PABNA UNIVERSITY OF SCIENCE AND TECHNOLOGY

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Department of Information and Communication Engineering

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Assignment: Student's t-Distribution and F-Distribution

Submitted By:

Name: Md. Uzzal Mia
Roll: 220605
Reg: 1065452
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Dept. of ICE, PUST
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Submitted To:

Dr. Md. Sarwar Hosain
Associate Professor
Department of Information and Communication Engineering
Pabna University of Science and Technology
Pabna, Bangladesh

Student's t-Distribution and F-Distribution

Student's t-Distribution

Let U be a standard normal variate ($N(0,1)$) and V be a chi-square (χ^2) variate with n degrees of freedom, where U and V are independent. The t-statistic is defined as:

$$t = \frac{U}{\sqrt{V/n}}$$

This follows a t-distribution with n degrees of freedom. The probability density function (pdf) is:

$$f(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{t^2}{n}\right)^{-(n+1)/2}, \quad -\infty < t < \infty$$

Properties of t-Distribution

1. Symmetric about $t = 0$ (even function)
2. Mean, median, and mode all equal to 0
3. Variance: $\frac{n}{n-2}$ for $n > 2$
4. Approaches standard normal as $n \to \infty$
5. Heavier tails than normal distribution

Additional Properties of t-Distribution

1. The total probability of t-density is equal to 1, i.e.

$$\int_{-\infty}^{\infty} f(t)\,dt = 1$$

2. For large n, t-distribution reduces to standard normal distribution.
3. All odd order raw moments are zero, i.e.

$$M'_{2r+1} = 0$$

4. Even order raw moments are found by the relation:

$$M'_{2r} = \frac{n^r \Gamma\left(r + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \frac{\Gamma\left(\frac{n}{2} - r\right)}{\Gamma\left(\frac{n}{2}\right)}, \quad r = 1, 2, 3, \ldots$$

5. Since $\beta_1 = 0$ and $\beta_2 = 3 + \frac{6}{n-4} > 3$, therefore, the distribution is symmetric ($\beta_1 = 0$) and leptokurtic ($\beta_2 > 3$).

6. It is a continuous type of distribution and its range extends from $-\infty$ to ∞, i.e.

$$-\infty < t < \infty$$

7. Moment generating function (MGF) of t-distribution does not exist.

Application or Uses of t-Distribution

1. To test if the sample mean ($\bar{x}$) differs significantly from the hypothetical value of μ (the population mean).

2. To test the significance of the difference between two sample means.

3. To test the significance of an observed sample correlation coefficient and sample regression coefficient.

4. To test the significance of an observed partial correlation coefficient.

5. To test the single population mean.

Derivation of t-distribution

Let $U \sim N(0, 1)$ and $V \sim \chi^2_n$. U and V are independent.

Now, we want to find the distribution of:

$$t = \frac{U}{\sqrt{V/n}}.$$

The probability density function (pdf) of U is given by:

$$f(U) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}U^2}, \quad -\infty < U < \infty.$$

The pdf of V is given by:

$$f(V) = \frac{1}{2^{n/2}\Gamma(n/2)} V^{(n/2)-1} e^{-V/2}, \quad 0 < V < \infty.$$

Since U and V are independent, the joint pdf is:

$$f(U, V) = f(U) f(V).$$

Rewriting,

$$f(U,V) = \frac{1}{\sqrt{2\pi}2^{n/2}\Gamma(n/2)} e^{-\frac{1}{2}U^2} V^{(n/2)-1} e^{-V/2}.$$

Now, define $t = \frac{U}{\sqrt{V/n}}$ and let $V = W$. Then,

$$U = t \cdot \sqrt{W/n}.$$

The Jacobian of the transformation is:

$$J = \begin{vmatrix} \frac{\partial U}{\partial t} & \frac{\partial U}{\partial W} \\ \frac{\partial V}{\partial t} & \frac{\partial V}{\partial W} \end{vmatrix} = \begin{vmatrix} \sqrt{W/n} & \frac{1}{2}t\sqrt{n/W} \\ 0 & 1 \end{vmatrix} = \sqrt{W/n}.$$

Thus, the joint pdf of t and W is:

$$g(t,W) = f(U,V) \cdot |J|.$$

Substituting values,

$$g(t,W) = \frac{1}{\sqrt{2\pi}2^{n/2}\Gamma(n/2)} e^{-\frac{1}{2}(t^2 W/n + W)} W^{(n/2)-1} \sqrt{W/n}.$$

Rearranging,

$$g(t,W) = \frac{1}{\sqrt{2\pi n}2^{n/2}\Gamma(n/2)} e^{-\frac{1}{2}(1+t^2/n)W} W^{(n+1)/2-1}.$$

Finally, integrating out W leads to the pdf of the t-distribution:

$$g(t) = \frac{1}{\sqrt{n\pi}\Gamma(n/2)} \frac{\Gamma((n+1)/2)}{(1+t^2/n)^{(n+1)/2}}.$$

This is the Student's t-distribution with n degrees of freedom.

Question:

Show that the total probability of t-density is equal to 1.

$$\int_{-\infty}^{\infty} f(t)\, dt = 1.$$

Proof:

Now,

$$\int_{-\infty}^{\infty} f(t)\, dt = \int_{-\infty}^{\infty} \frac{1}{\sqrt{n}\, p(1/2, n/2)} \left(1 + \frac{t^2}{n}\right)^{-\frac{n+1}{2}} dt$$

Let $w = \frac{t^2}{n}$, then $t = \sqrt{nw}$.

$$\Rightarrow t^2 = nw$$
$$\Rightarrow 2t\,dt = n\,dw$$
$$\Rightarrow dt = \frac{n}{2t}\,dw = \frac{n}{2\sqrt{nw}}\,dw$$

$$dt = \frac{\sqrt{n}}{2\sqrt{w}}\,dw$$

$$\int_{-\infty}^{\infty} f(t)\,dt = 2\int_0^{\infty} \frac{1}{\sqrt{n}\,p(1/2, n/2)}(1+w)^{-\frac{n+1}{2}} \cdot \frac{\sqrt{n}}{2\sqrt{w}}\,dw$$

[Since the integrand is an even function of t,]

$$\int_{-\infty}^{\infty} f(t)\,dt = \int_0^{\infty} \frac{1}{p(1/2, n/2)} \cdot \frac{w^{-1/2}}{(1+w)^{\frac{n+1}{2}}}\,dw$$
$$= \frac{1}{p(1/2, n/2)} \int_0^{\infty} \frac{w^{1/2-1}}{(1+w)^{1/2+n/2}}\,dw$$

$$= \frac{1}{p(1/2, n/2)} \cdot p(1/2, n/2) \cdot \left[\text{Note: } \int_0^{\infty} p(t,m) = \int_0^{\infty} \frac{x^{k-1}}{(1+x)^{k+m}}\,dx\right]$$

$$= 1$$

Question:

Find the mean and variance of the t-distribution.

Answer:

Mean:

$$E(t) = ?$$

We know that the pdf of the t-distribution is:

$$f(t) = \frac{1}{\sqrt{n} \cdot p(1/2, n/2)} \left(1 + \frac{t^2}{n}\right)^{-\frac{n+1}{2}}, \quad -\infty < t < \infty.$$

We also know:

$$E(t) = \int_{-\infty}^{\infty} t \cdot f(t)\,dt$$

$$= \int_{-\infty}^{\infty} t \cdot \frac{1}{\sqrt{n} \cdot p(1/2, n/2)} \left(1 + \frac{t^2}{n}\right)^{-\frac{n+1}{2}} dt$$

$$= \frac{1}{\sqrt{n} \cdot p(1/2, n/2)} \int_{-\infty}^{\infty} \frac{t}{\left(1 + \frac{t^2}{n}\right)^{\frac{n+1}{2}}} dt$$

Since the integrand is an odd function of t (i.e., $t \cdot g(t)$ where $g(t)$ is even), the integral evaluates to zero:

$$E(t) = \frac{1}{\sqrt{n} \cdot p(1/2, n/2)} \cdot 0 = 0.$$

Given that the mean of the t-distribution is:

$$\text{Mean} = E(t) = 0$$

Now, consider the v-th moment:

$$E(t^v) = \int_{-\infty}^{\infty} t^v f(t)\, dt$$

$$= \int_{-\infty}^{\infty} \frac{t^v}{\sqrt{n}\, p(1/2, n/2)} \left(1 + \frac{t^2}{n}\right)^{-\frac{n+1}{2}} dt$$

Let $w = \frac{t^2}{n}$, hence $t = \sqrt{nw}$.

$$\Rightarrow t^2 = nw$$
$$\Rightarrow 2t\, dt = n\, dw$$
$$\Rightarrow dt = \frac{n}{2t} dw = \frac{n}{2\sqrt{nw}} dw = \frac{\sqrt{n}}{2\sqrt{w}} dw$$

When $t = -\infty$, $w = \infty$, and when $t = \infty$, $w = \infty$.

$$E(t^v) = \int_{-\infty}^{\infty} \frac{(\sqrt{nw})^v}{\sqrt{n}\, p(1/2, n/2)} (1+w)^{-\frac{n+1}{2}} \cdot \frac{\sqrt{n}}{2\sqrt{w}} dw$$

$$= \frac{n^{v/2}}{2\, p(1/2, n/2)} \int_0^{\infty} \frac{w^{(v-1)/2}}{(1+w)^{\frac{n+1}{2}}} dw \quad \text{(using symmetry)}$$

[Since the integrand is an even function when v is even]

$$E(t^v) = \frac{n^{v/2}\, p\left(\frac{v+1}{2}, \frac{n-v}{2}\right)}{p(1/2, n/2)} \quad \text{for } v < n$$

$$= n^{v/2} \frac{\Gamma\left(\frac{v+1}{2}\right) \Gamma\left(\frac{n-v}{2}\right)}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{n}{2}\right)} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{v+1}{2}\right) \Gamma\left(\frac{n-v}{2}\right)}$$

$$= \text{Simplified expression for moments}$$

Special Cases:

- For $v = 2$ (Variance when $n > 2$):

$$E(t^2) = \frac{n}{n-2}$$

- For odd v, $E(t^v) = 0$ (by symmetry)

Question:

Show that the mean, median, and mode of the t-distribution are identical and equal to zero, i.e.,

$$\text{Mean} = \text{Median} = \text{Mode} = 0.$$

Answer:

1. Mean:

From previous results, we know that:

$$E(t) = \int_{-\infty}^{\infty} t f(t)\, dt = 0,$$

since the integrand $tf(t)$ is an odd function and the t-distribution is symmetric about zero. Thus,

$$\text{Mean} = 0.$$

2. Median:

Let M be the median of the distribution. By definition:

$$\int_{-\infty}^{M} f(t)\, dt = \frac{1}{2}.$$

Due to the symmetry of the t-distribution about zero:

$$\int_{-\infty}^{0} f(t)\, dt = \int_{0}^{\infty} f(t)\, dt = \frac{1}{2}.$$

Comparing these two results, we conclude:

$$M = 0.$$

Hence,

$$\text{Median} = 0.$$

3. Mode:

The mode is obtained by solving:

$$\frac{d\log f(t)}{dt} = 0, \quad \text{provided} \quad \frac{d^2 \log f(t)}{dt^2} < 0.$$

The pdf of the t-distribution is:

$$f(t) = \frac{1}{\sqrt{n}\, B(1/2, n/2)} \left(1 + \frac{t^2}{n}\right)^{-\frac{n+1}{2}}, \quad -\infty < t < \infty,$$

where B is the Beta function. Taking the logarithm:

$$\log f(t) = \log\left(\frac{1}{\sqrt{n}\, B(1/2, n/2)}\right) - \frac{n+1}{2} \log\left(1 + \frac{t^2}{n}\right).$$

Differentiating with respect to t:

$$\frac{d\log f(t)}{dt} = -\frac{n+1}{2} \cdot \frac{2t/n}{1 + t^2/n} = -\frac{t(n+1)}{n + t^2}.$$

Setting the derivative to zero:

$$-\frac{t(n+1)}{n + t^2} = 0 \implies t = 0.$$

To confirm this is a maximum, check the second derivative:

$$\left.\frac{d^2 \log f(t)}{dt^2}\right|_{t=0} = -\frac{(n+1)}{n} < 0.$$

Thus, the mode occurs at $t = 0$, and

$$\text{Mode} = 0.$$

Conclusion:

Since the mean, median, and mode are all equal to zero, we have shown that:

$$\text{Mean} = \text{Median} = \text{Mode} = 0.$$

Question:

Find the moments of t-distribution. Hence find mean, variance, skewness, kurtosis and comment on the shape of the distribution.

Odd order moments:

The moments are given by:

$$M_{2n+1} = \int_{-\infty}^{\infty} t^{2n+1} f(t)\,dt$$

$$= \int_{-\infty}^{\infty} \frac{t^{2n+1}}{\sqrt{n}\,p(y_2, y_3)\left(1 + \frac{t^2}{n}\right)^{\frac{n+1}{2}}}\,dt$$

$$= 0$$

Since the integrand is an odd function of t, and $(2n + 1)$ is an odd number,

$$\therefore M_{2n+1} = 0$$

Hence, we conclude that all odd order moments are zero.

Even order moments:

By the definition of moments about the origin, we have:

$$M'_{2n} = E[t^{2n}]$$

$$= \int_{-\infty}^{\infty} t^{2n} f(t)\,dt$$

$$= 2\int_{0}^{\infty} t^{2n} \frac{1}{\sqrt{n}\,p(\gamma_2, \gamma_3)\left(1 + \frac{t^2}{n}\right)^{\frac{n+1}{2}}}\,dt$$

Since the integrand is an even function of t,
Let $w = \frac{t^2}{n}$, then:

$$t = \sqrt{wn}$$

$$t^2 = wn$$

$$2t\,dt = n\,dw$$

$$dt = \frac{n}{2t}dw \Rightarrow dt = \frac{n}{2\sqrt{wn}}dw$$

$$dt = \frac{\sqrt{n}}{2\sqrt{w}}dw$$

When $t = 0$, then $w = 0$.
When $t = \infty$, then $w = \infty$.

$$\Rightarrow \mu'_{2r} = n\int_0^\infty \frac{(\sqrt{w}n)^{2r}}{\sqrt{\pi}\,\beta\left(\frac{1}{2},\frac{n}{2}\right)(1+w)^{\frac{n+1}{2}}}\cdot\frac{n}{2\sqrt{wn}}\,dw$$

$$= \int_0^\infty \frac{w^r n^{2r}}{\beta\left(\frac{1}{2},\frac{n}{2}\right)(1+w)^{\frac{n+1}{2}}}\cdot w^{-1/2}\,dw$$

$$= \frac{n^{2r}}{\beta\left(\frac{1}{2},\frac{n}{2}\right)}\int_0^\infty \frac{w^{r+\frac{1}{2}-1}}{(1+w)^{(n+1)/2}}\,dw$$

$$= \frac{n^{2r}}{\beta\left(\frac{1}{2},\frac{n}{2}\right)}\beta\left(r+\frac{1}{2},\frac{n}{2}-r\right) \quad \left[\text{as } \beta(\ell,m)=\int_0^\infty \frac{x^{\ell-1}}{(1+x)^{\ell+m}}dx\right]$$

$$\therefore \mu'_{2r} = \frac{n^{2r}}{\beta\left(\frac{1}{2},\frac{n}{2}\right)}\beta\left(r+\frac{1}{2},\frac{n}{2}-r\right)$$

$$= n^{2r}\cdot\frac{\sqrt{r+\frac{1}{2}}\sqrt{\frac{n}{2}-r}\sqrt{\left(\frac{n+1}{2}+r-r\right)}}{\sqrt{\frac{1}{2}}\sqrt{\frac{n}{2}}\sqrt{\left(\frac{1}{2}+\frac{n}{2}\right)}}$$

$$= n^{2r}\cdot\frac{\sqrt{r+\frac{1}{2}}\sqrt{\frac{n}{2}-r}}{\sqrt{\frac{n}{2}}\sqrt{\frac{1}{2}}}\cdot\frac{\sqrt{\frac{n}{2}+\frac{1}{2}}}{\sqrt{\frac{n}{2}+r}}$$

$$= n^{2r}\cdot\frac{\sqrt{r+\frac{1}{2}}\sqrt{\frac{n}{2}-r}}{\sqrt{\frac{1}{2}}\sqrt{\frac{n}{2}}}$$

$$\therefore \mu'_{2r} = \frac{n^{2r}\sqrt{r+\frac{1}{2}}\sqrt{\frac{n}{2}-r}}{\sqrt{\frac{1}{2}}\sqrt{\frac{n}{2}}}$$

—

Putting $r = 1, 2$, we get:

$$\mu'_2 = \frac{n\sqrt{1+\frac{1}{2}}\sqrt{\frac{n}{2}-1}}{\sqrt{\frac{1}{2}}\sqrt{\frac{n}{2}}} = \frac{n\sqrt{\frac{3}{2}}\sqrt{\frac{n}{2}-1}}{\sqrt{\frac{1}{2}}\sqrt{\frac{n}{2}}} = \frac{n\cdot\frac{1}{2}\cdot\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{n}{2}-1}}{\sqrt{\frac{1}{2}}\cdot\left(\frac{n}{2}-1\right)} = \frac{n}{n-2}$$

$$\therefore \mu'_2 = \frac{n}{n-2}$$

And

$$\mu'_4 = \frac{n^2\sqrt{2+\frac{1}{2}}\sqrt{\frac{n}{2}-2}}{\sqrt{\frac{1}{2}}\sqrt{\frac{n}{2}}} = \frac{n^2\sqrt{\frac{5}{2}}\sqrt{\frac{n}{2}-2}}{\sqrt{\frac{1}{2}}\sqrt{\frac{n}{2}}} = \frac{n^2\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{n}{2}-2}}{\sqrt{\frac{1}{2}}\cdot\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)} = \frac{3n^2}{(n-2)(n-4)}$$

$$\therefore \mu_4' = \frac{3n^2}{(n-2)(n-4)}$$

—

Central Moments:

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = \frac{n}{n-2} - 0 = \frac{n}{n-2} \Rightarrow \mu_2 = \text{variance} = \frac{n}{n-2}$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3 = 0 - 3\left(\frac{n}{n-2}\right) \cdot 0 + 2(0)^3 = 0 \Rightarrow \mu_3 = 0$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4 = \frac{3n^2}{(n-2)(n-4)} - 0 + 0 - 0 = \frac{3n^2}{(n-2)(n-4)}$$

—

Skewness:

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{0^2}{\left(\frac{n}{n-2}\right)^3} = 0$$

—

Kurtosis:

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3n^2}{(n-2)(n-4)} \cdot \frac{(n-2)^2}{n^2} = \frac{3(n-2)}{n-4} = \frac{3n-6}{n-4}$$

$$\Rightarrow \beta_2 = \frac{3n-6}{n-4} = \frac{3n-12+6}{n-4} = \frac{3(n-4)}{n-4} + \frac{6}{n-4} = 3 + \frac{6}{n-4}$$

$$\therefore \beta_2 = 3 + \frac{6}{n-4} > 3$$

F-Distribution

Definition

Let $U \sim \chi^2_{n_1}$ and $V \sim \chi^2_{n_2}$ be independent chi-square variables. The F-statistic is defined as:

$$F = \frac{U/n_1}{V/n_2} \sim F(n_1, n_2)$$

The probability density function is:

$$f(F) = \frac{\left(\frac{n_1}{n_2}\right)^{n_1/2} F^{n_1/2-1}}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1 + \frac{n_1}{n_2}F\right)^{(n_1+n_2)/2}}, \quad F > 0$$

Properties of F-distribution

1. The F-distribution is a continuous probability distribution with support $0 < F < \infty$.
2. It is derived from the chi-square (χ^2) distribution.
3. If $F \sim F(n_1, n_2)$, then:

$$\text{Mean} = \frac{n_2}{n_2 - 2}, \quad \text{for } n_2 > 2$$
$$\text{Variance} = \frac{2n_2^2(n_1 + n_2 - 2)}{n_1(n_2 - 2)^2(n_2 - 4)}, \quad \text{for } n_2 > 4$$

4. The mode of the distribution is:

$$\frac{n_2(n_1 - 2)}{n_1(n_2 + 2)}, \quad \text{for } n_1 > 2$$

5. If $F \sim F(n_1, n_2)$, then:
$$\frac{1}{F} \sim F(n_2, n_1)$$
6. The F-distribution is positively skewed.

Applications of F-distribution

1. F-distribution is used to test the equality of population variance.
2. It is used for testing the significance of an observed multiple correlation coefficient and sample correlation ratio.
3. It is used for testing the linearity of regression.
4. F-distribution is used to test the equality of several means.

Derivation of F-distribution

Let u and v be two independent χ^2 variables with n_1 and n_2 degrees of freedom, respectively. i.e., $u \sim \chi^2_{n_1}$ and $v \sim \chi^2_{n_2}$. u and v are independent.

Now we want to obtain the distribution of $F = \frac{u/n_1}{v/n_2}$.

Hence, the pdf of u is given by:

$$f(u) = \frac{1}{2^{n_1/2}\Gamma(n_1/2)} u^{n_1/2-1} e^{-u/2}, \quad 0 < u < \infty$$

The pdf of v is given by:

$$f(v) = \frac{1}{2^{n_2/2}\Gamma(n_2/2)} v^{n_2/2-1} e^{-v/2}, \quad 0 < v < \infty$$

Then the joint pdf of u and v is given by:

$$f(u,v) = f(u)f(v) \quad [\because u \text{ and } v \text{ are independent}]$$

$$\therefore f(u,v) = \frac{1}{2^{n_1/2}\Gamma(n_1/2)} u^{n_1/2-1} e^{-u/2} \cdot \frac{1}{2^{n_2/2}\Gamma(n_2/2)} v^{n_2/2-1} e^{-v/2}$$

$$\therefore 0 < u, v < \infty$$

Here, $F = \frac{u/n_1}{v/n_2}$, let $v = w$

$$\Rightarrow F = \frac{u/n_1}{w/n_2}$$

$$\Rightarrow \frac{u}{n_1} = F \cdot \frac{w}{n_2} \Rightarrow u = \frac{n_1}{n_2} F w$$

$$\therefore u = \frac{n_1}{n_2} F w \quad \text{and } v = w, \quad u + v = w\left(1 + \frac{n_1}{n_2} F\right)$$

Now, the Jacobian of the transformation is:

$$J = \begin{bmatrix} \frac{\partial u}{\partial F} & \frac{\partial u}{\partial w} \\ \frac{\partial v}{\partial F} & \frac{\partial v}{\partial w} \end{bmatrix} = \begin{bmatrix} \frac{n_1}{n_2} w & \frac{n_1}{n_2} F \\ 0 & 1 \end{bmatrix}$$

$$|J| = \frac{n_1}{n_2} w$$

Then the joint pdf of F and W is given by

$$g(F, W) = f(F|W) \cdot |J| \tag{1}$$

Thus,

$$g(F, W) = \frac{1}{2^{\frac{n_1+n_2}{2}} \Gamma\left(\frac{n_1}{2}\right) \Gamma\left(\frac{n_2}{2}\right)} \left(\frac{n_1}{n_2} F W\right)^{\frac{n_1}{2}-1} W^{\frac{n_2}{2}-1} e^{-\frac{1}{2}(1+\frac{n_1}{n_2}F)W} \tag{2}$$

Now, the pdf of F is given as

$$g(F) = \frac{n_1}{n_2}\left(\frac{n_1}{n_2}F\right)^{\frac{n_1}{2}-1}\int_0^\infty e^{-\frac{1}{2}(1+\frac{n_1}{n_2}F)W}W^{\frac{n_1+n_2}{2}-1}dW \tag{3}$$

Evaluating the integral,

$$g(F) = \frac{n_1}{n_2}\left(\frac{n_1}{n_2}F\right)^{\frac{n_1}{2}-1}\frac{1}{2^{\frac{n_1+n_2}{2}}}\frac{\Gamma\left(\frac{n_1+n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}(1+\frac{n_1}{n_2}F)^{-\frac{n_1+n_2}{2}} \tag{4}$$

which simplifies to the required pdf of the F-distribution:

$$g(F) = \frac{n_1}{n_2}\frac{\left(\frac{n_1}{n_2}F\right)^{\frac{n_1}{2}-1}}{B\left(\frac{n_1}{2},\frac{n_2}{2}\right)}\left(1+\frac{n_1}{n_2}F\right)^{-\frac{n_1+n_2}{2}}, \quad 0 < F < \infty \tag{5}$$

Proof that Total Probability Equals 1

We know that the pdf of the F-distribution is

$$f(F) = \frac{n_1}{n_2}\frac{\left(\frac{n_1}{n_2}F\right)^{\frac{n_1}{2}-1}}{B\left(\frac{n_1}{2},\frac{n_2}{2}\right)}\left(1+\frac{n_1}{n_2}F\right)^{-\frac{n_1+n_2}{2}}, \quad 0 < F < \infty \tag{6}$$

Now, we need to show:

$$\int_0^\infty f(F)dF = 1 \tag{7}$$

Substitute $w = \frac{n_1}{n_2}F$, then $F = \frac{n_2}{n_1}w$ and $dF = \frac{n_2}{n_1}dw$. The integral becomes:

$$\begin{aligned}\int_0^\infty f(F)dF &= \frac{1}{B\left(\frac{n_1}{2},\frac{n_2}{2}\right)}\int_0^\infty w^{\frac{n_1}{2}-1}(1+w)^{-\frac{n_1+n_2}{2}}dw \\ &= \frac{1}{B\left(\frac{n_1}{2},\frac{n_2}{2}\right)}\int_0^\infty \frac{w^{\frac{n_1}{2}-1}}{(1+w)^{\frac{n_1+n_2}{2}}}dw\end{aligned}$$

Recall that the beta function can be expressed as:

$$B(a,b) = \int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}}dt$$

Let $a = \frac{n_1}{2}$ and $b = \frac{n_2}{2}$, then:

$$\int_0^\infty \frac{w^{\frac{n_1}{2}-1}}{(1+w)^{\frac{n_1+n_2}{2}}}dw = B\left(\frac{n_1}{2},\frac{n_2}{2}\right)$$

Therefore:

$$\int_0^\infty f(F)dF = \frac{1}{B\left(\frac{n_1}{2},\frac{n_2}{2}\right)} \times B\left(\frac{n_1}{2},\frac{n_2}{2}\right) = 1$$

Thus, the total probability of the F-density is equal to 1.

Question: Find mean and variance of F-distribution

The pdf of F-distribution is:

$$f(t) = \frac{\frac{n_1}{n_2}\left(\frac{n_1}{n_2}t\right)^{n_1/2-1}}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}t\right)^{(n_1+n_2)/2}}$$

Mean:

$$E(F) = \int_0^\infty t \cdot f(t)\, dt$$

$$E(F) = \int_0^\infty F \cdot \frac{\left(\frac{n_1}{n_2}\right)\left(\frac{n_1}{n_2}F\right)^{\frac{n_1}{2}-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1+\frac{n_1}{n_2}F\right)^{\frac{n_1+n_2}{2}}}\, dF$$

Let, $w = \frac{n_1}{n_2}F \Rightarrow F = \frac{n_2}{n_1}w \Rightarrow dF = \frac{n_2}{n_1}dw$.
When $F = 0$, then $w = 0$, and when $F = \infty$, then $w = \infty$.

$$E(F) = \int_0^\infty \left(\frac{n_2}{n_1}w\right)\frac{n_1}{n_2}w^{\frac{n_1}{2}-1}\frac{n_2}{n_1}dw$$

$$= \frac{n_2}{n_1}\int_0^\infty \frac{w^{\left(\frac{n_1}{2}+1\right)-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)(1+w)^{\frac{n_1+n_2}{2}}}dw$$

$$= \frac{n_2}{n_1}\frac{1}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)}\int_0^\infty \frac{w^{\left(\frac{n_1}{2}+1\right)-1}}{(1+w)^{\left(\frac{n_1}{2}+1\right)+\left(\frac{n_2}{2}-1\right)}}dw$$

Using Beta function properties:

$$= \frac{n_2}{n_1}\frac{\beta\left(\frac{n_1}{2}+1, \frac{n_2}{2}-1\right)}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)}$$

$$= \frac{n_2}{n_1}\frac{\frac{\Gamma\left(\frac{n_1}{2}+1\right)\Gamma\left(\frac{n_2}{2}-1\right)}{\Gamma\left(\frac{n_1}{2}+\frac{n_2}{2}\right)}}{\frac{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}+\frac{n_2}{2}\right)}}$$

$$= \frac{n_2}{n_1}\frac{\Gamma\left(\frac{n_1}{2}+1\right)\Gamma\left(\frac{n_2}{2}-1\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}$$

$$= \frac{n_2}{n_1}\frac{\frac{n_1}{2}\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}-1\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}$$

$$= \frac{n_2}{n_1}\frac{n_1}{2}\frac{\Gamma\left(\frac{n_2}{2}-1\right)}{\Gamma\left(\frac{n_2}{2}\right)}$$

$$= \frac{n_2}{n_2 - 2}$$

Thus,

$$\mu_1' = E(F) = \frac{n_2}{n_2 - 2}$$

$$E(F) = \frac{n_2}{n_2 - 2}, \quad n_2 > 2$$

$$\text{Mean} = \frac{n_2}{n_2 - 2}, \quad n_2 > 2$$

Now,

$$E(F^r) = \int_0^\infty F^r \cdot f(F) \, dF$$

$$= \int_0^\infty F^r \cdot \frac{\Gamma\left(\frac{n_1+n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \left(\frac{n_1}{n_2}\right)^{n_1/2} F^{n_1/2-1} \left(1 + \frac{n_1}{n_2} F\right)^{-(n_1+n_2)/2} dF$$

Let $w = \frac{n_1}{n_2} F \Rightarrow F = \frac{n_2}{n_1} w \Rightarrow dF = \frac{n_2}{n_1} dw$
When $F = 0$, then $w = 0$, when $F = \infty$, then $w = \infty$

$$\Rightarrow E(F^r) = \int_0^\infty \left(\frac{n_2}{n_1} w\right)^r \cdot \frac{\Gamma\left(\frac{n_1+n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} w^{n_1/2-1} (1+w)^{-(n_1+n_2)/2} \frac{n_2}{n_1} dw$$

$$= \left(\frac{n_2}{n_1}\right)^r \frac{\Gamma\left(\frac{n_1+n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \int_0^\infty w^{r+n_1/2-1} (1+w)^{-(n_1+n_2)/2} dw$$

$$= \left(\frac{n_2}{n_1}\right)^r \frac{\Gamma\left(\frac{n_1+n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} B\left(r + \frac{n_1}{2}, \frac{n_2}{2} - r\right)$$

where $B(a, b)$ is the Beta function.

$$\Rightarrow E(\xi^y) = \frac{(\frac{n_2}{n_1})^r}{\beta(\frac{n_1}{2}, \frac{n_2}{2})} \cdot \beta\left(\frac{n_1}{2} + 2, \frac{n_2}{2} - 2\right) \int_0^t \rho(t/m) \frac{dv}{dt}$$

$$= \left(\frac{n_2}{n_1}\right)^r \frac{\left|\frac{n_1}{2} + 2\right| \frac{n_2}{2} - 2}{\left|\frac{n_1}{2}\right| \frac{n_2}{2} / \left|\frac{n_1}{2} + \frac{n_2}{2}\right|}$$

$$= \left(\frac{n_2}{n_1}\right)^r \frac{\left|\frac{n_1}{2} + 2\right| \frac{n_2}{2} - 2}{\left|\frac{n_1}{2}\right| \frac{n_2}{2} / \left|\frac{n_1}{2} + \frac{n_2}{2}\right|}$$

$$= \left(\frac{n_2}{n_1}\right)^r \frac{(\frac{n_1}{2} + 1) \cdot \frac{n_1}{2} \cdot \frac{n_1}{2}}{\left|\frac{n_1}{2}\right| \left|\frac{n_2}{2} + 1\right|} \cdot \frac{n_2}{2}$$

$$= \frac{\left(\frac{n_2}{n_1}\right)^r \left(\frac{n_1+2}{2}\right) \cdot \frac{n_1}{2}}{\left(\frac{n_2-2}{2}\right) \cdot \left(\frac{n_2-4}{2}\right)}$$

$$= \frac{n_2^r(n_1+2)}{n_1(n_2-2)(n_2-4)}$$

$$\therefore M_2 = E(\xi^y) = \frac{n_2^r(n_1+2)}{n_1(n_2-2)(n_2-4)}$$

Now, variance, $v(F) = E(F^y) - [E(F)]^r$

$$= \frac{n_2^r(n_1+2)}{n_1(n_2-2)(n_2-4)} - \frac{n_2^r}{(n_2-2)^r}$$

$$= \frac{n_2^\gamma(n_1+2)(n_2-2) - n_2^\gamma n_1(n_2-4)}{n_1(n_2-2)^\gamma(n_2-4)} = \frac{(n_2^3 - 2n_2^\gamma)(n_1+2) - n_1(n_2^3 - 4n_2^\gamma)}{n_1(n_2-2)^\gamma(n_2-4)} = \frac{n_1 n_2^3 + 2n_2^3 - 2n_1 n_2^\gamma + 4n_2^\gamma - n}{n_1(n_2-2)^\gamma(n_2-4}$$

$$\therefore \mathrm{Var}(F) = \frac{2n_2^\gamma(n_2+n_1-2)}{n_1(n_2-2)^\gamma(n_2-4)}$$

Therefore, the mean and *variance of F distribution are* $\frac{n_2}{n_2-2}$ and $\frac{2n_2^\gamma(n_2+n_1-2)}{n_1(n_2-2)^\gamma(n_2-4)}$ *respectively.*

Question: Find n-th moment of F-distribution.

Answer: We know that the pdf of F-distribution is:

$$f(f) = \frac{\left(\frac{n_1}{n_2}\right)^{n_1/2} f^{n_1/2-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1 + \frac{n_1}{n_2}f\right)^{(n_1+n_2)/2}}$$

The n-th moment about zero of F-distribution is given by

$$M_\phi = E\left[F^\phi\right] \quad \left[\because E[x^\phi] = \int x^\phi f(x)dx\right]$$

$$= \int_0^\infty f^\phi f(f) df$$

$$= \int_0^\infty f^\phi \frac{\left(\frac{n_1}{n_2}\right)^{n_1/2} f^{n_1/2-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1 + \frac{n_1}{n_2}f\right)^{(n_1+n_2)/2}} df$$

Let $w = \frac{n_1}{n_2}f \Rightarrow f = \frac{n_2}{n_1}w \Rightarrow df = \frac{n_2}{n_1}dw$
When $f = 0$, then $w = 0$; when $f = \infty$, then $w = \infty$.

$$\Rightarrow M_\phi = \int_0^\infty \left(\frac{n_2}{n_1}w\right)^\phi \frac{\left(\frac{n_1}{n_2}\right)^{n_1/2}\left(\frac{n_2}{n_1}w\right)^{n_1/2-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)(1+w)^{(n_1+n_2)/2}} \frac{n_2}{n_1} dw$$

$$= \left(\frac{n_2}{n_1}\right)^{\phi} \frac{1}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \int_0^{\infty} \frac{w^{(n_1/2+\phi)-1}}{(1+w)^{(n_1+n_2)/2}} dw$$

$$M_n = \frac{1}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \beta\left(\frac{n_1}{2}+n, \frac{n_2}{2}-n\right) [\cdots \beta(n)] = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{(n_1+n_2)/2}} dx$$

$$= \left(\frac{n_2}{n_1}\right)^{n} \frac{\Gamma\left(\frac{n_1}{2}+n\right)}{\Gamma\left(\frac{n_1}{2}\right)} \cdot \frac{\Gamma\left(\frac{n_2}{2}-n\right)}{\Gamma\left(\frac{n_2}{2}\right)}$$

$$\therefore M_n = \frac{\left(\frac{n_2}{n_1}\right)^{n} \Gamma\left(\frac{n_1}{2}+n\right) \Gamma\left(\frac{n_2}{2}-n\right)}{\Gamma\left(\frac{n_1}{2}\right) \Gamma\left(\frac{n_2}{2}\right)}$$

This gives the raw moments of the distribution with $n = 1, 2, 3, 4$.
Then we get M_1', M_2', M_3' and M_4'.
From these we can obtain the mean, variance, skewness, and other properties of the distribution.

Question:

Find the mode of F-distribution.

Guide:

Mode of the distribution will be obtained by solving the following equation:
We know that the pdf of F-distribution is given as:

$$f(f) = \frac{\left(\frac{n_1}{n_2}\right)^{n_1/2} f^{n_1/2-1}}{\beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right) \left(1+\frac{n_1}{n_2} f\right)^{(n_1+n_2)/2}}, \quad 0 < f < \infty$$

Taking the logarithm:

$$\log f(f) = \log\left(\frac{n_1}{n_2}\right)^{n_1/2} + \left(\frac{n_1}{2}-1\right) \log f - \log \beta\left(\frac{n_1}{2}, \frac{n_2}{2}\right) - \left(\frac{n_1+n_2}{2}\right) \log\left(1+\frac{n_1}{n_2} f\right)$$

Differentiating with respect to f:

$$\frac{d \log f(f)}{df} = 0 + \frac{\frac{n_1}{2}-1}{f} - \frac{\frac{n_1+n_2}{2}}{1+\frac{n_1}{n_2} f} \cdot \frac{n_1}{n_2}$$

$$= \frac{n_1-2}{2f} - \frac{n_1(n_1+n_2)}{2n_2(n_2+n_1 f)}$$

Setting the derivative equal to zero for finding the mode:

$$\frac{n_1 - 2}{2f} = \frac{n_1(n_1 + n_2)}{2n_2(n_2 + n_1 f)}$$

$$(n_1 - 2)(n_2 + n_1 f) = n_1(n_1 + n_2)f$$

$$n_1 n_2 + n_1^2 f - 2n_2 - 2n_1 f = n_1^2 f + n_1 n_2 f$$

$$n_1 n_2 - 2n_2 = n_1 n_2 f + 2n_1 f$$

$$n_2(n_1 - 2) = f(n_1 n_2 + 2n_1)$$

Therefore, the mode is:

$$f_{\text{mode}} = \frac{n_2(n_1 - 2)}{n_1(n_2 + 2)}$$

$$\begin{aligned} \therefore &\quad n_1 n_2 - 2n_2 - 2n_1 f - n_1 n_2 f = 0 \\ \therefore &\quad f = \frac{n_2(n_1 - 2)}{n_1(n_2 + 2)} \\ \therefore &\quad f = \frac{n_2(n_1 - 2)}{n_1(n_2 + 2)} \end{aligned}$$

The second derivative condition for maximum:

$$\left.\frac{d^2 \log f(f)}{df^2}\right|_{f = \frac{n_2(n_1-2)}{n_1(n_2+2)}} < 0$$

Therefore, $\frac{n_2(n_1-2)}{n_1(n_2+2)}$ is the mode of the distribution.

$$\therefore \text{Mode} = \frac{n_2(n_1 - 2)}{n_1(n_2 + 2)}$$

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Mean and Variance of F-Distribution

The pdf of F-distribution is:

$$f(F) = \frac{\left(\frac{n_1}{n_2}\right)^{n_1/2} F^{n_1/2 - 1}}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)\left(1 + \frac{n_1}{n_2}F\right)^{(n_1+n_2)/2}}, \quad F > 0$$

Mean Calculation

$$E(F) = \int_0^\infty F \cdot f(F)\, dF$$

Substitute $w = \frac{n_1}{n_2} F \Rightarrow F = \frac{n_2}{n_1} w$, $dF = \frac{n_2}{n_1} dw$:

$$\begin{aligned}
E(F) &= \frac{1}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \int_0^\infty \frac{n_2}{n_1} w \cdot w^{n_1/2-1} (1+w)^{-(n_1+n_2)/2} \frac{n_2}{n_1} dw \\
&= \frac{n_2}{n_1} \frac{B\left(\frac{n_1}{2}+1, \frac{n_2}{2}-1\right)}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \\
&= \frac{n_2}{n_1} \frac{\Gamma\left(\frac{n_1}{2}+1\right)\Gamma\left(\frac{n_2}{2}-1\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \\
&= \frac{n_2}{n_1} \frac{\frac{n_1}{2}\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}-1\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \\
&= \frac{n_2}{n_2-2}, \quad \text{for } n_2 > 2
\end{aligned}$$

Variance Calculation

First find $E(F^2)$:

$$\begin{aligned}
E(F^2) &= \left(\frac{n_2}{n_1}\right)^2 \frac{B\left(\frac{n_1}{2}+2, \frac{n_2}{2}-2\right)}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \\
&= \left(\frac{n_2}{n_1}\right)^2 \frac{\left(\frac{n_1}{2}+1\right)\frac{n_1}{2}}{\left(\frac{n_2}{2}-1\right)\left(\frac{n_2}{2}-2\right)} \\
&= \frac{n_2^2(n_1+2)}{n_1(n_2-2)(n_2-4)}, \quad \text{for } n_2 > 4
\end{aligned}$$

Then variance is:

$$\begin{aligned}
\mathrm{Var}(F) &= E(F^2) - [E(F)]^2 \\
&= \frac{n_2^2(n_1+2)}{n_1(n_2-2)(n_2-4)} - \left(\frac{n_2}{n_2-2}\right)^2 \\
&= \frac{2n_2^2(n_1+n_2-2)}{n_1(n_2-2)^2(n_2-4)}, \quad \text{for } n_2 > 4
\end{aligned}$$

n-th Moment of F-Distribution

The general formula for the n-th moment is:

$$E(F^n) = \left(\frac{n_2}{n_1}\right)^n \frac{\Gamma\left(\frac{n_1}{2}+n\right)\Gamma\left(\frac{n_2}{2}-n\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}, \quad \text{for } n_2 > 2n$$

Mode of F-Distribution

To find the mode, we maximize the pdf by solving $\frac{d}{dF} \log f(F) = 0$:

$$\begin{aligned}
\frac{d}{dF} \log f(F) &= \frac{n_1/2 - 1}{F} - \frac{(n_1 + n_2)/2}{1 + \frac{n_1}{n_2} F} \cdot \frac{n_1}{n_2} = 0 \\
\frac{n_1 - 2}{2F} &= \frac{n_1(n_1 + n_2)}{2n_2(1 + \frac{n_1}{n_2} F)} \\
(n_1 - 2)(n_2 + n_1 F) &= n_1(n_1 + n_2)F \\
n_1 n_2 - 2n_2 &= 2n_1 F + n_1 n_2 F \\
F_{\text{mode}} &= \frac{n_2(n_1 - 2)}{n_1(n_2 + 2)}, \quad \text{for } n_1 > 2
\end{aligned}$$

Summary of F-Distribution Properties

- **Mean**: $\frac{n_2}{n_2 - 2}$ for $n_2 > 2$
- **Variance**: $\frac{2n_2^2(n_1 + n_2 - 2)}{n_1(n_2 - 2)^2(n_2 - 4)}$ for $n_2 > 4$
- **Mode**: $\frac{n_2(n_1 - 2)}{n_1(n_2 + 2)}$ for $n_1 > 2$
- **n-th Moment**: $\left(\frac{n_2}{n_1}\right)^n \frac{\Gamma\left(\frac{n_1}{2} + n\right)\Gamma\left(\frac{n_2}{2} - n\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}$ for $n_2 > 2n$