Question: What do you mean by Likelihood funct

Answer:

Likelihood function: Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from a population with density f $f(x|\theta)$. If the joint pdf may be regarded as a function of θ, it is called the likelihood functio denoted by $L(\theta)$, defined as

$$L = f(x_1;\theta)\, f(x_2;\theta) \cdots f(x_n;\theta)$$
$$= \prod_{i=1}^{n} f(x_i:\theta)$$

Example: Let $x_1, x_2, \ldots, x_n$ be a random sampl from a distribution with p.d.f

$$f(x:\theta) = \theta^x (1-\theta)^{1-x} \quad ; \quad x = 0, 1 \quad 0 < \theta < 1$$

The joint pdf is

$$f(x_1, x_2, \ldots, x_n; \theta) = \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{n - \sum_{i=1}^{n} x_i}$$

If this joint p.d.f is a function of θ, then it is called likelihood function, denoted by $L(\theta)$, defi as

$$L(\theta) = \theta^{\sum x_i} (1-\theta)^{n - \sum x_i}$$

4a **Question:** Define maximum likelihood estimator with an example.

Answer:

Maximum Likelihood Estimator: Let $L(\theta) = \prod_{i=1}^{n} f(x$ be the likelihood function for the random variab

$x_1, x_2, \ldots, x_n$. Then the value of θ which maximize the likelihood function that the value of θ is called maximum likelihood estimator or MLE of θ. It is usually denoted by $\hat{\theta}$.

The MLE of θ is the solution of likelihood equation

$$\frac{\partial L(\theta)}{\partial \theta} = 0$$

If $\hat{\theta}$ is the MLE of θ. Then $\left.\frac{\partial L(\theta)}{\partial \theta}\right|_{\theta=\hat{\theta}} = 0$

and $\left.\frac{\partial^2 L(\theta)}{\partial \theta^2}\right|_{\theta=\hat{\theta}} < 0$

Example: Let $x_1, x_2, \ldots, x_n$ be a random sample drawn from a normal distribution with mean μ and variance 1. Then the pdf is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2}; \quad -\infty < x < \infty, \; -\infty < \mu < \infty.$$

Now the likelihood function is

$$L(\theta) = \left(\frac{1}{2\pi}\right)^{n/2} e^{-\frac{1}{2}\Sigma(x_i-\theta)^2}$$

$$\therefore \log L(\theta) = \frac{n}{2} \log\left(\frac{1}{2\pi}\right) - \frac{1}{2}\Sigma(x_i-\theta)^2$$

Now,

$$\frac{\partial \log L(\theta)}{\partial \theta} = 0$$

$$\Rightarrow \Sigma(x_i - \theta) = 0$$

$$\therefore \hat{\theta} = \bar{x}.$$

which maximizes the likelihood function. Hence $\hat{\theta} = \bar{x}$ is the MLE of θ.

Question: Describe the principle of maximum-likelihood estimation.

Answer:

The principle of maximum likelihood estimation in finding an estimator ~~of~~ the unknown parameter is the $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$, say which maximizes the likelihood function $L(\theta)$ for variations in parameter i.e we wish to find $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_k)$ so that

$$L(\hat{\theta}) = \sup L(\theta) \quad \forall\, \theta \in \Omega.$$

Let $x_1, x_2, \ldots, x_n$ be a random sample from the density $f(x;\theta)$, for a given sample, this can be treated as a function of θ. This function is called the likelihood function of θ. It is usually denoted by $L(\theta)$ defined as

$$L(\theta) = \prod_{i=1}^{n} f(x_i \mid \theta)$$

The estimator of θ which maximize the likelihood function is called the MLE of θ. Therefore the solution of MLE is

$\frac{\partial \log L}{\partial \theta} = 0$ given the MLE of θ for which

$$\frac{\partial^2 \log L}{\partial \theta^2} < 0$$

4b

Question: Write down the propertise of MLE.

Answer:

The propertise of MLE are given below:

(i) MLE are not generally unbiased but asymtotically unbiased.

(ii) MLE of θ is consistent under some conditions

(iii) MLE of θ is asymtotically efficient.

(iv) MLE of θ is invariant under functional transformation.

(v) MLE of θ is a function of sufficient statistic if exist.

(vi) If a MVB estimator exist, then the MLE is a MVB estimator.

(vii) MLE is asymtotically normally distributed as $N\left(\theta_0, \frac{1}{E\left(-\frac{\partial^2 \log L}{\partial \theta^2}\right)}\right)$

where θ_0 is the true value of θ.

Question: What are the advantages of MLE method over other method of point estimation.

Answer:

The advantage of MLE method over other method of point estimator are mentioned below:

(i) MLE methods obtained consistent estimator.

(ii) MLE methods' obtained most efficient estimator.

(iii) If sufficient statistic exists. MLE method obtained the sufficient estimator.

(iv) MLE has invariant property.

Regularity condition
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4c

Question: Is MLE always unbiased ?

Answer:

No, MLE is not always unbiased.

Because,

Let $x \sim N(\theta_1, \theta_2)$ and $x_1, x_2, \dots, x_n$ be a random sample. Then we know the MLE of θ_1 and θ_2 is

$$\hat{\theta}_1 = \bar{x} \quad \text{and} \quad \cdots\cdots \text{(i)}$$

$$\hat{\theta}_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 \quad \cdots \text{(ii)}$$

Taking expection of (i) on both sides, we have

$$E(\hat{\theta}_1) = E(\bar{x}) = E\left(\frac{1}{n}\sum x_i\right)$$
$$= \frac{1}{n}E\left(\sum x_i\right)$$
$$= \frac{1}{n} \cdot n\theta_1$$
$$= \theta_1$$
$$\therefore E(\hat{\theta}_1) = \theta_1$$

So, $\hat{\theta}_1$ is an unbiased estimator of θ_1

Again, taking expectation of (ii) on both sides, we ha

$$E(\hat{\theta}_2) = \frac{1}{n}E\left(\sum(x_i - \bar{x})^2\right)$$
$$= \frac{\theta_2}{n} E\left[\sum_{i=1}^{n}\frac{(x_i - \bar{x})^2}{\theta_2}\right]$$
$$= \frac{\sigma^2}{n} E\left[\chi^2_{n-1}\right]$$
$$= \frac{n-1}{n}\theta_2 \neq \theta_2$$

$\therefore E(\hat{\theta}_2) = \frac{n-1}{n}\sigma^2 \neq \sigma^2$

This means, $\hat{\theta}_2$ is a biased estimate of θ_2.

Hence, we say that MLE is not always ~~unique~~ unbiased.

Question: Is MLE unique? [MLE for a parameter need not be unique]

Answer:

No, MLE is not unique.

Let a density function

$$f(x;\theta) = 1 \quad ; \theta - \tfrac{1}{2} \le x \le \theta + \tfrac{1}{2}$$
$$= 0 \quad ; \text{otherwise.}$$

Let $x_1, x_2, \ldots, x_n$ be a random sample. Then the likelihood function is

$$L(\theta) = 1 \quad ; \theta - \tfrac{1}{2} \le x_i \le \theta + \tfrac{1}{2}$$
$$= 0 \quad ; \text{otherwise.}$$

If $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ is the ordered sample then

$$\theta - \tfrac{1}{2} \le x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)} \le \theta + \tfrac{1}{2}$$

Thus $L(\theta)$ attains the maximum if

$$\theta - \tfrac{1}{2} \le x_{(1)} \quad \text{and} \quad \theta + \tfrac{1}{2} \geq x_{(n)}$$

$$\Rightarrow \quad \theta \le x_{(1)} + \tfrac{1}{2} \quad \text{and} \quad x_{(n)} - \tfrac{1}{2} \le \theta$$

Hence, every statistic $T = t(x_1, x_2, \ldots, x_n)$ such that

$$x_{(n)} - \tfrac{1}{2} \le t(x_1, x_2, \ldots, x_n) \le x_{(1)} + \tfrac{1}{2},$$

provides an MLE for θ.

Hence, we say MLE is not unique.

Theorem: If $x_1, x_2, \ldots, x_n$ be a random sample from density $f(x;\theta)$ and if an MVB unbiased estimator of θ exist it is given by the maximum likelihood method. OR

If an MVB estimator exist, then the MLE is the MVB estimator.

Proof: Let $x_1, x_2, \ldots, x_n$ be a random sample of size n drawn from the density $f(x;\theta)$, where θ is a parameter. By definition of likelihood is

$$L = \prod_{i=1}^{n} f(x_i;\theta)$$

Let t be the MVB estimator of θ, we must have

$$\frac{\partial \log L}{\partial \theta} = A(t-\theta) \quad \text{where } A = \frac{1}{V(t)} \quad \text{---- (i)}$$

We know the likelihood equation

$$\frac{\partial \log L}{\partial \theta} = 0 \quad \text{---- (ii)}$$

Equation (i) and (ii) we have

$$A(t-\theta) = 0$$

Taking log on both sides, we have..

$$\log L = \frac{n}{2}\log\left(\frac{1}{2\pi}\right) - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\Sigma(x_i-\mu)^2 \quad \cdots\cdots (1)$$

Now, deffenentiating (1) w.r.to μ and set equal to zero i.e

$$\frac{\partial \log L}{\partial \mu} = 0$$

$$\Rightarrow \cancel{0} - \frac{1}{2\sigma^2}\cdot 2\Sigma(x_i-\mu)\cdot(-1) = 0$$

$$\Rightarrow \frac{1}{\sigma^2}\Sigma(x_i-\mu) = 0$$

$$\Rightarrow \Sigma x_i - n\mu = 0$$

$$\Rightarrow n\bar{x} - n\mu = 0$$

$$\Rightarrow \therefore \mu = \bar{x}$$

$$\therefore \hat{\mu} = \bar{x}$$

so, $\bar{x}$ is the MLE of μ.

Now differentiating (1) w.r.to σ^2 and set equal to zero i.e.

$$\frac{\partial \log L}{\partial \sigma^2} = 0$$

$$\Rightarrow -\frac{n}{2}\cdot\frac{1}{\sigma^2} + \frac{1}{2\sigma^4}\Sigma(x_i-\mu)^2 = 0$$

$$\Rightarrow -n\sigma^2 + \Sigma(x_i-\mu)^2 = 0$$

$$\Rightarrow n\sigma^2 = \Sigma(x_i-\mu)^2$$

$$\Rightarrow \sigma^2 = \frac{1}{n}\Sigma(x_i-\mu)^2$$

$$\therefore \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2$$

so, $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2$ is the MLE of σ^2.

Now, we have

$$\hat{\mu} = \bar{x}$$

$$\therefore E(\hat{\mu}) = E(\bar{x})$$

$$= \frac{1}{n}E(\Sigma x_i) \qquad \text{where } \bar{x} = \frac{1}{n}\Sigma x_i$$

$$= \frac{1}{n}\cdot n\mu$$

$$= \mu.$$

$$\therefore E(\hat{\mu}) = \mu.$$

Hence, $\hat{\mu}$ is an unbiased estimate of μ.

Again, we have

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\therefore E(\hat{\sigma}^2) = \frac{1}{n}E\left[\sum_{i=1}^{n}(x_i-\mu)^2\right]$$

$$= \frac{\sigma^2}{n}E\left[\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2\right]$$

$$= \frac{\sigma^2}{n}E\left[\chi^2_{n-1}\right]$$

$$= \frac{\sigma^2}{n}\cdot(n-1) \neq \sigma^2$$

$$\therefore E(\hat{\sigma}^2) = \frac{n-1}{n}\sigma^2 \neq \sigma^2.$$

so $\hat{\sigma}^2$ is not unbiased estimate of σ^2.
Hence, we say MLE is not always unbiased.

Question: Distinguish between joint density function and likelihood function.

Answer:
The difference between a joint density function and likelihood function are given below:

S.N	Joint density function	Likelihood function.
1.	The joint density function is a function of sample observations.	The likelihood function is a function of parameters of the distn.
2.	If $x_1, x_2, \cdots, x_n$ be a random sample then the joint density function is given by $f(x_1, x_2, \cdots, x_n)$	If $x_1, x_2, \cdots, x_n$ be a random sample then the likelihood function is given by $L(\theta \mid x_1, x_2, \cdots, x_n)$, which is function of θ
3.	The joint density satisfies the propertise as $f(x_1, x_2, \cdots, x_n) \geq 0$; $\int\int\cdots\int f(x_1, x_2, \cdots, x_n)\, dx_1, dx_2, \cdots, dx_n = 1$	It may or may not satisfy the condition $\int_\theta L(\theta \mid x)\, d\theta = 1$
4.	It cannot be used to estimate the MLE	It is used to estimate the MLE

Question: Let $x \sim N(\theta_1, \theta_2)$. Then show that MLE of θ_2 is biased but for large sample MLE of θ_2 is unbiased estimator of θ_2.

Solution:
Given that $x \sim N(\theta_1, \theta_2)$. Then the p.d.f is

$$f(x; \theta_1, \theta_2) = \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{1}{2\theta_2}\Sigma(x_i-\theta_1)^2}; \quad -\infty < x < \infty.$$

The likelihood function is

$$L\theta = \prod_{i=1}^{n} f(x_i; \theta_1, \theta_2)$$

$$= \left(\frac{1}{2\pi\theta_2}\right)^{n/2} e^{-\frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)^2}$$

$$= \left(\frac{1}{2\pi}\right)^{n/2} \left(\frac{1}{\theta_2}\right)^{n/2} e^{-\frac{1}{2\theta_2}\Sigma(x_i-\theta_1)^2}.$$

Taking log on both on both sides, we have.

$$\log L = \frac{n}{2}\log\left(\frac{1}{2\pi}\right) - \frac{n}{2}\log\theta_2 - \frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)^2$$

Now, taking derivative w.r.to θ_2; we have.

$$\frac{\partial \log L}{\partial \theta_2} = 0$$

$$\Rightarrow -\frac{n}{2}\cdot\frac{1}{\theta_2} + \frac{1}{2\theta_2^2}\Sigma(x_i-\theta_1)^2 = 0$$

$$\Rightarrow -n\theta_2 + \Sigma(x_i-\theta_1)^2 = 0$$

$$\Rightarrow n\theta_2 = \Sigma(x_i-\theta_1)^2$$

$$\Rightarrow \theta_2 = n^{-1}\Sigma(x_i-\theta_1)^2$$

$$\therefore \hat{\theta}_2 = \frac{1}{n}\Sigma(x_i-\theta_1)^2.$$

which is the MLE of θ_2.

$$\Rightarrow t - \theta = 0$$
$$\Rightarrow \theta = t$$
$$\therefore \hat{\theta} = t$$

which implies that MLE of θ is t

Hence, we conclude that if MVB estimator exist, then the MLE is the MVB estimator

(proved)

Theorem: If a sufficient statistic t exists for θ, any solution of the likelihood equation will be a function of t.

OR

Show that MLE is a function of sufficient statistic if exist.

Proof: Suppose t is sufficient for θ. Then by the factorization theorem, we can write.

$$L(\theta) = g(t;\theta)\, h(x) \quad \ldots\ldots (1)$$

Taking log on both sides we have

$$\log L(\theta) = \log\{g(t;\theta)\, h(x)\}$$
$$= \log\{g(t;\theta)\} + \log h(x)$$
$$\therefore \frac{\partial \log L(\theta)}{\partial \theta} = \frac{\partial \log\{g(t;\theta)\}}{\partial \theta} + 0$$
$$= \frac{\partial \log\{g(t;\theta)\}}{\partial \theta}.$$

From the likelihood ~~MLE~~ equation, we can write

$$\frac{\partial \log L(\theta)}{\partial \theta} = 0$$
$$\Rightarrow \frac{\partial \log\{g(t;\theta)\}}{\partial \theta} = 0$$
$$\Rightarrow f(t;\theta) = 0 \quad \text{where } f(t;\theta) = \frac{\partial \log\{g(t;\theta)\}}{\partial \theta}$$

is a function of t and θ

$$\therefore \hat{\theta} = h(t)$$

which implies that θ is a function of t.

The solution of θ from the equation $f(t;\theta) = 0$ will be a function of t only. Hence we can say, MLE is a function of sufficient statistic.

Question: Show by an example that Maximum likelihood estimator is not necessary unbiased.

Solution:

Let $x \sim N(\mu, \sigma^2)$. Then the p.d.f is

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \; ; \; -\infty < x < \infty$$

Now the likelihood function is

$$L = \prod_{i=1}^{n} f(x_i;\mu,\sigma^2)$$
$$= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

Now taking expectation on both sides, we have

$$E(\hat{\theta}_2) = \frac{1}{n} E\left[\Sigma (x_i-\theta_1)^2\right]$$

$$= \frac{1}{n}\theta_2 E\left[\sum_{i=1}^{n}\frac{(x_i-\theta_1)^2}{\theta}\right]$$

$$= \frac{\theta_2}{n} E\left[\chi^2_{n-1}\right]$$

$$= \frac{\theta_2}{n}(n-1)$$

$$= \frac{n-1}{n}\theta_2$$

$$\therefore E(\hat{\theta}_2) = \frac{n-1}{n}\theta_2 \neq \theta_2$$

So, $\hat{\theta}_2$ is a biased estimator of θ_2.

But for large sample as $n\to\infty$, then

$$E(\hat{\theta}_2) = \theta_2$$

Hence, $\hat{\theta}_2$ is asymtotically unbiased estimate of θ_2.

Question: Prove that MLE of θ obtained by ML method and log ML method are idential.

OR,

Show that at the same point the value of $L(\theta)$ and $\log L(\theta)$ are equal.

Proof: By graphically;

We know that the maximum value of θ would provide the maximum probability of $L(\theta)$

The likelihood function $L(\theta)$ and its logarithm $l(\theta) = \log L(\theta)$ are maximised for the same value of θ because $L(\theta)>0$ and $\log L(\theta)$ is a non-decreasing function or monotonic increasing function of $L(\theta)$. Therefore $L(\theta)$ and $\log L(\theta)$ attains their extreme (maximum or minimum) values for the same of θ.

So we can say, $L(\theta)$ and $\log L(\theta)$ possesses same maximum or minimum points.

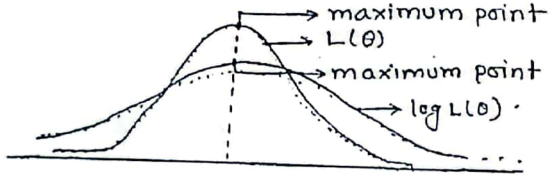

Fig-01.

Proof by example:

Let us consider an example that the p.d.f of a random variable x is

$$f(x;\theta) = \theta^{x}(1-\theta)^{1-x} \quad ; \quad x=0,1; \quad 0<\theta<1$$

Then the likelihood function is

$$L(\theta) = \prod_{i=1}^{n} f(x_i;\theta)$$

$$= \theta^{\Sigma x_i}(1-\theta)^{n-\Sigma x_i} \quad \text{——} \quad (1)$$

Now taking derivative of ① w.r.to θ, we have

$$\frac{\partial L(\theta)}{\partial \theta} = 0$$

$$\Rightarrow \Sigma x_i \theta^{\Sigma x_i - 1} \cdot (1-\theta)^{n-\Sigma x_i} - \theta^{\Sigma x_i}(n-\Sigma x_i)(1-\theta)^{n-\Sigma x_i-1} = 0$$

$$\Rightarrow \Sigma x_i \theta^{\Sigma x_i - 1}(1-\theta)^{n-\Sigma x_i} = \theta^{\Sigma x_i}(n-\Sigma x_i)(1-\theta)^{n-\Sigma x_i - 1}$$

$$\Rightarrow \frac{\Sigma x_i \theta^{\Sigma x_i - 1}(1-\theta)^{n-\Sigma x_i}}{\theta^{\Sigma x_i}(n-\Sigma x_i)(1-\theta)^{n-\Sigma x_i - 1}} = 1$$

$$\Rightarrow \frac{\Sigma x_i (1-\theta)}{(n-\Sigma x_i).\theta} = 1$$

$$\Rightarrow \Sigma x_i (1-\theta) = \theta(n-\Sigma x_i)$$

$$\Rightarrow \Sigma x_i - \theta\Sigma x_i - n\theta + \theta\Sigma x_i = 0$$

$$\Rightarrow n\bar{x} - n\theta = 0$$

$$\Rightarrow \theta = \bar{x}$$

$$\therefore \hat{\theta} = \bar{x}$$

so, $\hat{\theta} = \bar{x}$ is the MLE of θ.

Again taking log on both sides of ①. we get

$$\log L(\theta) = \Sigma x_i \log\theta + (n-\Sigma x_i)\log(1-\theta)$$

Now taking derivative with respect to θ. we have.

$$\frac{\partial \log L(\theta)}{\partial \theta} = \frac{\Sigma x_i}{\theta} + \frac{(n-\Sigma x_i)}{(1-\theta)}(-1) = 0$$

$$\Rightarrow \frac{\Sigma x_i}{\theta} - \frac{n-\Sigma x_i}{(1-\theta)} = 0$$

$$\Rightarrow \frac{\Sigma x_i}{\theta} = \frac{n-\Sigma x_i}{(1-\theta)}$$

$$\Rightarrow \Sigma x_i - \theta\Sigma x_i = n\theta - \theta\Sigma x_i$$

$$\Rightarrow \Sigma x_i - n\theta = 0$$

$$\Rightarrow \theta = \bar{x}$$

$$\therefore \hat{\theta} = \bar{x}$$

which is the MLE of θ.

Hence, we say MLE of θ obtained by ML method and log ML method are identical. (showed)

Question: what do you mean by invariance property of MLE?

Answer:

Invariance property: suppose that we wish to estimate MLE of a function of θ, say $g(\theta)$. Let $\hat{\theta}$ be the MLE of θ in the density $f(x;\theta)$. If $g(\theta)$ is one to one function then the MLE of $g(\theta)$ is $g(\hat{\theta})$. i.e $\widehat{g(\theta)} = g(\hat{\theta})$.

Example: If $\bar{x}$ is the mean of the sample from $B(1,\theta)$, so that $\hat{\theta} = \bar{x}$ and if $\widehat{g(\theta)} = \theta(1-\theta)$ then $\widehat{g(\theta)} = \bar{x}(1-\bar{x})$.

* If $\tau(\theta) = 1/\theta$ then

$\widehat{\tau(\theta)} = 1/\hat{\theta} = \tau(\hat{\theta})$ ($\hat{\theta} \doteq$ MLE of θ)

$\tau(\hat{\theta})$ is the MLE of $\tau(\theta)$.

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Question: State and prove the invariance property.

Statement: If $\hat{\theta}$ is the MLE of θ and $g(\theta)$ is a one to one function of θ, then MLE of $g(\theta)$ is $\widehat{g(\theta)} = g(\hat{\theta})$.

Proof:
Let the likelihood function is $L(\theta)$ and the reparameterized likelihood function $L^*(g(\theta))$.
Then the relationship is $L(\theta) = L^*(g(\theta))$.
Therefore

$$\frac{\partial L(\theta)}{\partial \theta} = \frac{\partial L^*(g(\theta))}{\partial \theta} = \frac{\partial L^*(g(\theta))}{\partial g(\theta)} \cdot \frac{\partial g(\theta)}{\partial \theta}$$

Note that if $\frac{\partial g(\theta)}{\partial \theta} \neq 0$ then.

$$\frac{\partial L^*(g(\theta))}{\partial \theta} = 0 \text{ whenever } \frac{\partial L(\theta)}{\partial \theta} = 0$$

Since $\hat{\theta}$ is the MLE of θ, then

$$\left.\frac{\partial L(\theta)}{\partial \theta}\right|_{\theta=\hat{\theta}} = 0 \text{ implies } \left.\frac{\partial L^* g(\theta)}{\partial g(\theta)}\right|_{\theta=\hat{\theta}} = 0$$

Therefor

$g(\hat{\theta})$ is the solution to $\frac{\partial L^*(g(\theta))}{\partial g(\theta)} = 0$

which is the equation that comes up in finding an MLE in the reparameterized case.

Again, by definition, $\widehat{g(\theta)}$ is the solution to $\frac{\partial L^*(\widehat{g(\theta)})}{\partial g(\theta)} = 0$

Therefore, $\widehat{g(\theta)} = g(\hat{\theta})$ (proved)

Example:
Let, $f(x;\theta) = \frac{1}{\theta} e^{-x/\theta}$; $x \geq 0$

The likelihood function is

$$L(\theta) = \frac{1}{\theta^n} e^{-\sum x_i/\theta} \quad \ldots\ldots \text{(1)}$$

Taking log on both sides of (1) we have

$$\log L = -n \log \theta - \sum x_i/\theta$$

We know the property of MLE is

$$\frac{\partial \log L}{\partial \theta} = 0$$

$$\Rightarrow -n \cdot \frac{1}{\theta} + \sum x_i/\theta^2 = 0$$

$$\Rightarrow -n\theta + \sum x_i = 0$$

$$\Rightarrow -n\theta + n\bar{x} = 0$$

$$\Rightarrow \theta = \bar{x}$$

$$\therefore \hat{\theta} = \bar{x}$$

which is the MLE of θ.

Now from (1), we have

$$L(g(\theta)) = [g(\theta)]^n e^{-g(\theta)\sum x_i} \qquad [g(\theta) = 1/\theta]$$

$$\therefore \log L[g(\theta)] = n \log[g(\theta)] - g(\theta)\sum x_i$$

Taking the first derivative w.r.to $g(\theta)$ and set equal zero. i.e

$$\frac{\partial \log L(g(\theta))}{\partial g(\theta)} = 0$$

$$\Rightarrow \frac{n}{g(\theta)} - \Sigma x_i = 0$$

$$\Rightarrow n - g(\theta)\Sigma x_i = 0$$

$$\Rightarrow g(\theta)\Sigma x_i = n$$

$$\Rightarrow g(\theta) = n/\Sigma x_i$$

$$= \frac{1}{\Sigma x_i/n}$$

$$\Rightarrow g(\theta) = \frac{1}{\bar{x}}$$

$$\Rightarrow g(\theta) = \frac{1}{\hat{\theta}} \quad \text{since } \hat{\theta} = \bar{x}$$

$$\therefore g(\theta) = g(\hat{\theta})$$

Hence $g(\hat{\theta})$ is the MLE of $g(\theta)$ which follows the Invariance property.

*Question: state and prove the limiting dist^n property of MLE.

statement: Let $\hat{\theta}$ is the MLE of θ, then

$$\hat{\theta} \sim AN\left(\theta, \frac{1}{nI(\theta)}\right)$$

Proof:

Let, $f(x;\theta)$ is the density function of x then

$$I(\theta) = E\left(\frac{\partial \ln f}{\partial \theta}\right) = -E\left(\frac{\partial^2 \ln f}{\partial \theta^2}\right)$$ is the fisher information for one observation.

Then, $L(\theta) = \prod_{i=1}^{n} f(x_i;\theta)$

is the likelihood function of θ based on the sample $x_1, x_2, \dots, x_n$.

Now, $\ln L(\theta) = \cancel{\prod_{i=1}^{n} \ln f(x_i;\theta)} = \sum_{i=1}^{n} \ln f(x_i;\theta)$

$$\therefore \frac{\partial \ln L(\theta)}{\partial \theta} = \sum_{i=1}^{n} \frac{\partial \ln f(x_i;\theta)}{\partial \theta} \quad \text{and}$$

$$\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} = \sum_{i=1}^{n} \frac{\partial^2 \ln f(x_i;\theta)}{\partial \theta^2}$$

Note that

$$E\left[\frac{\partial \ln f(x_i;\theta)}{\partial \theta}\right] = 0 = \text{mean}$$

and $$\text{var}\left[\frac{\partial \ln f(x_i;\theta)}{\partial \theta}\right] = I(\theta)$$

Therefore,

$$E\left[\frac{\partial \ln L(\theta)}{\partial \theta}\right] = 0 \quad \text{and} \quad \text{Var}\left[\frac{\partial \ln L(\theta)}{\partial \theta}\right] = nI(\theta)$$

i.e $Z = \frac{\partial \ln L(\theta)}{\partial \theta} = \sum_{i=1}^{n} \frac{\partial \ln f(x_i;\theta)}{\partial \theta}$ has zero mean and variance $nI(\theta)$.

If $\hat{\theta}$ is the MLE of θ, then $\frac{\partial \ln L(\theta)}{\partial \theta}\Big|_{\theta=\hat{\theta}} = 0$ which gives

$$\frac{\partial \ln L(\theta)}{\partial(\theta)} + (\hat{\theta} - \theta)\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} \approx 0 \quad \text{keeping in}$$

in the first two terms of the Taylors series expansion of

$$\frac{\partial \ln L(\theta)}{\partial \theta}\Big|_{\theta=\hat{\theta}} = 0$$

$$\Rightarrow \frac{\partial \ln L(\theta)}{\partial \theta} + (\hat{\theta}-\theta)\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} = 0$$

$$\Rightarrow (\hat{\theta}-\theta)\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} = -\frac{\partial \ln L(\theta)}{\partial \theta}$$

$$\Rightarrow (\hat{\theta}-\theta) = -\frac{\frac{\partial \ln L(\theta)}{\partial \theta}}{\frac{\partial^2 \ln L(\theta)}{\partial \theta^2}}$$

$$\Rightarrow (\hat{\theta}-\theta) = \frac{z}{-\frac{\partial^2 \ln L(\theta)}{\partial \theta^2}}$$

$$\Rightarrow \frac{\hat{\theta}-\theta}{\sqrt{1/nI(\theta)}} = \frac{z}{-\frac{\partial^2 \ln L(\theta)}{\partial \theta^2}/\sqrt{nI(\theta)}}$$

$$= \frac{z/\sqrt{nI(\theta)}}{-\frac{\partial^2 \ln L(\theta)}{\partial \theta^2}/nI(\theta)}$$

$$= \frac{z/\sqrt{nI(\theta)}}{-\sum_{i=1}^{n}\frac{\partial^2 \ln f(x_i;\theta)}{\partial \theta^2}/nI(\theta)} \quad \text{------ (1)}$$

Note that, $z/\sqrt{nI(\theta)} \sim AN(0,1)$ by central limit theorem and $-\sum_{i=1}^{n}\left(\frac{\partial \ln f(x_i;\theta)}{\partial \theta}\right)/I(\theta)n \xrightarrow{P} 1$

Therefore, by slutsky's theorem, the R.H.S of (1) is limiting $N(0,1)$

Hence L.H.S of (1) $\frac{\hat{\theta}-\theta}{\sqrt{1/nI(\theta)}} \sim AN(0,1)$

i.e $\hat{\theta} \sim AN\left(\theta, \frac{1}{nI(\theta)}\right)$ (proved)

4g

Question: Show that by an example, the property of unbiasedness not ingeneral property of MLE?
or,

$$f(x;\theta) = \frac{1}{\theta} \quad ; \quad 0<x<\theta.$$

Find the MLE and check the unbiasedness of this MLE.

Solution:

Let

$$f(x;\theta) = \frac{1}{\theta} \quad ; \quad 0<x<\theta$$
$$= 0 \quad ; \quad \text{otherwise}.$$

Let $x_1, x_2, \ldots, x_n$ be a random sample drawn from the distribution. Then the MLE is

$$L = \prod_{i=1}^{n} f(x_i;\theta) \quad \text{------ (1)}$$
$$= 1/\theta^n.$$

Taking log on both sides, we have

$$\log L = -n\log\theta.$$

The maximum likelihood equation is

$$\frac{\partial \log L}{\partial \theta} = 0$$

$$\Rightarrow -n \frac{1}{\theta} = 0$$

$$\Rightarrow \hat{\theta} = \infty$$, obviously an absurd result.

In this case we locate MLE as follows: We have to choose θ so that L in ① maximum. Now L is maximum if θ is minimum.

Let $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ be the ordered random sample of n independent observations from the given population so that

$$0 \le x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)} \le \theta$$

$$\Rightarrow \theta \geq x_{(n)}$$

Since the minimum value of θ consistent with the sample is $x_{(n)}$, the largest sample observation $\hat{\theta} = x_{(n)}$

$\therefore$ MLE for $\theta = x_{(n)}$
= the largest sample observation.

Then, we get

$$\hat{\theta} = x_{(n)}$$

$$\therefore E(\hat{\theta}) = E(x_n)$$

$$= \int_{x_{(n)}} x_{(n)} f(x_n)\, dx_{(n)}.$$

$$= \int_0^\theta x_{(n)} \frac{n}{\theta^n} x_{(n)}^{n-1}\, dx_{(n)} \cdot \left[f(x_n) = \frac{n}{\theta^n} x_{(n)}^{n-1} \right]$$

$$= \frac{n}{\theta^n} \int_0^\theta x_{(n)}^n\, dx_{(n)}.$$

$$= \frac{n}{\theta^n} \left. \frac{x_{(n)}^{n+1}}{n+1} \right|_0^\theta$$

$$= \frac{n}{\theta^n (n+1)} [\theta^{n+1} - 0]$$

$$= \frac{n\theta}{(n+1)}$$

$$\therefore E(\hat{\theta}) = \frac{n\theta}{n+1} \neq \theta$$

so, $\hat{\theta}$ is a biased estimator of θ. Hence we say the property of unbiased ~~MLE~~ is not in general property of MLE.

Problem: Obtain the MLE's of θ_1 and θ_2 from $N(\theta_1, \theta_2)$ based on a random sample $x_1, x_2, \ldots, x_n$. Also check the unbiasedness of the MLEs.

Solution:

Given that

$$X \sim N(\theta_1, \theta_2).$$

Then the p.d.f of x is

$$f(x; \theta_1, \theta_2) = \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{1}{2\theta_2}(x_i - \theta_1)^2}; \quad -\infty < x < \infty,\ -\infty < \theta_1 < \infty,\ \theta_2 \geq 0$$

P.T.O

The likelihood function is

$$L = \prod_{i=1}^{n} f(x_i; \theta_1, \theta_2)$$

$$= \left(\frac{1}{2\pi\theta_2}\right)^{n/2} e^{-\frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)^2}$$

$$= \left(\frac{1}{2\pi}\right)^{n/2} (\theta_2)^{-n/2} e^{-\frac{1}{2\theta_2}\sum(x_i-\theta_1)^2}$$

Taking log on both sides, we have

$$\log L = \frac{n}{2}\log\frac{1}{2\pi} - \frac{n}{2}\log\theta_2 - \frac{1}{2\theta_2}\sum(x_i-\theta_1)^2 \quad \text{------ (1)}$$

First derivative of (1) w.r.to θ_1 and set equal to zero i.e

$$\frac{\partial \log L}{\partial \theta_1} = 0$$

$$\Rightarrow -\frac{1}{2\theta_2} \cdot 2\sum(x_i-\theta_1)\cdot(-1) = 0$$

$$\Rightarrow \sum(x_i-\theta_1) = 0$$

$$\Rightarrow \sum x_i - n\theta_1 = 0$$

$$\Rightarrow n\bar{x} - n\theta_1 = 0$$

$$\Rightarrow \theta_1 = \bar{x}$$

$$\therefore \hat{\theta}_1 = \bar{x}$$

$$\frac{\partial^2 \log L}{\partial \theta_1^2} = -n/\theta_2$$

$$-E(\ ) = n/\theta_2$$

$$V(\hat{\theta}_1) = \theta_2/n.$$

$$\lim_{n\to\infty} V(\hat{\theta}_1) = 0$$

which is the MLE of θ_1

Again, the first derivative of (1) w.r.to θ_2 we have

$$\frac{\partial \log L}{\partial \theta_2} = 0$$

$$\Rightarrow -\frac{n}{2}\cdot\frac{1}{\theta_2} + \frac{1}{2\theta_2^2}\sum(x_i-\hat{\theta}_1)^2 = 0$$

$$\Rightarrow -n\theta_2 + \sum(x_i-\hat{\theta}_1)^2 = 0$$

$$\Rightarrow n\theta_2 = \sum(x_i-\hat{\theta}_1)^2$$

$$\Rightarrow \theta_2 = \frac{1}{n}\sum(x_i-\hat{\theta}_1)^2$$

$$\therefore \hat{\theta}_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\hat{\theta}_1)^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$$

which is the MLE of θ_2.

Unbiasedness:

We have

$$\hat{\theta}_1 = \bar{x}$$

$$\therefore E(\hat{\theta}_1) = E(\bar{x})$$

$$= \frac{1}{n}E(\sum x_i)$$

$$= \frac{1}{n}\cdot n\theta_1$$

$$= \theta_1$$

$$\therefore E(\hat{\theta}_1) = \theta_1$$

So, $\hat{\theta}_1$ is an unbiased estimator of θ_1

And, $\hat{\theta}_2 = \frac{1}{n}\sum(x_i-\bar{x})^2$

$$\therefore E(\hat{\theta}_2) = \frac{1}{n}E\left[\sum(x_i-\bar{x})^2\right]$$

$$= \frac{\theta_2}{n}E\left[\sum\frac{(x_i-\bar{x})^2}{\theta_2}\right]$$

$$= \frac{\theta_2}{n}E\left[\chi^2_{n-1}\right]$$

$$= \frac{n-1}{n}\cdot\theta_2$$

θ_2 এর efficiency:

CRLB এর সাথে $V(\sigma^2)$ compare করার সময়ে $V\left\{\frac{1}{n}\sum(x_i-\mu_i)^2\right\}$ use করব

সাথে consistency প্রুভ করা লাগবে

$\therefore E(\hat{\theta}_2) = \frac{n-1}{n}\theta_2 \neq \theta_2$

So, $\hat{\theta}_2$ is not unbiased estimate of θ_2.

4h

Problem:

Let $x_1, x_2, \dots, x_n$ be a random sample from the exponential distribution with p.d.f

$$f(x) = \theta e^{-\theta x} \quad ; \quad x \geq 0 \quad \theta \geq 0$$

(i) Find the MLE of θ.

(ii) show that MLE of θ is unbiased, sufficient and consistent estimator.

solution:

(i) Given that

$$f(x) = \theta . e^{-\theta x} \quad ; \quad x \geq 0 \; , \theta > 0$$

Then the MLE is ------(i)

$$L = \prod_{i=1}^{n} f(x_i ; \theta)$$

$$= \prod_{i=1}^{n} \theta e^{-\theta x}$$

$$= \theta^n e^{-\theta \Sigma x_i}$$

$$\therefore \log L = n\log\theta - \theta\Sigma x_i \text{ ------(ii)}$$

Now differentiate of (ii) w.r.to θ we have

$$\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} - \Sigma x_i = 0$$

$$\Rightarrow \frac{n}{\theta} - n\bar{x} = 0$$

$$\Rightarrow \frac{n}{\theta} = n\bar{x}$$

$$\Rightarrow \frac{1}{\theta} = \bar{x}$$

$$\therefore \hat{\theta} = \frac{1}{\bar{x}}$$

Again, $\left.\frac{\partial^2 \log L}{\partial \theta^2}\right|_{\theta=\hat{\theta}} = -n/\theta^2 < 0$

Hence, $\hat{\theta} = \frac{1}{\bar{x}}$ is the MLE of θ

(ii) Unbiasedness:

we have $\hat{\theta} = \frac{1}{\bar{x}}$

$$= n \cdot \frac{1}{\Sigma x_i}$$

$$\therefore E(\hat{\theta}) = n \cdot \frac{1}{E(\Sigma x_i)}$$

$$= n \cdot \frac{1}{n\theta^{-1}}$$

$$= \theta$$

$$\therefore E(\hat{\theta}) = \theta$$

Hence $\hat{\theta}$ is an unbiased estimate of θ.

Sufficient:

we have the likelihood equation is

$$\frac{\partial \log L}{\partial \theta} \cancel{=0}$$

$$\frac{\partial \log L}{\partial \theta} \Rightarrow \frac{n}{\theta} - \Sigma x_i \cancel{=0}$$

$$\Rightarrow \frac{n}{\theta} - n\bar{x} \cancel{=0}$$

$$\Rightarrow n\left(\frac{1}{\theta} - \bar{x}\right) \cancel{=0}$$

$$= \psi(\bar{x}, \theta)$$

is a function of $\bar{x}$ and θ only.

Hence $\bar{x}$ is a sufficient statistic for θ.

Consistency:

we have

$$\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} - \Sigma x_i$$

$$\therefore \frac{\partial^2 \log L}{\partial \theta^2} = -\frac{n}{\theta^2}$$

Now,

$$E\left(-\frac{\partial^2 \log L}{\partial \theta^2}\right) = \frac{n}{\theta^2}$$

we know,

$$v(\hat{\theta}) = \frac{1}{E\left(-\frac{\partial^2 \log L}{\partial \theta^2}\right)}$$

$$= \frac{\theta^2}{n}$$

Now, $\lim_{n\to\infty} v(\hat{\theta}) = \lim_{n\to\infty} \theta^2/n$

$$= 0.$$

Hence, $\bar{x}$ is also a consistent estimator of θ.

Problem: Let $x_1, x_2, \ldots, x_n$ be a random sample with the p.d.f is

$$f(x;\theta) = \frac{1}{\theta} e^{-x/\theta} \; ; \; x \geq 0 ; \theta > 0$$

(i) Find the MLE of θ

(ii) show that MLE of θ is sufficient; unbiased and consistent estimator.

SOLUTION:

(i) Given that $f(x;\theta) = \frac{1}{\theta} e^{-x/\theta} \; ; \; x \geq 0, \; \theta > 0$

The likelihood function is

$$L = \prod_{i=1}^{n} f(x_i;\theta)$$

$$= \frac{1}{\theta^n} e^{-\Sigma x_i/\theta}$$

Taking log on both sides, we have

$$\log L = -n \log \theta - \frac{1}{\theta} \Sigma x_i \quad \text{---} (1)$$

Now, differentiating (1) w.r.to θ, we have, and set equal to zero

$$\frac{\partial \log L}{\partial \theta} = 0$$

$$\Rightarrow -\frac{n}{\theta} + \frac{1}{\theta^2} \Sigma x_i = 0$$

$$\Rightarrow -\frac{n}{\theta} + \frac{n\bar{x}}{\theta^2} = 0$$

$$\Rightarrow \frac{n}{\theta} = \frac{n\bar{x}}{\theta^2}$$

$$\Rightarrow \frac{\bar{x}}{\theta} = 1$$

$$\Rightarrow \theta = \bar{x}$$

$$\therefore \hat{\theta} = \bar{x}.$$

And, $$\left.\frac{\partial^2 \log L}{\partial \theta^2}\right|_{\theta=\hat{\theta}} = \frac{n}{\theta^2} - \frac{2\Sigma x_i}{\theta^3} < 0$$

which is negative definite.

Hence, $\hat{\theta} = \bar{x}$ is the MLE of θ.

(ii) Unbiasedness:

we have

$$\hat{\theta} = \bar{x}$$
$$= \frac{1}{n}\Sigma x_i$$
$$\therefore E(\hat{\theta}) = \frac{1}{n} E[\Sigma x_i]$$
$$= \frac{1}{n} \cdot n\theta$$
$$= \theta$$
$$\therefore E(\hat{\theta}) = \theta$$

Hence, $\hat{\theta}$ is an unbiased estimate of θ.

Sufficiency:

we have

$$\frac{\partial \log L}{\partial \theta} = -\frac{n}{\theta} + \frac{n\bar{x}}{\theta^2}$$
$$= n\left[\frac{\bar{x}}{\theta^2} - \frac{1}{\theta}\right]$$
$$= \psi(\bar{x}, \theta)$$ is a function of $\bar{x}$ and θ only.

Hence, $\bar{x}$ is sufficient for θ.

Consistency:

we have,

$$\frac{\partial \log L}{\partial \theta} = -\frac{n}{\theta} + \frac{n\bar{x}}{\theta^2}$$
$$\therefore \frac{\partial^2 \log L}{\partial \theta^2} = \frac{n}{\theta^2} - \frac{2n\bar{x}}{\theta^3}$$

$$= \frac{n\theta^3 - 2n\bar{x}\theta^2}{\theta^3 \cdot \theta^2}$$
$$= \frac{\theta^2(2n\theta - 2n\bar{x})}{\theta^2 \cdot \theta^3}$$
$$= \frac{n(2\theta - 2\bar{x})}{\theta^3}$$
$$= -\frac{n(2\bar{x} - \theta)}{\theta^3}$$

$$= \frac{n[2\theta - \theta]}{\theta^3}$$
$$= \frac{n\theta}{\theta^3} = n/\theta^2$$

Then,

$$V(\hat{\theta}) = \frac{1}{E\left[-\frac{\partial^2 \log L}{\partial \theta^2}\right]} = \theta^2/n$$

So $\lim_{n\to\infty} V(\hat{\theta}) = \lim_{n\to\infty} \theta^2/n = 0$

Now,

$$E\left[-\frac{\partial^2 \log L}{\partial \theta^2}\right] = \frac{n[2E(\bar{x}) - \theta]}{\theta^3}$$ (problem)

Problem:

Let $x_1, x_2, \dots, x_n$ be a random sample with p.d.f

$$f(x;\theta) = \theta^x (1-\theta)^{1-x} \; ; \; x = 0, 1$$

The likelihood function is

$$L = \prod_{i=1}^{n} f(x_i;\theta)$$
$$= \theta^{\Sigma x_i} (1-\theta)^{n - \Sigma x_i}$$

Taking log on both sides, we have.

$$\log L = \Sigma x_i \log\theta + (n - \Sigma x_i)\log(1-\theta)$$

Now, differentiating w.r.to θ and set equal to zero i.e

$$\frac{\partial \log L}{\partial \theta} = 0$$
$$\Rightarrow \frac{\Sigma x_i}{\theta} + \frac{n - \Sigma x_i}{1-\theta}(-1) = 0$$

$$\Rightarrow \frac{\Sigma x_i}{\theta} - \frac{n-\Sigma x_i}{1-\theta} = 0.$$

$$\Rightarrow \frac{\Sigma x_i}{\theta} = \frac{n-\Sigma x_i}{1-\theta}$$

$$\Rightarrow n\theta - \theta\Sigma x_i = \Sigma x_i - \theta\Sigma x_i.$$

$$\Rightarrow n\theta - \theta\Sigma x_i - \Sigma x_i + \theta\Sigma x_i = 0$$

$$\Rightarrow n\theta - \Sigma x_i = 0$$

$$\Rightarrow n\theta - n\bar{x} = 0$$

$$\Rightarrow \theta = \bar{x}$$

$$\therefore \hat{\theta} = \bar{x}.$$

which is the MLE of θ.

And,

$$\frac{\partial^2 \log L}{\partial\theta^2}\Big|_{\theta=\hat{\theta}} = -\frac{\Sigma x_i}{\theta^2} + \frac{n-\Sigma x_i}{(1-\theta)^2}(-1).$$

$$= -\frac{\Sigma x_i}{\theta^2} - \frac{(n-\Sigma x_i)}{(1-\theta)^2} < 0$$

Hence, $\hat{\theta} = \bar{x}$ is the MLE of θ.

Unbiasedness:

We have

$$\hat{\theta} = \bar{x}$$

$$= \frac{1}{n}\Sigma x_i$$

$$\therefore E(\hat{\theta}) = \frac{1}{n}E[\Sigma x_i]$$

$$= \frac{1}{n}n\theta$$

$$= \theta$$

$$\therefore E(\hat{\theta}) = \theta$$

Hence, $\hat{\theta}$ is an unbiased estimate of θ.

Sufficiency:

We have

$$\frac{\partial \log L}{\partial\theta} = \frac{\Sigma x_i}{\theta} - \frac{n-\Sigma x_i}{1-\theta}$$

$$= \frac{n\bar{x}}{\theta} - \frac{n-n\bar{x}}{1-\theta}$$

$$= n\left[\frac{\bar{x}}{\theta} - \frac{(1-\bar{x})}{1-\theta}\right]$$

$= \psi(\bar{x}, \theta)$. is function of $\bar{x}$ and θ only.

Hence $\bar{x}$ is a sufficient for θ.

Consistency:

We have

$$\frac{\partial \log L}{\partial\theta} = \frac{\Sigma x_i}{\theta} - \frac{n-\Sigma x_i}{1-\theta}$$

$$\therefore \frac{\partial^2 \log L}{\partial\theta^2} = -\frac{\Sigma x_i}{\theta^2} + \frac{(n-\Sigma x_i)}{(1-\theta)^2}(-1)$$

$$= -\frac{\Sigma x_i}{\theta^2} - \frac{n-\Sigma x_i}{(1-\theta)^2}$$

$$= \frac{-\Sigma x_i(1-\theta)^2 - \theta^2(n-\Sigma x_i)}{\theta^2(1-\theta)^2}$$

$$= \frac{-\Sigma x_i + 2\Sigma x_i\theta \cancel{-\theta^2\Sigma x_i} - n\theta^2 \cancel{+\theta^2\Sigma x_i}}{\theta^2(1-\theta)^2}$$

$$= \frac{-n\bar{x} + 2n\bar{x}\theta - n\theta^2}{\theta^2(1-\theta)^2}$$

$$= \frac{-n(\bar{x} - 2\bar{x}\theta + \theta^2)}{\theta^2(1-\theta)^2}.$$

Now,

$$E\left(-\frac{\partial^2 \log L}{\partial \theta^2}\right) = \frac{n\,E(\bar{x}) - 2\theta E(\bar{x}) + \theta^2}{\theta^2(1-\theta)^2}$$

$$= \frac{n\theta - 2\theta^2 + \theta^2}{\theta^2(1-\theta)^2}$$

$$= \frac{n(\theta - \theta^2)}{\theta^2(1-\theta)^2}$$

$$= \frac{n(1-\theta)}{\theta(1-\theta)^2} = \frac{n}{\theta(1-\theta)}$$

we know

$$V(\hat{\theta}) = \frac{1}{E\left(-\frac{\partial^2 \log L}{\partial \theta^2}\right)}$$

$$= \frac{\theta(1-\theta)^2}{n(1-\theta)} = \frac{\theta(1-\theta)}{n}$$

Now,

$$\lim_{n\to\infty} V(\hat{\theta}) = \lim_{n\to\infty} \frac{\theta(1-\theta)^2}{n(1-\theta)} \text{ or } \frac{\theta(1-\theta)}{n}$$

$$= 0$$

Hence $\bar{x}$ is a consistent estimator of θ. (showed)

Problem:

Let $x_1, x_2, \ldots, x_n$ be a random sample with p.d.f is

$$f(x; n, \theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x}; \; x = 0, 1, 2, \ldots, N$$

(i) Find the MLE of θ.

(ii) Show that MLE of θ is unbiased and sufficient statistic of θ.

Solution:

Given that

$$f(x; n, \theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x}; \; x = 0, 1, 2, \ldots, N$$

The likelihood function is

$$L = \prod_{i=1}^{N} f(x_i; n, \theta)$$

$$= \prod_{i=1}^{N} \binom{n}{x} \theta^{\Sigma x_i} (1-\theta)^{nN - \Sigma x_i}$$

Taking log on both sides, we have

$$\log L = \log\left[\prod_{i=1}^{N} \binom{n}{x}\right] + \Sigma x_i \log\theta + nN - \Sigma x_i \log(1-\theta)$$

Now, differentiating w.r.to θ and set equal to zero i.e

$$\frac{\partial \log L}{\partial \theta} = 0$$

$$\Rightarrow \frac{\Sigma x_i}{\theta} + \frac{nN - \Sigma x_i}{(1-\theta)}(-1) = 0$$

$$\Rightarrow \frac{\Sigma x_i}{\theta} = \frac{nN - \Sigma x_i}{(1-\theta)}$$

$$\Rightarrow nN\theta - \theta\Sigma x_i = \Sigma x_i - \theta\Sigma x_i$$

$$\Rightarrow nN\theta = \Sigma x_i$$

$$\Rightarrow \theta = \frac{1}{n}\bar{x}$$

$$\therefore \hat{\theta} = \frac{\bar{x}}{n}.$$

And

$$\frac{\partial^2 \log L}{\partial \theta^2}\Big|_{\theta=\hat\theta} = -\frac{\Sigma x_i}{\hat\theta^2} - \frac{nN-\Sigma x_i}{(1-\hat\theta)^2} < 0$$

which is negative definite.

Hence, $\hat\theta = \frac{\bar x}{n}$ is ~~an unbiased~~ the MLE of θ.

Unbiasedness:

We have

$$\hat\theta = \frac{\bar x}{n}$$

$$= \frac{1}{nN}\Sigma x_i$$

$$\therefore E(\hat\theta) = \frac{1}{nN} E(\Sigma x_i)$$

$$= \frac{1}{nN} N \cdot n\theta$$

$$= \theta$$

$$\therefore E(\hat\theta) = \theta$$

consisten:

$$V(\hat\theta) = \frac{\theta(1-\theta)}{N\theta}$$

$$\text{as } n\to\infty \; V(\hat\theta) = 0$$

efficiency:

$$V(\bar\theta) = \frac{n\theta(1-\theta)}{N}$$

$$V(\hat\theta) = \frac{\theta(1-\theta)}{Nn}$$

$$E(\bar x) = \frac{V(\hat\theta)}{V(\bar x)} = 1/n^2$$

is not efficient

So, $\hat\theta$ is an unbiased estimate of θ.

Sufficiency:

We have

$$\frac{\partial \log L}{\partial\theta} = \frac{\Sigma x_i}{\theta} - \frac{nN-\Sigma x_i}{1-\theta}$$

$$= \frac{n\bar x}{\theta} - \frac{nN-n\bar x}{1-\theta}$$

$$= n\left[\frac{\bar x}{\theta} - \frac{N-\bar x}{1-\theta}\right]$$

$= \psi(\bar x, \theta)$ is a function of $\bar x$ and θ only.

Hence, $\bar x$ is a sufficient for θ.

(showed)

Q.

Problem:

Let $x_1, x_2, \ldots, x_n$ be a random sample with p.d.f is

$$f(x;\theta) = \frac{e^{-\theta}\cdot\theta^x}{\theta\, x!}\;;\quad x\geq 0,\ \theta>0$$

(i) Find the MLE of θ.

(ii) Show that MLE of θ is unbiased, sufficient and consistent estimator.

Solution:

(i) Given that

$$f(x;\theta) = \frac{e^{-\theta}\cdot\theta^x}{x!}\;;\quad x\geq 0$$

The likelihood function is

$$L = \prod_{i=1}^{n} f(x_i;\theta)$$

$$= \prod_{i=1}^{n} \frac{e^{-\theta}\cdot\theta^{x_i}}{x!}$$

$$= \frac{e^{-\theta n}\cdot\theta^{\Sigma x_i}}{\prod_{i=1}^{n}(x_i)!}$$

Taking log on both sides, we have

$$\log L = \log\left[\frac{1}{\prod(x_i)!}\right] - n\theta + \Sigma x_i \log\theta \;\cdots\cdots (1)$$

differentiating (1) with respect to θ and setequ to zero. i.e

$$\frac{\partial \log L}{\partial \theta} = 0$$

$$\Rightarrow 0 - n + \frac{\Sigma x_i}{\theta} = 0$$

$$\Rightarrow -n\theta + \Sigma x_i = 0$$

$$\Rightarrow -n\theta + n\bar{x} = 0$$

$$\Rightarrow \theta = \bar{x}$$

$$\therefore \hat{\theta} = \bar{x}$$

And,

$$\left.\frac{\partial^2 \log L}{\partial \theta^2}\right|_{\theta=\hat{\theta}} = -\frac{\Sigma x_i}{\theta^2} < 0$$

Hence, $\hat{\theta} = \bar{x}$ is the MLE of θ

(ii) Unbiasedness:

we have

$$\hat{\theta} = \bar{x}$$

$$= \frac{1}{n}\Sigma x_i$$

$$\therefore E(\hat{\theta}) = \frac{1}{n} E(\Sigma x_i)$$

$$= \frac{1}{n} n \cdot \theta$$

$$= \theta$$

$$\therefore E(\hat{\theta}) = \theta$$

Hence, $\hat{\theta}$ is an unbiased estimator of θ.

Sufficiency:

we have

$$\frac{\partial \log L}{\partial \theta} = -n + \frac{\Sigma x_i}{\theta}$$

$$= \frac{n\bar{x}}{\theta} - n$$

$$= n\left[\frac{\bar{x}}{\theta} - 1\right]$$

$= \psi(\bar{x}, \theta)$ is a function of $\bar{x}$ and θ only

So, $\bar{x}$ is sufficient for θ.

Consistency:

we have

$$\frac{\partial \log L}{\partial \theta} = -n + \frac{\Sigma x_i}{\theta}$$

$$\therefore \frac{\partial^2 \log L}{\partial \theta^2} = 0 - \frac{\Sigma x_i}{\theta^2}$$

$$= -\frac{n\bar{x}}{\theta^2}$$

Now,

$$E\left(-\frac{\partial^2 \log L}{\partial \theta^2}\right) = \frac{n\, E(\bar{x})}{\theta^2}$$

$$= \frac{n\theta}{\theta^2}$$

$$= n/\theta$$

we know,

$$V(\hat{\theta}) = \frac{1}{E\left(-\frac{\partial^2 \log L}{\partial \theta^2}\right)}$$

$$= \theta/n.$$

Now,

$$\lim_{n\to\infty} V(\hat{\theta}) = \lim_{n\to\infty} \theta/n$$

$$= 0$$

So, $\bar{x}$ is a consistent estimator of θ.

(1-0)

Problem:

Let $x_1, x_2, \ldots, x_n$ denote random sample of size n from a uniform population with p.d.f

$$f(x;\theta) = 1 \quad ; \quad \theta - \tfrac{1}{2} \le x \le \theta + \tfrac{1}{2} \; ; -\infty < \theta < \infty$$

Obtain the MLE of θ.

Solution:

Given that

$$f(x;\theta) = 1 \quad ; \theta - \tfrac{1}{2} < x < \theta + \tfrac{1}{2}$$

The ~~MLE~~ likelihood function is

$$L = \prod_{i=1}^{n} f(x_i ; \theta)$$

$$= 1 \qquad ; \theta - \tfrac{1}{2} < x_i < \theta + \tfrac{1}{2}$$

If $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ is the ordered sample then

$$\theta - \tfrac{1}{2} \le x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)} \le \theta + \tfrac{1}{2}$$

Thus, L attains the maximum if

$$\theta - \tfrac{1}{2} \le x_{(1)} \quad \text{and} \quad \theta + \tfrac{1}{2} \ge x_{(n)}$$

$$\Rightarrow \quad \theta \le \tfrac{1}{2} + x_{(1)} \quad \text{and} \quad \theta \ge x_{(n)} - \tfrac{1}{2}$$

Hence every statistic $t = t(x_1 x_2 \ldots, x_n)$ such that

$$x_{(n)} - \tfrac{1}{2} \le t(x_1 x_2 \cdots x_n) \le x_{(1)} + \tfrac{1}{2}$$

provides an MLE for θ.

Problem: Find the MLE of the parameter α and λ of the distribution

$$f(x;\alpha,\lambda) = \frac{1}{\Gamma\lambda}\left(\frac{\lambda}{\alpha}\right)^{\lambda} e^{-\lambda x/\alpha} \cdot x^{\lambda-1} \; ; \; 0 < x < \infty, \; \lambda > 0$$

Solution:

Given that

$$f(x;\alpha,\lambda) = \frac{1}{\Gamma\lambda} \cdot \left(\frac{\lambda}{\alpha}\right)^{\lambda} \cdot e^{-\lambda x/\alpha} x^{\lambda-1} \; ; \quad 0 < x < \infty$$

The likelihood function is

$$L = \prod_{i=1}^{n} f(x_i ; \alpha, \lambda)$$

$$= \left(\frac{1}{\Gamma\lambda}\right)^{n} \cdot \left(\frac{\lambda}{\alpha}\right)^{n\lambda} \cdot e^{-\lambda \Sigma x_i/\alpha} \cdot \prod_{i=1}^{n} (x_i^{\lambda-1})$$

Taking log on both sides, we have.

$$\log L = -n \log \Gamma\lambda + n\lambda[\log\lambda - \log\alpha] - \frac{\lambda}{\alpha}\Sigma x_i + (\lambda-1)\Sigma \log x_i$$

If G is the Geometric mean of $x_1 x_2 \cdots x_n$ th

$$\log G = \frac{1}{n}\Sigma \log x_i \Rightarrow n \log G = \Sigma \log x_i$$

$$\therefore \log L = -n \log \Gamma\lambda + n\lambda(\log\lambda - \log\alpha) - \frac{\lambda}{\alpha} n\bar{x} + (\lambda-1) n \log G \qquad (1)$$

where G is independent of λ and α.

Differentiating (1) w.r.to α and set equal to zero

$$\frac{\partial \log L}{\partial \alpha} = 0$$

$$\Rightarrow -n\lambda \cdot \frac{1}{\alpha} + \frac{\lambda n \bar{x}}{\alpha^2} = 0$$

(1-8)

$$\Rightarrow -\frac{1}{\alpha}+\frac{\bar{x}}{\alpha^2}=0$$

$$\Rightarrow -\alpha+\bar{x}=0$$

$$\Rightarrow \alpha=\bar{x}$$

$$\therefore \hat{\alpha}=\bar{x}.$$

Again, differentiating ① w.r.to λ and set equal to zero

$$\frac{\partial \log L}{\partial \lambda}=0$$

$$\Rightarrow -n\left(\log\lambda-\frac{1}{2\lambda}\right)+n\left\{1\cdot\log\lambda+\lambda\cdot\frac{1}{\lambda}-1\cdot\log\alpha\right\}-\frac{n\bar{x}}{\alpha}+n\log G=0$$

$$\Rightarrow -n\log\lambda+\frac{n}{2\lambda}+n\log\lambda+n-n\log\alpha-\frac{n\bar{x}}{\alpha}+n\log G=0$$

$$\Rightarrow n\left[\frac{1}{2\lambda}+1-\log\alpha-\frac{\bar{x}}{\alpha}+\log G\right]=0$$

$$\Rightarrow \frac{1}{2\lambda}+\left[1-\log\alpha+\log G-\bar{x}/\alpha\right]=0$$

$$\Rightarrow \frac{1}{2\lambda}+\left[1-\log\alpha+\log G-1\right]=0$$

$$\Rightarrow \frac{1}{2\lambda}+\left[\log G-\log\alpha\right]=0$$

$$\Rightarrow 1+2\lambda(\log G-\log\alpha)=0$$

$$\Rightarrow 1+2\lambda\log\left(\frac{G}{\alpha}\right)=0$$

$$\Rightarrow 1-2\lambda\log\left(\frac{\bar{x}}{G}\right)=0$$

$$\Rightarrow 2\lambda=\frac{1}{\log\left(\frac{\bar{x}}{G}\right)}$$

$$\therefore \hat{\lambda}=\frac{1}{2\log(\bar{x}/G)}.$$

Hence, $\hat{\alpha}=\bar{x}$

$$\hat{\lambda}=\frac{1}{2\log(\bar{x}/G)}$$

are the MLE of α and λ respectivily.

Problem:

Let $x_1, x_2, \ldots, x_n$ be a random sample from the uniform distribution with p.d.f

$$f(x;\theta)=\frac{1}{\theta} \quad ; \quad 0<x<\theta. \quad \theta>0$$

obtain the MLE for θ. (Before solution it).

Problem:

Let

$$f(x;\theta)=\frac{1}{1-\theta} \quad ; \quad \theta<x<1$$

Find the MLE of θ.

Solution:

Given that

$$f(x;\theta)=\frac{1}{1-\theta} \quad ; \quad \theta<x<1$$

Let $x_1, x_2, \ldots, x_n$ be a random sample from the given density. The likelihood function is

$$L=\prod_{i=1}^{n} f(x_i;\theta) \quad ; \quad \theta<x_i<1 \quad \text{①}$$

$$=\left(\frac{1}{1-\theta}\right)^n.$$

If $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ is orderd sample, then

$$\theta \le x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)} \le 1$$

Taking log on both sides, we have

$$\log L = -n \log(1-\theta)$$

$$\therefore \frac{\partial \log L}{\partial \theta} = -n \cdot \frac{1}{1-\theta}(-1) = 0$$

$$\Rightarrow \frac{n}{1-\theta} = 0$$

$$\Rightarrow n = 0 \quad \text{(non regular)}$$

In this case we locate MLE as follows:
we have choose θ so that L in (1) is maximum.
Now L is maximum if θ is minimum.
If $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ is the orderd sample then,

$$\theta \le x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)} \le 1$$

$$\Rightarrow \theta \le x_{(1)}.$$

Since the minimum value of θ consistent with the sample is $x_{(1)}$. so,

$$\hat{\theta} = x_{(1)}.$$

$\therefore$ MLE for $\theta = x_{(1)}$
$= $ smallest order statistic.

Here any value of statistic of θ in the range $\left(x_{(n)} - \frac{1}{2}, x_{(1)} + \frac{1}{2}\right)$ would be the MLE of θ.

Problem:
Let a random sample $x_1, x_2, \ldots, x_n$ taken from a density function

$$f(x|\theta) = e^{-(x-\theta)} \; ; \; 0 \le x \le \infty$$

Find the MLE of the given function.

Solution:
Given that, $f(x|\theta) = e^{-(x-\theta)} \; ; \; 0 \le x \le \infty$

The likelihood function is

$$L(\theta) = \prod_{i=1}^{n} f(x|\theta)$$

$$= \prod_{i=1}^{n} e^{-(x_i-\theta)}$$

$$= e^{-\Sigma (x_i - \theta)}$$

Taking log on both sides, we have

$$\log L(\theta) = -\Sigma (x_i - \theta)$$

$$= -\Sigma x_i + n\theta$$

$$\frac{\partial \log L(\theta)}{\partial \theta} = n$$

By the propertise of MLE is

$$\frac{\partial \log L(\theta)}{\partial \theta} = 0$$

$n = 0$

Now the ordered sample is

$$0 \le x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)} \le \infty$$

It is observed that the minimum posible value of θ is $x_{(1)}$ for which L is as maximum as posible. Therefore the MLE of θ is $x_{(1)}$. Hence MLE of θ is

$\hat{\theta} = x_{(1)}$ is the first order statistic.

$\hat{\theta} = x_{(1)}$ is the first order statistic i.e $L(\theta)$ is maximum if θ is minimum as possible with in the range

$$0 \le x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)} \le \infty.$$

Problem:
Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from a uniform distribution over the interval $(\mu-\sigma, \mu+\sigma)$. Obtain the MLE of μ and σ.

OR

Solution:
Given function is

$$f(x|\mu, \sigma^2) = \frac{1}{2\sigma} \; ; \; \mu-\sigma \le x \le \mu+\sigma$$

Obtain the MLE of μ and σ.

Solution:
We have

$$\because f(x|a, b) = \frac{1}{b} \; ; \; a \le x \le b.$$

$$f(x|\mu, \sigma) = \frac{1}{2\sigma} \; ; \; \mu-\sigma \le x \le \mu+\sigma$$

The likelihood function for a sample of size n is

$$L(\mu, \sigma^2) = \prod_{i=1}^{n} f(x_i|\mu, \sigma^2)$$
$$= \prod_{i=1}^{n} \frac{1}{2\sigma}$$
$$= \frac{1}{(2\sigma)^n} \; ; \; \mu-\sigma \le y_{(1)} \le \cdots \le y_{(n)} \le \mu+\sigma$$

where the LF is defined of this region. which is maximum for the minimum possible value

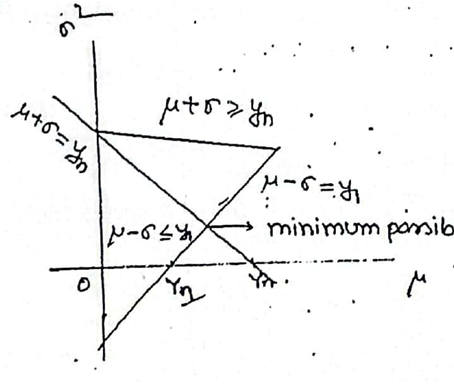
If $y_{(1)} \le y_{(2)} \le \cdots \le y_{(n)}$ are the ordend sample then

$$\mu-\sigma \le y_{(1)} \le y_{(2)} \le \cdots \le y_{(n)} \le \mu+\sigma$$

which is true iff $\mu-\sigma \le y_{(1)}$ and $\mu+\sigma \ge y_{(n)}$.

$$\mu - \sigma = y_1 \quad \text{——} (\mathrm{I})$$
$$\mu + \sigma = y_n \quad \text{——} (\mathrm{II})$$

Now we draw a graph from the above two equation (I) and (II)



The intersection of ① and ⑪ will give the minimum value for σ for which L is maximum.

Therefore the solution of the above two equations will give the MLE of μ and σ.

i.e $\hat{\sigma} = \dfrac{Y_{(n)} - Y_{(1)}}{2}$ and $\hat{\mu} = \dfrac{Y_{(n)} + Y_{(1)}}{2}$.

Problem:

Given that the function

$$f(x|\mu,\sigma) = \frac{1}{2\sqrt{3}\sigma} \; ; \; \mu - \sqrt{3}\sigma \le x \le \mu + \sqrt{3}\sigma$$
$$-\infty < \mu < \infty$$
$$\sigma > 0$$

obtain the MLE of μ and σ.

Solution:

We have

$$f(x|\mu,\sigma) = \frac{1}{2\sqrt{3}\sigma} \; ; \; \mu - \sqrt{3}\sigma \le x \le \mu + \sqrt{3}\sigma$$

The likelihood function for a sample of size n is

$$L(\mu,\sigma) = \prod_{i=1}^{n} f(x_i|\mu,\sigma)$$
$$= \prod_{i=1}^{n} \left(\frac{1}{2\sqrt{3}\sigma}\right)$$
$$= \left(\frac{1}{2\sqrt{3}\sigma}\right)^{n}.$$
$$\therefore L(\mu,\sigma) = \left(\frac{1}{2\sqrt{3}\sigma}\right)^{n}.$$

If $x_{(1)}, x_{(2)}, \dots, x_{(n)}$ are the ordered sample then

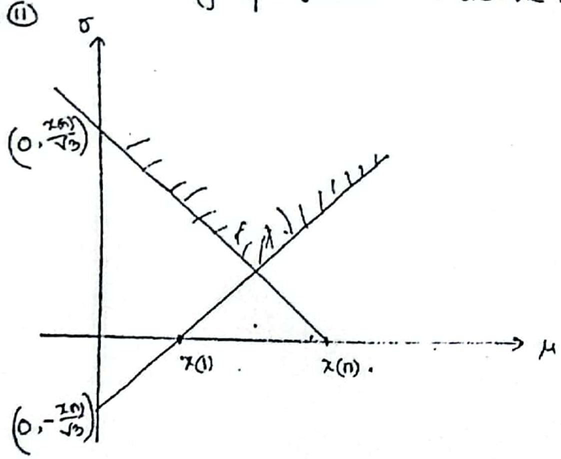
$$\mu - \sqrt{3}\sigma \le x_{(1)} \le x_{(2)} \le \dots \le x_{(n)} \le \mu + \sqrt{3}\sigma$$

$\therefore \mu - \sqrt{3}\sigma \le x_{(1)}$ —— ① and $\mu + \sqrt{3}\sigma \le x_{(n)}$ —— ⑪

$\Rightarrow -\sqrt{3}\sigma \le x_{(1)} - \mu$ $\qquad \Rightarrow \sqrt{3}\sigma \le x_{(n)} - \mu$

$\Rightarrow \sigma \ge \dfrac{\mu - x_{(1)}}{\sqrt{3}}$ $\qquad \Rightarrow \sigma \le \dfrac{x_{(n)} - \mu}{\sqrt{3}}$

Now we draw the graph from the above two equation ① and ⑪



The intersection of $\mu - \sqrt{3}\sigma = x_{(1)}$ and $\mu + \sqrt{3}\sigma = x_{(n)}$ with give the minimum value of σ for which L is minimum.

Therefore, the solution of the above two equations wi give the MLE of μ and σ

i.e $\hat{\mu} = \dfrac{x_{(n)} + x_{(1)}}{2}$ and $\hat{\sigma} = \dfrac{x_{(n)} - x_{(1)}}{2\sqrt{3}}$.