STAT-2201 Question and Answer

1a: What is sampling distribution, and why is it important in statistics?

A sampling distribution is the probability distribution of a statistic calculated from multiple sample taken from the same population. It shows how a statistic varies from sample to sample.

Importance in statistics:

1. Estimation of population parameters- Helps in estimating population characteristics (e.g. mean, variance) using sample data.
2. Basis for hypothesis testing- used to determine if sample results are statistically significant.
3. Standard error calculation- helps measure how much a sample statistic is expected to vary from the true population parameter.
4. Central limit Theorem Application- ensures that the sampling distribution of the mean is approximately normal, even if the population is not normal

In short, sampling distribution is important for making inferences about a population using sample data.

Q2: Difference between a sampling distribution and a parent distribution

sampling distribution	parent distribution
The distribution of a sample statistic	The distribution of all individual values ~~in the entire p~~ data in the population
Based on repeated samples of a fixed size from the population	Includes all elements of the population
shows how sample statistics vary from sample to sample	shows the natural variation of data in the population
often ~~normal~~ approaches normal distribution	can be normal skewed uniform etc
Helps estimate population parameters using sample data	Represents the actual data in the population.
Heights ~~of at~~ from multiple sample of 30 students each	Heights ~~of~~ all students in a school.

In short, sampling distribution is important for making inferences about a population using sample data.

Q2: Difference between a sampling distribution and a parent distribution

Sampling distribution	Parent distribution
The distribution of a sample statistic	The distribution of all individual values in the ~~entire p~~ data in the population
Based on repeated samples of a fixed size from the population	Includes all elements of the population
Shows how sample statistics vary from sample to sample	Shows the natural variation of data in the population
Often ~~normal~~ approaches normal distribution	Can be normal skewed uniform etc
Helps estimate population parameters using sample data	Represents the actual data in the population.
Heights ~~of at~~ from multiple sample of 20 students each	Heights of all students in a school.

1.C⇒ Define χ^2-distribution and explain how it is derived from standard normal variables.

Answers

The sum of squares of n independent standard normal distribution ~~variables~~ is called Chi-square (χ^2) distribution with n degree of freedom.

Let, $z_1, z_2, \ldots, z_n$ be n independent standard normal distribution. The chi-square denoted by χ^2, is defined as

$$\chi^2_n = \sum_{i=1}^{n} z_i^2$$

However, if $x_1, x_2, \ldots, x_n$ are n independently and identically distributed random variables each of which is normally distributed with mean μ and variance σ^2. Then

$$\chi^2_n = \sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma}\right)^2$$

is a chi-square distribution with n degree of freedom

1.d) What are the properties of the chi-square distribution? List at least five.

Answer:

i) χ^2 is a continuous type of distribution and it's range is 0 to ∞

ii) The distribution contains only one parameter which is the degree of freedom of the distribution.

iii) The mean and variance of χ^2-distribution for n d.f. is n and 2n respectively

iv) The mode-χ^2 distribution for n. d.f is $(n-2)$

v) The moment generating function of χ^2-distribution for n. d.f is $(1-2t)^{-n/2}$

vi) χ^2-distribution tends to normal distribution for large degree of freedom.

vii) It is positively skewed distribution for smaller values of n.

viii) The distribution becomes symmetrical as n tends to infinity $(n \to \infty)$

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1.e) How does the chi-square distribution behave as the degrees of freedom increase?

Answer:

As the df increase, the χ^2 distribution changes in shape and behavior.

when df is small, the distribution is highly skewed to the right. but as df increases, it becomes more symmetric and approaches a normal distribution.

The mean and variance of the distribution also depend on df. Then mean = df and variance = 2×df so The peak shifts right and the spread widens. When df is 30 or more it closely follows a normal distribution. where the ~~mean~~ standard deviation is $\sqrt{2df}$.

1.f)) How is the chi-square distribution used to test the goodness of fit.

Answer: The chi-square (χ^2) distribution is commonly used in goodness-of-fit-tests to determine how well observed data fit an expected distribution. Here's how it's used:

1) Hypothesis setup:

- Null hypothesis (H_0): The observed data fits the the expected distribution.
- Alternative hypothesis (H_1): The observed data does not fit the expected distribution.

2) The test statistic for the goodnes-of-fit test is calculated using the formula:

$$\chi_A^2 = \sum \frac{(O_i - E_i)^2}{E_i}$$

O_i = observed frequency

E_i = Expected "

3) <u>Decision:</u> Compare the calculated chi-square value to the critical value from the table.

- If the calculated value is larger than the critical value, reject the null hypothesis.
- If it's smaller do not reject the null hypothesis.

This test tells us if the observed data matches the expected data.

1.g) Derive the moment generating function (M.G.F) of the chi-square distribution.

Answer:

Let $x_1, x_2 \dots x_n$ be n independent random variables from $N(\mu, \sigma^2)$ i.e $x_i \sim N(\mu, \sigma^2)$, $i = 1, 2 \dots n$ x_i are independent

Now we want to find the distribution of $\chi^2 = \Sigma y_i = \Sigma \left(\frac{x_i - \mu}{\sigma}\right)^2$ by m.g.f technique.

Hence the m.g.f of χ^2 is given by $m_{\chi^2}(t) = m_{\Sigma y_i}(t)$

$$= \prod_{i=1}^{n} m_{y_i}(t)$$

$$\Rightarrow m_{\chi^2}(t) = \prod_{i=1}^{n} \left[m \left(\frac{x_i - \mu}{\sigma}\right)^2 t \right]$$

$$= \prod_{i=1}^{n} E\left[e^{t\left(\frac{x_i - \mu}{\sigma}\right)^2} \right]$$

$$= \prod_{i=1}^{n} E\left[e^{t u^2} \right] \quad \left[\text{let } u = \left(\frac{x_i - \mu}{\sigma}\right)\right]$$

$$= \prod_{i=1}^{n} \left\{ \int_{-\infty}^{\infty} e^{t u^2} f(u) \cdot du \right\}$$

$$= \prod_{i=1}^{n} \left\{ \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t u^2} \frac{1}{\sqrt{2\pi}} \cdot e^{-1/2 u^2} du \right\}$$

$$= \prod_{i=1}^{n} \left\{ \frac{2}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} e^{t u^2 - 1/2 u^2} \cdot du \right\}$$

since the integrand is an even function of u]

$$= \prod_{i=1}^{n} \left\{ \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty e^{-(1/2 - t)u^2} \cdot du \right\}$$

let $z = \left(\frac{1}{2} - t\right)u^2 \Rightarrow u^2 = \frac{z}{\frac{1-2t}{2}} \Rightarrow u = \sqrt{\frac{2z}{1-2t}}$

$dz = \left(\frac{1}{2} - t\right) 2u \cdot du$

$$\Rightarrow du = \frac{dz}{2u\left(\frac{1-2t}{2}\right)} = \frac{dz}{(1-2t)u}$$

$$\Rightarrow du = \frac{dz}{(1-2t)\sqrt{\frac{2z}{(1-2t)}}} = \frac{dz}{\sqrt{1-2t}\sqrt{2z}}$$

$$\therefore M_{x^2}(t) = \prod_{i=1}^{n} \left\{ \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty e^{-z} \cdot \frac{dz}{\sqrt{1-2t} \cdot \sqrt{2z}} \right\}$$

$$= \prod_{i=1}^{n} \left\{ \frac{\sqrt{2}}{\sqrt{\pi}\sqrt{1-2t}\sqrt{2}} \int_0^\infty e^{-z} \cdot z^{-1/2} \cdot dz \right\}$$

$$= \prod_{i=1}^{n} \left\{ \frac{1}{\sqrt{\pi}\sqrt{1-2t}} \int_0^\infty e^{-z} \cdot z^{1/2 - 1} \cdot dz \right\}$$

$$= \prod_{i=1}^{n} \left\{ \frac{1}{\sqrt{\pi}\sqrt{1-2t}} \Gamma 1/2 \right. \qquad \left[\because \Gamma n = \int_0^\infty e^{-k} u^{n-1} \cdot du \right]$$

$$= \prod_{i=1}^{n} \left\{ (1-2t)^{-1/2} \right\} = \prod_{i=1}^{n} \left\{ \frac{1}{\sqrt{\pi}\sqrt{1-2t}} \sqrt{\pi} \right\}$$

$$= \prod_{i=1}^{n} \left\{ (1-2t)^{-1/2} \right\} = (1-2t)^{-n/2}$$

$$\therefore M_{x^2}(t) = (1-2t)^{-n/2}$$

1.h) Using the MGF, find the mean and variance of the Chi-square distribution

$\Rightarrow$

The pdf of χ^2 distribution with n-degree of freedom is given by -

$$f(\chi^2) = \frac{1}{2^{n/2}\Gamma\frac{n}{2}} (\chi^2)^{n/2-1} e^{-\chi^2/2} \quad ; 0 < \chi^2 < \infty$$

<u>mean</u>

$$E(\chi^2) = \int_0^\infty \chi^2 \cdot f(\chi^2) \cdot d\chi^2$$

$$= \int_0^\infty \chi^2 \cdot \frac{1}{2^{n/2}\Gamma\frac{n}{2}} (\chi^2)^{n/2-1} e^{-\chi^2/2} \cdot d\chi^2$$

$$= \frac{1}{2^{n/2}\Gamma\frac{n}{2}} \int_0^\infty (\chi^2)^{(n/2+1)-1} e^{-1/2\chi^2} \cdot d\chi^2$$

$$= \frac{1}{2^{n/2}\Gamma\frac{n}{2}} \frac{\Gamma\frac{n}{2}+1}{\left(\frac{1}{2}\right)^{n/2+1}} \qquad \left[\because \frac{\Gamma n}{\alpha^n} = \int_0^\infty x^{n-1} e^{-\alpha x} \cdot dx\right]$$

$$E(\chi^2) = \frac{1}{2^{n/2}\Gamma\frac{n}{2}} \cdot \frac{n}{2} \cdot \Gamma\frac{n}{2} \cdot 2^{(n/2+1)}$$

$$= \frac{2n}{2} = n \quad \text{mean} = n$$

<u>Variance</u>

$$E[(\chi^2)^2] = \int_0^\infty (\chi^2)^2 f(\chi^2) \cdot d\chi^2$$

$$= \int_0^\infty (\chi^2)^2 \frac{1}{2^{n/2}\Gamma\frac{n}{2}} (\chi^2)^{n/2-1} \cdot e^{-1/2\chi^2} \cdot d\chi^2$$

$$= \frac{1}{2^{n/2}\Gamma\frac{n}{2}} \int_0^\infty (\chi^2)^{n/2+2-1} e^{-1/2\,\chi^2}\, d\chi^2$$

$$= \frac{1}{2^{n/2}\Gamma\frac{n}{2}} \frac{\Gamma\left(\frac{n}{2}+1\right)}{1/2\,\Gamma(n/2+2)}$$

$$= \frac{1}{2^{n/2}\Gamma\frac{n}{2}} \cdot \frac{\left(\frac{n}{2}+1\right)\frac{n}{2}\Gamma\frac{n}{2}}{\frac{1}{2}\Gamma(n/2+2)}$$

$$= \frac{1}{2^{n/2}\Gamma\frac{n}{2}} \cdot \left(\frac{n}{2}+1\right) n/2\, \Gamma\frac{n}{2} \cdot 2^{n/2} \cdot 2^2$$

$$= 4 \cdot \frac{n}{2}(n/2+1) = 2n(n/2+1) = n^2+2n$$

$$\cancel{= n^2}$$

$$V(\chi^2) = E[(\chi^2)^2] - [E(\chi^2)]^2$$

$$= n^2+2n-n^2$$

$$= 2n$$

$$V(\chi^2) = 2n.$$

Therefore the mean and variance of χ^2 distribution is n and $2n$ respectively.

2.b) What are the key properties of the t- distribution

Answer:

i) t- distribution is an even function

ii) t- distribution is symmetric about $t=0$

iii) mean = median = mode $=0$

iv) variance of the distribution is $\frac{n}{n-2}$; $n>2$

v) The total probability of t- density is equal to 1

i.e $\int_{-\infty}^{\infty} f(t) \cdot d(t) = 1$

vi) For large n t-distribution reduces to standard normal distribution.

vii) It is a continuous type of distribution and it's range extends from $-\infty$ to ∞

viii) mgf of t- distribution does not exist.

2.c) How is the t-distribution used to test the significance of a sample mean

Answer: The t- distribution is used to test the significance of a sample mean when the population standard deviation (σ) is

unknown and the sample size is small. The test is called one-sample t-test and follows these steps.

1) set up Hypotheses

Null Hypothesis: The sample mean is equal to the population mean μ_0 $H_0 : \mu = \mu_0$

H_1 : sample mean $\neq \mu_0$ (Two tailed test)

~~one~~ Left-tailed test $H_1 : \mu < \mu_0$

Right " " $H_1 : \mu > \mu_0$

2) The t-statistic is calculated using the formula

$$t = \frac{\bar{X} - \mu_0}{\frac{s}{\sqrt{n}}}$$

3) Find the critical value from the t-table using df. ($df = n-1$) and the chosen significance level (α, usually 0.05 or 0.01). Alternatively calculated P-value

4) If $|t_{calculated}| >$ critical or p value $< \alpha$, reject H_0 otherwise do not reject H_0

2.d) Derive the first four moments of the t-distribution

⇒ First four moments are

~~Mean~~

i) mean iii) skewness

ii) variance iv) kurtosis

Mean

We know that, the pdf of t-distribution is

$$f(t) = \frac{1}{\sqrt{n}\, B\left(\frac{1}{2}, n/2\right)\left(1+t^2/n\right)^{\frac{n+1}{2}}} \quad ; \quad -\infty < t < \infty$$

We know

$$E(t) = \int_{-\infty}^{\infty} t \cdot f(t) \cdot dt$$

$$= \int_{-\infty}^{\infty} t \cdot \frac{1}{\sqrt{n}\, B\left(\frac{1}{2}, \frac{n}{2}\right)\left(1+t^2/n\right)^{\frac{n+1}{2}}}\, dt$$

$$E(t) = \frac{1}{\sqrt{n}\, B(1/2, n/2)} \int_{-\infty}^{\infty} \frac{t}{\left(1+t^2/n\right)^{\frac{n+1}{2}}}\, dt$$

$$= \frac{1}{\sqrt{n} \cdot B(1/2, n/2)} \cdot 0$$

[since the integrand is an odd fn of t]

$$= 0$$

$\therefore$ mean $= E(t) = 0$

Variance

$$E(t^2) = \int_{-\infty}^{\infty} t^2 f(t)\, dt$$

$$= \int_{-\infty}^{\infty} \frac{t^2}{\sqrt{n}\, B(\frac{1}{2}, n/2) \left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}\, dt$$

let $w = \frac{t^2}{n}$ $\therefore t = \sqrt{nw}$

$t^2 = nw$

$2t\, dt = n \cdot dw \Rightarrow dt = \frac{n}{2t} dw = \frac{n}{2\sqrt{nw}} dw = \frac{\sqrt{n}}{2\sqrt{w}} dw$

$\therefore dt = \frac{\sqrt{n}}{2\sqrt{w}} dw$ when $t = -\infty$ then $w = -\infty$

" $t = \infty$ " $w = \infty$

$$\therefore E(t^2) = \int_{-\infty}^{\infty} \frac{nw}{\sqrt{n}\, B(1/2, n/2) (1+w)^{\frac{n+1}{2}}} \frac{\sqrt{n}}{2\sqrt{w}} \cdot dw$$

$$= \cancel{2} \int_{-\infty}^{\infty} \frac{nw}{B(1/2, n/2)} \cdot \frac{w^{-1/2}}{\cancel{2} (1+w)^{\frac{n+1}{2}}} \cdot dw$$

[Since the integrand is an even fn of w]

$$E(t^2) = \frac{n}{B(1/2, n/2)} \int_0^{\infty} \frac{w^{-1/2+1}}{(1+w)^{\frac{n+1}{2}}} dw$$

$$= \frac{n}{B(1/2, n/2)} \int_0^{\infty} \frac{w^{3/2-1}}{(1+w)^{3/2+\frac{n-2}{2}}} dw$$

$\left[\frac{n+1}{2} = \frac{3}{2} + \frac{n-2}{2} dw \quad n/2 = \frac{3}{2} - 1\right.$

$$= \frac{n}{B(1/2, n/2)} B\left(3/2, \frac{n-2}{2}\right)$$

$\left[\because B(l, m) = \int_0^{\infty} \frac{x^{l-1}}{(1+x)^{l+m}} d\right.$

$E(t^2) = \cancel{n}$

$$= \frac{n \frac{\Gamma\frac{3}{2} \Gamma\frac{n-2}{2}}{\Gamma\left(\frac{3}{2}+\frac{n-2}{2}\right)}}{\Gamma 1/2\, \Gamma n/2 \,/\, \Gamma\frac{n+1}{2}} = \frac{n \cdot \Gamma 3/2\, \Gamma\frac{n-2}{2} \,/\, \Gamma n+1/2}{\Gamma 1/2\, \Gamma n/2 \,/\, \Gamma\frac{n+1}{2}}$$

$$E(t^2) = \frac{n \cdot \Gamma(3/2)\,\Gamma\left(\frac{n-2}{2}\right)}{\Gamma(1/2)\,\Gamma(n/2)} = \frac{n \cdot \frac{1}{2}\Gamma(1/2)\,\Gamma(n/2-1)}{\Gamma(1/2)\,(n/2-1)\,\Gamma(n/2-1)}$$

$$= \frac{n}{2} \cdot \frac{2}{n-2} = \frac{n}{n-2}$$

$$E(t^2) = \frac{n}{n-2}$$

$$V(t) = E(t^2) - [E(t)]^2 = \frac{n}{n-2} - 0 = \frac{n}{n-2}, n>2$$

~~Then~~ The variance is $\frac{n}{n-2}$

Skewness and kurtosis

If $x \sim t_n$ then the raw moment of t- distribution is defined as $\mu'_r = E(x^r)$, $r = 1, 2, 3, 4 \ldots$

Case 1: odd ordered raw moments

$$\mu'_{2r+1} = \int_{-\infty}^{\infty} t^{2r+1} f(t)\, dt$$

$$= \int_{-\infty}^{\infty} \frac{t^{2r+1}}{\sqrt{n}\, B(1/2, n/2) \underbrace{\left(1 + \frac{t^2}{n}\right)^{\frac{n+1}{2}}}_{\text{pdf of t distribution}}}\, dt$$

$$= 0$$

since the integrand is an odd fn of t and $(2r+1)$ is an odd number

$\mu'_{2r+1} = \mu_{2r+1} = 0$

Now putting $r = 0, 1, 2, \ldots$ we have

$\mu_1' = 0, \ \mu_3' = 0 \ \ldots \ \mu'_{2r+1} = 0$

<u>Case-2:</u> Even order moments:

$$\mu'_{2r} = E(t^{2r}), \quad r = 1, 2 \ldots$$

$$= \int_{-\infty}^{\infty} t^{2r} \cdot f(t)\, dt$$

$$= 2\int_{0}^{\infty} \frac{t^{2r}}{\sqrt{n}\, B(1/2, n/2)\left(1 + t^2/n\right)^{\frac{n+1}{2}}} \cdot dt$$

since the integrand is an even function of t

let $w = \frac{t^2}{n}$, $\therefore t = \sqrt{wn}$

$\Rightarrow t^2 = wn$

$\Rightarrow 2t\, dt = n\, dw$

$\Rightarrow dt = \frac{n}{2t} dw \Rightarrow dt = \frac{n}{2\sqrt{wn}} dw$

when $t = 0$ then $w = 0$

when $t = \infty$ then $w = \infty$

Now,

$$\mu'_{2r} = 2\int_{0}^{\infty} \frac{(\sqrt{wn})^{2r}}{\sqrt{n}\, B\left(\frac{1}{2}, \frac{n}{2}\right)(1+w)^{\frac{n+1}{2}}} \cdot \frac{n}{2\sqrt{wn}}\, dw$$

$$= \int_{0}^{\infty} \frac{w^r n^r}{B(1/2, n/2)(1+w)^{\frac{n+1}{2}}} \cdot w^{-1/2}\, dw$$

$$= \frac{n^r}{B(1/2, n/2)} \int_0^\infty \frac{w^{r+1/2-1}}{(1+w)^{(n+1/2)+(n/2-r)}} \, dw$$

$$= \frac{n^r}{B(1/2, n/2)} B\left(r+\frac{1}{2}, \frac{n}{2}-r\right) \quad \left[\because \beta(l, m) = \int_0^\infty \frac{x^{l-1}}{(1+x)^{l+m}} dx\right]$$

$$\therefore \mu_{2r}' = \frac{n^r}{B\left(\frac{1}{2}, \frac{n}{2}\right)} B\left(r+\frac{1}{2}, \frac{n}{2}-r\right)$$

$$= n^r \frac{\Gamma\left(r+\frac{1}{2}\right)\Gamma\left(\frac{n}{2}-r\right) / \Gamma\left(r+\frac{1}{2}+\frac{n}{2}-r\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right) / \Gamma\left(\frac{1}{2}+\frac{n}{2}\right)}$$

$$= n^r \frac{\Gamma\left(r+\frac{1}{2}\right)\Gamma\left(\frac{n}{2}-r\right)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)} \times \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)}$$

$$\therefore \mu_{2r}' = \frac{n^r \Gamma\left(r+\frac{1}{2}\right)\Gamma\left(\frac{n}{2}-r\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)}$$

Putting $r = 1, 2$ we get

$$\mu_2' = \frac{n\,\Gamma\left(1+\frac{1}{2}\right)\Gamma\left(\frac{n}{2}-1\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)} = \frac{n\,\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{n}{2}-1\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)}$$

$$= \frac{n \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}-1\right)}{\Gamma\left(\frac{1}{2}\right)\left(\frac{n}{2}-1\right)\Gamma\left(\frac{n}{2}-1\right)} = \frac{n}{2} \cdot \frac{2}{n-2} = \frac{n}{n-2}$$

$$\therefore \mu_2' = \frac{n}{n-2}$$

and $\mu_4' = \dfrac{n^2\,\Gamma\left(2+\frac{7}{2}\right)\Gamma\left(\frac{n}{2}-2\right)}{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{n}{2}\right)} = \dfrac{n^2\,\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{n}{2}-2\right)}{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{n}{2}\right)}$

$$= \frac{n^2 \cdot \frac{3}{2} \cdot \frac{1}{2}\cancel{\Gamma\left(\frac{1}{2}\right)} \cdot \Gamma\left(\frac{n}{2}-2\right)}{\cancel{\Gamma\left(\frac{1}{2}\right)}\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)\Gamma\left(\frac{n}{2}-2\right)}$$

$$= \frac{\frac{3n^2}{4}}{\left(\frac{n-2}{2}\right)\left(\frac{n-4}{2}\right)} = \frac{3n^2}{\cancel{4}} \times \frac{\cancel{4}}{(n-2)(n-4)}$$

$$\therefore \mu_4' = \frac{3n^2}{(n-2)(n-4)}$$

Central moments:

If $X \sim t_n$ then the central moment of t-distribution is defined as $\mu_r = E[X-EX]^r$

$\mu_1 = 0$

$\mu_2 = \mu_2' - (\mu_1')^2 = \frac{n}{n-2} - 0 \Rightarrow \mu_2 = \text{variance} = \frac{n}{n-2}$

$\mu_3' = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

$= 0 - 3\left(\frac{n}{n-2}\right) \cdot 0 + 2 \cdot 0^3 = 0$

$\therefore \mu_3' = 0$

$\mu_4' = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$

$= \frac{3n^2}{(n-2)(n-4)} - 0 + 0 - 0$

$\mu_4' = \frac{3n^2}{(n-2)(n-4)}$

$\therefore$ Skewness $= \beta_1 = \dfrac{\mu_3^2}{\mu_2^3} = \dfrac{0}{\left(\frac{n}{n-2}\right)^3} = 0$

Kurtosis $\beta_2 = \dfrac{\mu_4}{\mu_2^2} = \dfrac{\frac{3n^2}{(n-2)(n-4)}}{\left(\frac{n}{n-2}\right)^2}$

$$= \frac{3n^2}{(n-2)(n-4)} \times \frac{(n-2)^2}{n^2}$$

$$= \frac{3(n-2)}{(n-4)} = \frac{3n-6}{n-4} = \frac{3n-12+6}{n-4}$$

$$= \frac{3(n-4)}{n-4} + \frac{6}{n-4} = 3 + \frac{6}{n-4} > 3$$

Comment. since $\beta_1 = 0$ and $\beta_2 = 3 + \frac{6}{n-4}$ then the distribution is symmetric and leptokurtic

2.e)) Define F- distribution ~~used to~~ and explain how it is derived from two independent chi-square variables.

Answer:

The F-distribution is the distribution of the ratio of two independent chi-square random variables divided by their respective degrees of freedom.

If χ_1^2 and χ_2^2 are two independent chi-square distributed variables with degrees of freedom n_1 and n_2 Then the F-statistic is defined as

$$F = \frac{\chi_1^2 / n_1}{\chi_2^2 / n_2}$$

where n_1 and n_2 are ~~two~~ degrees of freedom

It mathmatically $F \sim F(n_1, n_2)$

The denoity function of F is

$$f(F) = \frac{\frac{n_1}{n_2}\left(\frac{n_1}{n_2}F\right)^{n_1/2 - 1}}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1 + \frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}}, \quad F > 0$$

$$\therefore f(F) = \frac{\left(\frac{n_1}{n_2}\right)^{n_1/2} F^{n_1/2 - 1}}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1 + \frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}}$$

→ Proof pdf খাতায় করতে হবে।

24) How is the F-distribution used to test the equality of population variances?

The F-Test is used to compare the variances of two populations and determine whether they are significantly different.

1) Hypotheses for the F-Test

Null hypotheses: The population variances are equal

$$H_0: \sigma_1^2 = \sigma_2^2$$

Alternative hypothesis: $H_1: \sigma_1^2 \neq \sigma_2^2$

2) The F-statistic is calculated as

$$F = \frac{s_1^2}{s_2^2}$$

where s_1^2 = sample variance of population 1

s_2^2 = " " " " 2

The larger variance if numerator is $F \geq 1$

degree of freedom $d_1 = n_1 - 1$ and $d_2 = n_2 - 1$

3) Compare F-value with the critical value from the F-table at a given significance level α

i) If F is greater than the critical value, we reject H_0

ii) If F is less than or equal to the critical value, we fail to reject H_0

Example:

$s_1^2 = 25$ and $s_2^2 = 16$, sample size $n_1 = 11$, $n_2 = 9$

Degrees of freedom: $d_1 = 10$, $d_2 = 8$

Then $F = \frac{25}{16} = 1.5625$, $\alpha = 0.05$

Here $F > \alpha$

Hence Null hypothesis reject.

29) Explain how the t distribution converges to a normal distribution as the degress of freedom increase.

Answere

The t-distribution is a modified version of the normal distribution, mainly used for small samples. When the degrees of freedom are low, the t-distribution has heavier tails, meaning there is a higher chance of getting extreme values. This happens because in small samples, the sample standard deviation varies a lot and does not accurately estimate the true population standard deviation.

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As df increases, the sample standard deviation s becomes a better estimate of the population standard deviation (σ) reduce in the the extra variability.

Convergence to Normal distribution: As $df \to \infty$, the shape of the t-distribution becomes indentical to the standard normal distribution

when $df \geq 30$, the diffence between the t-distribution and the normal distribution becomes negligible

At last we can say that as df increases, the t-distribution gradually becomes more like the normal distribution, because the sample standard deviation becomes a better estimate of the population standard deviation.

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2.h) Maths based on T-distribution
2.i) Maths based on F-distribution

3

3.a)) Define: random sample, population, sample parameter, statistic.

Answer:

Random sample: A subset of a statistical population in which each member of the subset has an equal probability of being chosen.

Population: The collection of all units of a specific type in a given region at a particular time is termenate as a population or universe.

Example: A population of past students, a population of books in a library a population of tree of a certain region.

Sample: A sample is a representative part of the population that is taken and considered for study.

Parameter: A parameter is an index associated with population. Any numerical value describing a characteristic of a population is called a parameter.

example: population mean μ, variance σ^2 etc are population parameter

Statistic: A statistic is a characteristic of a sample. Any function of sample observations or items is called statistic. usually denoted by small english alphabet.

example: population ~~mean~~ p, sample variance s^2
sample mean p etc

3.b)) Explain the criteria of good estimator

Answer: পূর্বে লিখা আছে

3.c)) State and prove the Cramer-Rao Lower Bound theorem

Answer: আগে লেখা আছে

3.d)) Find the minimum variance bound unbiased estimator.

Answer:

From Cramer Rao inequality, we know

$$V(t) \geq \frac{[1+\gamma'(\theta)]}{E\left(\frac{\partial \log L}{\partial \theta}\right)^2}$$

or $$V(t) \geq \frac{[1+\gamma'(\theta)]^2}{V\left[\frac{\partial \log L}{\partial \theta}\right]}$$

or $$V(t) = \frac{[1+\gamma'(\theta)]^2}{V(\varphi)} \qquad \varphi = \frac{\partial \log L}{\partial \theta}$$

The condition for equality we have

$$[1+\gamma'(\theta)]^2 = [\mathrm{Cov}(t, \varphi)]^2$$

Also, $[1+\gamma'(\theta)]^2 \leq V(t) \cdot V(\varphi)$

Now, $[\mathrm{Cov}(t, \varphi)]^2 = V(t) \cdot V(\varphi)$

This is satisfied when, $\rho^2_{t,\varphi} = 1$, where t and φ

$\rho_{t\varphi}$ is the correlation coefficient between t and φ

If $\rho_{t\varphi} = 1$, t and φ are linearly related and relationship is of the form $\varphi = At + B$ —— (1.1)

Taking expectation on both sides in (1.1) we have

$$E(\varphi) = A E(t) + B$$

$$0 = A \cdot K(t) + B \quad \text{—— (1.2)}$$

Subtracting (1.2) from (1.1) we have

$$\varphi = A[t - E(t)]$$

or $$\varphi^2 = A^2 [t - E(t)]^2$$

$= A^2 V(t)$

or $V(t) = \frac{E(Q^2)}{A^2}$ $\because E(Q) = 0$

$= \frac{V(Q)}{A^2}$

$\therefore V(t) = \frac{V(Q)}{A^2}$ ——— (2.3)

Also, we know $V(t) = \frac{1}{V(Q)}$ since t is MVB estimator

$\therefore V(Q) = \frac{1}{V(t)}$

substituting this in (2.3) we have

$V(t) = \frac{1}{A^2 V(t)}$

or $[V(t)]^2 = \frac{1}{A^2}$

or $V(t) = \frac{1}{A}$

$\therefore V(t) = \frac{1}{A}$

Thus A is the reciprocal of the variance of MVBUE of t.

(showed)

3e) For Bernouli distribution, find MVBUE of θ.

Answer:

The pmf of bernouli distribution is

$$f(x|\theta) = \theta^x (1-\theta)^{1-x}, \; x=0,1$$

we know,

$$L = \prod_{i=1}^{n} f(x_i|\theta)$$

$$= \theta^{\sum_{i=0}^{n} x_i} (1-\theta)^{\sum_{i=1}^{n}(1-x_i)}$$

$$\log L = \sum x_i \log\theta + \sum (1-x_i)\log(1-\theta)$$

$$\therefore \frac{\partial \log L}{\partial \theta} = \sum x_i \frac{1}{\theta} + \sum (1-x_i)\frac{1}{1-\theta}(-1)$$

$$= \frac{\sum x_i}{\theta} - \frac{\sum(1-x_i)}{1-\theta}$$

$$= \frac{\sum x_i (1-\theta) - \theta \sum (1-x_i)}{\theta(1-\theta)}$$

$$= \frac{\sum x_i - \theta \sum x_i - \theta . n + \theta \sum x_i}{\theta . (1-\theta)}$$

$$= \frac{\sum x_i - \theta n}{\theta(1-\theta)}$$

$$= \frac{n\bar{x} - n\theta}{\theta(1-\theta)}$$

$$= \frac{n(\bar{x}-\theta)}{\theta(1-\theta)}$$

$$= \frac{n}{\theta(1-\theta)}(\bar{x}-\theta)$$

which can be expressed as $A[(t-\theta)]$ where $A=$ and variance $v(t) = \frac{\theta(1-\theta)}{n}$

Therefore we can say that $\bar{x}$ is the MVBUE of θ with variance $\theta(1-\theta)/n$

3.f)) if $X \sim N(\mu, \sigma^2)$ then find the MVBUE of μ

Answer:

Given $X \sim N(\mu, \sigma^2)$ then the density function

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2}\{(x-\mu)/\sigma\}^2\right], -\infty < x < \infty$$

We know,

$$L = \prod_{i=1}^{n} f(x_i|\mu, \sigma^2)$$

$$= (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

Taking log on bothsides, we have

$$\log L = \log c - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 \quad [c = (2\pi\sigma^2)^{-n/2}$$

$$\cancel{= \log c - \frac{1}{2}}$$

Now taking derivative w.r.t μ, we have

$$\frac{\partial \log L}{\partial \mu} = 0 - \frac{2}{2\sigma^2}\sum(x_i-\mu)^{2-1}(-1)$$

$$= \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)$$

$$= \frac{1}{\sigma^2}(\sum x_i - n\mu)$$

$$= \frac{1}{\sigma^2}(n\bar{x} - n\mu)$$

$$= \frac{n}{\sigma^2}(\bar{x}-\mu)$$

which can be expressed as $\frac{\partial \log L}{\partial \theta} = A(t-\theta)$ where $A = \frac{n}{\sigma^2}$ and variance $V(t) = A^{-1} = \frac{\sigma^2}{n}$.

Therefore, we can say that $\bar{x}$ is the MVBUE of μ with variance σ^2/n.

3.a)) if $X \sim N(\mu, \sigma^2)$ then find the MVBUE of σ^2

<u>Answer:</u>

Given that,

X is an $N(\mu, \sigma^2)$ variate when μ is kno

3.h) Establish the method of finding MVB for an unbiased estimator intended to estimate function of a parameter.

Answer: suppose we have found an MVB unbiased estimator θ, This case, we use MVB of unbiased estimator of g a function of θ.

let, $E(t') = g(\theta)$

Now, MVB of $t = \dfrac{\left[\frac{\partial E(t)}{\partial \theta}\right]^2}{nE\left[\frac{\partial \log f}{\partial \theta}\right]^2}$

$$= \frac{1}{nE\left[\frac{\partial \log f}{\partial \theta}\right]^2}$$

MVB of $t' = \dfrac{\left[\frac{\partial E(t')}{\partial g(\theta)}\right]^2}{nE\left[\frac{\partial \log f}{\partial g(\theta)}\right]^2}$

$$= \frac{1}{nE\left[\frac{\partial \log f}{\partial g(\theta)}\right]^2}$$

$$= \frac{1}{E\left[\frac{\partial \log L}{\partial g(\theta)}\right]^2}$$

$$= \frac{1}{E\left[\frac{\partial \log L}{\partial \theta} \cdot \frac{\partial \theta}{\partial g(\theta)}\right]^2} = \frac{1}{E\left(\frac{\partial \log L}{\partial \theta}\right)^2 \left(\frac{\partial \theta}{\partial g(\theta)}\right)^2}$$

$$= \frac{1}{\left(\frac{\partial \theta}{\partial g(\theta)}\right)^2 E\left(\frac{\partial \log L}{\partial \theta}\right)^2} = \frac{1}{\left(\frac{\partial \theta}{\partial g(\theta)}\right)^2 \cdot E\left(\frac{\partial \log L}{\partial \theta}\right)^2}$$

$$= \frac{1}{\left(\frac{\partial \theta}{\partial g(\theta)}\right)^2 E\left(\frac{\partial \log L}{\partial \theta}\right)^2} = \frac{1}{\left(\frac{\partial \theta}{\partial g(\theta)}\right)^2 E\left(\frac{\partial \log L}{\partial \theta}\right)^2}$$

$$= \frac{1}{\left(\frac{\partial \theta}{\partial g(\theta)}\right)^2} \, MVB(t)$$

$$= MVB(t) \cdot \left(\frac{\partial g(\theta)}{\partial (\theta)}\right)^2$$

$$\therefore MVB \text{ of } t' = MVB(t) \cdot \left(\frac{\partial g(\theta)}{\partial \theta}\right)^2$$

(Showed)

3i) Define estimation, Estimator and estimate. Show that sample mean is unbiased estimate of the population mean.

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4.a) Define maximum likelihood estimator with an example. পূর্বে লিখা আছে

4.b) write down the properties of MLE. পূর্বে লিখা আছে

4.c) Is MLE always unbiased? Distinguish between joint density and likelihood function.

Answer:

MLE is not always unbiased

let $x_1, x_2 \ldots x_n$ be a random sample of size n drawn from the density $f(x, \theta)$ where θ is a parameter By definition of likelihood is

$$L = \prod_{i=1}^{n} f(x_i, \theta).$$

let t be the MVB estimator of θ, we must have

$$\frac{\partial \log L}{\partial \theta} = A(t-\theta), \text{ where } A = \frac{1}{V(t)} \quad \text{—— (1)}$$

we know the likelihood equation

$$\frac{\partial \log L}{\partial \theta} = 0 \quad \text{—— (II)}$$

eqn (1) and (2) we have $A(t-\theta) = 0$

Taking log on both sides, we have

$$\log L = \frac{n}{2} \log\left(\frac{1}{2\pi}\right) - \frac{n}{2} \log \sigma^2 - \frac{1}{2\sigma^2} \Sigma (x_i - \mu)^2$$

Now differentiating (1) w.r.t. μ and set equal to zero

$$\frac{\partial \log L}{\partial \mu} = 0$$

$$\Rightarrow -\frac{1}{2\sigma^2} \cdot 2 \sum (x_i - \mu)(-1) = 0$$

$$\Rightarrow \frac{1}{\sigma^2} \sum (x_i - \mu) = 0$$

$$\Rightarrow \sum x_i - n\mu = 0$$

$$\Rightarrow n\bar{x} - n\mu = 0$$

$$\therefore \hat{\mu} = \bar{x}$$

So $\bar{x}$ is the MLE of μ

Now we have

$$\mu = \bar{x}$$

$$E(\mu) = E(\bar{x})$$

$$= \frac{1}{n} E\left(\sum x_i\right)$$

$$= \frac{1}{n} \cdot n\mu$$

$$= \mu$$

Hence $\bar{x}$ is an unbiased estimate of μ

Now differentiating (1) w.r.t σ^2 and set $\sigma^2 = 0$

$$\frac{\partial \log L}{\partial \sigma^2} = 0$$

$$\Rightarrow -\frac{n}{2} \cdot \frac{1}{\sigma^2} + \frac{1}{2\sigma^4} \sum (x_i - \mu)^2 = 0$$

$$\Rightarrow -n\sigma^2 + \sum (x_i - \mu)^2 = 0$$

$$n\sigma^2 = \sum (x_i - \mu)^2$$

$$\Rightarrow \sigma^2 = \frac{1}{n} \sum (x_i - \mu)^2$$

$$= \frac{1}{n} \sum (x_i - \mu)^2$$

$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2$ is the MLE of σ^2

we have

$$E(\sigma^2) = \frac{1}{n}E\left[\sum_{i=1}^{n}(x_i-\mu)^2\right]$$

$$= \frac{\sigma^2}{n}E\left[\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2\right]$$

$$= \frac{\sigma^2}{n}E\left[\chi^2_{n-1}\right]$$

$$= \frac{\sigma^2}{n}(n-1) \neq \sigma^2$$

$$E(\sigma^2) = \frac{n-1}{n}\sigma^2 = \sigma^2$$

so. σ^2 is not unbiased estimate of σ^2

Hence we say MLE is not always unbiased.

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4.d) State and prove the invariance property

Answer

Statement: If $\hat{\theta}$ is the MLE of θ and $g(\theta)$ is a one to one function of θ Then MLE of $g(\theta)$ is $\hat{g}(\theta) = g(\hat{\theta})$

Proof:

Let the likelihood function is $L(\theta)$ and the reparameterized likelihood function $L^*(g(\theta))$:

The the relationship is $L(\theta) = L^*(g(\theta))$

Therefore $\frac{\partial L(\theta)}{\partial \theta} = \frac{\partial L^* g(\theta)}{\partial \theta} = \frac{\partial L^*(g(\theta))}{\partial g(\theta)} = \frac{\partial g(\theta)}{\partial(\theta)}$

Note that if $\frac{\partial g(\theta)}{\partial \theta} \neq 0$. Then

$$\frac{\partial L^*(g(\theta))}{\partial \theta} = 0 \text{ whenever } \frac{\partial L(\theta)}{\partial \theta} = 0$$

Since $\hat{\theta}$ is the MLE of θ then

$$\frac{\partial L(\theta)}{\partial \theta}\Big|_{\theta=\hat{\theta}} = 0 \text{ implies } \frac{\partial L^*(g(\theta))}{\partial g(\theta)}\Big|_{\theta=\hat{\theta}} = 0$$

which is the equation that comes up in finding an MLE in the reparameterized case

Again by definition $\hat{g}(\theta)$ is the solution to $\frac{\partial L^*(g(\theta))}{\partial g(\theta)} = 0$

Therefore, $\hat{g}(\theta) = g(\hat{\theta})$ (proved)

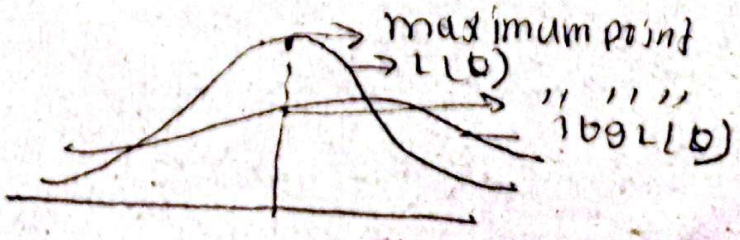
4.e)) Show that at the same point the value of $L(\theta)$ and $\log L(\theta)$ are equal

<u>Answere</u> By graphically, we know that the maximum value of θ would provide the maximum probability of $L(\theta)$. The likelihood function $L(\theta)$ and its logarithm $l(\theta) = \log L(\theta)$ are maximised for the same value of θ

because $L(\theta) > 0$ and $\log L(\theta)$ is a non-decreasing function or monotonic increasing fn of $L(\theta)$.

Therefore $L(\theta)$ and $\log L(\theta)$ attains their extreme (maximum or minimum) values for the same of θ.

So we can say, $L(\theta)$ and $\log L(\theta)$ posesses same maximum or minimum point



4.f)) State and prove the limiting distribution property of MLE.

Answer:

Statement: Let $\hat{\theta}$ is the MLE of θ. Then $\hat{\theta} \sim AN(\theta, \frac{1}{nI(\theta)})$

Proof:

Let $f(x;\theta)$ is the density function of x then

$$I(\theta) = E\left(\frac{\partial \ln f}{\partial \theta}\right) = -E\left(\frac{\partial^2 \ln f}{\partial \theta^2}\right)$$ is the fisher information for one observation

Then, $L(\theta) = \prod_{i=1}^{n} f(x_i, \theta)$ is the likelihood function of θ based on the sample $x_1, x_2, \ldots, x_n$.

Now, $\ln L(\theta) = \sum_{i=1}^{n} \ln f(x_i, \theta)$

$$\therefore \frac{\partial \ln L(\theta)}{\partial(\theta)} = \sum_{i=1}^{n} \frac{\partial \ln f(x_i, \theta)}{\partial(\theta)}$$ and

$$\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} = \sum_{i=1}^{n} \frac{\partial^2 \ln f(x_i, \theta)}{\partial \theta^2}$$

Note that,

$$E\left[\frac{\partial \ln f(x_i, \theta)}{\partial \theta}\right] = 0 = \text{mean}$$

and

$$\text{var}\left[\frac{\partial \ln f(x_i, \theta)}{\partial \theta}\right] = I(\theta)$$

Therefore, $E\left[\frac{\partial \ln L(\theta)}{\partial \theta}\right] = 0$ and $\text{var}\left[\frac{\partial \ln L(\theta)}{\partial \theta}\right] = nI(\theta)$

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i.e $z = \frac{\partial \ln L(\theta)}{\partial \theta} = \sum_{i=1}^{n} \frac{\partial \ln f(x_i, \theta)}{\partial \theta}$ has zero mean and variance $nI(\theta)$

If $\hat{\theta}$ is the MLE of θ, then $\left.\frac{\partial \ln L(\theta)}{\partial \theta}\right|_{\theta=\hat{\theta}} = 0$

which gives $\frac{\partial \ln L(\theta)}{\partial (\theta)} + (\hat{\theta}-\theta)\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} \approx 0$ keeping in the first two terms of the taylors series expansion of

$$\left.\frac{\partial \ln L(\theta)}{\partial \theta}\right|_{\theta=\hat{\theta}} = 0$$

$$\Rightarrow \frac{\partial \ln L(\theta)}{\partial \theta} + (\hat{\theta}-\theta)\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} = 0$$

$$\Rightarrow (\hat{\theta}-\theta)\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} = -\frac{\partial \ln L(\theta)}{\partial \theta}$$

$$\Rightarrow (\hat{\theta}-\theta) = -\frac{\partial \ln L(\theta)/\partial \theta}{\partial^2 \ln L(\theta)/\partial \theta^2}$$

$$\Rightarrow (\hat{\theta}-\theta) = \frac{\frac{\partial \ln L(\theta)}{\partial \theta}}{-\frac{\partial^2 \ln L(\theta)}{\partial \theta^2}}$$

$$\Rightarrow \frac{\hat{\theta}-\theta}{\sqrt{\frac{1}{n}I(\theta)}} = \frac{z}{-\frac{\partial^2 \ln L(\theta)}{\partial \theta^2}/\sqrt{nI(\theta)}}$$

$$= \frac{z/\sqrt{nI(\theta)}}{-\frac{\partial^2 \ln L(\theta)}{\partial \theta^2}/nI(\theta)}$$

$$= \frac{z/\sqrt{nI(\theta)}}{-\sum_{i=1}^{n}\frac{\partial^2 \ln f(x_i,\theta)}{\partial \theta^2}/nI(\theta)} \quad (1)$$

Note that $Z/\sqrt{nI(\theta)} \sim AN(0,1)$ by central limit theorem and $-\sum_{i=1}^{n}\left(\frac{\partial \ln f(x_i,\theta)}{\partial\theta}\right)/I(\theta)n \xrightarrow{P} 1$

Therefore, by Slutsky's theorem, the R.H.S of ① is limiting $N(0,1)$

Hence L.H.S of ① $\frac{\hat{\theta}-\theta}{\sqrt{nI(\theta)}} \sim AN(0,1)$

i.e. $\hat{\theta} \sim AN\left(\theta, \frac{1}{nI(\theta)}\right)$ (proved)

4, a) Show that by an example, the property of unbiasedness not in general property of MLE

or $f(x_i,\theta) = \frac{1}{\theta}$; $0 < x < \theta$

Find the MLE and check the unbiasedness of this MLE

Solution:

let $f(x_i,\theta) = \frac{1}{\theta}$; $0 < x < 1$

$= 0$ otherwise

let $x_1, x_2, \ldots x_n$ be a random sample drawn from the distribution. Then the MLE is

$$L = \prod_{i=1}^{n} f(x_i,\theta) \quad \ldots ① \quad = 1/\theta^n$$

Taking log on both sides, we have

$$\log L = -n\log\theta$$

The maximum likelihood equation is $\frac{\partial \log L}{\partial \theta} = 0$

$\Rightarrow -\frac{n}{\theta} = 0 \Rightarrow \hat{\theta} = \infty$. obviously an absurd result

In this case we locate MLE as follows. we have to choose θ so that L in maximum Now L is maximum if θ is minimum

let $x_1, x_2 \ldots x_n$ be the ordered random sample of n independent observations from the given population so that

$$0 \le x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)} \le \theta$$

$$\Rightarrow \theta \ge x_{(n)}$$

Since the minimum value of θ consistent with the sample is $x_{(n)}$. the largest sample observation $\hat{\theta} = x_{(n)}$

$\therefore$ MLE for $\theta = x_{(n)}$

= The largest sample observation

The we get $\hat{\theta} = x_{(n)}$

$$E(\hat{\theta}) = E(x_n) = \int_{x_{(n)}} x_n f(x_n)\, dx_n$$

$$= \int_0^\theta x_n \frac{n}{\theta^n} x_n^{n-1}\, dx_n \quad \left[f(x_n) = \frac{n}{\theta^n} x_n^{n-1} \right]$$

$$= \frac{n}{\theta^n} \int_0^\theta x_n^n\, dx_n$$

$$= \frac{n}{\theta^n} \left. \frac{x_n^{n+1}}{n+1} \right|_0^\theta = \frac{n}{\theta^n (n+1)} \left[\theta^{n+1} - 0 \right]$$

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$$= \frac{n\theta}{n+1}$$

$$E(\hat{\theta}) = \frac{n(\theta)}{n+1} \neq \theta$$

so, $\hat{\theta}$ is a biased estimator of θ. Hence we say the property of unbiased is not ingeneral property of MLE.

4.h)) let $x_1, x_2, \ldots x_n$ be a random sample from the exponential distribution with pdf $f(x) = \theta e^{-\theta x}$, $x \geq 0$, $\theta \geq 0$

i) Find the MLE of θ

ii) Show that MLE of θ is unbiased sufficient and consistent estimator.

Solution

i) Given that $f(x) = \theta e^{-\theta x}$, $x \geq 0$, $\theta > 0$

Then the MLE is

$$L = \prod_{i=1}^{n} f(x_i, \theta)$$

$$= \prod_{i=1}^{n} \theta e^{-\theta x}$$

$$= \theta^n e^{-\theta \sum x_i}$$

$$\log L = n \log \theta - \theta \sum x_i \quad \text{——} (1)$$

Now differentiate of (1) w.r.t θ we have

$$\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} - \sum x_i = 0$$

$$= \frac{n}{\theta} - n\bar{x} = 0$$

$$\Rightarrow \frac{n}{\theta} = n\bar{x}$$

$$\Rightarrow \frac{1}{\theta} = \bar{x}$$

$$\Rightarrow \hat{\theta} = \frac{1}{\bar{x}}$$

Again,

$$\left.\frac{\partial^2 \log L}{\partial \theta^2}\right|_{\theta=\hat{\theta}} = -\frac{n}{\hat{\theta}^2} < 0$$

Hence $\hat{\theta} = \frac{1}{\bar{x}}$ is the MLE of θ

ii) <u>Unbiasedness:</u>

We have $\hat{\theta} = \frac{1}{\bar{x}} = n \cdot \frac{1}{\sum x_i}$

$$E(\hat{\theta}) = n \cdot \frac{1}{E(\sum x_i)} = n \cdot \frac{1}{n\theta}$$

$$= \theta$$

$\therefore E(\hat{\theta}) = \theta$

Hence $\hat{\theta}$ is an unbiased estimate of θ

<u>Sufficient:</u>

We have the likelihood equation is

$$\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} - \sum x_i$$

$$\Rightarrow \frac{n}{\theta} - n\bar{x}$$

$$\Rightarrow n\left(\frac{1}{\theta} - \bar{x}\right)$$

$$\Rightarrow \psi(\bar{x}, \theta)$$

is a function of $\bar{x}$ and θ only.

Hence $\bar{x}$ is a sufficient statistic for θ

Consistency:

we have $\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} - \sum x_i$

$\therefore \frac{\partial^2 \log L}{\partial \theta^2} = -\frac{n}{\theta^2}$

Now, $E\left(-\frac{\partial^2 \log L}{\partial \theta^2}\right) = \frac{n}{\theta^2}$

we know,

$$V(\hat{\theta}) = \frac{1}{E\left(-\frac{\partial^2 \log L}{\partial \theta^2}\right)} = \frac{\theta^2}{n}$$

Now $E\left(-\frac{\partial^2 \log L}{\partial \theta^2}\right) = \frac{n}{\theta^2}$

we know, $V(\hat{\theta}) = \frac{1}{E\left(-\frac{\partial^2 \log L}{\partial \theta^2}\right)} = \frac{\theta^2}{n}$

Now, $\lim_{n\to\infty} V(\hat{\theta}) = \lim_{n\to\infty} \theta^2/n = 0$

Hence $\bar{x}$ is also a consistent estimator of θ

4.i) let $x_1, x_2 \ldots x_n$ be a random sample from the exponential distribution with pdf $f(x;\theta)$

$$= \frac{e^{-\theta} \cdot \theta^x}{x!}$$

i) Find the MLE of θ

ii) Show that MLE θ is unbiased, sufficient and consistent estimator

1) Given that, $f(x,\theta) = \frac{1}{\theta} e^{-x/\theta}$

The likelihood function is

$$L = \prod_{i=1}^{n} f(x_i, \theta)$$

$$= \frac{1}{\theta^n} e^{-\left(\sum x_i/\theta\right)}$$

Taking log on both sides, we have

$$\log L = -n\log\theta - \frac{1}{\theta}\sum x_i \quad \text{--- (1)}$$

Now, differentiating (1) w.r.t θ we have

$$\frac{\partial \log L}{\partial \theta} = 0$$

$$\Rightarrow -\frac{n}{\theta} + \frac{1}{\theta^2}\sum x_i = 0$$

$$\Rightarrow -\frac{n}{\theta} + \frac{n\bar{x}}{\theta^2} = 0$$

$$\Rightarrow \frac{n}{\theta} = \frac{n\bar{x}}{\theta^2}$$

$$\Rightarrow \frac{\bar{x}}{\theta} = 1$$

$$= \theta = \bar{x}$$

And $\left.\frac{\partial^2 \log L}{\partial \theta^2}\right|_{\theta=\bar{x}} = \frac{n}{\bar{x}^2} - \frac{2n\bar{x}}{\bar{x}^3} < 0$

which is negative definite

Hence $\hat{\theta} = \bar{x}$ is the MLE of θ

ii) unbiasedness

we have $\hat{\theta} = \bar{x} = \frac{1}{n}\sum x_i$

$\therefore E(\hat{\theta}) = \frac{1}{n} E[\sum x_i]$

$= \frac{1}{n} \cdot n\theta$

$= \theta$

$\therefore E(\hat{\theta}) = \theta$

Hence $\hat{\theta}$ is an unbiased estimator of θ

Sufficiency:

we have $\frac{\partial \log L}{\partial \theta} = -\frac{n}{\theta} + \frac{n\bar{x}}{\theta^2}$

$= n\left[\frac{\bar{x}}{\theta^2} - 1/\theta\right]$

$= \psi(\bar{x}, \theta)$ is a function of $\bar{x}$ and θ only

Hence $\bar{x}$ is sufficient for θ

Consistency:

we have, $\frac{\partial \log L}{\partial \theta} = -\frac{n}{\theta} + \frac{n\bar{x}}{\theta^2}$

$\therefore \frac{\partial^2 \log L}{\partial \theta^2} = \frac{n}{\theta^2} - \frac{2n\bar{x}}{\theta^3}$

$= \frac{n\theta^3 - 2n\bar{x}\theta^2}{\theta^3 \cdot \theta^2}$

$= \frac{\theta^2(n\theta - 2n\bar{x})}{\theta^2 \cdot \theta^3} = \frac{n(\theta - 2\bar{x})}{\theta^3}$

$= \frac{n(2\bar{x} - \theta)}{\theta^3}$

$$= \frac{n[2\theta - \theta]}{\theta^3}$$

$$= \frac{n\theta}{\theta^3}$$

$$= \frac{n}{\theta^2}$$

Then

$$V(\hat{\theta}) = \frac{1}{E\left[-\frac{\partial^2 \log L}{\partial \theta^2}\right]} = \theta^2/n$$

So, $\lim_{n\to\infty} V(\hat{\theta}) = \lim_{n\to\infty} \frac{\theta^2}{n} = 0$

Now $E\left[-\frac{\partial^2 \log L}{\partial \theta^2}\right] = \frac{n[2E(\bar{x}) - \theta]}{\theta^3}$ (Proved)

5th set

5.a) Explain the concept of hypothesis testing and it's importance is statistical analysis

Answer:

Hypothesis testing is a statistical method used to make decisions or inferences about a population based on sample data. It involves formulating to competing hypotheses

i) Null Hypothesis (H_0). Assumes there is no effect or difference. Example A new medicine has no effect on disease

ii) Alternative hypothesis: H_a: Suggests There is and effect or difference.
The new medicine helps cure the disease

Then we analyze sample data and if strong evidence is found, we reject H_0 and accept H_1

Importance of hypothesis testing

1. logical Decision-making: Helps make data-driven decisions instead of guessing
2. Used in research: Tests claims in medicine, business and technology
3. Reduces Errors: minimizes the chances of wrong conclusions.
4. Comparing Groups – helps in comparing two or more conditions.
5. predicting Future trends – helps in forecasting trends for business and research

5.b)) What are type I and type II errors in hypothesis testing? How are they related to the level of significance (α) and the power of the test?

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Type I error: The error of rejecting the null hypothesis when it is in fact true is called type I error. A type I error is also known as first kind of error.

Type II error: The error of accepting the null hypothesis when it is false is called type II error. A type II error is also known as second kind of error. That means type II error occurs when null hypothesis is not rejected wrongly.

Relationship with level of significance and power of the test:

Level of significance (α): This is the probability of committing a type I error. A lower α reduces the chances of rejecting a true null hypothesis but increases the risk of a type 2 error.

Power of the test ($1-\beta$): This represents the probability of correctly rejecting a false null hypothesis. A higher power means a lower probability of a type 2 error.

5.c) List the application of test statistics?

Answer:

A test statistic describes how closely the distribution of our data matches the distribution predicted under the null hypothesis of the statistical test.

Thus the application of test statistics are

1) Hypothesis testing: Determines whether to accept or reject a null hypothesis.

~~2) Or~~

2) Comparing Two Population — used in t-test and z-test to compare sample means or proportions.

3) Assessing variability - F-test is used to compare variances of two samples.

4) Correlation and Regression analysis: Helps in determining relationships betn variables.

5) Market Research: Helps in analyzing customer preferences and behavior.

6) Machine learning and AI → used in feature selecting, model evaluation and A/B testing.

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5.d) Explain the steps in hypothesis testing

⇒

1) Formulate the hypotheses:

- Null hypothesis (H_0): Assumes no effect or no difference
- Alternative hypothesis H_1: Assumes an effect or difference exists.

2) Select the significance Level (α)

- common values: 0.05 (5%) or 0.01 (1%)
- Determines the probability making a type I error

3) Select the suitable test statistic: Depends on data type and sample size (T-test, Z-test, χ^2-test)

4) Establish the critical region

- Find the critical value from statistical tables.
- Define the rejection region based on α.

5) Collect data and calculate the Test statistic

- Compute the test statistic using sample data

6) Compare ~~with the~~ observed and critical value:
1) If the test statistic falls in the rejection region, rejeject H_0
otherwise fail to reject H_0

7) make the decision

5. e) What is the difference between a one-tailed and a two-tailed test? When would you use each.

Answer:

One Tailed Test	Two-tailed Test
Tests if a value is greater or smaller than a given value	Tests if a value is different from a given value
Rejection region is on one side of the distribution	Rejection region is on both sides of the distributi
Hypothesis: $H_1: \mu > \mu_0$ (Right-tailed) or $H_1: \mu < \mu_0$ (Left-tailed) $H_0: \mu = \mu_0$	Hypothesis: $H_1: \mu \neq \mu_0$ $H_0: \mu = \mu_0$;
Used when we expect a specific directional change	Used when we only check for any difference without focusing on direction
Example: testing if a new drug is more effective than the old one	Testing if a new teaching method affects student scores.

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A one-tailed test is used when we need to determine if a value is greater or smaller than a given value. This test is appropriat when we have a clear expectation about the direction of chang.

On the other hand A two tailed test is used when we want to check if there is any difference without specifying whether the value is higher or lower. This test is used any kind of change is important.

Thus, one-tailed test is used when the expected change is in a specific direction, while a two tailed test is used when we only want to check for any difference

5.f) What is the p-value and critical value, and how are these interpreted in hypothesis testing

> The P-value is the probability of obtaing the observed result if the null hypothesis is true

~~That is observed significance~~.

P value of a statistical test is the smallest

value of the level of significance at which the null hypothesis can be rejected. It helps determine whether there is enough evidence to reject H_0

If $p \leq \alpha$: Reject H_0. If $p > \alpha$: Fail to reject H_0

<u>Example:</u> If a test gives $p = 0.03$ and $\alpha = 0.05$ since $0.03 < 0.05$, we reject H_0, meaning the result is significant

<u>Critical values:</u> The value of the sample statistic that separates acceptance region and rejection region is called critical value.

- If the test statistic is beyond the critical value: Reject H_0
- If the test statistic is within the critical value; Fail to reject H_0

<u>Example:</u> In a two tailed z-test $\alpha = 0.05$
The critical value = 1.96
Since $1.96 > 0.05$ we fail to reject H_0

Interpretation in hypothesis Testing:
The p-value tells us how strong the evidence is against H_0. The critical value sets the decision boundary for rejecting or accepting H_0
If $p \leq \alpha$ or test statistic exceeds the critical value we reject H_0 and result is statistically significant.

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5.9)) Explain the concept of the critical region and acceptance region in hypothesis testing. How is it determined?

Answer:

The critical region is the range of values where we reject the null hypothesis (H_0). If the test statistic falls in this region, it means the observed result is statistically significant. It is also called rejection region. On the other hand the acceptance region is the range of values where we fail to reject the null hypothesis. If the test statistic falls ~~the~~ this region, it means the observed result is not significantly different from what we expect under H_0, so we do not have enough evidence to reject it.

The critical region and acceptance region are determined based on the ~~four~~ following factors.

1) <u>Significance level (α):</u> commonly significance level is 0.05 or 0.01 that defines the probability of rejecting a true null hypothesis.

2) <u>Type of test:</u>

In a one-tailed test the critical ~~rgi~~ region is on one side while in a two-tailed test it is on both sides.

3) <u>Critical value:</u> The critical value is found from statistical tables like the z-table or t-table. If the test statistic falls within the critical region, the null hypothesis is rejected otherwise it is not rejected.

5.h)) <u>Example:</u> 16.6.1, 16.6.2, 16.16.3

5.i)) , , 16.6.4, 16.6.5, 16.6.6

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6

6.a)) Explain the hypothesis testing for single ~~pollution~~ / population mean

Answer:

Hypothesis testing for a single population mean involves testing an assumption about the population mean (μ) based on sample data. The hypothesis testing for a single population mean are following steps:

1) State the hypotheses
   - There is no effect: $H_0: \mu = \mu_0$ (Null hypothesis)
   - There is an effect: $H_1: \mu \neq \mu_0$ (Two-tailed test) (Alternative hypothesis)

2) Select the significance level (α): commonly values are 0.05 / 0.01

3) Select test:
   - Z-test: used when the sample size is large ($n > 30$) and the population standard deviation is known.

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T-Test: used when the sample size is small ($n<30$) or the population standard deviation is unknown.

4) Calculate test statistic:

- Z-test: $z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$
- T-test: $t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$

5) Compare test statistic to critical value or check p-value

6) Make Decision:

- if test statistic > critical value or p-value $\leq \alpha$: reject H_0
- Fail to reject H_0 if test statistic < critical value or p-value > α

Conclude if there is enough evidence to reject the null hypothesis.

6.b) Describe steps of mean test when variance unknown and sample size it small ($n<30$) (T-Test)

Answer: When the sample size is small and the population variance is unknown, we use the t-test to test the population mean. Here are the steps:

উপরের steps গুলো লিখলেই হবে।

6.c) What do you mean by Z-Test? mention the application of it

Answer: A Z-test is a statistical test used to determine whether there is a significant difference between the sample mean and the population population mean when the population variance is known, and the sample size is large (n>30). It is used for testing hypotheses about population parameters.

Application of Z-Test:

(I) Test of a single population mean
(II) Test of equality of two population means
(III) Test of a single population proportion
(IV) Test for difference between two population proportion
(V) Test of a specified correlation co-efficient
(VI) Test of equality of two-population correlation co-efficient.

6.d) What is test of hypothesis concerning attributes? Discuss test procedure of a hypothesis about proportion test of a population.

Answer:

A test of hypothesis concerning attributes involves making inferences about the proportions of a population, typically based on sample data. It's used when the data is categorical and the goal is to determine if a sample proportion differs significantly from a hypothesized population proportion.

Test procedure for Hypothesis about proportion of a population

① State the Hypotheses:

- Null Hypothesis: If $P = P_0$: Population proportion is equal to a specified value.
- Alternative hypothesis: $H_1 : P \neq P_0$
  - one-tailed test $H_1 : P > P_0$ greater
  - on-tailed " $H_1 : P < P_0$ smaller

② Choose the significance level (α):

select a significance level, commonly 0.05%

8) calculat the z-test statistic for proportion

$$z = \frac{p - \pi_0}{\sqrt{\frac{\pi_0(1-\pi_0)}{n}}}$$

9) Critical z-value: Find the critical z-value using the the z-table for the given significance level (α).

10) P-value: Find the p-value associated with the calculated z-value.

11) Make Decision:

Reject H_0: If the calculated z value is greater than the critical z-value or the p-value is less than or equal to α.

Fail to Reject H_0: If the calculated z-value is less than the critical z value or the p-value is greater than α.

6.e) What is the χ^2 test, and what are its application in hypothesis testing?

Answer: The chi-square Test is a statistical method used to test whether there is a significant difference between the observed and expected frequencies in categorical data.

Applications of χ^2 statistic

(I) Test of a population variance with specific value.

(II) Test of equality of several variances.

(III) Test of equality of several correlation co-efficient

(IV) Test of equality of several population proportions

(V) Tests of independence of attributes

(VI) Test of goodness of fit.

6.f) What is the role of the test statistic in hypothesis testing? How is it used to make decisions about the null hypothesis?

Answer: Role of the test statistic in Hypothesis testing:

1) Measure of Deviation: The Test statistic quantifies how much the observed data deviate from the null hypothesis

2) Standardization: It standardizes the sample data, allowing comparisons betn different types of statistical tests

③ Decision-making: It helps decide whether to reject or fail to reject the null hypothesis

It is used to make decision about null hypothesis following steps:

① calculate the test statistic based on the sample data.

② Determine critical value from statistical tables, based on the chosen significance (level).

③ If statistic value > critical value Reject H_0

or If statistic value < critical value - Fail to reject H_0

(4) <u>use p-value</u>: Reject H_0 if the p-value $\leq \alpha$
Fail to reject H_0 if the p-value $> \alpha$.

6.a)) Explain the test of hypothesis about difference between two population proportions.

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7

7.a)) What are non-parametric tests and how do they differ from parametric tests?

Answer

Non parametric tests are statistical tests that do not assume a specific distribution for the data. They are used when the data does not meet the assumptions required for parametric test. These tests are also known as distribution free-test. Example: χ^2-test

Parametric test: parametric tests are statistical tests that assume the data follows a specific distribution. These tests require the data to meet certain assumptions like normality and equal variances.

Difference

Parametric	Non-parametric
Assume normal distribution, equal variances	No distribution assumption
Interior ratio scal	Data type is ordinal or nominal scale
Sensitive to outliers	less sensitive to outliers
more powerfull when assumption are met	less powerful but more flexible
t-test.	chi-square

7.b)) List the advantages and disadvantages of using non-parametric tests?

Answer:

Advantages of Non-parametric test:

(i) No assumption is made about the distribution of the parent from which the sample is drawn

(ii) These tests are readily comprehensible and easy to apply

(iii) Since the data is social sciences, in general don not follow normal distribution.

(iv) Non-parametric methods are applicable to data which are given in grades like A+, A, B, B+, C etc.

(v) If the sample size small there is no alternative except to use non-parametric tests unless the nature of the population distribution is exactly known.

Disadvantages:

i) Non-parametric tests can result in loss of much of the information contained within the data.

ii) They are less efficient and less powerful than their counter parametric tests.

iii) The drawback of ranking, on which many non-parametric tests are based, is that the magnitude of the observation is lost.

4c)) What types ~~are~~ of data are suitable for non-parametric tests.

⇒ Types of data suitable for non-parametric tests

(1) Ordinal data — Ranked data without equal intervals

(2) Nominal data — categorical data without order

(3) small sample size — useful when $n<30$

(4) Non-Normal data — when data does not follow a normal distribution

(5) Data with outliers — less affected by extreme values.

(6) Ranked Data — based on relative position not numerical values.

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7.d)) What is the runs test for randomness and how is it used ~~to~~ to determine whether a sample is random?

Answer: The runs test for randomness is a non-parametric test used to check whether a sequence of observations is randomly distributed. It is not assume any specific distribution for the data.

The runs test for randomness helps determine whether a dataset follows a random pattern. It does this by analyzing the number of runs in the sequence.

Steps:

① Convert data:

- Binary data → uses 0 and 1
- continuous dat; calculate median
  "+" (above the median)
  "−" (below the median)

2. Count the number of Runs (R)

- A run is a sequence of consecutive identical values.

3. Calculate the expected Number of Runs (E[R]).

$$E[R] = \frac{2n_1n_2}{n_1+n_2} + 1$$

4) Calculate the standard deviation

$$\sigma_R = \sqrt{\frac{2n_1n_2(2n_1n_2 - n_1 - n_2)}{(n_1+n_2)^2(n_1+n_2+1)}}$$

5) Compute the z-score

$$z = \frac{R - E[R]}{\sigma_R}$$

- If z is outside this range the sample is not random and has a pattern.

Expected runs close to observed runs → The data is random

7.e) Explain the sign Test and its application as a non-parametric alternative to the one sample t-test

Answer: The sign test is used for testing the median rather than mean as location parameter of a population i.e. whether the ~~the~~ sample has been drawn from a population with the specified median m_0. It is a substitute of one sample t-test when the normality assumption of the parent population is not satisfied.

This test is simplest of all the non-parametric tests and its name comes from the fact that it is based upon the signs of the differences and not on their numerical magnitude.

Let $x_1, x_2 \cdots x_n$ be a random sample of size n from a population with unknown median m

Test the null hypothesis $H_0: m = m_0$ against ~~a two sided~~ $H_1: m \neq m_0$ (two-sided)

$H_1: m > m_0$ or $H_1: m < m_0$ - one sided.

Compare each sample value to the hypothesized median.

7.f) Describe the wilcoxon signed Ranks Test and explain how it improves upon the sign Test.

Answer

The wilcoxon signed Ranks Test is a non-parametric alternative to the one-sample test and paired t-test. It is used to test whether the median of a population differ from a specified value when the data does not follow a normal distribution.

Steps of the Test

1) Hypotheses:

H_0: The population median equals the hypothesized median

H_1: The population median is different.

2) Calculate differences:

- Compute the diffence betn each sample value
- Ignore differences that are exactly zero

$d_i = x_i - m_0$

3) Rank $|d_i|$ is ascending order an Assign original signs (+ or -) to ranks.

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5. Calculate Rank sumse

- compute T^+ and T^-
- Test statistic: $T = \min(T^+, T^-)$

6. compare with critical value:

If T is small enough, reject H_0

The wilcoxon signed Ranks Test is better than the sign Test because it considers both direction and magnitude of differences. The sign Test only checks +/- signs, ignoring size, making it less accurate. The wilcoxon test ranks differences improving detection of small effects.

It also works for both ordinal and numerical data and assumes symmetric distribution, making it more reliable for small or non-normal datasets.

8

8.a) What is the paired sample sign Test and when should it be used?

Answer: The paired sample sign test is a non-parametric statistical test used ~~to~~ for the comparison of two dependent samples. Here it is disired to test null hypothesis ~~but~~ that the two sample are drawn from the populations having the same median

$H_0: m_1 = m_2$ vs $H_1: m_1 \neq m_2$

It checks if the median difference b/n Paired values is zero.

This test is useful when the data doesn't follow a normal distribution and it often used when a paired t-test isn't suitable. It focuses on the direction of the differences rather than the magnitude

8.b)) What is the mann-whitney u-test and when is it used as an alternative to the two sample t-test?

Answer: The mann-whitney u test is non parametric statistical test used to compare two independent same distribution. It is an alternative to the two-sample t-test when the assumptions of normality and equal variances are not met.

It is used when the data is not normally distributed, and comparing two independent groups, when we want to test

if one group tends to have higher or lower values than the other but without assuming the data is normally distributed.

The mann-whitney u test ranks all the data points from both groups combined and then compares the sum of ranks between the two groups. It is particularly usefull when the data is ordinal or skewed.

8.e) How does the siegel-tukey test determine whether two populations have equal variability