Determination of Fourier Series Representation

$$x(t)=\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$ **Synthesis equation**

$$a_k=\frac{1}{T_0}\int_{<T_0>} x(t)\,e^{-jk\omega_0 t}\,dt$$ **Analysis equation**

a_k —— **Fourier Series Coefficients**
Spectral Coefficients

Fourier Series Co-efficient Calculation

$\rightarrow$ $$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$\rightarrow$ Multiply $e^{-jn\omega_0 t}$ both the side.

$$\Rightarrow x(t)\,e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}$$

$$\Rightarrow x(t)\,e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}$$

$\rightarrow$ let, $\int_0^T$ on both sides.

$$\Rightarrow \int_0^T x(t)\,e^{-jn\omega_0 t}\,dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}\,dt$$

$$\Rightarrow \int_0^T x(t)\,e^{-jn\omega_0 t}\,dt = \sum_{k=-\infty}^{\infty} a_k \underline{\int_0^T e^{j(k-n)\omega_0 t}\,dt}$$

→ From Eulers formula.

$$\Rightarrow \int_0^T e^{j(k-n)\omega_0 t}\,dt = \int_0^T \cos(k-n)\omega_0 t\,dt + j\int_0^T \sin(k-n)\omega_0 t\,dt$$

$$= \begin{cases} T & k=n \\ 0 & k\neq n \end{cases}$$

$$\Rightarrow \int_0^T x(t)\, e^{-jn\omega_0 t}\,dt = a_n T$$

$$\Rightarrow \boxed{a_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\,dt}$$

$$\rightarrow \boxed{x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}}$$

Problem 4.1 Determine the periodic signal whose fundamental frequency is 2π and Fourier coefficients are

$$a_0 = 1,\ a_2 = a_{-2} = \frac{1}{2};\ a_4 = a_{-4} = \frac{1}{4},\ a_6 = a_{-6} = \frac{1}{6}$$

Solution A periodic signal can be represented exponentially as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

For $\omega_0 = 2\pi$,

$$x(t) = \sum_{k=-6}^{6} a_k e^{jk2\pi t}$$

$$x(t) = a_{-6}e^{-j12\pi t} + a_{-4}e^{-j8\pi t} + a_{-2}e^{-j4\pi t} + a_0 + a_2 e^{j4\pi t} + a_4 e^{j8\pi t} + a_6 e^{j12\pi t}$$

$$x(t) = \frac{1}{6}e^{-j12\pi t} + \frac{1}{4}e^{-j8\pi t} + \frac{1}{2}e^{-j4\pi t} + 1 + \frac{1}{2}e^{j4\pi t} + \frac{1}{4}e^{j8\pi t} + \frac{1}{6}e^{j12\pi t}$$

$$x(t) = 1 + \frac{1}{3}\left[\frac{e^{j12\pi t} + e^{-j12\pi t}}{2}\right] + \frac{1}{2}\left[\frac{e^{j8\pi t} + e^{-j8\pi t}}{2}\right] + \left[\frac{e^{j4\pi t} + e^{-j4\pi t}}{2}\right]$$

$$x(t) = 1 + \frac{1}{3}\cos 12\pi t + \frac{1}{2}\cos 8\pi t + \cos 4\pi t$$

1. Periodic time function

$$x(t) = 1 + \mathsf{sin}\omega_0 t + 2\mathsf{cos}\omega_0 t + \mathsf{cos}(2\omega_0 t + \pi/4)$$

– Expand $x(t)$

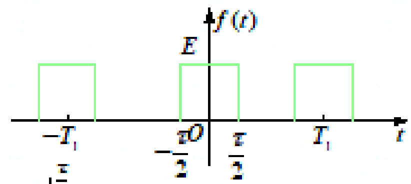
$$\begin{aligned} x(t) &= 1 + \frac{1}{2\mathsf{j}}\left(\mathsf{e}^{\mathsf{j}\omega_0 t} - \mathsf{e}^{-\mathsf{j}\omega_0 t}\right) \\ &\quad + \left(\mathsf{e}^{\mathsf{j}\omega_0 t} + \mathsf{e}^{-\mathsf{j}\omega_0 t}\right) + \frac{1}{2}\left(\mathsf{e}^{\mathsf{j}(2\omega_0 t + \pi/4)} + \mathsf{e}^{-\mathsf{j}(2\omega_0 t + \pi/4)}\right) \\ &= 1 + \left(1 + \frac{1}{2\mathsf{j}}\right)\mathsf{e}^{\mathsf{j}\omega_0 t} + \left(1 - \frac{1}{2\mathsf{j}}\right)\mathsf{e}^{-\mathsf{j}\omega_0 t} \\ &\quad + \left(\frac{1}{2}\mathsf{e}^{\mathsf{j}\pi/4}\right)\mathsf{e}^{2\mathsf{j}\omega_0 t} + \left(\frac{1}{2}\mathsf{e}^{-\mathsf{j}\pi/4}\right)\mathsf{e}^{-2\mathsf{j}\omega_0 t} \end{aligned}$$

– Fourier series coefficients

$$\begin{aligned} a_0 &= 1 \\ a_1 &= \left(1 + \frac{1}{2j}\right) \\ a_{-1} &= \left(1 - \frac{1}{2j}\right) \\ a_2 &= \frac{1}{2}e^{j\pi/4} \\ a_{-2} &= \frac{1}{2}e^{-j\pi/4} \\ a_k &= 0,\ |k| > 2 \end{aligned}$$

Example 1: Periodic Square Wave

$$a_0 = \frac{1}{T_1}\int_{T_1} x(t)dt = \frac{E\tau}{T}$$



$$a_k = \frac{1}{T_1}\int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} f(t)\mathrm{e}^{-\mathrm{j}k\omega_1 t}\,\mathrm{d}t$$

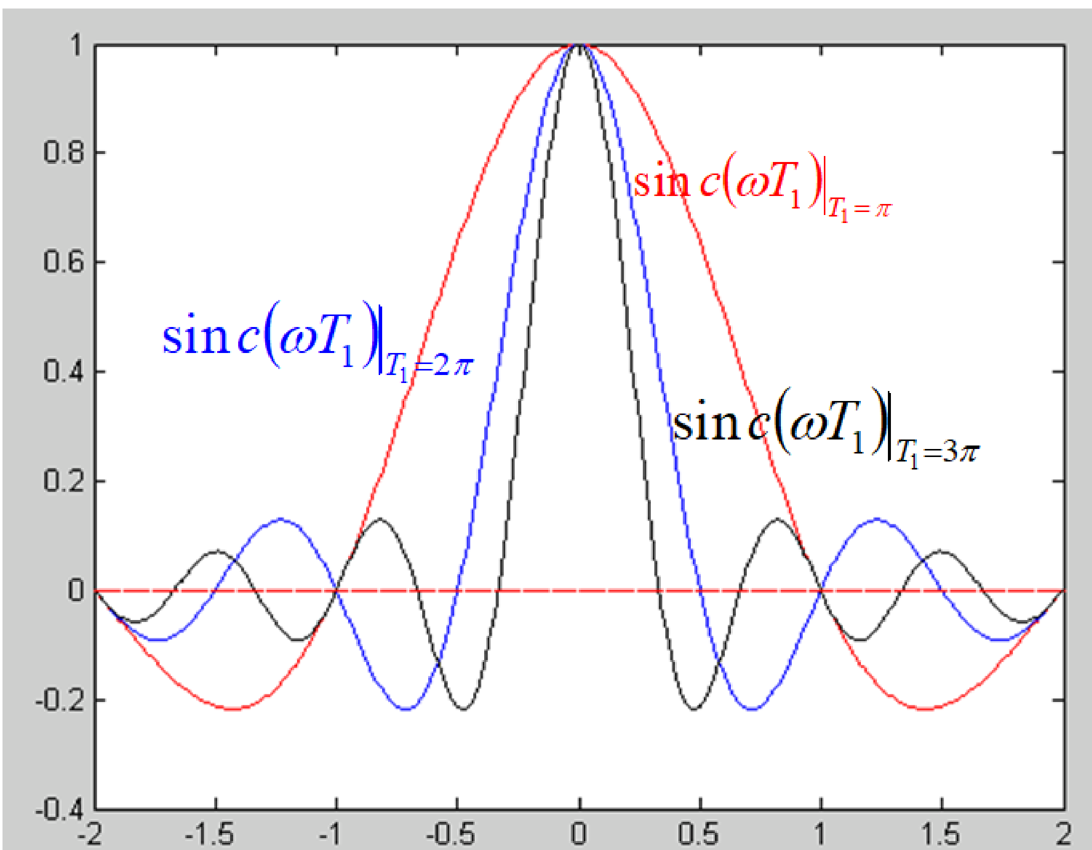
$$= \frac{1}{T_1}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} E\mathrm{e}^{-\mathrm{j}n\omega_1 t}\,\mathrm{d}t = \frac{E}{T_1}\frac{1}{-\mathrm{j}n\omega_1}\mathrm{e}^{-\mathrm{j}n\omega_1 t}\Big|_{-\frac{\tau}{2}}^{\frac{\tau}{2}}$$

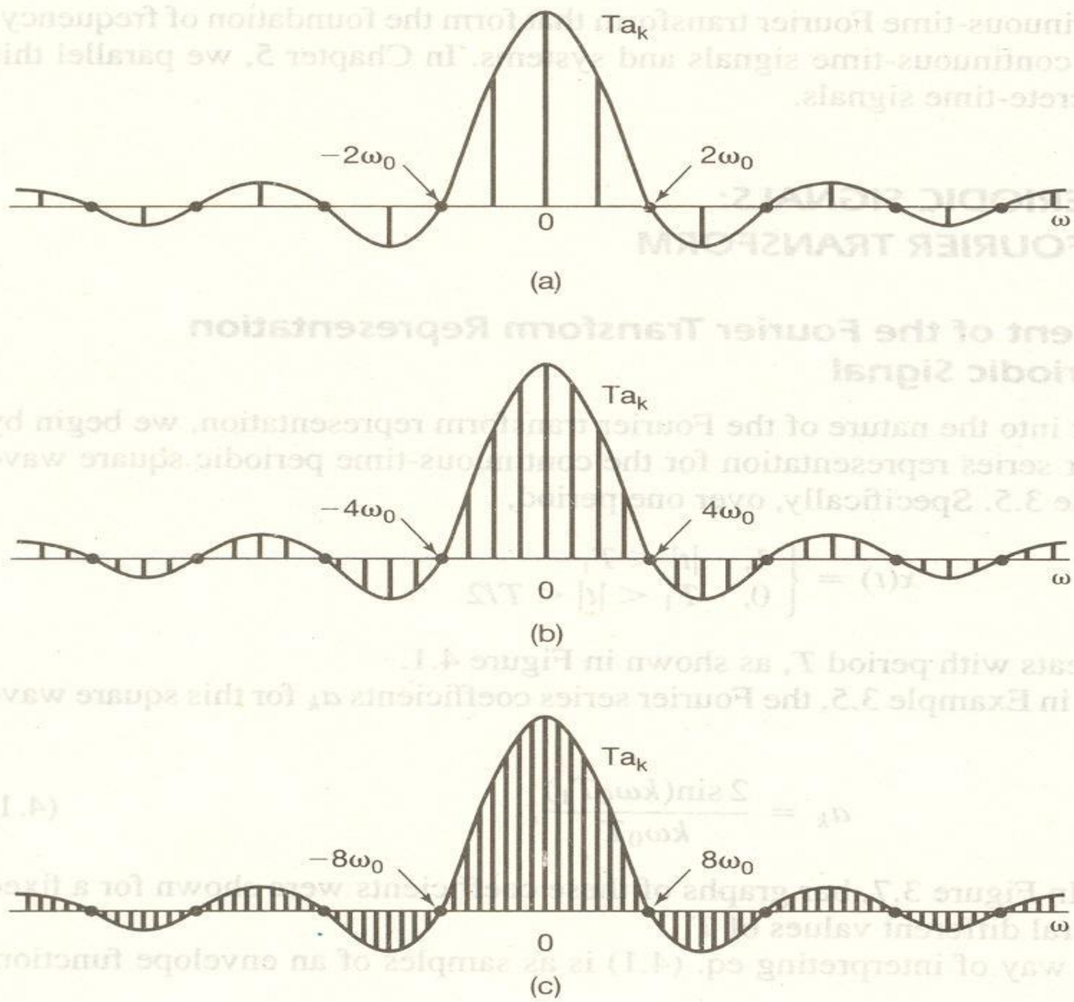
$$= \frac{-E}{\mathrm{j}n\omega_1 T_1}\left[\mathrm{e}^{-\mathrm{j}n\omega_1\frac{\tau}{2}} - \mathrm{e}^{\mathrm{j}n\omega_1\frac{\tau}{2}}\right]$$

$$= \frac{2E}{n\omega_1 T_1}\sin\left(n\omega_1\frac{\tau}{2}\right)$$

Defining $\sin c(x) = \dfrac{\sin x}{x}$

$$= \frac{E\tau}{T_1}\frac{\sin\left(n\omega_1\frac{\tau}{2}\right)}{n\omega_1\frac{\tau}{2}}$$





Ta_k
$-2\omega_0$
$2\omega_0$
0
ω
(a)
Ta_k
$-4\omega_0$
$4\omega_0$
0
ω
(b)
Ta_k
$-8\omega_0$
$8\omega_0$
0
ω
(c)