sqrt{n}} \text{ approx}, \text{ (since } \rho \text{ is now unknown).}$$

$$= \frac{1 - 0.5476}{\sqrt{900}} = .015$$

99% confidence limits for ρ are

$$r \pm 2.58 \text{ (S.E. of } r) = 0.74 \pm 2.58\ (.015)$$
$$= 0.74 \pm .04 = 0.78 \text{ and } 0.70$$

(2) **Test for 'significance' of observed correlation coefficient :**

Suppose, we have a random sample of n pairs of observations $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ from a bivariate normal population and let r be the observed correlation coefficient in the sample. It is required to test if this sample correlation coefficient is 'significant' of any correlation in the population (i.e. whether the value of the population correlation coefficient ρ is zero and the observed value of r has arisen due to fluctuations of sampling).

R. A. Fisher has shown that when the Null Hypothesis H_0 ($\rho = 0$) is true, the test statistic

$$t = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}} \qquad (14.9.3)$$

follows Student's t distribution with $(n-2)$ degrees of freedom. This statistic is used for testing the significance of r, as in the following examples.

Example 14 : 54 A random sample of 18 pairs of observations from a bivariate normal population gives a correlation coefficient of 0.3. Is it likely that the variables are uncorrelated in the population ? (At 5% level of significance the value of t for 16 d.f. is 2.12).

Solution : Here $n = 18$, $r = 0.3$.

Null Hypothesis H_0 ($\rho = 0$) ; Alternative Hypothesis H_1 ($\rho \neq 0$) When H_0 is true, the appropriate test statistic (*14.9.3*) follows Student's t distribution with $(n-2) = 16$ d.f.

$$t = \frac{0.3\sqrt{18-2}}{\sqrt{1-.09}} = \frac{0.3 \times 4}{0.95} = 1.26$$

Since the alternative hypothesis is both-sided (either ρ is less than or more than zero), the test is two-tailed.

Critical Region : $|t| > 2.12$

The observed value of t does not fall in the critical region (because $|t| = 1.26$ is not more than 2.12) the null hypothesis cannot be rejected at 5% level of significance and we conclude that the variables may be uncorrelated in the population.

Example 14 : 55 Find the least value of r in samples of 18 pairs of observations from a bivariate normal population, which is significant at 5% level. (Value of t at 5% level for 16 d.f. is 2.12).

Solution : The value of r for $n = 18$ will be significant at 5% level, if

$$|t| = \left| \frac{r\sqrt{n-2}}{\sqrt{1-r^2}} \right| \geqslant 2.12$$

Squaring both sides and putting $n = 18$,

$$\frac{r^2 (18-2)}{1-r^2} \geqslant 4.5$$

$$\begin{aligned}
\text{or,} \quad 16r^2 &\geqslant 4.5 - 4.5r^2 \\
\text{or,} \quad 20.5r^2 &\geqslant 4.5 \\
\text{or,} \quad r^2 &\geqslant 0.22 \\
\text{or,} \quad |r| &\geqslant 0.47
\end{aligned}$$

The least value of r is 0.47 (numerically).

Fisher's Z-transformation of Correlation Coefficient :

It has been stated that when $H_0(\rho = 0)$ is true, the statistic (*14.9.3*) follows t distribution, and t-test is used for the case $\rho = 0$. This is an exact test. If $\rho \neq 0$, the statistic (*14.9.3*) does not follow either normal or any other commonly-used distribution, even for large samples. In this case, R.A. Fisher has suggested the following transformation of r to a new variable

$$Z = \tfrac{1}{2} \log_e \frac{1+r}{1-r} \qquad (14.9.4)$$

This is known as **Fisher's Z-transformation** of correlation coefficient. Fisher has shown that even for small sample sizes, Z approximately follows normal distribution with mean μ and s.d. σ, given by

$$\mu = \tfrac{1}{2} \log_e \frac{1+\rho}{1-\rho}, \quad \sigma = \frac{1}{\sqrt{n-3}} \qquad (14.9.5)$$

i.e.

$$U = (Z - \mu)\sqrt{n-3} \qquad (14.9.6)$$

is approximately a standard normal variate.

Fisher's Z transformation is used to

(i) test for a specified value of $\rho \neq 0$, and find confidence limits for ρ ;

(ii) test for equality of two correlation coefficients.

(3) **Test for a specified correlation coefficient (any sample size) :**

A random sample of n pairs of observations from a bivariate normal population shows a correlation coefficient r. It is required to test the hypothesis that the population correlation coefficient ρ has a specified value. $H_0(\rho = \rho_0)$.

Using Fisher's Z-transformation (*14.9.4*) and when H_0 is true, the test statistic

$$U = (Z - \mu_0)\sqrt{n-3} \qquad (14.9.7)$$

follows Standard Normal distribution approximately, where μ_0 is the value of μ from (*14.9.5*) when $\rho = \rho_0$.

[**Note** :—(i) Since Fisher's Z-transformation uses logarithm to the base e, for practical calculations we may change the base to 10.

$$Z = \frac{1}{2}\log_e \frac{1+r}{1-r} = \frac{1}{2}\log_{10}\frac{1+r}{1-r} \times \log_e 10$$

$$= \frac{1}{2}\log_{10}\frac{1+r}{1-r} \times 2.30, \text{ since } \log_e 10 = 2.30 \text{ approx.}$$

$$= 1.15 \log_{10}\frac{1+r}{1-r} \text{ approximately} \qquad (14.9.8)$$

(*ii*) For testing H_0 $(\rho = 0)$, the t-test (*14.9.3*) is preferable, because it is an *exact* test. Fisher's Z-transformation may also be used, but this is an *approximate* test.]

Example 14 : 56 A random sample of 28 pairs of observations shows a correlation coefficient of 0.74. Is it reasonable to believe that the sample comes from a bivariate normal population with correlation coefficient 0.6 ?

Solution : Null Hypothesis $H_0(\rho = 0.6)$
Alternative Hypothesis $H_1(\rho \neq 0.6)$

Using Fisher's Z-transformation (*14.9.4*) and when H_0 is true (here $r = 0.74$, $\rho_0 = 0.6$ and $n = 28$) :

$$Z = 1.15 \log_{10}\frac{1+r}{1-r} = 1.15 \log_{10}\left(\frac{1.74}{0.26}\right)$$

$$= 1.15 \log_{10}\frac{174}{26} = 1.15 (\log 174 - \log 26)$$

$$= 1.15 (2.2405 - 1.4150) = 1.15 (0.8255) = 0.949$$

$$\mu_0 = 1.15 \log_{10}\frac{1+\rho_0}{1-\rho_0} = 1.15 \log_{10}\frac{1.6}{0.4}$$

$$= 1.15 \log 4 = 1.15 (0.6021) = 0.692$$

The test statistic is (*14.9.7*), which follows standard normal distribution under H_0.

$$U = (0.949 - 0.692)\sqrt{28-3} = 1.285$$

Since here $|U| < 1.96$, the observed value of the test statistic is not significant at 5% level. We therefore cannot reject H_0 and conclude that the sample might come from a bivariate normal population with correlation coefficient 0.6.

Example 14 : 57 A correlation coefficient of 0.2 is discovered in a sample of 28 pairs. Use Z test to find out if this is significantly different from zero. *[C.A., Nov. '82]*

Solution : Our null hypothesis is that the correlation coefficient in the population is zero, and the alternative hypothesis is that it is not zero. $H_0\ (\rho = 0)$ against $H_1\ (\rho \neq 0)$. We are given $r = 0.2$, $n = 28$, $\rho_0 = 0$.

Using Fisher's Z-transformation (*14.9.4*) and when H_0 is true,

$$Z = 1.15 \log \frac{1+0.2}{1-0.2} = 1.15 \log \frac{1.2}{0.8}$$

$$= 1.15 \log 1.5 \quad = 1.15\ (0.1761) = 0.203$$

$$\mu_0 = 1.15 \log \frac{1+0}{1-0} \quad = 1.15 \log 1$$

$$= 1.15 \times 0 = 0$$

The test statistic (*14.9.7*) has the observed value

$$U = (0.203 - 0)\sqrt{28-3} = 0.203 \times 5 = 1.015$$

Since $|U| < 1.96$, we cannot reject H_0 at 5% level of significance, and conclude that the population correlation coefficient may be zero; the observed correlation coefficient of *0.2* is not significantly different from zero.

[**Note :** The t-test (*14.9.3*) is more appropriate here, because it is an exact test; whereas Z-test is only an approximate test.]

(4) Test for equality of two correlation coefficients :

Suppose that two *independent* random samples of n_1 and n_2 pairs of observations, drawn from bivariate normal populations, show correlation coefficients r_1 and r_2 respectively. It is required to test the hypothesis that the correlation coefficients in the two populations are equal. $H_0\,(\rho_1 = \rho_2)$.

Using Fisher's Z-transformation (*14.9.4*), and when H_0 is true,

$$Z_1 = \tfrac{1}{2} \log_e \frac{1+r_1}{1-r_1}, \quad Z_2 = \tfrac{1}{2} \log_e \frac{1+r_2}{1-r_2}$$

are approximately normally distributed and independent with the same mean, but variances

$$\sigma_1^2 = 1/(n_1-3), \qquad \sigma_2^2 = 1/(n_2-3)$$

respectively. Consequently, (Z_1-Z_2) is normally distributed with mean 0 and variance $\frac{1}{n_1-3}+\frac{1}{n_2-3}$; i.e.

$$U = \frac{Z_1 - Z_2}{\sqrt{\left(\frac{1}{n_1-3}+\frac{1}{n_2-3}\right)}} \tag{14.9.9}$$

is approximately a standard normal variate. If the Alternative Hypothesis is $H_1(\rho_1 \neq \rho_2)$, the observed value of $|U|$ is compared with 1.96 or 2.58 for significance at 5% or 1% level. In case the observed value of U is significant, we conclude that the two population correlation coefficients are not equal.

Example 14 : 58 The correlation coefficients 0.45 and 0.70 were obtained from two independent random samples of 19 and 28 pairs of observations respectively drawn from bivariate normal populations. Do these results support the hypothesis that the correlation coefficients in the two populations are equal ?

Solution : The null hypothesis is that the correlation coefficients in the two populations are equal, and the alternative hypothesis is that they are not equal. $H_0(\rho_1 = \rho_2)$ against $H_1(\rho_1 \neq \rho_2)$.

Using Fisher's Z-transformation of correlation coefficient, the test statistic is (*14.9.9*). Here, $r_1 = 0.45$, $r_2 = 0.70$, $n_1 = 19$, $n_2 = 28$.

$$\begin{aligned} Z_1 &= 1.15 \log_{10}\left(\frac{1.45}{0.55}\right) = 1.15 \log_{10}\left(\frac{145}{55}\right) \\ &= 1.15\,(\log 145 - \log 55) = 1.15\,(2.1614 - 1.7404) \\ &= 1.15\,(0.4210) = 0.484 \end{aligned}$$

$$\begin{aligned} Z_2 &= 1.15 \log_{10}\left(\frac{1.70}{0.30}\right) = 1.15 \log_{10}\left(\frac{17}{3}\right) \\ &= 1.15\,(\log 17 - \log 3) = 1.15\,(1.2304 - 0.4771) \\ &= 1.15\,(0.7533) = 0.866 \end{aligned}$$

$$\therefore\ U = \frac{0.484 - 0.866}{\sqrt{1/16 + 1/25}} = \frac{-0.382 \times 20}{\sqrt{41}} = -1.19$$

Since here $|U| < 1.96$, the observed value of the test statistic is not significant at 5% level. The results therefore do not provide any evidence against H_0, and we conclude that the samples might come from populations with equal correlation coefficients.

TABLE 14.10—**Formulae for Confidence Limits**

Confidence limits for (Parameter) (1)	Confidence coefficient (2)	Confidence limits (3)	Reference to formula (4)
(Large Samples)—Approximate Limits			
(1) Population Mean (μ) :			
	95%	$\bar{x} \pm 1.96$ (S.E.)	
	99%	$\bar{x} \pm 2.58$ (S.E.)	*(14.5.2)*
	Almost sure	$\bar{x} \pm 3$ (S.E.)	
where S.E. $= \sigma/\sqrt{n}$ or $S/\sqrt{n}$ approx.			
(2) Population Proportion (P) :			
	95%	$p \pm 1.96$ (S.E.)	
	99%	$p \pm 2.58$ (S.E.)	*(14.5.3)*
	Almost sure	$p \pm 3$ (S.E.)	
where S.E. $= \sqrt{pq/n}$ (approx.), $p+q=1$			
[Note : In sampling without replacement from a finite population, the formulae for S.E. above should be multiplied by the correction factor $\sqrt{(N-n)/(N-1)}$.			
(3) Difference of Population Means ($\mu_1-\mu_2$) :			
	95%	$(\bar{x}_1-\bar{x}_2) \pm 1.96$ (S.E.)	
	99%	$(\bar{x}_1-\bar{x}_2) \pm 2.58$ (S.E.)	*(14.5.4)*
	Almost sure	$(\bar{x}_1-\bar{x}_2) \pm 3$ (S.E.)	
where S.E. $= \sqrt{\sigma_1^2/n_1+\sigma_2^2/n_2}$ or $\sqrt{S_1^2/n_1+S_2^2/n_2}$ approx.			
Note : If $\sigma_1^2=\sigma_2^2$ (but unknown), they are estimated by $S^2=(n_1S_1^2+n_2S_2^2)/(n_1+n_2)$			
(4) Difference of Population Proportions (P_1-P_2) :			
	95%	$(p_1-p_2) \pm 1.96$ (S.E.)	
	99%	$(p_1-p_2) \pm 2.58$ (S.E.)	*(14.5.5)*
	Almost sure	$(p_1-p_2) \pm 3$ (S.E.)	
where S.E. $= \sqrt{p_1q_1/n_1+p_2q_2/n_2}$ approx.			
(5) Population Correlation Coefficient (ρ) :			
	95%	$r \pm 1.96$ (S.E.)	
	99%	$r \pm 2.58$ (S.E.)	*(14.9.2)*
	Almost sure	$r \pm 3$ (S.E.)	
where S.E. $= (1-r^2)/\sqrt{n}$ approx.			

(Small Samples)—Exact Limits
Assumption : Normal Population

(6) Population Mean (μ) :

(a) σ known

95% $\bar{x} \pm 1.96$ (S.E.)
99% $\bar{x} \pm 2.58$ (S.E.) *(14.5.6)*

where S.E. $= \sigma / \sqrt{n}$

(b) σ unknown

95% $\bar{x} \pm t_{.025}\, S / \sqrt{n-1}$
99% $\bar{x} \pm t_{.005}\, S / \sqrt{n-1}$ *(14.5.8a,b)*

where S = sample s.d. ; t has d.f. $(n-1)$

(7) Difference of Population Means ($\mu_1 - \mu_2$) :

(a) σ_1, σ_2 known

95% $(\bar{x}_1 - \bar{x}_2) \pm 1.96$ (S.E.)
99% $(\bar{x}_1 - \bar{x}_2) \pm 2.58$ (S.E.) *(14.5.7)*

where S.E. $= \sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}$

(b) $\sigma_1 = \sigma_2$ unknown

95% $(\bar{x}_1 - \bar{x}_2) \pm t_{.025}\, s\sqrt{1/n_1 + 1/n_2}$
99% $(\bar{x}_1 - \bar{x}_2) \pm t_{.005}\, s\sqrt{1/n_1 + 1/n_2}$ *(14.5.9a, b)*

where $s^2 = (n_1 S_1^2 + n_2 S_2^2) / (n_1 + n_2 - 2)$; t has d.f. $(n_1 + n_2 - 2)$

(8) Population Variance (σ^2) :

(a) μ known

95% $\Sigma(x_i - \mu)^2/\chi^2_{.025} \leqslant \sigma^2 \leqslant \Sigma(x_i - \mu)^2/\chi^2_{.975}$
99% $\Sigma(x_i - \mu)^2/\chi^2_{.005} \leqslant \sigma^2 \leqslant \Sigma(x_i - \mu)^2/\chi^2_{.995}$
(χ^2 has d.f. n). *(14.5.10)*

(b) μ unknown

95% $nS^2/\chi^2_{.025} \leqslant \sigma^2 \leqslant nS^2/\chi^2_{.975}$
99% $nS^2/\chi^2_{.005} \leqslant \sigma^2 \leqslant nS^2/\chi^2_{.995}$ *(14.5.11)*
(χ^2 has d.f. $n-1$)

(9) Ratio of Population Variances (σ_1^2/σ_2^2) : μ_1, μ_2 unknown

95% $$\frac{s_1^2}{s_2^2 \,.\, F_{.025}} \leqslant \frac{\sigma_1^2}{\sigma_2^2} \leqslant \frac{s_1^2}{s_2^2 \,.\, F_{.975}}$$

99% $$\frac{s_1^2}{s_2^2 \,.\, F_{.005}} \leqslant \frac{\sigma_1^2}{\sigma_2^2} \leqslant \frac{s_1^2}{s_2^2 \,.\, F_{.995}}$$

(14.5.13)

where $s_1^2 = n_1 S_1^2/(n_1 - 1)$, $s_2^2 = n_2 S_2^2/(n_2 - 1)$; S_1, S_2 being sample standard deviations ; F has d.f. $(n_1 - 1, n_2 - 1)$; $s_1^2 > s_2^2$

TABLE 14.11—Test of Significance (Summary)

Test for Parameter & Null Hypothesis H_0 (1)	Test Statistic (2)	Sampling Distribution under H_0 (3)	Alternative Hypothesis H_1 (4)	Level of Signifi-cance (5)	Reject H_0 if (i.e. Critical Region) (6)
Large Sample Tests (Approximate)					
(1) Test for a specified Proportion : H_0 ($P = P_0$)					
	$z = \dfrac{p - P_0}{\sqrt{P_0 Q_0 / n}}$	Standard Normal	$P \neq P_0$	.05 .01	$\|z\| \geq 1.96$ $\|z\| \geq 2.58$
		(see *14.7.1*)	$P > P_0$	.05 .01	$z \geq 1.645$ $z \geq 2.33$
			$P < P_0$	.05 .01	$z \leq -1.645$ $z \leq -2.33$
(2) Test for Equality of two Proportions : H_0 ($P_1 = P_2$)					
	$z = \dfrac{p_1 - p_2}{\sqrt{pq\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$	Standard Normal	$P_1 \neq P_2$	.05 .01	$\|z\| \geq 1.96$ $\|z\| \geq 2.58$
		(see *14.7.4*)	$P_1 > P_2$	.05 .01	$z \geq 1.645$ $z \geq 2.33$
	where $p = \dfrac{n_1 p_1 + n_2 p_2}{n_1 + n_2}$		$P_1 < P_2$	.05 .01	$z \leq -1.645$ $z \leq -2.33$
(3) Test for a specified Mean : H_0 ($\mu = \mu_0$)					
	$z = \dfrac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$	Standard Normal	$\mu \neq \mu_0$	.05 .01	$\|z\| \geq 1.96$ $\|z\| \geq 2.58$
	(σ known)	(see *14.7.6*)	$\mu > \mu_0$	.05 .01	$z \geq 1.645$ $z \geq 2.33$
	or $= \dfrac{\bar{x} - \mu_0}{S / \sqrt{n}}$ (σ unknown)		$\mu < \mu_0$	.05 .01	$z \leq -1.645$ $z \leq -2.33$
(4) Test for Equality of two Means : H_0 ($\mu_1 = \mu_2$)					
	$z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$	Standard Normal	$\mu_1 \neq \mu_2$	.05 .01	$\|z\| \geq 1.96$ $\|z\| \geq 2.58$
	(σ_1, σ_2 known)	(see *14.7.9*)	$\mu_1 > \mu_2$	.05 .01	$z \geq 1.645$ $z \geq 2.33$
	or $= \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}}}$ (σ_1, σ_2 unknown)		$\mu_1 < \mu_2$	.05 .01	$z \leq -1.645$ $z \leq -2.33$

(1)	(2)	(3)	(4)	(5)	(6)
(5)	**Test for Goodness of Fit :** H_0 (*Data support theory*)				
	(Pearsonian χ^2) $\chi^2=\sum\frac{(f_o-f_e)^2}{f_e}$	χ^2 d.f. $= k-1$ (see *14.7.11*)	Data do not support theory	.05 .01	$\chi^2 \geqslant \chi^2_{.05}$ $\chi^2 \geqslant \chi^2_{.01}$
(6)	**Test for Independence of Attributes :** H_0 (*Attributes are Independent*)				
	(Contingency χ^2) $\chi^2=\sum\frac{(f_o-f_e)^2}{f_e}$ where $f_e=\frac{(A_i)(B_j)}{N}$	χ^2 d.f. $=$ $(m-1)(n-1)$ (see *14.7.12*)	Attributes are associated	.05 .01	$\chi^2 \geqslant \chi^2_{.05}$ $\chi^2 \geqslant \chi^2_{.01}$
	For 2×2 table, apply Yates' correction : $\chi^2 = \frac{N(\mid ad-bc \mid -N/2)^2}{R_1\ R_2\ C_1\ C_2}$; d.f. $= 1$ (see *14.7.14*)				
(7)	**Test for a specified Correlation Coefficient :** H_0 $(\rho=\rho_0)$				
	(a) **for very large sample** $z=\frac{r-\rho_0}{\frac{1-\rho_0^2}{\sqrt{n}}}$	Standard Normal (see *14.9.1*)	$\rho\neq\rho_0$	.05 .01	$\mid z \mid \geqslant 1.96$ $\mid z \mid \geqslant 2.58$
	(b) **using Fisher's Z-transformation** $U=(Z-\mu_0)\sqrt{n-3}$ where $Z=\frac{1}{2}\log_e\frac{1+r}{1-r}$ $\mu_0=\frac{1}{2}\log_e\frac{1+\rho_0}{1-\rho_0}$	Standard Normal (see *14.9.7*)	$\rho\neq\rho_0$	.05 .01	$\mid z \mid \geqslant 1.96$ $\mid z \mid \geqslant 2.58$
(8)	**Test for Equality of two Correlation Coeffs. :** H_0 $(\rho_1=\rho_2)$				
	$U=\frac{Z_1-Z_2}{\sqrt{\frac{1}{n_1-3}+\frac{1}{n_2-3}}}$ where $Z_1=\frac{1}{2}\log_e\frac{1+r_1}{1-r_1}$; $Z_2=\frac{1}{2}\log_e\frac{1+r_2}{1-r_2}$	Standard Normal (see *14.9.9*)	$\rho_1\neq\rho_2$	.05 .01	$\mid z \mid \geqslant 1.96$ $\mid z \mid \geqslant 2.58$

(1)	(2)	(3)	(4)	(5)	(6)
	Small Sample Tests (Exact) *(Assumption : Normal Population)*				
(9)	**Test for a specified Mean :** H_0 $(\mu = \mu_0)$				
(a)	σ **known**				
	$z = \dfrac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$	Standard Normal	$\mu \neq \mu_0$	.05	$\lvert z \rvert \geqslant 1.96$
				.01	$\lvert z \rvert \geqslant 2.58$
		(see *14.8.1*)	$\mu > \mu_0$	.05	$z \geqslant 1.645$
				.01	$z \geqslant 2.33$
			$\mu < \mu_0$	.05	$z \leqslant -1.645$
				.01	$z \leqslant -2.33$
(b)	σ **unknown** (Student's t)				
	$t = \dfrac{\bar{x}-\mu_0}{S/\sqrt{n-1}}$	t d.f. $= n-1$ (see *14.8.6*)	$\mu \neq \mu_0$	.05	$\lvert t \rvert \geqslant t_{.025}$
				.01	$\lvert t \rvert \geqslant t_{.005}$
			$\mu > \mu_0$	.05	$t \geqslant t_{.05}$
				.01	$t \geqslant t_{.01}$
			$\mu < \mu_0$	.05	$t \leqslant -t_{.05}$
				.01	$t \leqslant -t_{.01}$
(10)	**Test for Equality of two Means :** H_0 $(\mu_1 = \mu_2)$				
(a)	σ_1, σ_2 **known (two independent samples)**				
	$z = \dfrac{\bar{x}_1-\bar{x}_2}{\sqrt{\dfrac{\sigma_1^2}{n_1}+\dfrac{\sigma_2^2}{n_2}}}$	Standard Normal	$\mu_1 \neq \mu_2$	.05	$\lvert z \rvert \geqslant 1.96$
				.01	$\lvert z \rvert \geqslant 2.58$
		(see *14.8.4*)	$\mu_1 > \mu_2$	.05	$z \geqslant 1.645$
				.01	$z \geqslant 2.33$
			$\mu_1 < \mu_2$	.05	$z \leqslant -1.645$
				.01	$z \leqslant -2.33$
(b)	$\sigma_1 = \sigma_2$ **unknown (two independent samples)** (Fisher s t)				
	$t = \dfrac{\bar{x}_1-\bar{x}_2}{s\sqrt{\dfrac{1}{n_1}+\dfrac{1}{n_2}}}$	t d.f. $= n_1+n_2-2$ (see *14.8.9*)	$\mu_1 \neq \mu_2$	.05	$\lvert t \rvert \geqslant t_{.025}$
				.01	$\lvert t \rvert \geqslant t_{.005}$
	where		$\mu_1 > \mu_2$	.05	$t \geqslant t_{.05}$
				.01	$t \geqslant t_{.01}$
	$s^2 = \dfrac{n_1S_1^2+n_2S_2^2}{n_1+n_2-2}$		$\mu_1 < \mu_2$	.05	$t \leqslant -t_{.05}$
				.01	$t \leqslant -t_{.01}$
(c)	**From correlated pairs :** H_0 $(\mu_x = \mu_y)$ (Paired t)				
	$t = \dfrac{\bar{d}}{S/\sqrt{n-1}}$	t d.f. $= n-1$	$\mu_x \neq \mu_y$	.05	$\lvert t \rvert \geqslant t_{.025}$
				.01	$\lvert t \rvert \geqslant t_{.005}$

(1)	(2)	(3)	(4)	(5)	(6)
	where $d = x - y$	(see *14.8.10a*)	$\mu_x > \mu_y$	.05	$t \geqslant t_{.05}$
				.01	$t \geqslant t_{.01}$
			$\mu_x < \mu_y$	.05	$t \leqslant -t_{.05}$
				.01	$t \leqslant -t_{.01}$
(11)	**Test for 'significance' of observed Correlation Coeff. : H_0 ($\rho = 0$)**				
	$t = \dfrac{r\sqrt{n-2}}{\sqrt{1-r^2}}$	t d.f. $= n-2$ (see *14.9.3*)	$\rho \neq 0$	.05	$\lvert t \rvert \geqslant t_{.025}$
				.01	$\lvert t \rvert \geqslant t_{.005}$
(12)	**Test for a specified s.d. : H_0 ($\sigma = \sigma_0$)**				
	(a) μ known				
	$\chi^2 = \dfrac{\Sigma (x_i - \mu)^2}{\sigma_0^2}$	χ^2 d.f. $= n$	$\sigma > \sigma_0$	.05	$\chi^2 \geqslant \chi^2_{.05}$
				.01	$\chi^2 \geqslant \chi^2_{.01}$
	(b) μ unknown				
	$\chi^2 = \dfrac{\Sigma (x_i - \bar{x})^2}{\sigma_0^2}$	χ^2 d.f. $= n-1$	$\sigma > \sigma_0$	.05	$\chi^2 \geqslant \chi^2_{.05}$
				.01	$\chi^2 \geqslant \chi^2_{.01}$
	$= \dfrac{nS^2}{\sigma_0^2}$	(see *14.8.11*)			
(13)	**Test for Equality of two s.d. (μ_1, μ_2 unknown) : H_0 ($\sigma_1 = \sigma_2$)**				
	$F = \dfrac{s_1^2}{s_2^2}$; $(s_1 > s_2)$	F	$\sigma_1 > \sigma_2$	.05	$F \geqslant F_{.05}$
				.01	$F \geqslant F_{.01}$
	$= \dfrac{n_1 S_1^2/(n_1-1)}{n_2 S_2^2/(n_2-1)}$	d.f. $= (n_1-1, n_2-1)$ (see *14.8.12*)			
	where S_1, S_2 are sample s. d. ;				

Exercises 14

1. State the criteria of a good estimator. Give illustrations.

2. Distinguish between *point estimation* and *interval estimation*. [*C.U., B.Sc.(Econ) '81*]

3. On the basis of the following random sample, obtain unbiased estimates of the mean μ and variance σ^2 in the population: 38, 32, 39, 37, 27, 35, 28, 32, 30, 35.

4. Define a *consistent estimator* and give an example.

5. Show that in random samples from a normal population with a known variance, the sample mean $\bar{x}$ is unbiased and consistent for estimating the population mean μ.

6. What is a '*sufficient statistic*'? Why is it so called? Give an example. [*C.U., B.Sc. '77*]

7. Explain the terms '*unbiased estimator*' and '*minimum variance estimator*'. [*W.B.H.S. '78, '83*]

8. (a) Let T_1 and T_2 be statistics with expectations $E(T_1) = \theta_1 + \theta_2$ and $E(T_2) = \theta_1 - \theta_2$. Find unbiased estimators of θ_1 and θ^2 [*W.B.H.S. '78*]

(b) If T_1 and T_2 be statistics with expectations $E(T_1) = 2\theta_1 + 3\theta_2$ and $E(T_2) = \theta_1 + \theta_2$, find unbiased estimators of parameters θ_1 and θ_2. [*W.B.H.S. '83*]

9. What do you understand by Best Linear Unbiased Estimator (*BLUE*)? Give an example.

10. Describe the '*method of maximum likelihood*' for the estimation of unknown parameters. State the important properties of maximum likelihood estimators.

11. If $x_1, x_2, \ldots x_n$ is a random sample from a normal population with zero mean, obtain the maximum likelihood estimator of the variance, and show that it is unbiased.

12. On the basis of a random sample from a binomial population with parameters (n, P), find the maximum likelihood estimator of P. Show that this estimator is unbiased.

13. Suppose $x_1, x_2, \ldots x_n$ are independent random observations from a univariate normal population with mean μ and variance one. Find the maximum likelihood estimator for μ. [*C.U., B.Sc. '78*]

14. Show that the estimates of the Poisson parameter obtained by the method of moments and by the method of maximum likelihood are identical.

15. Show that both the Method of Moments and the Method of Maximum Likelihood give identical estimates of the parameters of a normal population. [*C.U., B.Sc. (Econ) '82*]

16. Explain the concept of *confidence interval, confidence limits* and *confidence coefficient*.

17. The mean height obtained from a sample of size 100 taken randomly from a population is 64 inches. If the s.d. of the height distribution of the population is 3 inches, set up probable limits to the mean height of the population.

18. A random sample of 400 is taken from a large number of coins. The mean weight of the coins in the sample is 28.57 gms. and the s.d. is 1.25 gms. What are the limits which have a 95% chance of including the mean weight of all the coins?

19. A random sample of 100 days shows an average daily sale of Rs. 50/- with a s.d. of Rs. 10/- in a particular shop. Assuming a

normal distribution, construct a 95% confidence interval for the expected sale per day. [C.U., B.Sc.(Econ) '82]

20. In a random sample of size 100 taken from a population of size 1,000, the mean and S.D. of a sample characteristic are found to be 4.8 and 1.1 respectively. Find the 95% confidence interval for population mean.

21. A random sample of size 1000 selected from a large bulk of mass produced machine parts contains 6% defectives. What information can be inferred about the percentage of defective in the bulk? [C.U., M.Com, '73]

22. A random sample of 500 pineapples was taken from a large consignment and 65 of them were found to be bad. Show that the Standard Error of the proportion of bad ones in a sample of this size is 0.015 and deduce that the percentage of bad pineapples in the consignment certainly lies between 8.5 and 17.5. [I.C.W.A., Dec. '78; June '82]

23. In a sample of 400 oranges from a large consignment, 40 fruits were considered defective. Estimate the percentage of defective oranges in the whole consignment and assign limits within which the percentage will probably lie. [C.A., Nov. '75]

24. A factory is producing 50,000 pairs of shoes daily. From a sample of 500 pairs, 2% were found to be of sub-standard quality. Estimate the number of pairs that can be reasonably expected to be spoiled in the daily production and assign limits at 95% level of confidence. [C.A., May '79]

25. Out of 20,000 customers' ledger accounts, a sample of 600 accounts was taken to test the accuracy of posting and balancing, and 45 mistakes were found. Assign limits within which the number of defective cases can be expected at 5% level. ($z = 2.00$). [C.A., May '76]

26. (a) Under what circumstances can the normal distribution be used to find confidence limits of the population mean?

(*b*)) What is Student's t distribution? When is it used to construct a confidence interval estimate of the population mean?

27. On the basis of a random sample from a normal population with a known variance σ^2, obtain 99% confidence limits for the population mean μ. What will be the confidence limits, if the variance is unknown?

28. A sample of 16 items is randomly drawn from a normal population with unknown mean and s.d. = 12. If the mean of the sample is 40, find 95% interval estimate of the population mean.

29. Suppose that a random sample of size 10, drawn from a normal population, has mean 40 and s.d. 12. Find 99% confidence limits for the population mean. (Given $t_{.005} = 3.25$ for 9 d.f.).

30. The marks obtained by 17 candidates in an examination have a mean 57 and variance 64. Find 99% confidence limits for the mean of the population of marks, assuming it to be normal. For 16 degrees of freedom $P(|t| > 2.9) = 0.01$ [*C.U., B.Sc.(Math) '77*]

31. Two independent random samples of sizes 8 and 6 are drawn from normal populations with unknown means μ_1, μ_2 and variances 32 and 30 respectively. If the sample means are 68.1 and 60.4 respectively, find 95% confidence limits for the difference of population means.

32. Two independent random samples of sizes 8 and 6 are drawn from two normal populations whose means and variances are unknown. If the samples have means 28.3 and 20.8, and standard deviations 6 and 5 respectively, find 95% confidence limits for the difference of population means. State the necessary assumption. (Value of t for 12 d.f. is $t_{.025} = 2.18$).

33. The following random sample was obtained from a normal population :— 13, 10, 11, 20, 8, 16. Find the 95% confidence limits for the population standard deviation, when the population mean is (i) known to be 14, (ii) unknown. Given

d.f.	$\chi^2_{.975}$	$\chi^2_{.025}$
5	0.83	12.8
6	1.24	14.4

34. For two independent random samples of sizes $n_1 = 8$ and $n_2 = 10$, the standard deviations are respectively $S_1 = 7$, $S_2 = 6$. Assuming that the populations are normal, find the 95% confidence limits for the variance-ratio σ_1^2/σ_2^2. Given $F_{.995} = 0.117$ and $F_{.005} = 6.88$ for (7, 9) d.f.

35. (*a*) What is meant by a test of a null hypothesis? What are Type I and Type II errors? [*W.B.H.S. '79*]

(*b*) Explain the terms :—Tests of Significance, Null hypothesis, Critical region, Level of Significance. The explanations must be as brief as possible. [*I.C.W.A., Dec. '78*]

36. Explain clearly the following terms in connection with testing of hypothesis :— (i) Region of Acceptance, (ii) First kind of error, (iii) Second kind of error. [*W.B.H.S. '83*]

37. In order to test whether a coin is perfect, I shall toss it six times. I shall reject the null hypothesis of perfectness if and only if I get no head or 6 heads. What is then the probability of Type I error for my test? [*W.B.H.S. '79*]

38. A professor is concerned with the effectiveness of a given teaching technique. What null hypothesis is he testing if he is committing a Type I error when he erroneously concludes that the particular teaching technique is effective? [*W.B.H.S. '80*]

39. In order to test whether a coin is perfect, the coin is tossed 5 times. The null hypothesis of perfectness is rejected if and only if more than 4 heads are obtained. What is the probability of Type I error? Find the probability of Type II error when the corresponding probability of head is 0.2. [*W.B.H.S.* '82]

40. The proportion of defective items in a large lot of items is p. To test the hypothesis $p = 0.2$, we take a random sample of 8 items and accept the hypothesis if the number of defectives in the sample is 6 or less. Find the probability of type I error of the test. What is the type II error if $p = 0.3$? [*W.B.H.S.* '78]

41. A coin is tossed 900 times and heads appear 490 times. Does this result support the hypothesis that the coin is unbiased?
[*I.C.W.A., Dec.* '74]

42. A sample of size 600 persons selected at random from a large city shows that the percentage of male in the sample is 53%. It is believed that male to total population ratio in the city is 1/2. Test whether this belief is confirmed by the observation. Also, find 95% confidence limits for the percentage of males in the whole city.

43. A person threw 10 dice 500 times and obtained 2560 times 4, 5 or 6. Can this be attributed to fluctuations of sampling?
[*C.A., May* '75]

44. In a sample of 500 people in Kerala, 280 are tea drinkers and the rest are coffee drinkers. Can we assume that both coffee and tea are equally popular in this State at 1% level of significance?
[*C.A., May* '78]

45. Over a long period, a travelling salesman's records show a proportion of successful calls of 1/3. He tries a new technique and achieves 19 successes in 40 calls. Does this signify that the new technique is effective?

46. In a sample of 400 parts manufactured by a factory, the number of defective parts was found to be 30. The company however claims that only 5% of their product is defective. Is the claim tenable?
[*I.C.W.A., Dec.* '75]

47. A manufacturer claimed that at least 95% of the equipment which he supplied to a factory conformed to specifications. An examination of a sample of 200 pieces of equipment revealed that 18 were faulty. Test his claim at a significance level of (i) .05, (ii) .01.
[*I.C.W.A., Dec.* '77; *C.A., Nov.* '79]

48. A manufacturer claimed that at least 90% of the components which he supplied, conformed to specifications. A random sample of 200 components showed that only 164 were upto the standard. Test this claim at 5% level of significance. [*C.U., B.Sc.(Econ)* '81]

49. A Life Insurance Company has 1500 policies averaging Rs. 2,000 on lives at age 30. From the experience table, it is found that of 100,000 alive at age 30, only 99,000 are alive at age 31. Find the lower and the upper values of the amount that Company will have to pay out in insurance during the year. [*C.A., May '82*]

50. In a certain district *A*, 450 persons were considered regular consumers of Tea out of a sample of 1000 persons. In another district *B*, 400 were regular consumers of Tea out of a sample of 800 persons. Do these facts reveal a significant difference between the two districts as far as tea-drinking habit is concerned? (Use 5% level). [*C.A., Nov. '75*]

51. In a sample of 600 students of a certain college, 400 are found to use dot pens. In another college from a sample of 900 students 450 were found to use dot pens. Test whether the two colleges are significantly different with respect to the habit of using dot pens. (Null and alternative hypotheses should be stated clearly). [*I.C.W.A., Dec. '82*]

52. Before an increase in excise duty on tea, 400 people out of a sample of 500 persons were found to be tea drinkers. After an increase in duty, 400 people were tea drinkers in a sample of 600 people. Using standard error of proportion, state whether there is a significant decrease in the consumption of tea. [*C.A., May '77*]

53. In a large city, 20% of a random sample of 1100 school boys had a certain physical defect. In another large city, 200 out of a random sample of 900 school boys had the same defect. Do you think that the percentage is less in the former city? Calculate the 95% confidence limits for the difference.

54. Is it likely that a sample of 300 items whose mean is 16.0 is a random sample from a large population whose mean is 16.8 and standard deviation 5.2? [*I.C.W.A., June '73; M.B.A. '78*]

55. An automatic machine fills tea in sealed tins with mean weight of tea 1 kg. and Standard Deviation = 1 gm. A random sample of 50 tins was examined, and it was found that their mean weight was 999.50 gms. Using standard error of the mean, state whether the machine is working properly or not. [*C.A., Nov. '73*]

56. The mean lifetime of a sample of 100 fluorescent-light bulbs produced by a company is computed to be 1570 hours with a standard deviation of 120 hours. The company claims that the average life of the bulbs is 1600 hours. Using a level of significance of .05 is the claim acceptable? [*I.C.W.A., June, '76*]

57. A sample of 900 members has a mean 3.4 cm. and s.d. 2.61 cm. Can the sample be regarded as drawn from a population with mean 3.25 cm.? Find the 95% confidence limits for the population mean. [*C.U., B.Sc. '78*]

58. A new variety of potato grown on 250 plots gave rise to a mean yield of 82.7 mds. per acre with a s.d. of 14.6 mds. per acre. Is it reasonable to assert that the new variety is superior in yield to the standard variety with an established yield of 80.2 mds. per acre?

59. The mean breaking strength of the cables supplied by a manufacturer is 1800 with a standard deviation of 100. By a new technique in the manufacturing process it is claimed that the breaking strength of the cables has increased. In order to test this claim a sample of 50 cables is tested. It is found that the mean breaking strength is 1850. Can we support the claim at a 0.01 level of significance? [*I.C.W.A., Dec. '80*]

60. The average number of defective articles per day in a certain factory is claimed to be less than the average for all the factories. The average for all the factories is 30.5. A random sample of 100 days showed the following distribution :—

Class limits :	16-20	21-25	26-30	31-35	36-40	Total
No. of days :	12	22	20	30	16	100

Calculate the mean and standard deviation of the sample and use it to claim that the average is less than the figure for all factories, at 5% level of significance. Given $Z(1.645) = 0.95$. [*C.A., May '79*]

61. Random samples of size 500 and 400 have means 11.5 and 10.9 respectively. Can the samples be regarded as drawn from the same population of s.d. 5? Find 99% confidence limits for the difference of means.

62. Two salesmen *A* and *B* are working in a certain district. From a sample survey conducted by the Head Office, the following results were obtained. State whether there is any significant difference in the average sales between the two salesmen :—

	A	*B*
No. of sales	20	18
Average sales in Rs.	170	205
S.D. in Rs.	20	25

[*C.A., Nov. '77*]

63. Intelligence tests on two groups—one group consisting of 121 girls and the other group consisting of 81 boys—give the following results :

Group of Girls : Mean 84, Standard Deviation 10.
Group of Boys : Mean 81, Standard Deviation 12.

Examine if the difference is significant. [*I.C.W.A., Dec. '81*]

64. A simple sample of heights of 6400 Englishmen has a mean of 67.85 inches and a s.d. of 2.56 inches, while a simple sample of heights of 1600 Australians has a mean of 68.55 and a s.d. of 2.52

inches. Do the data indicate that Australians are on the average taller than the Englishmen? [C.U., B.Sc.(Math) '67]

65. Define Pearsonian χ^2 and discuss its uses in testing of hypothesis. [W.B.H.S. '82]

66. Of 160 offsprings of a certain cross between guinea pigs, 102 were red, 24 were black and 34 were white. According to a genetic model the probabilities of red, black and white are respectively 9/16, 3/16 and 1/4. Test at 5% significance level, if the data are consistent with the model. For 2 degrees of freedom $P(\chi^2 > 5.99) = 0.05$. [C.U., B.Sc.(Math) '77, '79]

67. 200 digits from 0 to 9 are taken at random from a page of a certain random number table. The frequency distribution of the digits is given:

Digit :	0	1	2	3	4	5	6	7	8	9
Frequency :	18	19	13	21	16	25	22	20	21	25

Can this table be regarded as random? (Given $\chi^2_{\cdot 05, 9} = 16.92$). [W.B.H.S. '81]

68. In 60 throws of a die, face one turned up 6 times, face two or three 18 times, face four or five 24 times, and face six 12 times. Test at 10% significance level if the die is honest, it being given that $P(\chi^2 > 6.25) = 0.1$ for 3 degrees of freedom. [C.U., B.Sc.(Math) '75]

69. Discuss the importance of χ^2 test. How is it used to test the association between attributes? [C.A., Nov. '82]

70. A random sample of 200 students in a Calcutta college were asked the question: "Do you think scientists are slightly unbalanced people?" The number of students of each class saying 'yes' and 'no' are:

	Class in College			
	P.U.	1st year	2nd year	3rd year
Yes	15	8	5	2
No	55	42	35	38

Test whether there is any association between opinion and class in College. (Given $\chi^2_{\cdot 05} = 7.81$ for 3 d.f.)

71. Random samples from lots supplied by four manufacturers A, B, C, D were examined, and the following defectives were detected:

Manufacturer	A	B	C	D
Sample size	100	200	150	250
Number of defectives	20	35	40	45

Do you consider that the lots supplied by the different manufacturers are of the same quality? Given, $\chi^2_{\cdot 05} = 7.81$ for 3 degrees of freedom.

72. A company using door-to-door sales procedure is testing a new sales approach and has the following results on a comparative test under otherwise identical conditions:

	Sales	No Sales
Old approach	8	12
New approach	10	10

Use χ^2 test to determine the significance of the observed difference. [Given that the values of χ^2 from statistical tables for 1 degree of freedom are 3.8 and 6.6 at significance levels .05 and .01 respectively.]

73. In an infantile paralysis epidemic, 500 persons contracted the disease. 200 received no serum treatment and of these 75 became paralysed. Of those who did receive serum treatment 65 became paralysed. Was the serum treatment effective? $\chi^2_{.01} = 6.64$ for 1 d.f. [*C.A., Nov. '79*]

74. From the following results regarding eye-colour of fathers and sons, test if the colour of son's eyes is associated with that of father's.

Father's eye-colour	Son's eye-colour Light	Not-light
Light	47	16
Not-light	14	23

(Given $\chi^2_{.05} = 3.84$ for 1 d.f.)

75. Out of 8,000 graduates in a town 800 are females, out of 1,600 graduate employees 120 are females. Use χ^2 test to determine if any distinction is made in appointment on the basis of sex. Value of χ^2 for 5% level for one degree of freedom is 3.84. [*C.A., Nov. '81*]

76. A variable x is normally distributed in the population with mean 20 and standard deviation 5. If a random sample of size 25 is drawn, what is the probability that the sample mean $\bar{x}$, will be greater than 21? (Probability that the normal deviate lies between 0 and 1 is .3413). [*B.U., B.A.(Econ) '71*]

77. The mean of a random sample of size 10 is 12.1. It is known that the population s.d. is 3.2. Can it be concluded that the sample came from a normal population with mean 14.5? Find the 99% confidence limits for the population mean.

78. A sample of 100 iron bars is said to be drawn from a large number of bars whose lengths are normally distributed with mean 4 ft. and s.d. 0.6 ft. If the sample mean is 4.2 ft., can the sample be regarded as a truly random sample? (Null hypothesis and assumptions should be stated clearly). [*I.C.W.A., June '82*]

79. A manufacturer of string has found from past experience that samples of a certain type have a mean breaking strength of 15.6 kg. and a s.d. of 2.2 kg. A time-saving change in the manufacturing process of this string is tried. A sample of 50 pieces is then taken, for which the mean breaking strength turns out to be 15.5 kg. On the basis of this sample can it be concluded that the new process has a harmful effect on the strength of the string? (Assume that breaking strength of string is normally distributed). [*C.U., B.Sc.(Math) '65*]

80. Independent samples of sizes 30 and 55 from two normal populations having a common variance 17.6 were found to have means 23.0 and 21.9 respectively. Test at 5% level of significance whether the populations have also the same mean. Also, find the 95% confidence limits for the difference of population means. Given $P(z>1.96) = 0.025$.

81. Two independent random samples of sizes 10 and 25 from two normal populations having variances 9.61 and 7.29 were found to have means 23.0 and 20.3 respectively. Test at 1% significance level the hypothesis that mean of the first population is larger. Calculate also the 99% confidence limits for the difference of means.

82. For a random sample of size 10 the mean is 12.1 and the standard deviation is 3.2. Is it reasonable to suppose that this sample came from a normal population with mean 14.5? [Given $P(|t| > 2.26) = .05$ for 9 d.f.]. Also, find 95% confidence interval for mean of the population.

83. A Company has been producing steel tubes of mean inner diameter 2.00 cms. A sample of 10 tubes gives a mean inner diameter of 2.01 cms. and a variance of .004 sq. cm. Is the difference in the value of mean significant? (Value of t for d.f. 9 at 5% level = 2.262). [*C.A., Nov. '81*]

84. The heights of 10 males of a given locality are found to be 70, 67, 62, 68, 61, 68, 70, 64, 64, 66 inches. Is it reasonable to believe that the average height is greater than 64 inches? Test at 5% significance level, assuming that for 9 degrees of freedom $P(t > 1.83) = .05$. [*C.U., B.Sc.(Math) '78*]

85. It is claimed that students entering a college have an average I.Q. higher than 100. A random sample of 16 is taken and the sample mean is found to be 106. The sample s.d. is 10. Is the claim supportable? (It is assumed that the I.Q. are normally distributed. and given $t_{.01} = 2.82$ for 9 d.f.). [*C.U., B.Sc.(Econ) '81*]

86. The heights of 10 adult males selected at random from a given locality had a mean 63.2 inches and variance 6.25 sq. inches. Test at 5% significance level the hypothesis that the adult males of the given locality are on the average less than 65 inches tall. [Given, for 9 degrees of freedom $P(t > 1.83) = 0.05$]

87. (a) How do you test the significance of the difference between the means of two samples?

(b) Two samples of 6 and 5 items respectively gave the following data:

Mean of the first sample	=	40
S. D. of the first sample	=	8
Mean of the second sample	=	50
S. D. of the second sample	=	10

Is the difference of the means significant? (The value of t for 9 d.f. at 5% level is 2.26). [*C.A., May '81*]

88. The means of two independent random samples of sizes 8 and 6 are 38.4 and 33.7 respectively. The sum of the squares of deviations from the respective sample means are 20.8 and 15.7 Do you think that mean of the first population is the larger? (Assume Normal population with the same standard deviation). Also find 99% confidence limits for the difference of the population means. Given $t_{.01} = 2.68$ and $t_{.005} = 3.06$ for 12 d.f.

89. The heights of 6 randomly chosen Sailors are (in inches): 63, 65, 68, 69, 72, 71, and those of 10 Soldiers are 66, 62, 69, 61, 65, 72, 69, 73, 70, 71. In the light of these data, discuss the suggestion that Sailors are on the average taller than Soldiers. (Given, the 5% value of t for 14 d.f. is 1.76).

90. Ten soldiers visit a rifle range for two consecutive weeks. Their scores are

Soldier :	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*
1st week :	67	24	57	55	63	54	56	68	33	43
2nd week :	70	38	58	58	56	67	68	72	42	38

Examine if there is any significant improvement in their performance. Given $t_{.01} = 2.82$ for 9 d.f.

91. Two laboratories carry out independent estimates of a particular chemical in a medicine produced by a certain firm. A sample is taken from each batch, halved and the separate halves sent to the two laboratories. The following data are obtained:

Number of samples	...	10
Mean value of the difference of estimates	...	0.6
Sum of the squares of the differences (from their mean)	...	20

Is the difference significant? (Value of t at 5% level for 9 d.f. is 2.26). [*C.A., May '82*]

92. Nine patients to whom a certain drug was administered, registered the following rise in blood pressure: 3, 7, 4, −1, −3, 6, −4, 1, 5. Test the hypothesis that the drug does not raise the blood pressure at 10% level of significance. Assume that the sample is from a normal population and $P(t > 1.397) = 0.10$ for 8 degrees of freedom. [*C.U., B.Sc.(Math) '67*]

93. A random sample is drawn from a normal population. The data give sample size and sample variance only. What statistic would you use to test the hypothesis that the population variance has a particular value? Give reasons. *[C.U., B.Sc.(Econ) '82]*

94. Obtain formulae for 95% confidence limits of the variance of a normal population, when the mean is (i) known, (ii) unknown.

95. Given that x is normally distributed and that a sample of 20 yields a mean of 42 and a variance of 25, test the hypothesis that the population standard deviation is $\sigma = 8$ at 5% level of significance. (Given the value of χ^2 for d.f. 19 is 30.14 at 5% level.)

96. A random sample of 12 values gave an unbiased estimate s^2 of the population variance equal to 10.62. May the sample be reasonably regarded as drawn from a normal population with variance 7? Calculate the 95% confidence limits for the population s.d. Given that for 11 degrees of freedom, $\chi^2_{\cdot 05} = 19.68$, $\chi^2_{\cdot 025} = 21.92$, $\chi^2_{\cdot 975} = 3.82$.

97. State the assumptions underlying Student's t-test and Snedecor's F-test when applied to both single and two-sample problems. *[C.A., May '80]*

98. The standard deviations calculated from two independent random samples of sizes 9 and 10 were found to be 2.4 and 1.8 respectively. Can the samples be regarded to have been drawn from equally variable normal populations? (Given that 5% value of F from the tables is 3.23 for degrees of freedom 8 and 9 respectively).

99. A random sample of size 8, drawn from a normal population, shows that the sum of observations is 9.6 and the sum of the squares of observations is 60.52. An independent sample of size 11 taken from another normal population, gave these figures as 16.5 and 64.75 respectively. Test whether the two populations may have equal means and variances. Given that $F_{\cdot 05} = 3.14$ for degrees of freedom (7, 10), and $t_{\cdot 025} = 2.11$ for 17 d.f.

100. A correlation coefficient of 0.22 is obtained from a random sample of 1024 pairs of observations from a bivariate normal population. Find 95% confidence limits for the correlation coefficient in the population.

101. A random sample of 27 pairs of observations from a bivariate normal population gives a correlation coefficient of 0.42. Is it likely that the variables are uncorrelated in the population? (Given $t_{\cdot 025} = 2.06$ for 25 d.f.).

102. Find the least value of r in a sample of 27 pairs from a bivariate normal population which is significant at 1% level. (Given $t_{\cdot 005} = 2.79$ for 25 d.f.)

103. Explain Fisher's Z-transformation of correlation coefficient and indicate its uses in test of significance.

104. The value of r obtained from a random sample of 19 pairs of observations from a bivariate normal population is 0.8. Is this value consistent with the hypothesis that the population correlation coefficient is 0.55?

105. The first of two samples consisting of 19 pairs of observations gives a correlation coefficient of 0.5, while an independent sample of 23 pairs shows a correlation coefficient of 0.6. Are the two correlation coefficients significantly different at 1% level?

Answers

3. 33.3 & 17.3 (see Ex. 14 : 3).
8. (a) $(T_1+T_2)/2$, $(T_1-T_2)/2$; (b) $3T_2-T_1$, T_1-2T_2.
11. $\sigma_0^2 = \Sigma x_i^2/n$ 12. $\bar{x}/n$ 13. $\Sigma x_i/n$
17. 63.1 and 64.9 inches (use $\bar{x} \pm 3\sigma/\sqrt{n}$)
18. 28.45 to 28.69 gms. (use $\bar{x} \pm 1.96\, S/\sqrt{n}$)
19. Rs. 48.04 to 51.96 (see *14.5.6*).
20. 4.60 to 5.00 (see Ex. 14 : 12).
21. 3.7% to 8.3% (see Ex. 14 : 11). 22. (see Ex. 14 : 11).
23. 10%; 5.5% to 14.5% (see Ex. 14 : 11).
24. 400 to 1600 (see Ex. 14 : 14). 25. 1070 to 1930 (see Ex. 14 : 14).
26. (a) Normal population with σ *known*, or Large sample from any population; (b) (see p. 196); Normal population with σ *unknown*.
27. (see *14.5.6* & *14.5.8b*). 28. 34.12 to 45.88 (see *14.5.6*).
29. 27 and 53 (see *14.5.8b*).
30. 51.2 and 62.8 (see *14.5.8b*). 31. 1.82 to 13.58 (see *14.5.7*)
32. 0.39 to 14.61 (see *14.5.9a*). Populations have equal s.d.
33. (i) 2.66 to 9.07 (see *14.5.10*); (ii) 2.74 to 10.8 (see *14.5.11*)
34. 0.20 to 12.0 (see *14.5.13*). 37. 1/32
38. The teaching technique is *not* effective.
39. $(1/2)^5 = 0.03125$; $1-(0.2)^5 = 0.99968$.
40. .0000,8448; 0.9987,0967.
41. No; $|z| = 2.68 > 1.96$, $H_0(P=\frac{1}{2})$, $H_1(P\neq\frac{1}{2})$; (see Ex. 14 : 20).
42. Yes; $|z| = 1.47 < 1.96$, " " "
Confidence limits for P : 46% to 54%
43. Yes; $|z| = 1.70 < 1.96$, $H_0(P=\frac{1}{2})$, $H_1(P\neq\frac{1}{2})$; (see Ex. 14 : 20).
44. No; $|z| = 2.70 > 2.58$, " " "
45. Yes; $z = 1.90 > 1.645$; $H_0(P=1/3)$, $H_1(P>1/3)$; see Ex. 14 : 21

46. No; $z = 2.29 > 1.645$; $H_0(P=.05)$, $H_1(P>.05)$; see Ex. 14:21
47. $z = -2.60$; $H_0(P = .95)$; $H_1(P < .95)$; see Ex. 14 : 22.
    (i) since $z < -1.645$, claim not justified at .05 level.
    (ii) since $z < -2.33$, claim not justified at .01 level.
48. Not tenable; $z = -3.77 < -1.645$; $H_0(P = 0.9)$, $H_1(P<0.9)$; see Ex. 14 : 22.
49. Rs. 6,000 to Rs. 54,000 (see Ex. 14:14).
50. Yes; $|z| = 2.11 > 1.96$; $H_0(P_1=P_2)$, $H_1(P_1 \neq P_2)$; see Ex. 14:23
51. Yes; $|z| = 6.38 > 2.58$; H_0 (Proportion of students using dot pens equal in two colleges), H_1(Proportions different); see Ex. 14:23.
52. Yes; $z = 4.94 > 2.33$; $H_0(P_1=P_1)$, $H_1(P_1>P_2)$; see Ex. 14:24
53. No; $z=-1.21>-1.645$; $H_0(P_1=P_2)$, $H_1(P_1<P_2)$; see Ex. 14:25
    Confidence limits for P_1-P_2 : -5.8% to 1.3%.
54. No; $|z|=2.67>2.58$; $H_0(\mu=16.8)$, $H_1(\mu \neq 16.8)$; see Ex. 14 : 26
55. Not; $|z|=3.54>2.58$; $H_0(\mu=1$ kg$)$, $H_1(\mu \neq 1$ kg$)$; see Ex. 14:26
56. No; $|z|=2.5>1.96$; $H_0(\mu=1600)$, $H_1(\mu \neq 1600)$; see Ex. 14 : 27
57. Yes; $|z|=1.72<1.96$; $H_0(\mu=3.25)$, $H_1(\mu \neq 3.25)$; see Ex. 14:27
    Confidence limits for μ : 3.23 to 3.57 (cm.)
58. Yes; $z = 2.71>2.33$; $H_0(\mu=80.2)$, $H_1(\mu >80.2)$
59. Yes; $z = 3.54>2.33$; $H_0(\mu=1800)$, $H_1(\mu >1800)$.
60. $\bar{x} = 28.8$, $S = 6.35$; Claim justified; $z = -2.68<-2.33$; $H_0(\mu = 30.5)$, $H_1(\mu < 30.5)$; see Ex. 14:28
61. Yes; $|z| = 1.79<1.96$; $H_0(\mu_1=\mu_2)$, $H_1(\mu_1 \neq \mu_2)$; see Ex. 14:30
    Confidence limits for $\mu_1-\mu_2$: -0.27 to 1.47.
62. Yes; $|z| = 4.73>1.96$; $H_0(\mu_1=\mu_2)$, $H_1(\mu_1 \neq \mu_2)$; see Ex. 14:29
63. No; $|z| = 1.86<1.96$; $H_0(\mu_1=\mu_2)$, $H_1(\mu_1 \neq \mu_2)$; see Ex. 14:29
64. Yes; $z = -9.9<-2.33$; $H_0(\mu_1=\mu_2)$, $H_1(\mu_1<\mu_2)$: see Ex. 14:29
65. see *(14.7.11)*
66. Yes; $\chi^2=3.7<5.99$; H_0(Data agree with model); see Ex. 14:32
67. Yes; $\chi^2 = 6.3<16.92$; H_0(Table is random; i.e. probability of each digit appearing is 1/10); see Ex. 14:33
68. Yes; $\chi^2 = 3.0<6.25$; H_0(Die is honest; i.e. probabilities of the 4 classes are 1/6, 1/3, 1/3, 1/6 respectively); see Ex. 14:32
69. see *(14.7.12)*
70. No; $\chi^2 = 5.64 < 7.81$; H_0 (Attributes are independent); see Ex. 14:35
71. (Form a 2×4 contingency table showing 'No. of defectives' and 'No. of non-defectives' against A, B, C, D). Yes; $\chi^2 = 5.57 <7.81$; H_0(Lots are of same quality); see Ex. 14:35

72. Not significant; $\chi^2 = 0.101<3.8$ (apply Yates' correction); H_0 ('Procedure' and 'Sales' independent); see Ex. 14:36

73. (Form a 2×2 contingency table showing treatment with 'serum' and 'no serum' against 'paralysed' and 'not paralysed'). Yes; $\chi^2=14.1>6.64$); H_0(Attributes are not independent); see Ex. 14:36

74. Yes; $\chi^2=11.74>3.84$; H_0 (Attributes are not independent); see Ex. 14:36

75. (Form a 2×2 table showing 'males' and 'females' against 'employed' & 'unemployed'). Yes; $\chi^2 = 13.54>3.84$; H_0(Attributes are not independent); see Ex. 14:36

76. $P(\bar{x}>21) = P(z>1) = 0.1587$; see Ex. 13:26 & 27 and 14:38

77. No; $|z| = 2.37>1.96$; $H_0(\mu=14.5)$, $H_1(\mu\neq 14.5)$; σ known; Confidence limits for μ : 9.5 & 14.7; see Ex. 14:37

78. No; $|z| = 3.33 > 2.58$; H_0(Sample randomly drawn); see Ex. 14:37

79. No; $z = -0.32>-1.645$; $H_0(\mu = 15.6)$, $H_1(\mu < 15.6)$; see Ex. 14:37

80. Yes; $|z| = 1.16<1.96$; $H_0(\mu_1 = \mu_2)$, $H_1(\mu_1 \neq \mu_2)$; see Ex. 14:39. Confidence limits for $\mu_1-\mu_2$: −0.77 & 2.97.

81. Yes, larger; $z = 2.41>2.33$; $H_0(\mu_1 = \mu_2)$, $H_1(\mu_1 > \mu_2)$; see Ex. 14:39. Confidence limits for $\mu_1-\mu_2$: −0.19 & 5.59.

82. Yes; Student's t-test; $|t| = 2.25<2.26$; $H_0(\mu = 14.5)$, $H_1(\mu\neq 14.5)$; see Ex. 14:40. Confidence limits for μ : 9.69 & 14.51.

83. No; Student's t-test, $|t| = 0.47<2.262$; $H_0(\mu=2.00)$, $H_1(\mu \neq 2.00)$; Assumption : Normal population. See Ex. 14:40

84. Yes; Student's t-test, $t = 2>1.83$; $H_0(\mu=64)$, $H_1(\mu>64)$; $\bar{x} = 66$, $S = 3$; see Ex. 14:41 & 42.

85. No; Student's t-test, $t = 2.32<2.82$; $H_0(\mu = 100)$, $H_1(\mu>100)$; see Ex. 14:42

86. Yes; Student's t-test, $t=-2.16 <-1.83$; $H_0(\mu=65)$; $H_1(\mu<65)$; see Ex. 14:40 to 42.

87. (a) (i) For large samples use approximate z-test (*14.7.9*); (ii) For normal populations with *known s.d.*, use exact z-test (*14.8.4*); (iii) For normal populations with *unknown s.d.* (but assumed equal), use Fisher's t-test (*14.8.9*).
(b) No; Fisher's t-test, $|t| = 1.67 < 2.26$; $H_0(\mu_1 = \mu_2)$, $H_1(\mu_1 \neq \mu_2)$; see Ex. 14:43

88. Yes; Fisher's t-test, $t = 4.99>2.68$; $H_0(\mu_1=\mu_2)$, $H_1(\mu_1>\mu_2)$; see Ex. 14:43 & 44; $s^2 = (20.8+15.7)/12$; Confidence limits for $\mu_1-\mu_2$: 1.82 & 7.58

89. No; Fisher's t-test, $t= -0.57>-1.76$; $H_0(\mu_1=\mu_2)$, $H_1(\mu_1<\mu_2)$ $n_1 = 6$, $\bar{x}_1 = 68$, $S_1^2 = 10$, $n_2 = 10$, $\bar{x}_2 = 69$, $S_2^2 = 10$; see Ex. 14 : 44

90. No; Paired t-test, $t = 2.04<2.82$; $H_0(\mu_x = \mu_y)$, $H_1(\mu_x > \mu_y)$; (x = final score, $d = x-y$, $\Sigma d = 47$, $\Sigma d^2 = 699$); see Ex. 14:45A

91. No ; Paired t-test, $|t| = 1.27<2.26$; $H_0(\mu_x = \mu_y)$, $H_1(\mu_x \neq \mu_y)$, ($\Sigma d = 6, \Sigma(d-\bar{d})^2 = 20$); see Ex. 14:45

92. Yes, it raises; Paired t-test, $t = 1.51>1.397$ (8 d.f.); $H_0(\mu_x = \mu_y)$, $H_1(\mu_x > \mu_y)$; (x = final score); see Ex. 14:45A

93. Statistic : see (*14.8.11*); Reason : see (*13.8.12*)

94. see (*14.5.10*) & (*14.5.11*)

95. σ may be 8; χ^2-test, $\chi^2 = 7.81 <30.14$; $H_0(\sigma = 8)$, $H_1(\sigma > 8)$; see Ex. 14:46–47

96. Yes; χ^2-test, $\chi^2 = 16.7 < 19.68$; use $nS^2 = (n-1)s^2$; $H_0(\sigma^2 = 7)$, $H_1(\sigma^2 > 7)$; see Ex. 14 : 46-47; Confidence limits for σ : 2.31 & 5.53; see Ex. 14 : 18

97. (a) Student's t-test : (i) In *single sample problem* of testing for a specified mean $H_0(\mu = \mu_0)$, the sample must be drawn *randomly* from *Normal* population, with *unknown s.d.*; see (*14.8.6*). (ii) In *two sample* problem of testing for equality of two population means, i.e. $H_0(\mu_1 = \mu_2)$, the two samples must be *indedependent*, and *randomly* drawn from two *Normal populations* with *equal s.d., but unknown;* see (*14.8.9*)

    (b) F-test : (i) Not applicable in *single sample* problems; (ii) In *two sample* problem of testing for equality of two standard deviations $H_0(\sigma_1 = \sigma_2)$, the two samples must be *independent* and drawn *randomly* from *Normal populations* (any or both the means may be known or unknown); see (*14.8.12*) & (*13.8.21* to *24*)

98. Yes; F-test, $F = 1.8<3.23$; $H_0(\sigma_1 = \sigma_2)$, $H_1(\sigma_1 > \sigma_2)$; see Ex. 14:48–50

99. Yes; t and F-tests, $F = 1.75<3.14$; $H_0(\sigma_1 = \sigma_2)$, $H_1(\sigma_1 > \sigma_2)$; see Ex. 14:52. Use $S^2 = \Sigma x^2/n - (\Sigma x/n)^2$ and $(n-1)s^2 = nS^2$. Also $|t| = 0.28<2.11$; $H_0(\mu_1 = \mu_2)$, $H_1(\mu_1 \neq \mu_2)$.

100. 0.16 & 0.28; see Ex. 14:53

101. No; $|t| = 2.31>2.06$; $H_0(\rho = 0)$, $H_1(\rho \neq 0)$; see Ex. 14:54

102. $|r| = 0.487$; see Ex. 14:55

103. see (*14.9.4*)

104. Yes; Using Fisher's Z-transformation, calculate (*14.9.6*): $|U| = 1.92<1.96$; $H_0(\rho = 0.55)$, $H_1(\rho \neq 0.55)$; see Ex. 14:56

105. No; Using Fisher's Z-transformation, calculate (*14.9.6*) : $|U| = 0.43<1.96$; $H_0(\rho_1 = \rho_2)$, $H_1(\rho_1 \neq \rho_2)$; see Ex. 14:58