Question: Derive the Cramer Rao lower bound (CRLB) for the variance of an unbiased estimator t of the parameter θ.
or
State and prove the Cramer Rao Lower bound

Proof:

Statement: suppose

(i) $x_1, x_2, \ldots, x_n$ are independent random variables each with density $f(x|\theta)$. $\theta \in \Omega$ an open interval on the real Line.

(ii) t is an estimator of θ.

(iii) $E(t) = \theta + b(\theta)$ where $b(\theta)$ is the biase of t and is a differentiable function of θ.

(iv) The following regularity conditions hold.

(a) for almost all x, $\frac{\partial L}{\partial \theta}$ (L is a likely hood function) must exist for all $\theta \in \Omega$.

(b) $\frac{\partial}{\partial \theta}\int\cdots\int L = \int\cdots\int \frac{\partial L}{\partial \theta}$ which is possible when the Limits of integration are independent of θ

(c) $E\left[\frac{\partial \log L}{\partial \theta}\right]^2 > 0$ for $\theta \in \Omega$

(d) $\frac{\partial}{\partial \theta}\int\cdots\int tL = \int\cdots\int t\frac{\partial L}{\partial \theta}$.

Then for all $\theta \in \Omega$

$$V(t) \geq \frac{[1+b'(\theta)]^2}{nE\left[\frac{\partial \log f}{\partial \theta}\right]^2}$$

$$= \frac{[1+b'(\theta)]^2}{E\left[\frac{\partial \log L}{\partial \theta}\right]^2}$$

$$= -\frac{[1+b'(\theta)]^2}{E\left[\frac{\partial^2 \log L}{\partial \theta^2}\right]}$$

where, $b'(\theta)$ is the first derivative of $b(\theta)$ w.r.to θ

Proof:

we know,

$$L = \prod_{i=1}^{n} f(x_i|\theta) \quad \cdots\cdots\cdots (i)$$

Since L is the joint density of the observation

$$\int\cdots\int L.\,dx_1\,dx_2\ldots dx_n = 1 \quad \cdots\cdots (ii)$$

Now, suppose the first and second differentials of L exist. Then taking the first derivation of (ii) w.r.to θ on both sides.

$$\int\cdots\int \frac{\partial L}{\partial \theta}\,dx_1\,dx_2\ldots dx_n = 0$$

$$\text{or } \int\cdots\int \frac{\partial L}{\partial \theta}\cdot\frac{1}{L}\cdot L\,dx_1\,dx_2\ldots dx_n = 0$$

$$\text{or } \int\cdots\int \frac{\partial \log L}{\partial \theta}\cdot L\,dx_1\,dx_2\ldots dx_n = 0 \quad (iii)$$

$$\text{or } E\left[\frac{\partial \log L}{\partial \theta}\right] = 0 \quad \cdots\cdots\cdots (iii)$$

$$\text{or } E(\phi) = 0 \quad \text{where } \phi = \frac{\partial \log L}{\partial \theta}$$

Again ~~the~~ differentiating (iii) w.r.to θ.

$$\int\cdots\int\left[\frac{\partial \log L}{\partial\theta}\cdot\frac{\partial L}{\partial\theta}+L\frac{\partial^2\log L}{\partial\theta^2}\right]dx_1\,dx_2\cdots dx_n=0$$

or, $$\int\cdots\int\left[\frac{\partial \log L}{\partial\theta}\cdot\frac{\partial L}{\partial\theta}\cdot\frac{1}{L}\cdot L+L\cdot\frac{\partial^2\log L}{\partial\theta^2}\right]dx_1\,dx_2\cdots dx_n=0$$

or, $$\int\cdots\int\left[\left(\frac{\partial \log L}{\partial\theta}\right)^2\cdot L+L\cdot\frac{\partial^2\log L}{\partial\theta^2}\right]dx_1\,dx_2\cdots dx_n=0$$

or, $$\int\cdots\int\left(\frac{\partial \log L}{\partial\theta}\right)^2\cdot L\,dx_1\,dx_2\cdots dx_n+\int\cdots\int\frac{\partial^2\log L}{\partial\theta^2}\cdot L\,dx_1\,dx_2\cdots dx_n=0$$

or, $$E\left(\frac{\partial \log L}{\partial\theta}\right)^2+E\left(\frac{\partial^2\log L}{\partial\theta^2}\right)=0$$

or $$E\left(\frac{\partial \log L}{\partial\theta}\right)^2=-E\left(\frac{\partial^2\log L}{\partial\theta^2}\right).$$

Now,

$$E(t)=\theta+b(\theta)$$
$$=\int\cdots\int tL\cdot dx_1\,dx_2\cdots dx_n.$$

$$\therefore\ \frac{\partial E(t)}{\partial\theta}=[1+b'(\theta)]=\int\cdots\int t\,\frac{\partial L}{\partial\theta}\,dx_1\,dx_2\cdots dx_n$$
$$=\int\cdots\int t\cdot\frac{\partial\log L}{\partial\theta}\cdot L\,d\underset{\sim}{x}$$
$$=E\left(t,\frac{\partial\log L}{\partial\theta}\right)$$
$$=E(t,\phi)\quad\text{since }\phi=\frac{\partial\log L}{\partial\theta}.$$

$$=Cov(t,\phi)$$
since $E(\phi)=0$.

or, $$[1+b'(\theta)]^2=[Cov(t,\phi)]^2$$
$$\le V(t).V(\phi)$$
by schwartz's inequality.

Therefore,

$$V(t)\geq\frac{[1+b'(\theta)]^2}{V(\phi)}$$
$$=\frac{[1+b'(\theta)]^2}{E\left[\frac{\partial\log L}{\partial\theta}\right]^2}$$
$$=-\frac{[1+b'(\theta)]^2}{E\left[\frac{\partial^2\log L}{\partial\theta^2}\right]}$$
$$=-\frac{[1+b'(\theta)]^2}{nE\left[\frac{\partial\log f}{\partial\theta}\right]^2}$$

In case, t is an unbaised estimator of θ. i.e $E(t)=$.
Then,

$$V(t)\geq\frac{1}{V(\phi)}$$

(proved)

Also, we know

$$v(t) = \frac{1}{v(\phi)} \quad \text{since } t \text{ is MVB estimator}$$

$$\therefore v(\phi) = \frac{1}{v(t)}$$

Substituting this in (1.3), we have.

$$v(t) = \frac{1}{A^2 \cdot v(t)}$$

$$\text{or, } [v(t)]^2 = \frac{1}{A^2}$$

$$\text{or } \quad v(t) = \frac{1}{A}$$

$$\therefore v(t) = \frac{1}{A}$$

Thus, A is the reciprocal of the variance of MVBUE of t.

(showed)

3f

Question: Let $x \sim N(\mu, \sigma^2)$. Find the MVBUE of μ.

Answer:

Given, $x \sim N(\mu, \sigma^2)$, then the density function.

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2}\{(x-\mu)/\sigma\}^2\right] \; ; -\infty < x < \infty$$

We know

$$L = \prod_{i=1}^{n} f(x_i|\mu, \sigma^2)$$

$$= (2\pi\sigma^2)^{-n/2} \, e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

Taking log on bothsides, we have

$$\log L = \log c - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 \qquad \text{where } c = (2\pi\sigma^2)^{-n/2}$$

$$= \log c - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

Now, taking derivative w.r.to μ, we have.

$$\frac{\partial \log L}{\partial \mu} = 0 - \frac{2}{2\sigma^2}\sum(x_i-\mu)^{2-1}(-1)$$

$$= \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)$$

$$= \frac{1}{\sigma^2}(\sum x_i - n\mu)$$

$$= \frac{1}{\sigma^2}(n\bar{x} - n\mu)$$

$$= \frac{n}{\sigma^2}(\bar{x} - \mu)$$

which can be expressed as $\frac{\partial \log L}{\partial \theta} = A(t-\theta)$ where $A = n/\sigma^2$ and variance $v(t) = A^{-1} = \sigma^2/n$.

Therefore, we can say that $\bar{x}$ is the MVBUE of μ with variance σ^2/n.

Question: Let $x \sim E(\theta)$. Find the MVBUE of θ.

Answer:

Given, $x \sim E(\theta)$, then the density function.

$$= \frac{n\bar{x} - \theta n^2}{\theta(1-\theta)}$$

$$= \frac{n(\bar{x} - n\theta)}{\theta(1-\theta)}$$

$$= \frac{n}{\theta(1-\theta)}(\bar{x} - n\theta) = \frac{n}{\theta(1-\theta)} \cdot n\left(\frac{\bar{x}}{n} - \theta\right)$$

which can be expressed in the for $A(t-\theta)$ where $A = n/\theta(1-\theta)$ and variance $v(t) = \theta(1-\theta)/n$.

Therefore we can say that $\bar{x}$ is the MVBUE of $n\theta$ with variance $(\theta(1-\theta)/n)$.

Question: For Bernoulli distribution. Find MVBUE of θ.

Answer:

The pmf of bernoulli distribution is,

$$f(x|\theta) = \theta^x (1-\theta)^{1-x} \quad ; x = 0, 1$$

We know,

$$L = \prod_{i=1}^{n} f(x_i|\theta)$$

$$= \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{\sum_{i=1}^{n}(1-x_i)}$$

$$\log L = \sum x_i \log\theta + \sum(1-x_i)\log(1-\theta)$$

$$\therefore \frac{\partial \log L}{\partial\theta} = \sum x_i \frac{1}{\theta} + \sum(1-x_i)\frac{1}{(1-\theta)} \cdot (1)$$

$$= \frac{\sum x_i}{\theta} - \frac{\sum(1-x_i)}{(1-\theta)}$$

$$= \frac{\sum x_i(1-\theta) - \theta\sum(1-x_i)}{\theta(1-\theta)}$$

$$= \frac{\sum x_i - \theta\sum x_i - \theta . n + \theta\sum x_i}{\theta(1-\theta)}$$

$$= \frac{\sum x_i - \theta n}{\theta(1-\theta)}$$

$$= \frac{n\bar{x} - n\theta}{\theta(1-\theta)}$$

$$= \frac{n(\bar{x} - \theta)}{\theta(1-\theta)}$$

$$= \frac{n}{\theta(1-\theta)}(\bar{x} - \theta)$$

which can be expressed as $A(t-\theta)$ where $A = \frac{n}{\theta(1-\theta)}$ and variance $v(t) = \frac{\theta(1-\theta)}{n}$.

Therefore we can say that $\bar{x}$ is the MVBUE of θ with variance $\theta(1-\theta)/n$.

Ans

Question: Find MVBUE of θ for poission distribution.

Answer:

We know the pmf of poission distribution.

$$f(x|\theta) = \frac{e^{-\theta}\theta^x}{x!} \quad ; x = 0, 1, 2, \ldots$$

we know

$$L = \prod_{i=1}^{n} f(x_i|\theta)$$

$$= \frac{e^{-n\theta} \cdot \theta^{\Sigma x_i}}{\prod_{i=1}^{n} (x_i)!}$$

$$\log L = -n\theta + \Sigma x_i \log\theta - \log\left[\prod_{i=1}^{n} (x_i)!\right]$$

$$\log L = -n\theta + \Sigma x_i \log\theta - \log c$$

$$\therefore \frac{\partial \log L}{\partial\theta} = -n + \Sigma x_i \frac{1}{\theta}$$

$$= -n + \frac{n\bar{x}}{\theta}$$

$$= \frac{-n\theta + n\bar{x}}{\theta}$$

$$= \frac{n}{\theta}(\bar{x}-\theta)$$

$$\therefore \frac{\partial \log L}{\partial\theta} = \frac{n}{\theta}(\bar{x}-\theta)$$

Therefore, we can say that $\bar{x}$ is the MVBUE of θ with variance θ/n.

3g. ~~Problem~~: If $x \sim N(0, \sigma^2)$. Then, find the MVBUE of σ^2.

or

A random sample $x_1, x_2, \ldots, x_n$ is taken from a normal population with mean 0 and variance σ^2. Examine if $\Sigma x_i^2/n$ is a MVBUE of σ^2.

Solution:

Since $x \sim N(0, \sigma^2)$

Then, the density function

$$f(x|\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}x^2} \; ; \; -\infty < x < \infty$$

We know,

$$L = \prod_{i=1}^{n} f(x|\sigma^2)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}x^2}$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\Sigma x_i^2}$$

Taking log on both sides, we have

$$\log L = \frac{n}{2}\log\left(\frac{1}{2\pi\sigma^2}\right) - \frac{1}{2\sigma^2}\Sigma x_i^2 \cdot \log e$$

$$= \frac{n}{2}\log\frac{1}{2\pi} - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\Sigma x_i^2$$

Now, differentiating, we get

$$\frac{\partial \log L}{\partial \sigma^2} = 0 - \frac{n}{2}\left(\frac{1}{\sigma^2}\right) + \frac{\Sigma x_i^2}{2\sigma^4}$$

$$= \frac{\Sigma x_i^2}{2\sigma^4} - \frac{n}{2}\,{}^{1}\!/_{\sigma^2}$$

$$= \frac{n}{2\sigma^4}\left[\frac{\Sigma x_i^2}{n} - \sigma^2\right]$$

We can write the form in the following term

$$\frac{\partial \log L}{\partial\theta} = A[t-\theta]$$

where, $A = \frac{n}{2\sigma^4}$, and variance $v(t) = \frac{2\sigma^4}{n}$.

Therefore, we can say that $\Sigma x_i^2/n$ is an MVBUE of σ^2 with variance $\frac{2\sigma^4}{n}$.

Hence,

The MVB of t' where t' is an unbiased estimator of σ is given by

$$(\text{MVB of } \sigma^2) \cdot \left(\frac{\partial g(\theta)}{\partial \theta}\right)^2$$

$$= \frac{2\sigma^4}{n} \cdot \frac{1}{4\sigma^2} \qquad \because \frac{\partial g(\theta)}{\partial \sigma} = \frac{1}{2\sigma}.$$

$$= \frac{\sigma^2}{2n}.$$

Thus, an MVB of σ is $\frac{\sigma^2}{2n}$ which is not attain.

⇒ x is an $N(\mu, \sigma^2)$ variate. Find the MVB of unbaised estimator of σ^2 when μ is known.

Soln. Given that

x is an $N(\mu, \sigma^2)$ variate when μ is known (here, $\mu = 0$) then, $N(0, \sigma^2)$.

Then the pdf of x is

$$f(x|\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2} \quad ; -\infty < x < \infty$$

we know,

$$L = \prod_{i=1}^{n} f(x|\sigma^2)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2} x^2}$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2} \Sigma x_i^2}$$

Taking log on both we have.

$$\log L = \frac{n}{2} \log\left(\frac{1}{2\pi\sigma^2}\right) - \frac{1}{2\sigma^2} \Sigma x_i^2 \cdot \log e.$$

$$= \frac{n}{2} \log \frac{1}{2\pi} - \frac{n}{2} \log \sigma^2 - \frac{1}{2\sigma^2} \Sigma x_i^2$$

$$\therefore \frac{\partial \log L}{\partial \sigma^2} = 0 - \frac{n}{2}\left(\frac{1}{\sigma^2}\right) + \frac{\Sigma x_i^2}{2\sigma^4}$$

$$= \frac{\Sigma x_i^2}{2\sigma^4} - \frac{n}{2} \cdot (1/\sigma^2)$$

$$= \frac{n}{2\sigma^4}\left[\frac{\Sigma x_i^2}{n} - \sigma^2\right]$$

Therefore, we can say that $\Sigma x_i^2/n$ is the MVBUE of σ^2 with variance $2\sigma^4/n$.

Problem: x is an $N(\mu, \sigma^2)$ variate. Find an MVB of unbiased estimator of σ^2 when μ is unknown.

Answer:

Given that

x is an $N(\mu, \sigma^2)$, when μ is unknown.

Then the density function of x is

$$f(x|\mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} ; -\infty < x < \infty.$$

Now, the likehood function is

$$L = \prod_{i=1}^{n} f(x|\mu, \sigma^2)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2}\frac{\sum_{i=1}^{n}(x-\mu)^2}{\sigma^2}}$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left[\frac{-1}{2\sigma^2}\Sigma(x-\mu)^2\right]$$

$$\log L = \frac{n}{2}\log\frac{1}{(2\pi\sigma^2)} - \frac{1}{2\sigma^2}\Sigma(x_i-\mu)^2 \log e$$

$$= \frac{n}{2}\log\frac{1}{(2\pi\sigma^2)} - \frac{1}{2\sigma^2}\Sigma(x_i-\mu)^2$$

$$\frac{\partial \log L}{\partial \sigma^2} = 0 - \frac{n}{2}\left(\frac{1}{\sigma^2}\right) + \frac{\Sigma(x_i-\mu)^2}{2\sigma^4}$$

$$= \frac{\Sigma(x_i-\mu)^2}{2\sigma^4} - \frac{n}{2}\cdot 1/\sigma^2$$

$$= \frac{n}{2\sigma^4}\left[\frac{\Sigma(x_i-\mu)^2}{n} - \sigma^2\right]$$

$$\therefore \frac{\partial \log L}{\partial \sigma^2} = \frac{n}{2\sigma^4}\left[\frac{\Sigma(x_i-\mu)^2}{n} - \sigma^2\right]$$

which can be expressed in the form as $\frac{\partial \log L}{\partial \theta} = A[t-\theta]$. where $A = n/2\sigma^4$.

Therefore, we can say that $\frac{\Sigma(x_i-\mu)}{n}$ is a the MVBUE of σ^2 with variance $2\sigma^4/n$.

3h

Question: Estiblish the method of finding MVB for a unbiased estimator intended to estimate function of a parameter.

Answer:

Suppose we have found an MVB unbiased estimator θ. This case, we use MVB of unbiased estimator of g a function of θ.

Let, $E(t') = g(\theta)$.

Now,

$$\text{MVB of } t = \frac{\left[\frac{\delta E(t)}{\partial\theta}\right]^2}{nE\left[\frac{\partial \log f}{\partial\theta}\right]^2}$$

$$= \frac{1}{nE\left[\frac{\partial \log f}{\partial\theta}\right]^2} \qquad \because E(t)=\theta,\ \frac{\partial E(t)}{\partial\theta}=1$$

and

$$\text{MVB of } t' = \frac{\left[\frac{\partial E(t')}{\delta g(\theta)}\right]^2}{nE\left[\frac{\partial \log f}{\partial g(\theta)}\right]^2}$$

$$= \frac{1}{nE\left[\frac{\partial \log f}{\partial g(\theta)}\right]^2} \qquad \because E(t') = g(\theta),\ \frac{\partial E(t')}{\partial g(\theta)} = 1$$

$$= \frac{1}{E\left[\frac{\partial \log L}{\partial g(\theta)}\right]^2}$$

$$= \frac{1}{E\left[\frac{\partial \log L}{\partial\theta}\cdot\frac{\partial\theta}{\partial g(\theta)}\right]^2}$$

$$= \frac{1}{E\left(\frac{\partial \log L}{\partial \theta}\right)^2 \left(\frac{\partial \theta}{\partial g(\theta)}\right)^2}$$

$$= \frac{1}{\left(\frac{\partial \theta}{\partial g(\theta)}\right)^2 E\left(\frac{\partial \log L}{\partial \theta}\right)^2}$$

$$= \frac{1}{\left(\frac{\partial \theta}{\partial g(\theta)}\right)^2 E\left(\frac{\partial \log L}{\partial \theta}\right)^2}$$

$$= \frac{1}{\left(\frac{\partial \theta}{\partial g(\theta)}\right)^2} \cdot \frac{1}{E\left(\frac{\partial \log L}{\partial \theta}\right)^2}$$

$$= \frac{1}{\left(\frac{\partial \theta}{\partial g(\theta)}\right)^2} \cdot MVB(t)$$

$$= MVB(t) \cdot \left(\frac{\partial g(\theta)}{\partial \theta}\right)^2$$

$$\therefore MVB \text{ of } t' = MVB(t) \cdot \left(\frac{\partial g(\theta)}{\partial \theta}\right)^2$$

(showed)

Example: x is an $N(0, \sigma^2)$ variate. Find an MVB unbiased estimator of σ.

Answer:

Given that:

x is an $N(0, \sigma^2)$ variate.

Let, $\theta = \sigma^2$ and $g(\theta) = \sigma$

Now, we know the likelyhood function

$$L = \prod_{i=1}^{n} f(x|\sigma^2)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}(x-\mu/\sigma)^2};$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2}$$

Taking log on both sides. we have.

$$\log L = \frac{n}{2}\log\left(\frac{1}{2\pi\sigma^2}\right) - \frac{1}{2\sigma^2}\sum(x_i-\mu)^2$$

$$= \frac{n}{2}\log\frac{1}{2\pi} - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum(x_i-\mu)^2$$

Now, differentiating ~~lot~~ $\log L$ with respect to σ^2,

$$\frac{\partial \log L}{\partial \sigma^2} = 0 - \frac{n}{2}\frac{1}{\sigma^2} + \frac{1}{2\sigma^4}\sum(x_i-\mu)^2$$

$$= \frac{1}{2\sigma^4}\sum(x_i-\mu)^2 - \frac{n}{2\sigma^2}$$

$$= \frac{1}{2\sigma^4}\sum x_i^2 - \frac{n}{2\sigma^2} \quad \text{Since } \mu = 0$$

$$= \frac{n}{2\sigma^4}\left[\frac{\sum x_i^2}{n} - \sigma^2\right]$$