201

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Sheet – 02

Topic:

(i). χ^2 distribution
(ii). t-distribution
(iii). F-distribution

"Sampling Distribution"

1.a

Sampling distribution:

"A sampling distribution is a probability distribution of a statistic obtained through a large number of samples drawn from a specific population."

The sampling distribution of a given population is the distribution of frequencies of a range of different outcomes that could possibly occur for a statistic of a population.

Sampling distributions are important in statistics because they provide a major simplification enroute to statistical inference.

More specifically, they allow analytical considerations to be based on the probability distribution of a statistic, rather than on the joint probability distribution of all the individual sample values.

1b

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For example:

χ^2, F and t distributions are sampling distribution

"The distribution of sample statistics is called sampling distribution."

parent distribution:

"Measurement of any physical quantity is always affected by uncontrollable random ("stochastic") processes. These produce a statistical scatter in the values measured.

1.C

The parent distribution for a given measurement gives the probability of obtaining a particular result from a single measure."

"The probability distribution of parameter is called parent distribution."

for example:

Normal, Binomial distributions are parent distribution.

* Distinguish between sampling distribution and parent distribution is given in previous lecture.

~~Discuss about χ^2, t and t~~

* Discuss about χ^2, t and F distribution

1.D

χ^2 (chi-square) distribution:

The sum of squares of n independent standard normal variates is called chi-squares (χ^2) variate with n degrees of freedom.

Let $z_1, z_2, \cdots, z_n$ be n independent standard normal variates, then chi-square denoted by χ^2, is defined as

$$\chi_n^2 = \sum_{i=1}^{n} z_i^2$$

However, if $x_1, x_2, \cdots, x_n$ are n independently and identically distributed random variables each of which is normally distributed with mean μ and variance σ^2. Then

$$\chi_n^2 = \sum_{i=1}^{n} \left(\frac{x_i-\mu}{\sigma}\right)^2$$

is a chi-square (χ^2) variate with n degree of freedom.

Properties of χ^2 distribution:

(i). χ^2 is a continuous type of distribution and its range is 0 to ∞ i.e. $0 < \chi^2 < \infty$.

(ii). The distribution contains only one parameter which is the degree of freedom of the distribution.

(iii). The mean and variance of χ^2 distribution for n d.f. ~~is~~ is n and 2n respectively.

(iv). The mode of χ^2 distribution for n d.f. is $(n-2)$.

(v). The moment generating function of χ^2-distribution for n d.f. is $(1-2t)^{-n/2}$.

(vi). χ^2 distribution tends to normal distribution for large degree of freedom.

(vii). It is positively skewed distribution for smaller values of n.

(viii). The distribution becomes symmetrical as n tends to infinity $(n \to \infty)$.

Application /uses of chi-square (χ^2) distribution:

(i). To test if the hypothetical value of the population variance is $\sigma^2 = \sigma_0^2$ (say).

(ii). To test the goodness of fit.

(iii). To test the independence of attributes.

(iv). To test the homogeneity of independent estimates of the population variance.

(v). To test the homogeneity of independent estimates of the population correlation coefficient.

(vi). To combine various probabilities obtained from independent experiments to give a single test of significance.

Problem:

Suppose, $X \sim N(0,1)$. Obtain the pdf of $Y = X^2$ by m.g.f technique.

Solution:

Here, $X \sim N(0,1)$. Then the pdf of x is as.

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \quad ; \quad -\infty < x < \infty$$

Now, the mgf of Y is given by

$$M_Y(t) = M_{X^2}(t) = E[e^{tX^2}]$$

$$= \int_{-\infty}^{\infty} e^{tx^2} f(x) \cdot dx$$

$$\Rightarrow M_Y(t) = \int_{-\infty}^{\infty} e^{tx^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \cdot dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(\frac{1}{2}-t)x^2} \cdot dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(\frac{1}{2}-t)x^2} \cdot dx \quad \left[\because \text{the integrand is an even function of } x\right]$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{1}{2}(1-2t)x^2} \cdot dx$$

$$\cancel{= \frac{2}{\sqrt{2\pi}} \cdot \frac{1}{2} \cdot \frac{\sqrt{\frac{1}{2}}}{\frac{1}{2}(1-2t)}}$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{1}{2}(1-2t)x^2} \cdot x^{2\cdot\frac{1}{2}-1} \cdot dx$$

$$= \frac{2}{\sqrt{2\pi}} \frac{1}{2} \frac{\sqrt{\frac{1}{2}}}{\left[\frac{1}{2}(1-2t)\right]^{\frac{1}{2}}} \quad \left[\because \frac{\frac{1}{2}\sqrt{n}}{a^n} = \int_0^{\infty} e^{-ax^2} x^{2n-1} \cdot dx\right]$$

$$= \frac{1}{\sqrt{2}\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{\frac{1}{\sqrt{2}} \cdot (1-2t)^{\frac{1}{2}}} = \frac{1}{\sqrt{2}\sqrt{\pi}} \cdot \frac{\sqrt{2}\sqrt{\pi}}{(1-2t)^{\frac{1}{2}}}$$

$$\Rightarrow M_Y(t) = \frac{1}{(1-2t)^{\frac{1}{2}}} = (1-2t)^{-\frac{1}{2}}$$

$$\therefore M_Y(t) = (1-2t)^{-\frac{1}{2}} \quad \left[\alpha = \tfrac{1}{2},\ \beta = 2\right]$$

which is the mgf of gamma random variable with shape parameter $\alpha = \frac{1}{2}$ and scale parameter $\beta = 2$.

Therefore, the distribution of Y is gamma with shape parameter $\alpha = \frac{1}{2}$ and scale parameter $\beta = 2$.

i.e., $g(y) = \frac{1}{2^{\frac{1}{2}} \sqrt{\frac{1}{2}}}\, y^{\frac{1}{2}-1} e^{-y/2}$; $y > 0$

Question :

Derivation of the chi-square (χ^2) distribution by the method of moment generating function.

Derivation :

Let $x_1, x_2, \cdots, x_n$ be n independent random variable from $N(\mu, \sigma^2)$ i.e., $x_i \sim N(\mu, \sigma^2)$; $i = 1, 2, 3, \cdots, n$

x_i's are independent.

Now we want to find the distribution of

$\chi^2 = \sum y_i = \sum \left(\frac{x_i - \mu}{\sigma}\right)^2$ by m.g.f technique.

Hence, the mgf of χ^2 is given by.

$$M_{\chi^2}(t) = M_{\sum y_i}(t) = \prod_{i=1}^{m} M_{y_i}(t) \quad \left[\because y_i\text{'s are independent}\right]$$

$$\Rightarrow M_{\chi^2}(t) = \prod_{i=1}^{n} \left[M_{\left(\frac{x_i-\mu}{\sigma}\right)^2}(t) \right]$$

$$= \prod_{i=1}^{n} E\left[e^{t\left(\frac{x_i-\mu}{\sigma}\right)^2} \right]$$

$$= \prod_{i=1}^{n} E\left[e^{tu^2} \right] \qquad \left[\text{let } u = \frac{x_i-\mu}{\sigma}\right]$$

$$= \prod_{i=1}^{n} \left\{ \int_{-\infty}^{\infty} e^{tu^2} f(u) \cdot du \right\}$$

$$= \prod_{i=1}^{n} \left\{ \int_{-\infty}^{\infty} e^{tu^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-1/2 u^2} du \right\}$$

$$= \prod_{i=1}^{n} \left\{ \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{tu^2 - 1/2 u^2} du \right\}$$

[Since the integrand is an even function of u]

$$= \prod_{i=1}^{n} \left\{ \frac{\sqrt{2}}{\sqrt{\pi}} \int_{0}^{\infty} e^{-(1/2 - t)u^2} du \right\}$$

Let, $z = (1/2 - t)u^2 \Rightarrow u^2 = \frac{z}{\left(\frac{1-2t}{2}\right)} \Rightarrow u = \sqrt{\frac{2z}{1-2t}}$

$\Rightarrow dz = (1/2 - t)\, 2u \cdot du$

$\Rightarrow 2u \cdot du = \frac{dz}{\left(\frac{1-2t}{2}\right)}$

$\Rightarrow du = \frac{dz}{2u\left(\frac{1-2t}{2}\right)} \Rightarrow du = \frac{dz}{(1-2t)u}$

$$\Rightarrow du = \frac{dz}{(1-2t)\sqrt{\frac{2z}{(1-2t)}}} = \frac{dz}{\sqrt{1-2t}\sqrt{2z}}$$

$$\therefore M_{\chi^2}(t) = \prod_{i=1}^{n} \left\{ \frac{\sqrt{2}}{\sqrt{\pi}} \int_{0}^{\infty} e^{-z} \cdot \frac{dz}{\sqrt{1-2t}\sqrt{2z}} \right\}$$

$$= \prod_{i=1}^{n} \left\{ \frac{\sqrt{2}}{\sqrt{\pi}\sqrt{1-2t}\sqrt{2}} \int_{0}^{\infty} e^{-z} z^{-1/2} \cdot dz \right\}$$

$$= \prod_{i=1}^{n} \left\{ \frac{1}{\sqrt{\pi}\sqrt{1-2t}} \int_{0}^{\infty} e^{-z} z^{1/2-1} \cdot dz \right\}$$

$$= \prod_{i=1}^{n} \left\{ \frac{1}{\sqrt{\pi}\sqrt{1-2t}} \Gamma 1/2 \right\} \qquad \left[\because \Gamma n = \int_0^\infty e^{-x} x^{n-1} \cdot dx\right]$$

$$= \prod_{i=1}^{n} \left\{ \frac{1}{\sqrt{\pi}\sqrt{1-2t}} \sqrt{\pi} \right\}$$

$$= \prod_{i=1}^{n} \left\{ (1-2t)^{-1/2} \right\}$$

$$= (1-2t)^{-n/2}$$

$$\therefore M_{\chi^2}(t) = (1-2t)^{-n/2}$$

Which is the mgf of gamma distribution with shape parameter $\alpha = n/2$ and scale parameter $\beta = 2$.

Therefore, the pdf χ^2 distribution is as-

$$f(\chi^2) = \frac{1}{2^{n/2}\sqrt{n/2}} (\chi^2)^{n/2-1} e^{-\chi^2/2} \;;\quad \chi^2 > 0 \; (0 < \chi^2 < \infty)$$

This is the pdf of χ^2 variate with n degree of freedom.

Question:

Show that the total probability of chi-square (χ^2)-distribution is unity.

i.e., $\int_0^\infty f(\chi^2)\, d\chi^2 = 1$.

Proof:

The pdf of χ^2 distribution with n degree of freedom is given by

$$f(\chi^2) = \frac{1}{2^{n/2}\sqrt{n/2}} (\chi^2)^{n/2-1} e^{-\chi^2/2} \;;\quad 0 < \chi^2 < \infty$$

Thus,

$$\int_0^\infty f(\chi^2).d\chi^2 = \int_0^\infty \frac{1}{2^{n/2}\sqrt{n/2}} (\chi^2)^{n/2-1} e^{-\chi^2/2}.d\chi^2$$

$$= \frac{1}{2^{n/2}\sqrt{n/2}} \int_0^\infty e^{-\frac{1}{2}\chi^2} (\chi^2)^{n/2-1}.d\chi^2$$

$$= \frac{1}{2^{n/2}\sqrt{n/2}} \cdot \frac{\sqrt{n/2}}{(1/2)^{n/2}} \qquad \left[\because \frac{\sqrt{n}}{\alpha^n} = \int_0^\infty e^{-\alpha x} x^{n-1}.dx\right]$$

$$\Rightarrow \int_0^\infty f(\chi^2)\, d\chi^2 = \frac{1}{2^{n/2}} \cdot 2^{n/2} = 1$$

$$\therefore \int_0^\infty f(\chi^2)\, d\chi^2 = 1$$

Thus, the total probability of chi-square (χ^2) distribution is unity. (showed)

Question: Find mean and variance of chi-square (χ^2) distribution.

Solution:

The pdf of χ^2 distribution with n degree of freedom is given by-

$$f(\chi^2) = \frac{1}{2^{n/2}\sqrt{n/2}} (\chi^2)^{n/2-1} e^{-\chi^2/2} \;;\quad 0 < \chi^2 < \infty$$

Mean:

$$E(\chi^2) = \int_0^\infty \chi^2 \cdot f(\chi^2).d\chi^2$$

$$= \int_0^\infty \chi^2 \cdot \frac{1}{2^{n/2}\sqrt{n/2}} (\chi^2)^{n/2-1} e^{-\chi^2/2}.d\chi^2$$

$$= \frac{1}{2^{n/2}\sqrt{n/2}} \int_0^\infty (\chi^2)^{(n/2+1)-1} e^{-\frac{1}{2}\chi^2}.d\chi^2$$

$$= \frac{1}{2^{n/2}\sqrt{n/2}} \cdot \frac{\sqrt{n/2+1}}{(1/2)^{(n/2+1)}} \qquad \left[\because \frac{\sqrt{n}}{\alpha^n} = \int_0^\infty x^{n-1} e^{-\alpha x}.dx\right.$$

$$\Rightarrow E(\chi^2) = \frac{1}{2^{n/2}\Gamma\frac{n}{2}} \cdot \frac{n}{2}\,\Gamma\frac{n}{2} \cdot 2^{\left(\frac{n}{2}+1\right)}$$

$$= \frac{1}{2^{n/2}\Gamma\frac{n}{2}} \cdot \frac{n}{2}\,\Gamma\frac{n}{2} \cdot 2^{n/2} \cdot 2$$

$$= \frac{2n}{2}$$

$$\therefore E(\chi^2) = n$$

$\therefore$ Mean $= n$

$$\therefore E[(\chi^2)^2] = \int_0^\infty (\chi^2)^2 f(\chi^2)\, d\chi^2$$

$$= \int_0^\infty (\chi^2)^2 \frac{1}{2^{n/2}\Gamma\frac{n}{2}} (\chi^2)^{\frac{n}{2}-1} e^{-\frac{1}{2}\chi^2}\, d\chi^2$$

$$= \frac{1}{2^{n/2}\Gamma\frac{n}{2}} \int_0^\infty (\chi^2)^{\left(\frac{n}{2}+2\right)-1} e^{-\frac{1}{2}\chi^2}\, d\chi^2$$

$$= \frac{1}{2^{n/2}\Gamma\frac{n}{2}} \frac{\Gamma\left(\frac{n}{2}+2\right)}{\left(\frac{1}{2}\right)^{\left(\frac{n}{2}+2\right)}} \qquad \left[\because \frac{\Gamma n}{\alpha^n} = \int_0^\infty x^{n-1} e^{-\alpha x}\, dx\right]$$

$$= \frac{1}{2^{n/2}\Gamma\frac{n}{2}} \frac{\left(\frac{n}{2}+1\right)\frac{n}{2}\,\Gamma\frac{n}{2}}{\frac{1}{2^{\left(\frac{n}{2}+2\right)}}}$$

$$= \frac{1}{2^{n/2}\Gamma\frac{n}{2}} \cdot \left(\frac{n}{2}+1\right)\frac{n}{2}\,\Gamma\frac{n}{2} \cdot 2^{n/2} \cdot 2^2$$

$$\Rightarrow E[(\chi^2)^2] = 4\cdot\frac{n}{2}\left(\frac{n}{2}+1\right) = 2n\left(\frac{n}{2}+1\right) = n^2+2n$$

$$\therefore E[(\chi^2)^2] = n^2+2n$$

$$\therefore V(\chi^2) = E[(\chi^2)^2] - [E(\chi^2)]^2$$

$$= n^2+2n-n^2$$

$$\therefore V(\chi^2) = 2n$$

Therefore, the mean and variance of χ^2 distribution is n and $2n$ respectively.

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Question:

Find moment generating function of χ^2 distribution and hence find mean, variance, skewness and kurtosis of the distribution and comment shape of the distribution.

Answer:

The pdf of χ^2-distribution with n d.f. is given by

$$f(\chi^2) = \frac{1}{2^{n/2}\Gamma\frac{n}{2}} (\chi^2)^{\frac{n}{2}-1} e^{-\chi^2/2} \quad ; 0<\chi^2<\infty$$

Hence, the moment generating function of χ^2 distribution is as:

$$M_{\chi^2}(t) = E[e^{t\chi^2}]$$

$$= \int_0^\infty e^{t\chi^2} f(\chi^2)\, d\chi^2$$

$$\Rightarrow M_{\chi^2}(t) = \int_0^\infty e^{t x^2} \frac{1}{2^{n/2}\Gamma(n/2)} (x^2)^{n/2-1} e^{-\frac{1}{2}x^2}\, dx^2$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\infty e^{t x^2 - \frac{1}{2}x^2} \cancel{dx^2} (x^2)^{n/2-1}\, dx^2$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\infty e^{-\frac{1}{2}x^2(1-2t)} \cdot (x^2)^{n/2-1}\, dx^2$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\infty (x^2)^{n/2-1} \cdot e^{-\left(\frac{1-2t}{2}\right)x^2} \cdot dx^2$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)} \frac{\Gamma(n/2)}{\left(\frac{1-2t}{2}\right)^{n/2}} \qquad \left[\because \frac{\Gamma n}{\alpha^n} = \int_0^\infty x^{n-1} e^{-\alpha x} \cdot dx\right]$$

$$= \frac{1}{2^{n/2}} \cdot \frac{2^{n/2}}{(1-2t)^{n/2}} = \frac{1}{(1-2t)^{n/2}} = (1-2t)^{-n/2}$$

$$\therefore M_{\chi^2}(t) = (1-2t)^{-n/2}$$

This is the moment generating function of χ^2 distribution.

Cumulant generating function (CGF):

Now, cumulant generating function of χ^2-distribution is -

$$K_{\chi^2}(t) = \log[M_{\chi^2}(t)]$$
$$= \log\left[(1-2t)^{-n/2}\right]$$

$$\Rightarrow K_{\chi^2}(t) = \frac{n}{2}\left(2t + \frac{(2t)^2}{2} + \frac{(2t)^3}{3} + \frac{(2t)^4}{4} + \cdots\right)$$

$$= \frac{n}{2}\left(2t + \frac{4t^2}{2} + \frac{8t^3}{3} + \frac{16t^4}{4} + \cdots\right)$$

$$= \left(\frac{t}{1!} n + \frac{t^2}{2!} \cdot 2n + \frac{t^3}{3!} 8n + \frac{t^4}{4!} 48n + \cdots\right)$$

By the defination, we know that

$$K_x(t) = \frac{t}{1!}\kappa_1 + \frac{t^2}{2!}\kappa_2 + \frac{t^3}{3!}\kappa_3 + \frac{t^4}{4!}\kappa_4 + \cdots$$

Thus, κ_r = Coefficient of $\frac{t^r}{r!}$ in $K(t)$.

Therefore, Comparing the coefficients we get-

$\kappa_1 = \text{mean} = n$, $\kappa_2 = \text{variance} = \mu_2 = 2n$, $\kappa_3 = \mu_3 = 8n$,

$\kappa_4 = 48n$, $\mu_4 = \kappa_4 + 3\kappa_2^2 = 48n + 3\cdot 4n^2 = 48n + 12n^2$

$$\therefore \mu_4 = 48n + 12n^2$$

Skewness:

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{64n^2}{8n^3} = \frac{8}{n} > 0$$

Kurtosis:

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{48n + 12n^2}{4n^2} = \frac{48n}{4n^2} + \frac{12n^2}{4n^2}$$
$$= \frac{12}{n} + 3$$

$$\therefore \beta_2 = 3 + \frac{12}{n} > 3$$

Comment:

The χ^2 distribution is positively skewed (since $\beta_1 > 0$) and leptokurtic (since $\beta_2 > 3$).

$\therefore$ Mean $= n$, variance $= 2n$, $\beta_1 = 8/n$, $\beta_2 = 3 + \frac{12}{n}$

Question:

Find the mode of χ^2 distribution.

Mode:

The mode of the distribution will be obtained by the solution of the equation.

$$\frac{d \log f(\chi^2)}{d\chi^2} = 0 \text{ ; provided } \frac{d^2 \log f(\chi^2)}{d(\chi^2)^2} < 0.$$

$\therefore$ we know that, the pdf of χ^2-distribution with n degree of freedom is

$$f(\chi^2) = \frac{1}{2^{n/2}\Gamma n/2} (\chi^2)^{n/2 - 1} e^{-\chi^2/2}$$

$$\Rightarrow \log f(\chi^2) = \log \frac{1}{2^{n/2}\Gamma n/2} + (n/2 - 1)\log \chi^2 - \frac{1}{2}\chi^2$$

$$\therefore \frac{d \log f(\chi^2)}{d\chi^2} = 0 + (n/2 - 1)\cdot\frac{1}{\chi^2} - 1/2$$

Now $\frac{d \log f(\chi^2)}{d\chi^2} = 0$

$$\Rightarrow (n/2 - 1)\frac{1}{\chi^2} - 1/2 = 0$$

$$\Rightarrow \frac{n-2}{2\chi^2} - 1/2 = 0$$

$$\Rightarrow \frac{n-2-\chi^2}{2\chi^2} = 0 \Rightarrow n-2-\chi^2 = 0 \Rightarrow \chi^2 = n-2 \text{ ; } n > 2$$

$\therefore \chi^2 = n-2$; $n > 2$

Now $\frac{d^2 \log f(\chi^2)}{d(\chi^2)^2} = -\left(\frac{n}{2} - 1\right)\frac{1}{(\chi^2)^2} < 0$

$\cancel{\text{so} \left[\frac{d^2 \log f(\chi^2)}{d(\chi^2)^2}\right]_{\chi^2 = n-2} < 0}$

So, the mode of χ^2 distribution is, $\chi^2 = n-2$.

Note: we know for χ^2_n, $\beta_1 = 8/n$ and $\beta_2 = 3 + \frac{12}{n}$

As, $n \to \infty$, $\beta_1 \to 0$ and $\beta_2 \to 3$, then chi-square (χ^2_n) distribution tends to normal distribution.

Problem:

Suppose, $U_i \sim \chi^2_{n_i}$; $i = 1, 2, 3, \cdots, K$. U_i's are independent.

Obtain the pdf of $Y = \sum_{i=1}^{K} U_i$ by mgf technique.

or, sum of ~~χ^2 vari~~ ~~K~~ independent χ^2 variates is also χ^2 variate.

Solution:

We know that, the mgf of χ^2 distribution with n degree of freedom is

$$M_{\chi^2}(t) = (1-2t)^{-n/2}$$

Now, we want to find the pdf of $Y = \sum_{i=1}^{k} U_i$ by mgf technique.

Hence, the mgf of Y is given by

$$M_Y(t) = M_{\sum_{i=1}^{k} U_i}(t)$$

$$= \prod_{i=1}^{k} M_{U_i}(t)$$

$$= M_{U_1}(t) \cdot M_{U_2}(t) \cdots M_{U_k}(t)$$

$$= (1-2t)^{-n_1/2} \cdot (1-2t)^{-n_2/2} \cdots (1-2t)^{-n_k/2}$$

$$= \prod_{i=1}^{k} (1-2t)^{-n_i/2}$$

$$\therefore M_Y(t) = (1-2t)^{-\sum_{i=1}^{k} n_i/2} \qquad \left[\text{where } n = \sum_{i=1}^{k} n_i\right]$$

which is the mgf of χ^2 variate with $\sum_{i=1}^{k} n_i$ degree of freedom.

Therefore, the distribution of Y is χ^2 with $\sum_{i=1}^{k} n_i$ degree of freedom.

Problem: Suppose, $U \sim \chi^2_n$ and $U_1 \sim \chi^2_1$. $U = U_1 + U_2$. Obtain the U_2 degree of freedom.

Solution:

Let $U = U_1 + U_2$; where $U \sim \chi^2_n$ and $U_1 \sim \chi^2_1$. U_1 and U_2 are independent.

Now the mgf of U is given by

$$M_U(t) = M_{U_1+U_2}(t) = E\left[e^{tU_1} \cdot e^{tU_2}\right]$$

$$\Rightarrow M_U(t) = E\left[e^{tU_1}\right] \cdot E\left[e^{tU_2}\right] \qquad \left[\because U_1 \text{ and } U_2 \text{ are independent}\right]$$

$$\Rightarrow M_U(t) = M_{U_1}(t) \cdot M_{U_2}(t)$$

$$\Rightarrow (1-2t)^{-n/2} = (1-2t)^{-1/2} M_{U_2}(t)$$

$$\Rightarrow M_{U_2}(t) = \frac{(1-2t)^{-n/2}}{(1-2t)^{-1/2}} = (1-2t)^{-n/2+1/2}$$

$$\Rightarrow M_{U_2}(t) = (1-2t)^{-\left(\frac{n-1}{2}\right)}$$

$$\therefore M_{U_2}(t) = (1-2t)^{-\frac{(n-1)}{2}}$$

which is the mgf of χ^2 variate with $(n-1)$ degree of freedom.

Therefore, the degree of freedom of U_2 is $(n-1)$.

Problem:

$X_1 \sim \chi^2_{n_1}$ and $X_2 \sim \chi^2_{n_2}$. X_1 and X_2 are independent or, $X_i \sim \chi^2_{n_i}$; $i = 1, 2$. X_i's are independent. Obtain the pdf of $Y = X_1 + X_2$ by mgf technique.

Solution:

We know that, the mgf of χ^2-distribution with n degree of freedom is

$$M_{\chi^2}(t) = (1-2t)^{-n/2}$$

Now, we want to find the pdf of $Y = X_1 + X_2$ by mgf technique.

Hence, the mgf of Y is given by

$$M_Y(t) = E[e^{tY}] = E[e^{t(X_1+X_2)}]$$
$$= E[e^{tX_1} \cdot e^{tX_2}]$$
$$= E[e^{tX_1}] \cdot E[e^{tX_2}] \quad [\because X_1 \text{ and } X_2 \text{ are independent}]$$
$$= M_{X_1}(t) \cdot M_{X_2}(t)$$
$$= (1-2t)^{-n_1/2} (1-2t)^{-n_2/2}$$
$$= (1-2t)^{-(n_1/2 + n_2/2)}$$
$$\therefore M_Y(t) = (1-2t)^{-\frac{(n_1+n_2)}{2}}$$

which is the mgf of χ^2 variate with (n_1+n_2) degree of freedom i.e. $\chi^2_{n_1+n_2}$

Therefore, the distribution of Y is χ^2 with (n_1+n_2) degree of freedom ($\chi^2_{n_1+n_2}$).

$\Rightarrow$ The sum of two independent χ^2 variate is also a χ^2 variate.

Problem:

Let X_1 and X_2 be two independent χ^2 variates with n_1 and n_2 degrees of freedom respectively. or, $X_1 \sim \chi^2_{n_1}$ and $X_2 \sim \chi^2_{n_2}$. X_1 and X_2 are independent.

Prove that, $U = X_1 + X_2$ and $V = \frac{X_1}{X_1+X_2}$ are independent. Hence obtain the pdf of U and V.

Solution:

The pdf of X_1 is as-

$$f(x_1) = \frac{1}{2^{n_1/2}\,\Gamma n_1/2}\, x_1^{n_1/2 - 1} e^{-x_1/2} \quad ; x_1 > 0$$

The pdf of X_2 is as-

$$f(x_2) = \frac{1}{2^{n_2/2}\,\Gamma n_2/2}\, x_2^{n_2/2 - 1} e^{-x_2/2} \quad ; x_2 > 0$$

Now, the joint pdf of x_1 and x_2 is given as

$$f(x_1, x_2) = f(x_1) \cdot f(x_2) \qquad \left[\because x_1 \text{ and } x_2 \text{ are independent}\right]$$

$$= \frac{1}{2^{n_1/2}\,\overline{|n_1/2}}\, x_1^{n_1/2 - 1}\, e^{-x_1/2} \cdot \frac{1}{2^{n_2/2}\,\overline{|n_2/2}}\, x_2^{n_2/2-1}\, e^{-x_2/2}$$

$$\therefore f(x_1, x_2) = \frac{1}{2^{\frac{n_1+n_2}{2}} \cdot \overline{|n_1/2}\,\overline{|n_2/2}} \cdot x_1^{n_1/2-1} \cdot x_2^{n_2/2-1} \cdot e^{-\left(\frac{x_1+x_2}{2}\right)} \quad ; \quad x_1, x_2 > 0$$

Here, $u = x_1 + x_2$

$\Rightarrow x_2 = u - x_1$

$\Rightarrow x_2 = u - uv$

$\Rightarrow x_2 = u(1-v)$

and $v = \frac{x_1}{x_1+x_2}$

$\Rightarrow v = \frac{x_1}{u} \quad [\because u = x_1 + x_2]$

$\Rightarrow x_1 = uv$

Then the Jacobian of the transformation is

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ (1-v) & -u \end{vmatrix} = -uv - u(1-v)$$
$$= -uv - u + uv$$
$$= -u$$

$\Rightarrow J = -u$.

$\therefore |J| = u$

Now, the joint pdf of u and v is given as

$$g(u,v) = \cancel{f(x_1, x_2) \cdot |J|} = f(u,v) \cdot |J|$$

$$\Rightarrow g(u,v) = \frac{1}{2^{\frac{n_1+n_2}{2}}\,\overline{|n_1/2}\,\overline{|n_2/2}} \cdot (uv)^{n_1/2-1}\,[u(1-v)]^{n_2/2-1}\, e^{-u/2} \cdot u$$

$$= \frac{1}{2^{\frac{n_1+n_2}{2}}\,\overline{|n_1/2}\,\overline{|n_2/2}} \cdot u^{n_1/2 - 1 + n_2/2 - \cancel{1} + \cancel{1}} \cdot v^{n_1/2-1}\,(1-v)^{n_2/2-1} \cdot e^{-u/2}$$

$$= \frac{1}{2^{\frac{n_1+n_2}{2}} \cdot \overline{|n_1/2}\,\overline{|n_2/2}}\, u^{\frac{n_1+n_2}{2}-1}\, v^{n_1/2-1}\,(1-v)^{n_2/2-1} \cdot e^{-u/2}$$

$$= \frac{1}{2^{\frac{n_1+n_2}{2}}\,\overline{|\frac{n_1+n_2}{2}}} \cdot u^{\frac{n_1+n_2}{2}-1} \cdot e^{-u/2} \cdot \frac{v^{n_1/2-1}(1-v)^{n_2/2-1}}{\frac{\overline{|n_1/2}\,\overline{|n_2/2}}{\overline{|\frac{n_1+n_2}{2}}}}$$

$$= \frac{1}{2^{\frac{n_1+n_2}{2}}\,\overline{|\frac{n_1+n_2}{2}}}\, u^{\frac{n_1+n_2}{2}-1}\, e^{-u/2} \cdot \frac{1}{\beta(n_1/2, n_2/2)} \cdot v^{n_1/2-1} \cdot (1-v)^{n_2/2-1}$$

$$= g(u) \cdot g(v)$$

$$= \chi^2_{n_1+n_2} \cdot \beta_1(n_1/2, n_2/2)$$

$\Rightarrow g(u,v) = g(u) \cdot g(v)$; u and v are independent (proved)

Here, $g(u,v)$ can be expressed as ~~theirs~~ the product

of their marginal pdf. Hence U and V are independently distributed (proved).
Also it is seen that U is a χ^2 variate with (n_1+n_2) degree of freedom and V is a beta variate of the first kind with parameters $n_1/2$ and $n_2/2$.

Problem :
$X_1 \sim \chi^2_{n_1}$ and $X_2 \sim \chi^2_{n_2}$. X_1 and X_2 are independent, obtain the distribution of $U = \frac{X_1}{X_2}$.

Solution :
If X_1 and X_2 are two independent χ^2 variates with n_1 and n_2 degree of freedom respectively. Then the pdf of X_1 and X_2 are.

$$f(x_1) = \frac{1}{2^{n_1/2}\Gamma n_1/2} \cdot x_1^{n_1/2-1} \cdot e^{-x_1/2} \quad ; \quad x_1 > 0$$

$$f(x_2) = \frac{1}{2^{n_2/2}\Gamma n_2/2} x_2^{n_2/2-1} e^{-x_2/2} \quad ; \quad x_2 > 0$$

Then the joint pdf of X_1 and X_2 is given by-

$$f(x_1, x_2) = f(x_1) \cdot f(x_2) \quad [\text{since, } x_1 \text{ and } x_2 \text{ are independent}]$$

$$= \frac{1}{2^{n_1/2}\Gamma n_1/2}(x_1)^{n_1/2-1} e^{-x_1/2} \cdot \frac{1}{2^{n_2/2}\Gamma n_2/2} x_2^{n_2/2-1} e^{-x_2/2}$$

$$\therefore f(x_1, x_2) = \frac{1}{2^{\frac{n_1+n_2}{2}}\Gamma n_1/2 \, \Gamma n_2/2} \cdot x_1^{n_1/2-1} x_2^{n_2/2-1} e^{-\left(\frac{x_1+x_2}{2}\right)} \quad ; \quad x_1, x_2 > 0$$

Here, $U = \frac{X_1}{X_2}$ let $V = X_2$

$\Rightarrow U = \frac{X_1}{V}$

$\therefore X_1 = UV$ and $X_2 = V$

Then the Jacobian of the transformation is-

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$$

$\therefore |J| = v$

Then the joint pdf of U and V is as

$$g(u, v) = f(u, v) \cdot |J|$$

$$= \frac{1}{2^{\frac{n_1+n_2}{2}}\Gamma n_1/2 \, \Gamma n_2/2} \cdot (uv)^{n_1/2-1} (v)^{n_2/2-1} e^{-\left(\frac{uv+v}{2}\right)} \cdot v$$

$$\Rightarrow g(u,v) = \frac{1}{2^{\frac{n_1+n_2}{2}}\,\Gamma n_1/2\,\Gamma n_2/2}\, u^{n_1/2-1} v^{n_1/2-1} \cdot v^{n_2/2-1} e^{-v\frac{(u+1)}{2}} \cdot v$$

Now the pdf of u is given by

$$g(u) = \frac{u^{n_1/2-1}}{2^{\frac{n_1+n_2}{2}}\,\Gamma n_1/2\,\Gamma n_2/2} \int_0^\infty v^{\frac{n_1+n_2}{2}-1} e^{-v\frac{(u+1)}{2}} \cdot dv$$

$$= \frac{u^{n_1/2-1}}{2^{\frac{n_1+n_2}{2}}\,\Gamma n_1/2\,\Gamma n_2/2} \frac{\Gamma \frac{n_1+n_2}{2}}{\left(\frac{1+u}{2}\right)^{\frac{n_1+n_2}{2}}} \quad \left[\because \frac{\Gamma n}{\alpha^n} = \int_0^\infty x^{n-1} e^{-\alpha x} \cdot dx\right.$$

$$= \frac{u^{n_1/2-1}}{2^{\frac{n_1+n_2}{2}}\,\Gamma n_1/2\,\Gamma n_2/2} \cdot \frac{\Gamma \frac{n_1+n_2}{2} \cdot 2^{\frac{n_1+n_2}{2}}}{(1+u)^{\frac{n_1+n_2}{2}}}$$

$$= \frac{1}{\frac{\Gamma n_1/2\,\Gamma n_2/2}{\Gamma \frac{n_1+n_2}{2}}} \cdot \frac{u^{n_1/2-1}}{(1+u)^{\frac{n_1+n_2}{2}}}$$

$$\therefore g(u) = \frac{1}{\beta(n_1/2, n_2/2)} \cdot \frac{u^{n_1/2-1}}{(1+u)^{\frac{n_1+n_2}{2}}} = \beta_2\left(\frac{n_1}{2}, \frac{n_2}{2}\right) \; ; \; 0 < u < \infty$$

which is the pdf of beta second kind distribution.

$\therefore u \sim \beta_2\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$

It is seen that u is a beta variate of the 2nd kind with parameters $n_1/2$ and $n_2/2$.

Question:

Prove that, for large degree of freedom χ^2- tends to normal distribution.

Solution:

For standard χ^2 variate : $z = \frac{x^2-n}{\sqrt{2n}}$

Now, we want to find the pdf of $z = \frac{x^2-n}{\sqrt{2n}}$

Hence, the mgf of z is given by

$$M_z(t) = E\left[e^{tz}\right]$$

$$= E\left[e^{t\left(\frac{x^2-n}{\sqrt{2n}}\right)}\right]$$

$$= E\left[e^{\frac{tx^2}{\sqrt{2n}}} \cdot e^{-nt/\sqrt{2n}}\right]$$

$$= e^{-nt/\sqrt{2n}} \cdot E\left[e^{\frac{tx^2}{\sqrt{2n}}}\right]$$

$$\therefore M_z(t) = e^{\frac{-nt}{\sqrt{2n}}} \left(1 - 2\cdot\frac{t}{\sqrt{2n}}\right)^{-n/2} \quad \left[\because M_{x^2}(t) = (1-2t)^{-n/2}\right]$$

$$\Rightarrow \log M_z(t) = -\frac{nt}{\sqrt{2n}} - n/2 \log\left(1 - \frac{2t}{\sqrt{2n}}\right)$$

$$\Rightarrow k_2(t) = -\frac{nt}{\sqrt{2n}} + \frac{n}{2}\left[\frac{2t}{\sqrt{2n}} + \frac{(2t)^2}{(\sqrt{2n})^2 \cdot 2} + \frac{(2t)^3}{(\sqrt{2n})^3 \cdot 3} + \cdots\right]$$

$$= -\frac{nt}{\sqrt{2n}} + \frac{nt}{\sqrt{2n}} + \frac{n}{2}\cdot\frac{4t^2}{4n} + 0\cdot(n^{-1/2})$$

$$\therefore k_2(t) = -t\sqrt{\frac{n}{2}} + t\cdot\sqrt{n/2} + \frac{t^2}{2} + 0\cdot(n^{-1/2})$$

where, $0\cdot(n^{-1/2})$ terms are containing $n^{1/2}$ and higher powers of n in the denominator.

$$\therefore \lim_{n\to\infty} k_2(t) = \frac{t^2}{2} \Rightarrow M_2(t) = e^{t^2/2} \text{ as } n\to\infty$$

$\therefore M_2(t) = e^{t^2/2}$ as $n\to\infty$

which is the mgf of standard normal variate.

Therefore, for large degree of freedom χ^2 distribution tends to normal distribution.

(showed)

= 0 =

Some formula:

(i). $\beta(l,m) = \int_0^1 x^{l-1}(1-x)^{m-1}.dx$ [1st kind beta]

(ii). $\beta(l,m) = \int_0^\infty \frac{x^{l-1}}{(1+x)^{l+m}}dx$ [2nd kind beta]

(iii). $\beta(l,m) = \frac{\Gamma l\,\Gamma m}{\Gamma(l+m)}$

(iv). $\Gamma n = \int_0^\infty x^{n-1}e^{-x}.dx$

(v). $\frac{\Gamma n}{\alpha^n} = \int_0^\infty x^{n-1}e^{-\alpha x}.dx$

(vi). $\Gamma n = (n-1)\Gamma(n-1)$

(vii). $\Gamma n = (n-1)!$

(viii). $\Gamma\frac{1}{2} = \sqrt{\pi}$

(ix). $\Gamma n = (n-1)(n-2)\cdots\Gamma 1$

(x). $\Gamma 4 = (4-1)! = 3! = 3\times2\times1 = 6$

"t- distribution"

2a

Student's t distribution:

Let U be a $N(0,1)$ variate and V be a chi-square (χ^2) variate with n degree of freedom. Also U and V are independent.

Define $t = \frac{U}{\sqrt{V/n}}$. Then t will follow t distribution with n degree of freedom.

The form of t distribution with n degree of freedom is given below:

$$f(t) = \frac{1}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})\left(1+t^2/n\right)^{\frac{n+1}{2}}}, \quad -\infty < t < \infty$$

Properties of t- distribution:

(i). t- distribution is an even function.

(ii). t- distribution is symmetric about $t=0$.

(iii). Mean = Median = mode = 0

(iv). Variance of the distribution is $\frac{n}{n-2}$; $n>2$

2b

(v). The total probability of t-density is equal to 1.

i.e. $\int_{-\infty}^{\infty} f(t) \cdot dt = 1$

(vi). For large n t-distribution reduces to standard normal distribution.

(vii). All odd order raw moments are zero.

i.e. $\mu'_{2r+1} = 0$

(viii). Even order raw moments are found by the relation:

$$\mu'_{2r} = \frac{n^r \Gamma(r+\frac{1}{2}) \Gamma(\frac{n}{2}-r)}{\Gamma\frac{1}{2} \Gamma\frac{n}{2}} ; r = 1, 2, 3, \ldots$$

(ix). Since, $\beta_1 = 0$ and $\beta_2 = 3 + \frac{6}{n-4} > 3$, therefore, the distribution is symmetric ($\beta_1 = 0$) and leptokurotic ($\beta_2 > 3$).

(x). It is a continuous type of distribution and its range extends from $-\infty$ to ∞

i.e. $-\infty < t < \infty$

(xi). Mgf of t-distribution does not exist.

Application or uses of t-distribution:

(i). To test if the sample mean ($\bar{x}$) differs significantly from the hypothetical value of μ of the population mean.

(ii). To test the significance of the difference between two sample mean.

(iii). To test the significance of an observed sample correlation coefficient and sample regression coefficient.

(iv). To test the significance of an observed partial correlation coefficient.

(v) To test the single population mean.

Distinguish between t and normal distribution:

t-distribution	Normal distribution
(i). The pdf of t-distribution is: $f(t) = \frac{1}{\sqrt{n}\, \beta(\frac{1}{2}, \frac{n}{2})(1+t^2/n)^{\frac{n+1}{2}}}$; $-\infty < t < \infty$	(i). The pdf of normal distn is: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$; $-\infty < x < \infty$
(ii). Mean, median, Mode of this distribution concide at zero.	(ii). Mean, median, mode of this distribution ~~is~~ are not zero.

t-distribution	Normal distribution
(iii). It is an exact sampling distribution.	(iii). It is a parent distribution.
(iv). The distribution is symmetric and leptokurtic since, $\beta_1 = 0$ and $\beta_2 > 3$.	(iv). The distribution is symmetric and mesokurtic (normal curve) since, $\beta_1 = 0$ and $\beta_2 = 3$

Derivation of t-distribution:

Let $U \sim N(0,1)$ and $V \sim \chi^2_n$. U and V are independent.

Now, we want to find the distribution of

$$t = \frac{U}{\sqrt{V/n}}.$$

The pdf of U is given by

$$f(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} \quad ; -\infty < u < \infty$$

The pdf of V is given by

$$f(v) = \frac{1}{2^{n/2}\sqrt{n/2}} v^{n/2-1} \cdot e^{-v/2} \quad ; \quad 0 < v < \infty$$

Then, the joint pdf of U and V is given as—

$$f(u,v) = f(u) \cdot f(v) \quad [\because U \text{ and } V \text{ are independent}]$$

$$\Rightarrow f(u,v) = \frac{1}{\sqrt{2\pi}\, 2^{n/2}\sqrt{n/2}} \cdot e^{-\frac{1}{2}(u^2+v)} \cdot v^{n/2-1} \; ; \quad \begin{matrix} -\infty < u < \infty \\ 0 < v < \infty \end{matrix}$$

Here, $t = \frac{u}{\sqrt{v/n}}$ and Let $v = w$.

$$\Rightarrow t = \frac{u}{\sqrt{w/n}}$$

$$\therefore u = t \cdot \sqrt{w/n}$$

Then the Jacobian of the transformation is:

$$J = \begin{vmatrix} \frac{\partial u}{\partial t} & \frac{\partial u}{\partial w} \\ \frac{\partial v}{\partial t} & \frac{\partial v}{\partial w} \end{vmatrix} = \begin{vmatrix} \sqrt{w/n} & \frac{1}{2}\frac{t}{\sqrt{n}} w^{\frac{1}{2}-1} \\ 0 & 1 \end{vmatrix} = \sqrt{w/n}$$

$$\Rightarrow |J| = \sqrt{w/n}$$

Then the joint pdf of t and w is given by

$$g(t,w) = f(t,w) \cdot |J|$$

$$= \frac{1}{\sqrt{2\pi}\, 2^{n/2}\sqrt{n/2}} e^{-\frac{1}{2}\left(\frac{t^2 w}{n}+w\right)} w^{n/2-1} \cdot \sqrt{w/n} \quad ; -\infty < t < \infty, w > 0$$

$$= \frac{1}{\sqrt{2\pi n}\, 2^{n/2}\sqrt{n/2}} \cdot e^{-\frac{1}{2}w\left(t^2/n+1\right)} w^{n/2-1+1/2}$$

$$\Rightarrow g(t,w) = \frac{1}{\sqrt{2\pi n}\, 2^{n/2}\sqrt{n/2}} e^{-\frac{1}{2}(1+t^2/n)w} \cdot w^{\frac{n+1}{2}-1}$$

$$\therefore g(t,w) = \frac{1}{\sqrt{2\pi n}\, 2^{n/2}\sqrt{n/2}} e^{-\frac{1}{2}(1+t^2/n)w} \cdot w^{\frac{n+1}{2}-1}.$$

Now, the pdf of t is given as—

$$g(t) = \frac{1}{\sqrt{2\pi n}\, 2^{n/2} \Gamma\frac{n}{2}} \int_0^\infty e^{-\frac{1}{2}(1+t^2/n)w} \cdot w^{\frac{n+1}{2}-1} \cdot dw$$

$$= \frac{1}{\sqrt{2\pi n}\, 2^{n/2}\Gamma\frac{n}{2}} \frac{\Gamma\frac{n+1}{2}}{\left[\frac{1}{2}(1+t^2/n)\right]^{\frac{n+1}{2}}} \qquad \left[\because \frac{\Gamma n}{\alpha^n} = \int_0^\infty x^{n-1} e^{-\alpha x} \cdot dx\right]$$

$$= \frac{\Gamma\frac{n+1}{2} \cdot 2^{\frac{n+1}{2}}}{\sqrt{n}\sqrt{2\pi}\, 2^{n/2}\Gamma\frac{n}{2} \left(1+t^2/n\right)^{\frac{n+1}{2}}}$$

$$= \frac{\Gamma\frac{n+1}{2} \cdot 2^{n/2}\sqrt{2}}{\sqrt{n}\sqrt{2\pi}\, 2^{n/2}\Gamma\frac{n}{2}\left(1+t^2/n\right)^{\frac{n+1}{2}}} \qquad \left[\because 2^{1/2} = \sqrt{2}\right]$$

$$= \frac{1}{\sqrt{n}\, \frac{\Gamma\frac{1}{2}\Gamma\frac{n}{2}}{\Gamma\frac{n+1}{2}} \left(1+t^2/n\right)^{\frac{n+1}{2}}} \qquad \left[\because \sqrt{\pi} = \Gamma\tfrac{1}{2}\right]$$

$$\cancel{\therefore g(t) = \frac{1}{\sqrt{n}\beta(\frac{1}{2}, }}$$

$$\therefore g(t) = \frac{1}{\sqrt{n}\, \beta\left(\frac{1}{2}, \frac{n}{2}\right)\left(1+t^2/n\right)^{\frac{n+1}{2}}} \;; \; -\infty < t < \infty$$

which is the pdf of t-distribution.

Question:
Show that, the total probability of t-density is equal to 1. i.e. $\int_{-\infty}^{\infty} f(t) \cdot dt = 1$.

proof:
Now, $$\int_{-\infty}^{\infty} f(t)\cdot dt = \int_{-\infty}^{\infty} \frac{1}{\sqrt{n}\, \beta(\frac{1}{2}, \frac{n}{2})\left(1+t^2/n\right)^{\frac{n+1}{2}}} \cdot dt$$

Let, $w = \frac{t^2}{n}$ $\qquad \therefore t = \sqrt{nw}$

$\Rightarrow t^2 = nw$

$\Rightarrow 2t \cdot dt = n dw$ $\quad \Rightarrow dt = \frac{n}{2t} \cdot dw = \frac{n}{2\sqrt{nw}} dw$

$\therefore dt = \frac{\sqrt{n}}{2\sqrt{w}} \cdot dw$

$$\therefore \int_{-\infty}^{\infty} f(t) \cdot dt = 2\int_0^\infty \frac{1}{\sqrt{n}\, \beta(\frac{1}{2}, \frac{n}{2})(1+w)^{\frac{n+1}{2}}} \cdot \frac{\sqrt{n}}{2\sqrt{w}} \cdot dw$$

[Since, the integrand is an even function of t]

$$\Rightarrow \int_{-\infty}^{\infty} f(t) \cdot dt = \int_0^\infty \frac{1}{\beta(\frac{1}{2}, \frac{n}{2})} \frac{w^{-1/2}}{(1+w)^{\frac{n+1}{2}}} \cdot dw$$

$$= \frac{1}{\beta(\frac{1}{2}, \frac{n}{2})} \int_0^\infty \frac{w^{1/2 - 1}}{(1+w)^{1/2 + n/2}} \cdot dw$$

$$= \frac{1}{\beta(\frac{1}{2}, \frac{n}{2})} \cdot \beta\left(\tfrac{1}{2}, \tfrac{n}{2}\right) \qquad \left[\because \beta(l, m) = \int_0^\infty \frac{x^{l-1}}{(1+x)^{l+m}} \cdot dx\right]$$

$$= 1$$

$$\therefore \int_{-\infty}^{\infty} f(t)\cdot dt = 1$$

Therefore, the total probability of t-density is equal to 1. (showed).

Question:
Find mean, variance of t-distribution.

Answer:
Mean:

~~E(t) =~~

We know that, the pdf of t-distribution is:

$$f(t) = \frac{1}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} \; ; -\infty < t < \infty$$

We know

$$E(t) = \int_{-\infty}^{\infty} t\cdot f(t)\cdot dt$$

$$= \int_{-\infty}^{\infty} t \cdot \frac{1}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}\, dt$$

$$= \int_{-\infty}^{\infty} \frac{t}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}\cdot dt$$

$$\Rightarrow E(t) = \frac{1}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})}\int_{-\infty}^{\infty} \frac{t}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}\cdot dt$$

$$= \frac{1}{\sqrt{n}\cdot\beta(\frac{1}{2},\frac{n}{2})}\cdot 0 = 0$$ [Since, the integrand is an odd function of t]

$$\therefore \text{Mean} = E(t) = 0$$

Now,

$$E(t^2) = \int_{-\infty}^{\infty} t^2 f(t)\cdot dt$$

$$= \int_{-\infty}^{\infty} \frac{t^2}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}\cdot dt$$

Let, $w = \frac{t^2}{n}$ $\quad \therefore t = \sqrt{nw}$

$$\Rightarrow t^2 = nw$$

$$\Rightarrow 2t\cdot dt = n\,dw \Rightarrow dt = \frac{n}{2t}dw = \frac{n}{2\sqrt{nw}}\cdot dw = \frac{\sqrt{n}}{2\sqrt{w}}\cdot dw$$

$$\therefore dt = \frac{\sqrt{n}}{2\sqrt{w}}\,dw$$

When, $t = -\infty$, then $w = -\infty$

When $t = \infty$, then $w = \infty$

$$\Rightarrow E(t^2) = \int_{-\infty}^{\infty} \frac{nw}{\cancel{\sqrt{n}}\,\beta(\frac{1}{2},\frac{n}{2})(1+w)^{\frac{n+1}{2}}}\cdot\frac{\cancel{\sqrt{n}}}{2\sqrt{w}}\,dw$$

$$= \cancel{2}\int_{0}^{\infty} \frac{nw}{\beta(\frac{1}{2},\frac{n}{2})}\cdot\frac{w^{-1/2}}{\cancel{2}(1+w)^{\frac{n+1}{2}}}\cdot dw$$

[Since, the integrand is an even function of w]

$$\Rightarrow E(t^2) = \frac{n}{\beta(\frac{1}{2},\frac{n}{2})}\cdot\int_{0}^{\infty}\frac{w^{-\frac{1}{2}+1}}{(1+w)^{\frac{n+1}{2}}}\,dw$$

$$= \frac{n}{\beta(\frac{1}{2},\frac{n}{2})}\int_{0}^{\infty}\frac{w^{3/2-1}}{(1+w)^{3/2+\frac{n-2}{2}}}\,dw$$ $\left[\frac{n+1}{2} = \frac{3}{2}+\frac{n-2}{2} \text{ and } \frac{1}{2} = \frac{3}{2}-1\right]$

$$= \frac{n}{\beta(\frac{1}{2},\frac{n}{2})}\,\beta\left(\frac{3}{2},\frac{n-2}{2}\right)$$ $\left[\because \beta(l,m) = \int_0^{\infty}\frac{x^{l-1}}{(1+x)^{l+m}}dx\right]$

$$\Rightarrow E(t^2) = n\,\frac{\dfrac{\Gamma\frac{3}{2}\,\Gamma\frac{n-2}{2}}{\Gamma\left(\frac{3}{2}+\frac{n-2}{2}\right)}}{\dfrac{\Gamma\frac{1}{2}\,\Gamma\frac{n}{2}}{\Gamma\frac{n+1}{2}}} = \frac{n.\Gamma\frac{3}{2}\,\Gamma\frac{n-2}{2}\Big/\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\frac{1}{2}\,\Gamma\frac{n}{2}\Big/\Gamma\left(\frac{n+1}{2}\right)}$$

$$\Rightarrow E(t^2) = n.\frac{\Gamma\frac{3}{2}\,\Gamma\frac{n-2}{2}}{\Gamma\frac{1}{2}\,\Gamma\frac{n}{2}} = n.\frac{\frac{1}{2}\Gamma\frac{1}{2}\;\Gamma\left(\frac{n}{2}-1\right)}{\Gamma\frac{1}{2}\;\left(\frac{n}{2}-1\right)\Gamma\left(\frac{n}{2}-1\right)}$$

$$= \frac{n}{2}\cdot\frac{2}{n-2} = \frac{n}{n-2}$$

$$\therefore E(t^2) = \frac{n}{n-2}$$

$$\therefore V(t) = E(t^2) - [E(t)]^2 = \frac{n}{n-2} - 0 = \frac{n}{n-2}$$

∴ the mean and variance of the distribution is 0 and $\frac{n}{n-2}$ respectively.

Question:

Show that, mean, median and mode of t-distribution are identical or equal and hence its zero. i.e. Mean = Median = Mode = 0

Answer:

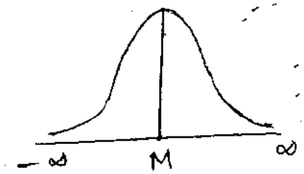
Mean:

We have already got it in the last question.

i.e. $E(t) = 0$

∴ Mean = 0

Median:



Let, M be the median of the distribution.

$$\therefore \int_{-\infty}^{M} f(t)\cdot dt = \frac{1}{2} = \int_{M}^{\infty} f(t)\cdot dt$$

$$\text{Now, } M = \int_{M}^{\infty} f(t)\cdot dt = \frac{1}{2}$$

$$\Rightarrow \int_{M}^{\infty} f(t)\cdot dt = \frac{1}{2} \quad \cdots (i)$$

We know that, the total probability of t-density is equal to 1.

$$\text{i.e. } \int_{-\infty}^{\infty} f(t)\cdot dt = 1$$

$$\Rightarrow 2\int_{0}^{\infty} f(t)\cdot dt = 1$$

$$\Rightarrow \int_{0}^{\infty} f(t)\cdot dt = \frac{1}{2} \quad \cdots (ii)$$

Comparing (i) and (ii) we get

$$M = 0$$

∴ Median = 0

Hence, the median of t-distribution is zero.

Mode of t-distribution :

Mode will be obtained by the solution of the equation.

$$\frac{d \log f(t)}{dt} = 0 \;;\; \text{provided } \frac{d^2 \log f(t)}{dt^2} < 0$$

Now, the pdf of t distribution is –

$$f(t) = \frac{1}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})\,(1+t^2/n)^{\frac{n+1}{2}}} \;;\; -\infty < t < \infty$$

$$\Rightarrow \log f(t) = \log \frac{1}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})} + \log\left(1+t^2/n\right)^{-\left(\frac{n+1}{2}\right)}$$

Now $$\frac{d\log f(t)}{d(t)} = 0 - \left(\frac{n+1}{2}\right)\frac{1}{(1+t^2/n)}\cdot\frac{2t}{n}$$

$$\Rightarrow \frac{d\log f(t)}{dt} = -\frac{t(n+1)}{n(1+t^2/n)}$$

Hence, $$\frac{d\log f(t)}{dt} = 0$$

$$\Rightarrow -\frac{t(n+1)}{n(1+t^2/n)} = 0$$

$$\Rightarrow -t(n+1) = 0$$

$$\therefore t = 0$$

It ~~can be easily~~ is easy to verify that $\frac{d^2\log f(t)}{dt^2} < 0$ at $t=0$.

Hence, $t=0$ is the mode of the distribution.

$\therefore$ Mode $= 0$

Hence, Mean = Median = Mode = 0. (showed)

Question :

Find the moments of t-distribution. Hence find mean, variance, skewness, kurtosis and comment on the shape of the distribution.

Odd order raw moments :

$$\mu'_{2r+1} = \int_{-\infty}^{\infty} t^{2r+1}\, f(t)\, dt$$

$$= \int_{-\infty}^{\infty} \frac{t^{2r+1}}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})(1+t^2/n)^{\frac{n+1}{2}}}\, dt$$

$$= 0$$

Since, the integrand is an odd function of t. and $(2r+1)$ is an odd number.

$$\therefore \mu'_{2r+1} = \mu_{2r+1} = 0$$

Hence, we conclude that all odd order raw moments are zero.

$\mu'_{2r+1} = 0$

Now, putting $r = 0, 1, 2, 3, \ldots$; we have

$\mu_1' = 0,\ \mu_3' = 0,\ \ldots,\ \mu'_{2r+1} = 0$

Even order moments:

By the defination of raw moments we have 2rth raw moment about origin is given by-

$$\mu'_{2r} = E[t^{2r}]$$

$$= \int_{-\infty}^{\infty} t^{2r} f(t) \cdot dt$$

$$= 2\int_{0}^{\infty} t^{2r} \cdot \frac{1}{\sqrt{n}\,\beta(\frac{1}{2}, \frac{n}{2})\,(1 + t^2/n)^{\frac{n+1}{2}}} \cdot dt$$

[Since, the integrand is an ~~odd~~ even function of t.]

Let, $w = \frac{t^2}{n}$ $\therefore t = \sqrt{wn}$

$\Rightarrow t^2 = wn$

$\Rightarrow 2t \cdot dt = n\,dw$

$\Rightarrow dt = \frac{n}{2t} \cdot dw \Rightarrow dt = \frac{n}{2\sqrt{wn}} \cdot dw$

$\therefore dt = \frac{n}{2\sqrt{wn}} \cdot dw$

when $t = 0$, then $w = 0$

when $t = \infty$, then $w = \infty$.

$$\Rightarrow \mu'_{2r} = \cancel{2}\int_{0}^{\infty} \frac{(\sqrt{wn})^{2r}}{\sqrt{n}\,\beta(\frac{1}{2}, \frac{n}{2})\,(1+w)^{\frac{n+1}{2}}} \cdot \frac{n}{\cancel{2}\sqrt{wn}}\,dw$$

$$= \int_{0}^{\infty} \frac{w^r n^r}{\cancel{\sqrt{n}}\,\beta(\frac{1}{2}, \frac{n}{2})\,(1+w)^{\frac{n+1}{2}}} \cdot w^{-1/2}\,dw$$

$$= \frac{n^r}{\beta(\frac{1}{2}, \frac{n}{2})} \int_{0}^{\infty} \frac{w^{r + \frac{1}{2} - 1}}{(1+w)^{(r+\frac{1}{2}) + (\frac{n}{2} - r)}} \cdot dw$$

$$= \frac{n^r}{\beta(\frac{1}{2}, \frac{n}{2})}\,\beta\left(r + \tfrac{1}{2}, \tfrac{n}{2} - r\right) \quad \left[\because \beta(l, m) = \int_0^\infty \frac{x^{l-1}}{(1+x)^{l+m}}\,dx\right]$$

$$\therefore \mu'_{2r} = \frac{n^r}{\beta(\frac{1}{2}, \frac{n}{2})}\,\beta\left(r + \tfrac{1}{2}, \tfrac{n}{2} - r\right)$$

$$= n^r \cdot \frac{\Gamma(r+\frac{1}{2})\,\Gamma(\frac{n}{2} - r) \big/ \Gamma(r + \frac{1}{2} + \frac{n}{2} - r)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{n}{2}) \big/ \Gamma(\frac{1}{2} + \frac{n}{2})}$$

$$= n^r \cdot \frac{\Gamma(r+\frac{1}{2})\,\Gamma(\frac{n}{2} - r)}{\Gamma(\frac{n+1}{2})} \times \frac{\Gamma(\frac{n}{2} + \frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(\frac{n}{2})}$$

$$= n^r \frac{\Gamma(r+\frac{1}{2})\,\Gamma(\frac{n}{2} - r)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{n}{2})}$$

$$\therefore \mu'_{2r} = \frac{n^r\,\Gamma(r+\frac{1}{2})\,\Gamma(\frac{n}{2} - r)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{n}{2})}\ ; \ \cancel{r =}$$

putting $r = 1, 2,$ we get.

$$\mu_2' = \frac{n\,\Gamma(1+\frac{1}{2})\,\Gamma(\frac{n}{2}-1)}{\Gamma\frac{1}{2}\,\Gamma\frac{n}{2}}$$

$$= \frac{n\,\Gamma\frac{3}{2}\,\Gamma(\frac{n}{2}-1)}{\Gamma\frac{1}{2}\,\Gamma\frac{n}{2}} = \frac{n\cdot\frac{1}{2}\,\cancel{\Gamma\frac{1}{2}}\,\cancel{\Gamma(\frac{n}{2}-1)}}{\cancel{\Gamma\frac{1}{2}}\,(\frac{n}{2}-1)\,\cancel{\Gamma(\frac{n}{2}-1)}}$$

$$= \frac{n}{2}\cdot\frac{2}{n-2} = \frac{n}{n-2}$$

$$\therefore \mu_2' = \frac{n}{n-2}$$

and $$\mu_4' = \frac{n^2\,\Gamma(2+\frac{1}{2})\,\Gamma(\frac{n}{2}-2)}{\Gamma\frac{1}{2}\,\Gamma\frac{n}{2}}$$

$$= \frac{n^2\,\Gamma\frac{5}{2}\,\Gamma(\frac{n}{2}-2)}{\Gamma\frac{1}{2}\,\Gamma\frac{n}{2}} = \frac{n^2\,\frac{3}{2}\cdot\frac{1}{2}\,\cancel{\Gamma\frac{1}{2}}\,\cancel{\Gamma(\frac{n}{2}-2)}}{\cancel{\Gamma\frac{1}{2}}\,(\frac{n}{2}-1)(\frac{n}{2}-2)\,\cancel{\Gamma(\frac{n}{2}-2)}}$$

$$= \frac{\frac{3n^2}{4}}{(\frac{n-2}{2})\cdot(\frac{n-4}{2})} = \frac{3n^2}{4}\times\frac{4}{(n-2)(n-4)} = \frac{3n^2}{(n-2)(n-4)}$$

$$\therefore \mu_4' = \frac{3n^2}{(n-2)(n-4)}$$

Central moments:

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = \frac{n}{n-2} - 0 \qquad [\because \mu_{2r+1}' = 0]$$

$$\therefore \mu_2 = \text{variance} = \frac{n}{n-2}$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 \qquad [\because \mu_{2r+1}' = 0]$$

$$= 0 - 3\left(\frac{n}{n-2}\right)\cdot 0 + 2\cdot(0)^3$$

$$= 0$$

$$\therefore \mu_3 = 0$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$

$$= \frac{3n^2}{(n-2)(n-4)} - 0 + 0 - 0 \qquad [\because \mu_{2r+1}' = 0]$$

$$\therefore \mu_4 = \frac{3n^2}{(n-2)(n-4)}$$

Skewness:

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{0^2}{\left(\frac{n}{n-2}\right)^3} = 0 \qquad [\because \mu_3 = 0]$$

Kurtosis:

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3n^2}{(n-2)(n-4)} \times \frac{(n-2)^2}{n^2}$$

$$= \frac{3(n-2)}{n-4} = \frac{3n-6}{n-4}$$

$$\Rightarrow \beta_2 = \frac{3n-6}{n-4} = \frac{3n-12+6}{n-4} = \frac{3(n-4)}{(n-4)} + \frac{6}{n-4}$$

$$= 3 + \frac{6}{n-4}$$

$$\therefore \beta_2 = 3 + \frac{6}{n-4} > 3$$

<u>Comment</u>: Since, $\beta_1 = 0$ and $\beta_2 = 3 + \frac{6}{n-4}$, then the distribution is symmetric $(\beta_1 = 0)$ and leptokurtic $(\beta_2 > 3)$.

<u>Question</u>:
Establish the relationship between t-distribution and cauchy distribution.

<u>Answer</u>:
The relationship between t-distribution and cauchy distribution are given as follows:

We know, the pdf of t-distribution is as

$$f(t) = \frac{1}{\sqrt{n}\,\beta(\frac{1}{2}, \frac{n}{2})\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} \quad ; \; -\infty < t < \infty$$

If $n = 1$, then we get the form of above equation

$$f(t) = \frac{1}{\sqrt{1}\,\beta(\frac{1}{2}, \frac{1}{2})(1+t^2)^{\frac{1+1}{2}}}$$

$$\Rightarrow f(t) = \frac{1}{\beta(\frac{1}{2}, \frac{1}{2})(1+t^2)^1}$$

$$= \frac{1}{\frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+\frac{1}{2})}(1+t^2)} = \frac{1}{\sqrt{\pi}\,\sqrt{\pi}\,(1+t^2)} \quad \left[\because \Gamma\tfrac{1}{2} = \sqrt{\pi}\right] \left[\Gamma 1 = 1\right]$$

$$\therefore f(t) = \frac{1}{\pi(1+t^2)} \quad ; \; -\infty < t < \infty$$

which is the pdf of standard cauchy distribution. Which is the relationship between t-distribution and cauchy distribution.

<u>Question</u>: 29
Show that, for large degree of freedom t-distribution tends to normal distribution.

<u>Proof</u>:
We know that, the pdf of ~~d~~ t-distribution is as:

$$f(t) = \frac{1}{\sqrt{n}\,\beta(\frac{1}{2}, \frac{n}{2})\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} \quad ; \; -\infty < t < \infty$$

$$\therefore f(t) = \frac{1}{\sqrt{n}\,\beta(\frac{1}{2}, \frac{n}{2})} \cdot \frac{1}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}$$

Taking limit on both sides, we have–

$$\lim_{n\to\infty} f(t) = \lim_{n\to\infty}\left\{\frac{1}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})}\cdot\frac{1}{(1+t^2/n)^{\frac{n+1}{2}}}\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{1}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})}\right\}\cdot \lim_{n\to\infty}\left\{\frac{1}{(1+t^2/n)^{\frac{n+1}{2}}}\right\}$$

$$\therefore \lim_{n\to\infty}\left\{\frac{1}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})}\right\} = \lim_{n\to\infty}\frac{1}{\sqrt{n}\cdot\frac{\Gamma\frac{1}{2}\,\Gamma\frac{n}{2}}{\Gamma\left(\frac{1}{2}+\frac{n}{2}\right)}}$$

$$= \lim_{n\to\infty}\frac{\Gamma\left(\frac{1}{2}+\frac{n}{2}\right)}{\sqrt{n}\,\sqrt{\pi}\,\Gamma\frac{n}{2}} \qquad \left[\because \Gamma\tfrac{1}{2}=\sqrt{\pi}\right]$$

$$= \cancel{\lim_{n\to\infty}}\ \frac{1}{\sqrt{n}\sqrt{\pi}}\lim_{n\to\infty}\frac{\Gamma\left(\frac{1}{2}+\frac{n}{2}\right)}{\Gamma\frac{n}{2}}$$

$$= \frac{1}{\sqrt{n\pi}}\left(\frac{n}{2}\right)^{1/2}$$

$$= \frac{1}{\sqrt{n}\sqrt{\pi}}\cdot n^{1/2}\,2^{-1/2}$$

$$= \frac{1}{\sqrt{\pi}\,2^{1/2}} = \frac{1}{\sqrt{\pi}\sqrt{2}} = \frac{1}{\sqrt{2\pi}}$$

$$\therefore \lim_{n\to\infty}\left\{\frac{1}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})}\right\} = \frac{1}{\sqrt{2\pi}}$$

Also,

$$\lim_{n\to\infty}\left\{\left(1+\frac{t^2}{n}\right)^{-\left(\frac{n+1}{2}\right)}\right\} = \lim_{n\to\infty}\left(1+\frac{t^2}{n}\right)^{-n/2}\cdot\lim_{n\to\infty}\left(1+\frac{t^2}{n}\right)^{-1/2}$$

$$= \lim_{n\to\infty}\left\{\left(1+\frac{t^2}{n}\right)^{n}\right\}^{-1/2}\cdot\lim_{n\to\infty}\left(1+\frac{t^2}{n}\right)^{-1/2}$$

$$= e^{-t^2/2}\cdot 1$$

Now,

$$\lim_{n\to\infty}\left(1+\frac{t^2}{n}\right)^{-\left(\frac{n+1}{2}\right)} = e^{-t^2/2}$$

Hence, $\lim_{n\to\infty} f(t) = \frac{1}{\sqrt{2\pi}}\,e^{-t^2/2}\ ; -\infty<t<\infty$

which is the pdf of standard normal distribution.

Therefore, for large degree of freedom t-distribution tends to normal distribution.

(Showed)

Problem :

let $f(t) = \dfrac{1}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})\left(1+t^2/n\right)^{\frac{n+1}{2}}}\ ; -\infty<t<\infty.$

Then obtain the pdf of $z=t^2$.

=0=

"F- distribution"

F- distribution:

22 "The F-distribution is the distribution of the ratio of two independent chi-square (χ^2) random variables divided by their respective degrees of freedom."

If χ_1^2 and χ_2^2 are two independent chi-square variates having n_1 and n_2 degrees of freedom respectively, then the statistic is given as-

$$F = \frac{\chi_1^2/n_1}{\chi_2^2/n_2}$$

has the F-distribution with n_1 and n_2 degrees of freedom.

In mathematically, $F \sim F(n_1, n_2)$

The density function of F is -

$$f(F) = \frac{\frac{n_1}{n_2}\left(\frac{n_1}{n_2}F\right)^{n_1/2-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}}; \quad F>0$$

$$\therefore f(F) = \frac{\left(\frac{n_1}{n_2}\right)^{n_1/2} F^{n_1/2-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}}; \quad F>0$$

Properties of F-distribution:

(i). F-distribution is a continuous type of distribution and its range is 0 to ∞. i.e., $0<F<\infty$

(ii). It is an exact sampling distribution.

(iii). It is derived from chi-square (χ^2) distribution.

(iv). If $F \sim F(n_1, n_2)$, then the mean and variance is $\frac{n_2}{n_2-2}$ and variance $\frac{2n_2^2(n_1+n_2-2)}{n_1(n_2-2)^2(n_2-4)}$ respectively.

(v). The mode of the distribution is $\frac{n_2(n_1-2)}{n_1(n_2+2)}$.

(vi). If $F \sim F(n_1, n_2)$, then $\frac{1}{F} \sim F(n_2, n_1)$.

(vii). If $F \sim F(n_1, n_2)$, then $\frac{n_1}{n_2}F \sim \beta_2\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$.

(viii). If $F \sim F(n_1, n_2)$, then $\frac{1}{1+\frac{n_1}{n_2}F} \sim \beta_1\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$.

(ix). If n_1 and n_2 are very large, then F-distribution tends to normal distribution.

(x). The distribution is positively skewed.

Application or uses of F-distribution:

(i). F-distribution is used to test the equality of population variance.

(ii). It is used for testing the significance of ~~and~~ observed multiple correlation coefficient and sample correlation ratio.

(iii). It is used for testing the linearity of regression.

(iv). F-distribution is used to test the equality of several means.

Derivation of F-distribution: 2e

Let U and V are two independent χ^2 variates with n_1 and n_2 degrees of freedom, respectively.

i.e. $U \sim \chi^2_{n_1}$ and $V \sim \chi^2_{n_2}$. U and V are independent.

Now we want to obtain the distribution of

$$F = \frac{U/n_1}{V/n_2}.$$

Hence, the pdf of U is given by

$$f(u) = \frac{1}{2^{n_1/2}\Gamma n_1/2} u^{n_1/2-1} e^{-u/2} \quad ; \quad 0<u<\infty$$

The pdf of v is given by-

$$f(v) = \frac{1}{2^{n_2/2}\Gamma n_2/2} v^{n_2/2-1} e^{-v/2} \quad ; \quad 0<v<\infty$$

Then the joint pdf of u and v is given as.

$$f(u,v) = f(u) \cdot f(v) \qquad [\because U \text{ and } V \text{ are independent}]$$

$$\therefore f(u,v) = \frac{1}{2^{n_1/2}\Gamma n_1/2} u^{n_1/2-1} e^{-u/2} \cdot \frac{1}{2^{n_2/2}\Gamma n_2/2} v^{n_2/2-1} \cdot e^{-v/2} \quad ; \quad 0<u,v<\infty$$

Here, $F = \dfrac{u/n_1}{v/n_2}$. let $v = w$

$$\Rightarrow F = \frac{u/n_1}{w/n_2}$$

$$\Rightarrow \frac{u}{n_1} = F \cdot \frac{w}{n_2} \Rightarrow u = \frac{n_1}{n_2} F w$$

$$\therefore u = \frac{n_1}{n_2} F w \text{ and } v = w, \quad u+v = w\left(1+\frac{n_1}{n_2}F\right)$$

Now, the Jacobian of the transformation is-

$$J = \begin{vmatrix} \frac{\partial u}{\partial F} & \frac{\partial u}{\partial w} \\ \frac{\partial v}{\partial F} & \frac{\partial v}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{n_1}{n_2} w & \frac{n_1}{n_2} F \\ 0 & 1 \end{vmatrix} = \frac{n_1}{n_2} w$$

$$\therefore |J| = \frac{n_1}{n_2} w$$

Then the joint pdf of F and w is given by

$$g(F,w) = f(F,w)\cdot|J|$$

$$\therefore g(F,w) = \frac{1}{2^{\frac{n_1+n_2}{2}}\Gamma\frac{n_1}{2}\Gamma\frac{n_2}{2}}\cdot\left(\frac{n_1}{n_2}Fw\right)^{n_1/2-1}\cdot w^{\cancel{n_2}-1}\cdot e^{-\frac{1}{2}\left(1+\frac{n_1}{n_2}F\right)w}\cdot\frac{n_1}{n_2}w$$

Now, the pdf of F is given as

$$g(F) = \frac{\frac{n_1}{n_2}\cdot\left(\frac{n_1}{n_2}F\right)^{n_1/2-1}}{2^{\frac{n_1+n_2}{2}}\Gamma\frac{n_1}{2}\Gamma\frac{n_2}{2}}\int_0^\infty e^{-\frac{1}{2}\left(1+\frac{n_1}{n_2}F\right)w}\cdot w^{n_1/2-1+\frac{n_2}{2}-\cancel{1}+\cancel{1}}\cdot dw$$

$$= \frac{\frac{n_1}{n_2}\left(\frac{n_1}{n_2}F\right)^{n_1/2-1}}{2^{\frac{n_1+n_2}{2}}\Gamma\frac{n_1}{2}\Gamma\frac{n_2}{2}}\int_0^\infty w^{\frac{n_1+n_2}{2}-1}\cdot e^{-\frac{1}{2}\left(1+\frac{n_1}{n_2}F\right)w}\cdot dw$$

$$= \frac{\frac{n_1}{n_2}\left(\frac{n_1}{n_2}F\right)^{n_1/2-1}}{2^{\frac{n_1+n_2}{2}}\Gamma\frac{n_1}{2}\Gamma n_2/2}\cdot\frac{\Gamma\frac{n_1+n_2}{2}}{\left[\frac{1}{2}\left(1+\frac{n_1}{n_2}F\right)\right]^{\frac{n_1+n_2}{2}}}\left[\because\frac{\Gamma n}{\alpha^n}=\int_0^\infty x^{n-1}e^{-\alpha x}\cdot dx\right]$$

$$= \frac{\frac{n_1}{n_2}\left(\frac{n_1}{n_2}F\right)^{n_1/2-1}}{2^{\frac{n_1+n_2}{2}}\frac{\Gamma n_1/2\,\Gamma n_2/2}{\Gamma\frac{n_1+n_2}{2}}}\cdot\frac{2^{\frac{n_1+n_2}{2}}}{\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}}$$

$$\therefore g(F) = \frac{\frac{n_1}{n_2}\left(\frac{n_1}{n_2}F\right)^{\frac{n_1}{2}-1}}{\beta\left(\frac{n_1}{2},\frac{n_2}{n_2}\right)\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}}\quad;\quad 0<F<\infty\ (F>0)$$

Which is the required pdf of F-distribution.

Question:

Show that, the total probability of F-density is equal to 1. i.e., $\int_0^\infty f(F)\cdot dF = 1$

Proof:

We know that, the pdf of F-distribution is as.

$$f(F) = \frac{\frac{n_1}{n_2}\left(\frac{n_1}{n_2}F\right)^{n_1/2-1}}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}}\quad;\quad 0<F<\infty\ (F>0)$$

Now, $$\int_0^\infty f(F)\cdot dF = \int_0^\infty\frac{\frac{n_1}{n_2}\left(\frac{n_1}{n_2}F\right)^{\frac{n_1}{2}-1}}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}}\cdot dF$$

Let, $w = \frac{n_1}{n_2}F$

$\Rightarrow F = \frac{n_2}{n_1}w \quad \Rightarrow dF = \frac{n_2}{n_1}dw$

When, $F=0$, then $w=0$

When $F=\infty$, then $w=\infty$

$$\Rightarrow \int_0^\infty f(F)\cdot dF = \frac{\frac{\cancel{n_1}}{\cancel{n_2}}\cdot\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}-1}}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)}\int_0^\infty \frac{\left(\frac{n_2}{n_1}w\right)^{n_1/2-1}}{(1+w)^{\frac{n_1+n_2}{2}}}\cdot\frac{\cancel{n_2}}{\cancel{n_1}}\cdot dw$$

$$= \frac{\left(\frac{n_1}{n_2}\right)^{n_1/2-1}\cdot\left(\frac{n_2}{n_1}\right)^{n_1/2-1}}{\beta\left(\frac{n_1}{2},n_2/2\right)}\int_0^\infty \frac{w^{n_1/2-1}}{(1+w)^{n_1/2+n_2/2}}\cdot dw$$

$$= \frac{1}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)}\cdot\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right) \quad \left[\because \beta(l,m)=\int_0^\infty \frac{x^{l-1}}{(1+x)^{l+m}}\cdot dx\right]$$

$$\therefore \int_0^\infty f(F)dF = 1$$

Therefore, the total probability of F-density is equal to 1. i.e. $\int_0^\infty f(F)\cdot dF = 1$ (showed)

Question:

Find mean and variance of F-distribution.

Answer:

we know that, the pdf of F-distribution is-

$$f(F) = \frac{\frac{n_1}{n_2}\left(\frac{n_1}{n_2}F\right)^{n_1/2-1}}{\beta\left(n_1/2,\frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}} \quad ; 0<F<\infty \ (F>0)$$

Mean: $E(F) = \int_0^\infty F\cdot f(F)\cdot dF$

$$\Rightarrow E(F) = \int_0^\infty F\cdot\frac{\left(\frac{n_1}{n_2}\right)\left(\frac{n_1}{n_2}F\right)^{n_1/2-1}}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}}\cdot dF$$

Let, $w = \frac{n_1}{n_2}F \Rightarrow F = \frac{n_2}{n_1}w \Rightarrow dF = \frac{n_2}{n_1}dw$

when, $F=0$, then $w=0$, when $F=\infty$, then $w=\infty$.

$$\Rightarrow E(F) = \int_0^\infty \frac{\left(\frac{n_2}{n_1}\cdot w\right)\frac{\cancel{n_1}}{\cancel{n_2}}\cdot w^{n_1/2-1}}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)(1+w)^{\frac{n_1+n_2}{2}}}\cdot\frac{\cancel{n_2}}{\cancel{n_1}}\cdot dw$$

$$= \frac{\frac{n_2}{n_1}}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)}\int_0^\infty \frac{w^{(n_1/2+1)-1}}{(1+w)^{\frac{n_1+n_2}{2}}}\cdot dw$$

$$= \frac{\frac{n_2}{n_1}}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)}\int_0^\infty \frac{w^{(n_1/2+1)-1}}{(1+w)^{(n_1/2+1)+(n_2/2-1)}}\cdot dw$$

$$= \frac{n_2/n_1}{\beta(n_1/2,n_2/2)}\beta\left(n_1/2+1,\ \frac{n_2}{2}-1\right) \quad \left[\because \beta(l,m)=\int_0^\infty \frac{x^{l-1}}{(1+x)^{l+m}}\right]$$

$$= \frac{n_2}{n_1}\cdot\frac{\Gamma(n_1/2+1)\,\Gamma(n_2/2-1)\Big/\Gamma\left(\frac{n_1}{2}+\cancel{1}+n_2/2-\cancel{1}\right)}{\Gamma(n_1/2)\,\Gamma(n_2/2)\Big/\Gamma(n_1/2+n_2/2)}$$

$$= \frac{n_2}{n_1}\cdot\frac{\Gamma(n_1/2+1)\,\Gamma(n_2/2-1)}{\Gamma(n_1/2)\,\Gamma(n_2/2)} = \frac{n_2}{n_1}\cdot\frac{\frac{n_1}{2}\cancel{\Gamma(n_1/2)}\ \cancel{\Gamma(n_2/2-1)}}{\cancel{\Gamma(n_1/2)}\left(\frac{n_2}{2}-1\right)\cancel{\Gamma\left(\frac{n_2}{2}-1\right)}}$$

$$= \frac{n_2}{\cancel{n_1}}\cdot\frac{\cancel{n_1}}{\cancel{2}}\cdot\frac{\cancel{2}}{n_2-2} = \frac{n_2}{n_2-2} \qquad \therefore \mu_1' = E(F) = \frac{n_2}{n_2-2}$$

$$\therefore E(F) = \frac{n_2}{n_2-2}$$

$$\text{Mean} = \frac{n_2}{n_2-2} \quad ; \quad n_2 > 2$$

Now,

$$E(F^2) = \int_0^\infty F^2 f(F)\, dF$$

$$= \int_0^\infty F^2 \cdot \frac{\frac{n_1}{n_2}\left(\frac{n_1}{n_2}F\right)^{n_1/2-1}}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}}\, dF$$

Let $w = \frac{n_1}{n_2}F \Rightarrow F = \frac{n_2}{n_1}w \Rightarrow dF = \frac{n_2}{n_1}\, dw$

When $F=0$, then $w=0$, when $F=\infty$, then $w=\infty$

$$\Rightarrow E(F^2) = \int_0^\infty \frac{\left(\frac{n_2}{n_1}w\right)^2 \frac{\cancel{n_1}}{\cancel{n_2}}\left(\frac{n_1}{n_2}F\right)^{n_1/2-1}}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)(1+w)^{\frac{n_1+n_2}{2}}} \cdot \cancel{dF}\ \frac{\cancel{n_2}}{\cancel{n_1}}\, dw$$

$$= \frac{(n_2/n_1)^2}{\beta\left(\frac{n_1}{2},\frac{n_2}{n_1}\right)} \int_0^\infty \frac{w^2\ \cancel{\frac{n_1}{n_2}}\ w^{n_1/2-1}}{(1+w)^{\frac{n_1+n_2}{2}}}\, dw$$

$$= \frac{\left(\frac{n_2}{n_1}\right)^2}{\beta\left(\frac{n_1}{2},n_2/2\right)} \int_0^\infty \frac{w^{(n_1/2+2)-1}}{(1+w)^{\left(\frac{n_1}{2}+2\right)+\left(\frac{n_2}{2}-2\right)}}\, dw$$

$$= \frac{\left(\frac{n_2}{n_1}\right)^2}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)} \cdot \beta\left(\frac{n_1}{2}+2, \frac{n_2}{2}-2\right)$$

$$\Rightarrow E(F^2) = \frac{\left(\frac{n_2}{n_1}\right)^2}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)} \cdot \beta\left(\frac{n_1}{2}+2, \frac{n_2}{2}-2\right) \quad \left[\because \beta(l,m) = \int_0^\infty \frac{x^{l-1}}{(1+x)^{l+m}}\,dx\right]$$

$$= \left(\frac{n_2}{n_1}\right)^2 \frac{\Gamma\left(\frac{n_1}{2}+2\right)\Gamma\left(\frac{n_2}{2}-2\right) \Big/ \Gamma\left(\frac{n_1}{2}+2+\frac{n_2}{2}-2\right)}{\Gamma(n_1/2)\,\Gamma(n_2/2) \Big/ \Gamma\left(\frac{n_1}{2}+\frac{n_2}{2}\right)}$$

$$= \left(\frac{n_2}{n_1}\right)^2 \frac{\Gamma\left(\frac{n_1}{2}+2\right)\Gamma\left(\frac{n_2}{2}-2\right)}{\Gamma(n_1/2)\,\Gamma\left(\frac{n_2}{2}\right)}$$

$$= \left(\frac{n_2}{n_1}\right)^2 \frac{\left(\frac{n_1}{2}+1\right)\cdot\frac{n_1}{2}\,\cancel{\Gamma\frac{n_1}{2}}\ \cancel{\Gamma\left(\frac{n_2}{2}-2\right)}}{\cancel{\Gamma\frac{n_1}{2}}\left(\frac{n_2}{2}-1\right)\left(\frac{n_2}{2}-2\right)\cancel{\Gamma\left(\frac{n_2}{2}-2\right)}}$$

$$= \frac{\frac{n_2^2}{n_1^2}\left(\frac{n_1+2}{2}\right)\cdot\frac{n_1}{2}}{\left(\frac{n_2-2}{2}\right)\cdot\left(\frac{n_2-4}{2}\right)} = \frac{\frac{n_2^2}{n_1^2}(n_1+2)\,n_1}{(n_2-2)(n_2-4)}$$

$$= \frac{n_2^2(n_1+2)}{n_1(n_2-2)(n_2-4)}$$

$$\therefore \mu_2' = E(F^2) = \frac{n_2^2(n_1+2)}{n_1(n_2-2)(n_2-4)}$$

Now, variance, $V(F) = E(F^2) - [E(F)]^2$

$$= \frac{n_2^2(n_1+2)}{n_1(n_2-2)(n_2-4)} - \frac{n_2^2}{(n_2-2)^2}$$

$$= \frac{n_2^2(n_1+2)(n_2-2) - n_2^2 n_1 (n_2-4)}{n_1(n_2-2)^2(n_2-4)}$$

$$= \frac{(n_2^3 - 2n_2^2)(n_1+2) - n_1(n_2^3 - 4n_2^2)}{n_1(n_2-2)^2(n_2-4)}$$

$$= \frac{\cancel{n_1 n_2^3} + 2n_2^3 - 2n_1 n_2^2 + 4n_2^2 - \cancel{n_1 n_2^3} + 4n_1 n_2^2}{n_1(n_2-2)^2(n_2-4)}$$

$$= \frac{2n_2^3 + 2n_1 n_2^2 + 4n_2^2}{n_1(n_2-2)^2(n_2-4)}$$

$$= \frac{2n_2^2(n_2+n_1-2)}{n_1(n_2-2)^2(n_2-4)}$$

$$\therefore \text{ variance } = \frac{2n_2^2(n_2+n_1-2)}{n_1(n_2-2)^2(n_2-4)}$$

Therefore, The mean and variance of F distribution is $\frac{n_2}{n_2-2}$ and $\frac{2n_2^2(n_2+n_1-2)}{n_1(n_2-2)^2(n_2-4)}$ respectively.

"তুমি ছাড়া আ'বুদ আমি ঠিকানা বিহীন
যদিও তুমি দিয়েছো পুরোটা জমিন।"

Question:

Find r-th raw moments of F-distribution.

Answer:

We know that, the pdf of F-distribution is as:

$$f(F) = \frac{\frac{n_1}{n_2}\left(\frac{n_1}{n_2}F\right)^{n_1/2-1}}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}} \quad ; \quad 0<F<\infty \ (F>0)$$

The r-th raw moments about zero of F-distribution is given by-

$$\mu_r' = E\left[F^r\right] \qquad \left[\because E[x^r] = \int x^r f(x)\,dx\right]$$

$$= \int_0^\infty F^r f(F)\,dF$$

$$= \int_0^\infty F^r \frac{\frac{n_1}{n_2}\left(\frac{n_1}{n_2}F\right)^{n_1/2-1}}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}}\,dF$$

Let, $w = \frac{n_1}{n_2}F \Rightarrow F = \frac{n_2}{n_1}w \Rightarrow dF = \frac{n_2}{n_1}dw$

when $F=0$, then $w=0$; when $F=\infty$, then $w=\infty$.

$$\Rightarrow \mu_r' = \int_0^\infty \frac{\left(\frac{n_2}{n_1}w\right)^r \frac{\cancel{n_1}}{\cancel{n_2}}\, w^{n_1/2-1}}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)(1+w)^{\frac{n_1+n_2}{2}}}\cdot\frac{\cancel{n_2}}{\cancel{n_1}}\,dw$$

$$= \frac{\left(\frac{n_2}{n_1}\right)^r}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)} \int_0^\infty \frac{w^{n_1/2+r-1}}{(1+w)^{\left(n_1/2+r\right)+\left(\frac{n_2}{2}-r\right)}}\,dw$$

$$\Rightarrow \mu_r' = \frac{\left(\frac{n_2}{n_1}\right)^r}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)} \beta\left(\frac{n_1}{2}+r, \frac{n_2}{2}-r\right) \quad \left[\because \beta(l,m) = \int_0^\infty \frac{x^{l-1}}{(1+x)^{l+m}}dx\right]$$

$$= \left(\frac{n_2}{n_1}\right)^r \frac{\Gamma\left(\frac{n_1}{2}+r\right)\Gamma\left(\frac{n_2}{2}-r\right) / \Gamma\left(\frac{n_1}{2}+\cancel{r}+\frac{n_2}{2}-\cancel{r}\right)}{\Gamma\frac{n_1}{2}\,\Gamma\frac{n_2}{2} / \Gamma\left(\frac{n_1}{2}+\frac{n_2}{2}\right)}$$

$$\therefore \mu_r' = \frac{\left(\frac{n_2}{n_1}\right)^r \Gamma\left(\frac{n_1}{2}+r\right)\Gamma\left(\frac{n_2}{2}-r\right)}{\Gamma\frac{n_1}{2}\,\Gamma\frac{n_2}{2}}$$

his is the r-th raw moments of F-distribution

utting $r = 1, 2, 3, 4$

en we get, μ_1', μ_2', μ_3' and μ_4'

en we can get mean, variance, skewness

d kurtosis of the distribution.

uestion:

nd the mode of F-distribution.

ode:

de of the distribution will be obtained by

solution of the following equation.

$$\frac{d\log f(F)}{dF} = 0 \ ; \text{ provided } \frac{d^2\log f(F)}{dF^2} < 0 .$$

We know that, the pdf of F distribution is given as –

$$f(F) = \frac{\frac{n_1}{n_2}\left(\frac{n_1}{n_2}F\right)^{n_1/2-1}}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}} \ ; \quad 0<F<\infty \ (F>0)$$

$$\therefore \log f(F) = \log\frac{\frac{n_1}{n_2}}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)} + \left(\frac{n_1}{2}-1\right)\log\left(\frac{n_1}{n_2}F\right) - \left(\frac{n_1+n_2}{2}\right)\log\left(1+\frac{n_1}{n_2}F\right)$$

$$\therefore \frac{d\log f(F)}{dF} = 0 + \frac{\frac{n_1}{2}-1}{\frac{\cancel{n_1}}{\cancel{n_2}}F}\cdot\frac{\cancel{n_1}}{\cancel{n_2}} - \frac{\frac{n_1+n_2}{2}}{1+\frac{n_1}{n_2}F}\cdot\frac{n_1}{n_2}$$

$$= \frac{\frac{n_1}{2}-1}{F} - \frac{n_1^2+n_1n_2}{2\cancel{n_2}\left(\frac{n_2+n_1F}{\cancel{n_2}}\right)}$$

$$= \frac{n_1-2}{2F} - \frac{n_1^2+n_1n_2}{2(n_2+n_1F)}$$

$$= \frac{(n_1-2)(n_2+n_1F) - F(n_1^2+n_1n_2)}{2F(n_2+n_1F)}$$

$$= \frac{n_1n_2+\cancel{n_1^2F}-2n_2-2n_1F-\cancel{n_1^2F}-n_1n_2F}{2F(n_2+n_1F)}$$

$$= \frac{n_1n_2-2n_2-2n_1F-n_1n_2F}{2F(n_2+n_1F)}$$

$$\because \frac{d\log f(F)}{dF} = 0$$

$$\Rightarrow \frac{n_1n_2-2n_2-2n_1F-n_1n_2F}{2F(n_2+n_1F)} = 0$$

$$\therefore \Rightarrow n_1 n_2 - 2n_2 - 2n_1 F - n_1 n_2 F = 0$$

$$\Rightarrow -F(2n_1 + n_1 n_2) = -n_1 n_2 + 2n_2$$

$$\Rightarrow -F(2n_1 + n_1 n_2) = -n_2(n_1 - 2)$$

$$\Rightarrow F = \frac{n_2(n_1-2)}{n_1(n_2+2)}$$

$$\therefore F = \frac{n_2(n_1-2)}{n_1(n_2+2)}$$

It is easy to verify that $\frac{d^2 \log f(F)}{dF^2} < 0$ at $F = \frac{n_2(n_1-2)}{n_1(n_2+2)}$

Therefore, $\frac{n_2(n_1-2)}{n_1(n_2+2)}$ is the mode of the distribution.

$$\therefore \text{Mode} = \frac{n_2(n_1-2)}{n_1(n_2+2)}$$

uestion :

elation between F and χ^2 distribution.

r $F(n_1, n_2)$ distribution and let $n_2 \to \infty$, then $\chi^2 = n_1 F$ follow χ^2 distribution with n_1 d.f.

olution :

e know that, the pdf of F distribution is –

$$f(F) = \frac{\frac{n_1}{n_2}\left(\frac{n_1}{n_2}F\right)^{n_1/2 - 1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1 + \frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}} \quad ; 0 < F < \infty \ (F > 0)$$

$$= \frac{\frac{n_1}{n_2}\left(\frac{n_1}{n_2}\right)^{n_1/2 - 1} F^{n_1/2 - 1}}{\frac{\Gamma\frac{n_1}{2}\,\Gamma\frac{n_2}{2}}{\Gamma\frac{n_1+n_2}{2}}\left(1 + \frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}}$$

$$= \frac{(n_1/n_2)^{n_1/2} F^{n_1/2 - 1}\,\Gamma\frac{n_1+n_2}{2}}{\Gamma\frac{n_1}{2}\,\Gamma\frac{n_2}{2}\left(1 + \frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}}$$

$$= \frac{(n_1)^{n_1/2}(n_2)^{-n_1/2} F^{n_1/2 - 1}\,\Gamma\frac{n_1+n_2}{2}}{\Gamma\frac{n_1}{2}\,\Gamma\frac{n_2}{2}\left(1 + \frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}}$$

$$\therefore f(F) = \frac{\Gamma\frac{n_1+n_2}{2}}{\Gamma\frac{n_2}{2}\,(n_2)^{n_1/2}} \cdot \frac{(n_1)^{n_1/2} F^{n_1/2 - 1}}{\Gamma\frac{n_1}{2}\left(1 + n_1/n_2 F\right)^{\frac{n_1+n_2}{2}}}$$

In the limit as $n_2 \to \infty$, we have –

$$\lim_{n_2 \to \infty} \frac{\Gamma\frac{n_1+n_2}{2}}{\Gamma\frac{n_2}{2}\,(n_2)^{n_1/2}} = \lim_{n_2 \to \infty} \frac{\Gamma\left(\frac{n_1}{2} + \frac{n_2}{2}\right)}{(n_2)^{n_1/2}\,\Gamma\frac{n_2}{2}}$$

$$= \frac{\left(\frac{n_2}{2}\right)^{n_1/2}}{(n_2)^{n_1/2}} = \frac{(n_2)^{n_1/2} \cdot 2^{-n_1/2}}{(n_2)^{n_1/2}} = \frac{1}{2^{n_1/2}}$$

$$\therefore \lim_{n_2\to\infty} \frac{\sqrt{\frac{n_1}{2}+\frac{n_2}{2}}}{(n_2)^{n_1/2}\sqrt{n_2/2}} = \frac{1}{2^{n_1/2}} \qquad \left[\because \frac{\sqrt{n+k}}{\sqrt{n}} \to n^k \text{ as } n\to\infty\right]$$

Also,

$$\lim_{n_2\to\infty}\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}} = \lim_{n_2\to\infty}\left(1+\frac{n_1}{n_2}F\right)^{n_1/2} \cdot \lim_{n_2\to\infty}\left(1+\frac{n_1}{n_2}F\right)^{n_2/2}$$

$$= \lim_{n_2\to\infty}\left[\left(1+\frac{n_1}{n_2}F\right)^{n_2}\right]^{1/2} \cdot \lim_{n_2\to\infty}\left(1+\frac{n_1}{n_2}F\right)^{n_1/2}$$

$$= e^{\frac{n_1F}{2}}\cdot 1 \qquad \left[\because \lim_{n\to\infty}\left\{\left(1+\frac{a}{n}\right)^n\right\}^{1/2} = e^{a/2}\right]$$

$$\therefore \lim_{n_2\to\infty}\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}} = e^{n_1F/2}$$

Hence, $$f(F) = \frac{(n_1)^{n_1/2}\, F^{n_1/2-1}}{2^{n_1/2}\sqrt{n_1/2}\; e^{n_1F/2}}$$

et $\chi^2 = n_1F \Rightarrow F = \frac{\chi^2}{n_1} \Rightarrow \frac{dF}{d\chi^2} = \frac{1}{n_1} = J$

$\therefore |J| = \left|\frac{dF}{d\chi^2}\right| = \frac{1}{n_1} = n_1^{-1}$

ow the pdf of χ^2 is given as –

$$f(\chi^2) = \frac{(n_1)^{n_1/2}\,(\chi^2/n_1)^{n_1/2-1}}{2^{n_1/2}\sqrt{n_1/2}\; e^{\chi^2/2}}\cdot|J|$$

$$\Rightarrow f(\chi^2) = \frac{(n_1)^{n_1/2}\, e^{-\chi^2/2}\,(\chi^2)^{n_1/2-1}\,(n_1)^{-(n_1/2-1)}}{2^{n_1/2}\sqrt{n_1/2}}\cdot n_1^{-1}$$

$$= \frac{(n_1)^{n_1/2-n_1/2+1-1}\, e^{-\chi^2/2}\,(\chi^2)^{n_1/2-1}}{2^{n_1/2}\sqrt{n_1/2}}$$

$$= \frac{e^{-\chi^2/2}\cdot(\chi^2)^{n_1/2-1}}{2^{n_1/2}\sqrt{n_1/2}}$$

$$\therefore f(\chi^2) = \frac{1}{2^{n_1/2}\sqrt{n_1/2}}\cdot(\chi^2)^{n_1/2-1}\, e^{-\chi^2/2} \quad ; 0<\chi^2<\infty$$

which is the pdf of χ^2 distribution with n_1 degree of freedom.

This is the relation between F and χ^2 distribution.

Question :

Establish the relationship between t and F distribution.

or, if F has on $F(n_1,n_2)$, then $t^2 \approx F \sim t^2_{n_2}$ if $n_1=1$ and $n_2=n$

Solution :

we know, the pdf of F-distribution with n_1 and n_2 degree of freedom is –

$$\therefore f(F) = \frac{\frac{n_1}{n_2}\left(\frac{n_1}{n_2}F\right)^{n_1/2-1}}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}} \; ; \; 0<F<\infty$$

Now, putting $n_1 = 1$ and $n_2 = n$, then we get

$$\cancel{f(F) = \frac{\frac{1}{n}\left(\frac{1}{n}\right)^{n_1/2-1} \cdot F^{n_1/2-1}}{\beta}}$$

$$f(F) = \frac{\frac{1}{n}\left(1/n\right)^{1/2-1} F^{1/2-1}}{\beta\left(1/2,\frac{n}{2}\right)\left(1+\frac{1}{n}F\right)^{\frac{1+n}{2}}}$$

$$= \frac{\left(\frac{1}{n}\right)^{1/2} F^{1/2-1}}{\beta\left(1/2, n/2\right)\left(1+\frac{1}{n}F\right)^{\frac{1+n}{2}}}$$

$$\therefore f(F) = \frac{F^{1/2-1}}{\sqrt{n}\,\beta\left(1/2, n/2\right)\left(1+1/n\,F\right)^{\frac{1+n}{2}}}$$

Let, $t^2 = F \Rightarrow 2t \cdot dt = dF \Rightarrow \frac{dF}{dt} = 2t = J$

$\therefore |J| = \left|\frac{dF}{dt}\right| = 2t$

Hence, the pdf of t distribution is –

$$f(t) = \frac{(t^2)^{-1/2}}{\sqrt{n}\,\beta\left(1/2, n/2\right)\left(1+t^2/n\right)^{\frac{n+1}{2}}} \cdot |J|$$

$$= \frac{2t\,t^{-1}}{\sqrt{n}\,\beta\left(1/2, n/2\right)\left(1+t^2/n\right)^{\frac{n+1}{2}}}$$

$$\therefore f(t) = \frac{2}{\sqrt{n}\,\beta\left(1/2, n/2\right)\left(1+t^2/n\right)^{\frac{n+1}{2}}}$$

This function is not one to one, then the function is an even function.

Therefore, the pdf of t distribution is –

$$f(t) = \frac{1}{\sqrt{n}\,\beta\left(1/2, n/2\right)\left(1+t^2/n\right)^{\frac{n+1}{2}}} \; ; \; -\infty<t<\infty$$

which is the pdf of t-distribution with n degrees of freedom.

Hence $t^2 \sim F(1, n)$ (showed)

This is the relationship between t_n and $F_{1,n}$ distribution.

Question:

Beta distribution of 1st kind tends to F distribution.

or, Relation between F distribution and beta distribution of 1st kind.

or,

Let x be a beta variate of 1st kind with parameters n_1 and n_2. Find the distribution of

$$F = \frac{n_2 x}{n_1(1-x)}$$

Answer:

The pdf of beta distribution with parameters $\frac{n_1}{2}$ and $\frac{n_2}{2}$ is given by-

$$f(x) = \frac{1}{\beta(\frac{n_1}{2}, \frac{n_2}{2})} x^{n_1/2 - 1} (1-x)^{n_2/2 - 1} \quad ; \; 0<x<1$$

Let $x = \frac{n_1 F}{n_2 + n_1 F}$

$\Rightarrow n_1 F = n_2 x + n_1 F x$

$\Rightarrow n_1 F - n_1 F x = n_2 x$

$\Rightarrow n_1 F(1-x) = n_2 x$

$\therefore F = \frac{n_2 x}{n_1 (1-x)}$

when $x=0$ then $F=0$; when $x=1$, then $F=\infty$.

$\therefore$ Jacobian of the transformation is

$$|J| = \left|\frac{dx}{dF}\right| = \left|\frac{d}{dF}\left(\frac{n_1 F}{n_2 + n_1 F}\right)\right|$$

$$= \left|\frac{(n_2+n_1F)n_1 - n_1 F(0+n_1)}{(n_2+n_1F)^2}\right|$$

$$= \left|\frac{n_1 n_2 + n_1^2 F - n_1^2 F}{(n_2+n_1F)^2}\right|$$

$$= \left|\frac{n_1 n_2}{(n_2+n_1F)^2}\right|$$

$$\therefore |J| = \left|\frac{dx}{dF}\right| = \frac{n_1 n_2}{(n_2+n_1F)^2}$$

Now, the pdf of F is as-

$$f(F) = f(x)\cdot|J|$$

$$= \frac{1}{\beta(\frac{n_1}{2}, \frac{n_2}{2})} \left(\frac{n_1 F}{n_2+n_1F}\right)^{n_1/2-1} \left(1-\frac{n_1 F}{n_2+n_1F}\right)^{n_2/2-1} \cdot \frac{n_1 n_2}{(n_2+n_1F)^2}$$

$$= \frac{1}{\beta(\frac{n_1}{2}, \frac{n_2}{2})} (n_1)^{\frac{n_1}{2}-1} \left(\frac{1}{n_2+n_1F}\right)^{\frac{n_1}{2}-1} \left(\frac{n_2}{n_2+n_1F}\right)^{\frac{n_2}{2}-1} \cdot \frac{n_1 n_2}{(n_2+n_1F)^2} \cdot F^{n_1/2-}$$

$$= \frac{(n_1)^{\frac{n_1}{2}-1+1} \cdot F^{n_1/2-1} (n_2)^{n_2/2-1+1}}{\beta(\frac{n_1}{2}, \frac{n_2}{2}) (n_2+n_1F)^{\frac{n_1}{2}-1+\frac{n_2}{2}-1+2}}$$

$$= \frac{(n_1)^{n_1/2} (n_2)^{n_2/2} \cdot F^{n_1/2-1}}{\beta(\frac{n_1}{2}, \frac{n_2}{2}) \left[n_2\left(1+\frac{n_1}{n_2}F\right)\right]^{\frac{n_1}{2}+n_2/2}}$$

$$= \frac{(n_1)^{n_1/2} (n_2)^{n_2/2} F^{n_1/2-1}}{\beta(\frac{n_1}{2}, \frac{n_2}{2}) (n_2)^{\frac{n_1+n_2}{2}} \left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}}$$

$$= \frac{(n_1)^{n_1/2} (n_2)^{\cancel{n_2/2}-\frac{n_1}{2}-\cancel{\frac{n_2}{2}}} \cdot F^{n_1/2-1}}{\beta(\frac{n_1}{2}, \frac{n_2}{2}) \left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}}$$

$$= \frac{\left(\frac{n_1}{n_2}\right)^{n_1/2} \cdot F^{n_1/2-1}}{\beta(\frac{n_1}{2}, \frac{n_2}{2}) \left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}}$$

$$\therefore f(F) = \frac{\frac{n_1}{n_2}\left(\frac{n_1}{n_2}F\right)^{n_1/2-1}}{\beta(n_1/2, n_2/2) \left(1+n_1/n_2 F\right)^{\frac{n_1+n_2}{2}}}$$

$$\therefore f(F) = \frac{\frac{n_1}{n_2}\left(\frac{n_1}{n_2}F\right)^{n_1/2-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}} \; ; \; 0<F<\infty$$

which is the pdf of F distribution with n_1 and n_2 degree of freedom.

Therefore the beta distribution of 1st kind tends to F distribution.

Question:

Beta distribution of 2nd kind tends to F distribution.

or, Relation between F distribution and beta distribution of 2nd kind.

or, $F \sim F(n_1, n_2)$, then show that the statistic $\frac{n_1}{n_2}F \sim \beta_2$ (Beta distribution of 2nd kind).

nswer:

e pdf of beta distribution of 2nd kind with $n_1/2$ and $n_2/2$ degrees of freedom is given as -

$$f(x) = \frac{1}{\beta\left(n_1/2, n_2/2\right)} \cdot \frac{x^{n_1/2-1}}{(1+x)^{\frac{n_1+n_2}{2}}} \; ; \; 0<x<\infty$$

If $x = \frac{n_1}{n_2}F \Rightarrow dx = \frac{n_1}{n_2}dF \Rightarrow \frac{dx}{dF} = \frac{n_1}{n_2}$

$$\therefore |J| = \left|\frac{dx}{dF}\right| = \frac{n_1}{n_2}$$

when $x=0$, then $F=0$; when $x=\infty$, then $F=\infty$.

Now, the pdf of F ~~dis~~ is given by -

$$f(F) = f(x) \cdot |J|$$

$$= \frac{1}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \cdot \frac{\left(n_1/n_2 F\right)^{n_1/2-1}}{\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}} \cdot \frac{n_1}{n_2}$$

$$= \frac{1}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \cdot \frac{n_1/n_2\left(\frac{n_1}{n_2}F\right)^{n_1/2-1}}{\left(1+n_1/n_2 F\right)^{\frac{n_1+n_2}{2}}}$$

$$\therefore f(F) = \frac{\frac{n_1}{n_2}\left(\frac{n_1}{n_2}F\right)^{n_1/2-1}}{\beta\left(n_1/2, n_2/2\right)\left(1+n_1/n_2 F\right)^{\frac{n_1+n_2}{2}}} \; ; \; 0<F<\infty$$

which is the pdf of F distribution with n_1 and n_2 degrees of freedom.

So, beta distribution of 2nd kind tends to F-distribution (Showed).

Problem:

If x_1 and x_2 be two independent random variables from $f(x) = e^{-x}$; $0<x<\infty$.

obtain the pdf of $U = \frac{x_1}{x_2}$

or, show that $U = \frac{x_1}{x_2}$ has an F distribution.

Solution:

The pdf of x_1 is - $f(x_1) = e^{-x_1}$; $0 < x_1 < \infty$

The pdf of x_2 is - $f(x_2) = e^{-x_2}$; $0 < x_2 < \infty$

Now, the joint pdf of x_1 and x_2 is given by

$f(x_1, x_2) = f(x_1) \cdot f(x_2)$ [$\because$ x_1 and x_2 are independent]

$= e^{-x_1} \cdot e^{-x_2}$

$\therefore f(x_1, x_2) = e^{-(x_1+x_2)}$; $0 < x_1, x_2 < \infty$.

Here, $u = \frac{x_1}{x_2}$ let $v = x_1 + x_2$

$\Rightarrow x_1 = u x_2$ $\Rightarrow v = u x_2 + x_2$

$\Rightarrow v = x_2(1+u)$

$\Rightarrow x_1 = u \cdot \frac{v}{1+u}$ $\therefore x_2 = \frac{v}{1+u}$

$\therefore x_1 = \frac{uv}{1+u}$

Now, the Jacobian of the transformation is -

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{(1+u)v - uv \cdot 1}{(1+u)^2} & \frac{u}{1+u} \\ \frac{(1+u)\cdot 0 - v \cdot 1}{(1+u)^2} & \frac{1}{1+u} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{(1+u)v - uv}{(1+u)^2} & \frac{u}{1+u} \\ \frac{-v}{(1+u)^2} & \frac{1}{1+u} \end{vmatrix}$$

$$\Rightarrow J = \frac{(1+u)v - uv}{(1+u)^3} + \frac{uv}{(1+u)^3}$$

$$= \frac{v + uv - uv + uv}{(1+u)^3} = \frac{v+uv}{(1+u)^3} = \frac{v(1+u)}{(1+u)^3}$$

$$= \frac{v}{(1+u)^2}$$

$\therefore |J| = \frac{v}{(1+u)^2}$; $0 < v < \infty$, $0 < u < \infty$

Now, the joint pdf of u and v is -

$g(u, v) = e^{-v} |J|$ [$\because x_1 + x_2 = v$]

$\therefore g(u, v) = e^{-v} \frac{v}{(1+u)^2}$; $0 < v < \infty$, $0 < u < \infty$

Now, the pdf of u is given by -

$$g(u) = \int_0^\infty e^{-v} \frac{v}{(1+u)^2} \cdot dv$$

$$= \frac{1}{(1+u)^2} \int_0^\infty e^{-v} \cdot v \cdot dv$$

$$= \frac{1}{(1+u)^2} \int_0^\infty e^{-v} v^{2-1} \cdot dv$$

$$= \frac{1}{(1+u)^2} \Gamma 2 \qquad \left[\because \Gamma n = \int_0^\infty x^{n-1} e^{-x} . dx\right]$$

$\therefore g(u) = \frac{1}{(1+u)^2}$; $0 < u < \infty$ $\left[\because \Gamma 2 = (2-1)! = 1! = 1;\ \Gamma n = (n-1)!\right]$

$$= \frac{(2/2)^{2/2} u^{2/2 - 1}}{\beta(2/2, 2/2)\left(1 + \frac{2}{2}u\right)^{\frac{2+2}{2}}}$$

$$\therefore\ g(u) = \frac{\left(\frac{2}{2}\right)^{2/2}\, u^{2/2-1}}{\beta\left(\frac{2}{2}, \frac{2}{2}\right)\left(1+\frac{2}{2}u\right)^{\frac{2+2}{2}}} \ ; \ 0<u<\infty$$

which is the pdf of $F_{2,2}$.

Therefore $v = \frac{x_1}{x_2}$ has an F-distribution with 2 and 2 degree of freedom. (Showed)

Problem:

If x is a chi-square variate with n d.f., then prove that for large n, $\sqrt{2x} \sim N(\sqrt{2n}, 1)$.

Proof:

Since, x is a chi-square variate with n d.f.

then mean $E(x) = n$, $V(x) = \sigma_x^2 = 2n$; $\sigma_x = \sqrt{2n}$

$$\therefore z = \frac{x - E(x)}{\sigma_x} = \frac{x-n}{\sqrt{2n}} \sim N(0,1) \text{ for large } n \quad \text{(i)}$$

Consider, $P = \left(\frac{x-n}{\sqrt{2n}} \leq z\right)$

$= P(x \leq n + z\sqrt{2n})$

$= P(2x \leq 2n + 2z\sqrt{2n})$ [Multiply by 2 and square root both sides]

$= P\left[\sqrt{2x} \leq (2n + 2z\sqrt{2n})^{1/2}\right]$

$= P\left[\sqrt{2x} \leq \sqrt{2n}\left(1 + z\sqrt{2/n}\right)^{1/2}\right]$

$= P\left[\sqrt{2x} \leq \sqrt{2n}\left(1 + \frac{z}{\sqrt{2n}} + \frac{z^2}{4n} + \cdots\right)\right]$

$[\because (1+x)^n = 1 + n_{c_1} + n_{c_2} + \cdots]$

$= P\left[\sqrt{2x} \leq \sqrt{2n} + z\right]$; for large n

$= P\left[\sqrt{2x} - \sqrt{2n} \leq z\right]$; for large n ... (i)

Since for large n, $\frac{x-n}{\sqrt{2n}} \sim N(0,1)$, from (i) we conclude that

$$\sqrt{2x} - \sqrt{2n} \sim N(0,1) \text{ for large } n$$

$\therefore \sqrt{2x}$ is asymptotically $N(\sqrt{2n}, 1)$

Therefore, $\sqrt{2x} \sim N(\sqrt{2n}, 1)$ (proved)

Problem:

Let. $$f(F) = \frac{\frac{n_1}{n_2}\left(\frac{n_1}{n_2}F\right)^{n_1/2-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}} \ ; \ 0<F<\infty$$

Then obtain the pdf of $z = \frac{n_1}{n_2}F$.

Solution:

The pdf of F distribution with n_1 and n_2 degrees of freedom is-

$$f(F) = \frac{\frac{n_1}{n_2}\left(\frac{n_1}{n_2}F\right)^{n_1/2-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}} \ ; \ 0<F<\infty$$

Here, $z = \frac{n_1}{n_2} F \Rightarrow \cancel{dz = \frac{n_1}{n_2} dF \Rightarrow \frac{dz}{dF} = \frac{n_1}{n_2} = J}$

$\therefore \cancel{|J| = \left|\frac{dz}{dF}\right| = \frac{n_1}{n_2}}$ $\quad F = \frac{n_2}{n_1} z \Rightarrow dF = \frac{n_2}{n_1} dz$

$\therefore |J| = \left|\frac{dF}{dz}\right| = \frac{n_2}{n_1} \quad \therefore \frac{dF}{dz} = \frac{n_2}{n_1} = J$

Then the pdf of z is given as -

$$g(z) = \frac{\frac{n_1}{n_2} (z)^{n_1/2 - 1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right) (1+z)^{\frac{n_1+n_2}{2}}} \cdot |J|$$

$$= \frac{\cancel{\frac{n_1}{n_2}} (z)^{n_1/2-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right) (1+z)^{\frac{n_1+n_2}{2}}} \cdot \cancel{\frac{n_2}{n_1}}$$

$$= \frac{z^{n_1/2 - 1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right) (1+z)^{\frac{n_1+n_2}{2}}}$$

$$g(z) = \frac{1}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \cdot \frac{z^{n_1/2 - 1}}{(1+z)^{n_1/2 + n_2/2}}$$

Which is the pdf of beta distribution of 2nd kind. $\therefore \cancel{g(z)}.\ z = \frac{n_1}{n_2} F \sim \beta_2\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$.

Problem:

$$f(x) = \frac{1}{\beta(n_1/2, n_2/2)} \cdot \frac{x^{n_1/2 - 1}}{(1+x)^{\frac{n_1+n_2}{2}}} \; ; \; 0 < x < \infty$$

and $x = \frac{n_1}{n_2} F$. Find the distribution of F.

Solution:

Given that, $f(x) = \frac{1}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \cdot \frac{x^{n_1/2 - 1}}{(1+x)^{\frac{n_1+n_2}{2}}}$; $0 < x < \infty$

Here, $x = \frac{n_1}{n_2} F \Rightarrow dx = \frac{n_1}{n_2} dF \Rightarrow \frac{dx}{dF} = \frac{n_1}{n_2} = J$

$\therefore |J| = \left|\frac{dx}{dF}\right| = \frac{n_1}{n_2}$

The pdf of F is given as -

$$f(F) = f(x) \cdot |J|$$

$$= \frac{1}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \cdot \frac{\left(\frac{n_1}{n_2} F\right)^{n_1/2 - 1}}{\left(1 + \frac{n_1}{n_2} F\right)^{\frac{n_1+n_2}{2}}} \cdot \frac{n_1}{n_2}$$

$$\therefore f(F) = \frac{\frac{n_1}{n_2} \left(\frac{n_1}{n_2} F\right)^{n_1/2 - 1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right) \left(1 + \frac{n_1}{n_2} F\right)^{\frac{n_1+n_2}{2}}} \; ; \; 0 < F < \infty$$

Which is the pdf of F distribution.

Problem:

If $F \sim F(n_1, n_2)$. Then obtain the pdf of $z = \frac{1}{F}$.

Solution:

We know that, the pdf of F-distribution is –

$$f(F) = \frac{\frac{n_1}{n_2}\left(\frac{n_1}{n_2}F\right)^{n_1/2-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}} \quad ; \quad 0 < F < \infty$$

Here, $z = \frac{1}{F} \Rightarrow F = \frac{1}{z} \Rightarrow dF = -\frac{1}{z^2}dz$

$\Rightarrow \frac{dF}{dz} = -1/z^2 = J \qquad \therefore |J| = \left|\frac{dF}{dz}\right| = \frac{1}{z^2}$

Then the pdf of z is given by –

$$f(z) = f(F) \cdot |J|$$

$$= \frac{\frac{n_1}{n_2}\left(\frac{n_1}{n_2}\cdot\frac{1}{z}\right)^{n_1/2-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}\cdot\frac{1}{z}\right)^{\frac{n_1+n_2}{2}}} \cdot \frac{1}{z^2}$$

$$= \frac{\frac{n_1}{n_2}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \cdot \frac{\left(\frac{n_1}{n_2}\cdot 1/z\right)^{n_1/2-1}}{\left[\left(\frac{n_1}{n_2}\cdot\frac{1}{z}\right)\left(\frac{n_2}{n_1}z+1\right)\right]^{\frac{n_1+n_2}{2}}} \cdot \frac{1}{z^2}$$

$$= \frac{\frac{n_1}{n_2}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \quad \frac{\left(\frac{n_1}{n_2}\cdot\frac{1}{z}\right)^{n_1/2-1}}{\left(\frac{n_1}{n_2}\cdot\frac{1}{z}\right)^{\frac{n_1+n_2}{2}}\left(1+\frac{n_2}{n_1}z\right)^{\frac{n_1+n_2}{2}}} \cdot \frac{1}{z^2}$$

$$= \frac{\frac{n_1}{n_2}\left(\frac{n_2}{n_1}z\right)^{\frac{n_1+n_2}{2}} \cdot \left(\frac{n_2}{n_1}\cdot z\right)^{-n_1/2+1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1+\frac{n_2}{n_1}z\right)^{\frac{n_1+n_2}{2}}} \cdot \frac{1}{z^2}$$

$$= \frac{\frac{n_1}{n_2}\cdot\left(\frac{n_2}{n_1}\right)^{n_1/2+n_2/2-n_1/2+1} \cdot z^{n_1/2+n_2/2-\frac{n_1}{2}+1-2}}{\beta\left(n_1/2, \frac{n_2}{2}\right)\left(1+\frac{n_2}{n_1}z\right)^{\frac{n_1+n_2}{2}}}$$

$$= \frac{\frac{n_1}{n_2}\left(\frac{n_2}{n_1}\right)^{n_2/2+1} z^{n_2/2-1} \cdot \left(\frac{n_2}{n_1}\right)^{-2}\left(\frac{n_2}{n_1}\right)^2}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1+\frac{n_2}{n_1}z\right)^{\frac{n_1+n_2}{2}}}$$

$$= \frac{\frac{n_1}{n_2}\left(\frac{n_2}{n_1}\right)^{n_2/2-1} z^{n_2/2-1}\left(\frac{n_2}{n_1}\right)^2}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1+\frac{n_2}{n_1}z\right)^{\frac{n_1+n_2}{2}}}$$

$$= \frac{\frac{n_2}{n_1}\left(\frac{n_2}{n_1}z\right)^{n_2/2-1}}{\beta\left(n_1/2, n_2/2\right)\left(1+\frac{n_2}{n_1}z\right)^{\frac{n_1+n_2}{2}}}$$

$$\therefore f(z) = \frac{\frac{n_2}{n_1}\left(\frac{n_2}{n_1}z\right)^{n_2/2-1}}{\beta\left(\frac{n_2}{2}, \frac{n_1}{2}\right)\left(1+\frac{n_2}{n_1}z\right)^{\frac{n_1+n_2}{2}}} \quad ; \quad 0 < z < \infty$$

Which is the pdf of F distribution with

n_2 and n_1 degrees of freedom.

$\therefore \quad Z = \frac{1}{F} \sim F_{n_2, n_1}$