Fourier analysis

Mainly two type signals are present in the world such as analog or continuous signal and digital or discrete signal. Both signals have two forms such as periodic and aperiodic. Fourier analysis is an excellent method to analysis every type of signal either continuous time or discrete time signal.

Fourier analysis for following case such as:

1. Continuous Time Periodic Signal
2. Discrete Time Periodic Signal
3. Continuous Time Aperiodic Signal
4. Discrete Time Aperiodic Signal

1. FOURIER SERIES FOR CONTINUOUS TIME PERIODIC

1.1 Fourier series for Continuous Time Periodic Signal in Trigonometric Form

Fourier given the concept all real periodic signal consists of dc signal which frequency is zero and one to infinity number of sinusoidal signal. Each sinusoidal wave has different amplitude and different frequency. In real world, sinusoidal signal means sine wave signal and cosine wave signal. This statement can be represented by mathematical form which is called Fourier series as given below.

$$f(t) = a_0 + (a_1 cos\omega_0 t + a_2 cos2\omega_0 t + a_3 cos3\omega_0 t + \ldots \ldots \ldots \ldots \ldots \ldots \infty) \quad + (b_1 sin\omega_0 t$$
$$+ b_2 sin2\omega_0 t + b_3 sin3\omega_0 t + \ldots \ldots \ldots \ldots \ldots \ldots \ldots \infty)$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n cosn\omega_0 t + b_n sinn\omega_0 t \ \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1)$$
$$Or \ f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n cosn\theta + b_n sinn\theta \ldots \ldots \ldots \ldots \ldots \ldots \ldots .. (2)$$

Calculation to find out the value of a_0:

We know that from equation –(2),

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n cosn\theta + b_n sinn\theta$$

Or $\int_0^{2\pi} f(\theta)d\theta = \int_0^{2\pi} a_0 d\theta + \sum_{n=1}^{+\infty}(a_n\int_0^{2\pi} cosn\theta\ d\theta + b_n\int_0^{2\pi} sin\ n\theta\ d\theta)$

Or $\int_0^{2\pi} f(\theta)d\theta = \int_0^{2\pi} a_0 d\theta + 0 + 0$

Or $\int_0^{2\pi} f(\theta)d\theta = a_0 2\pi$

$Or,$ $a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta$ -- (3)

Calculation to find out the value of a_n:

We know that from initial (first) equation

$f(\theta) = a_0 + a_1 cos\ \theta + a_2 cos2\theta +$ --- $+a_n cosn\theta$

$+ b_1 sin\ \theta + b_2 sin2\theta +$ --- $+ b_n sinn\theta$

Or $\int_0^{2\pi} f(\theta)\ cos\ n\theta\ d\theta = \int_0^{2\pi} a_0 cos\ n\theta\ d\theta + a_1\int_0^{2\pi} cos\theta\ cos\ n\theta\ d\theta$

$+a_2\int_0^{2\pi} cos2\theta\ cos\ n\theta\ d\theta +$ --- $+$

$a_n\int_0^{2\pi} cosn\theta cosn\theta\ d\theta$

$+ b_1\int_0^{2\pi} cosn\theta\ sin\theta\ d\theta + b_2\int_0^{2\pi} sin2\theta\ cos2\theta\ d\theta +$ ------------------------------------- $+$

$b_n\int_0^{2\pi} sin\ n\theta\ cos\ n\theta\ d\theta$ -- (4)

(We know that from geometrical function in **Appendix –A**, $\int_0^{2\pi}$ cos nθ cos mθ dθ=0 [if

m $\neq$ n] and $\int_0^{2\pi}$ cos nθ cos mθ dθ=π [if m=n] $\int_0^{2\pi}$ sin nθ cos mθ dθ=0 [either m $\neq$ n

nor m=n]) Now apply the above this geometrical function in equation-(4) then we get

Or $\int_0^{2\pi} f(\theta)\ cos\ n\theta\ d\theta = 0 + 0 + 0 +$ -----------------$a_n\int_0^{2\pi}$ cos nθ cos nθ dθ

$+ 0 + 0 + 0 +\text{--}+$

$Or \quad \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta = a_n \int_0^{2\pi} \cos n\theta \cos n\theta \, d\theta$

$Or \quad \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta = a_n \pi$

$a_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta \text{--}(5)$

Calculation to find out the value of bn:

We know that from initial (first) equation

$f(\theta) = a_0 + a_1 \cos\theta + a_2 \cos2\theta + \text{----} \text{----------------------------------}+a_n\cos n\theta$

$\qquad + b_1 \sin\theta + b_2 \sin2\theta + \text{-------------------------------------}+b_n\sin n\theta$

Now we are multiply both side of above equation by $\sin n\theta$ and apply integration on both side. So we get following equation

$Or \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta = \int_0^{2\pi} a_0 \sin n\theta \, d\theta + a_1 \int_0^{2\pi} \cos\theta \sin n\theta \, d\theta$

$\qquad\qquad +a_2 \int_0^{2\pi} \cos2\theta \sin n\theta \, d\theta + \text{------------} + a_n \int_0^{2\pi} \cos n\theta \sin n\theta \, d\theta$

$\qquad\qquad + b_1 \int_0^{2\pi} \cos\theta \sin n\theta \, d\theta + b_2 \int_0^{2\pi} \sin2\theta \sin n\theta \, d\theta + \text{-----------}+$

$b_n \int_0^{2\pi} \sin n\theta \sin n\theta \, d\theta + \text{--} (6)$

(We know that from geometrical function $\int_0^{2\pi} \sin n\theta \cos m\theta \, d\theta = 0$ [if $m \neq$ n] and $\int_0^{2\pi} \sin n\theta \sin m\theta \, d\theta = \pi$ [if m=n] $\int_0^{2\pi} \sin n\theta \cos m\theta \, d\theta = 0$ [either $m \neq n$ nor m=n]). Now apply the above this geometrical function in equation-(6) then we get

$Or \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta = 0+0+0+0+0+0+\text{-----------------------------}+ b_n \int_0^{2\pi} \sin^2 n\theta d\theta$

$Or \quad \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta = b_n \pi$

$b_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta \text{--}(7)$

Summary 1.1(A): Fourier series for Continuous Time Periodic Signal

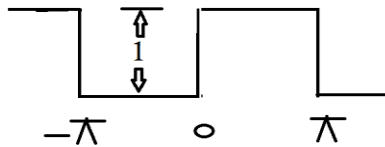$$f(\theta) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta \, d\theta$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta \, d\theta$$

Example-01: Find the mathematical equation for given signal using Fourier series as trigonometric form.

$$f(x) = \begin{cases} 0 & -\pi \le x \le 0 \\ 1 & 0 \le x \le \pi \end{cases}$$

Solution:

We know that,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{2\pi}[\int_{-\pi}^{0} 0 \, dx + \int_0^{\pi} 1 \, dx$$

$$= \frac{1}{2}$$

$$Again \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx$$

[sin nπ=0 for n=1,2.....] [cos nπ =-1 for n=1,3,5…….. and =1 for n=2,4,….]

$$= \frac{1}{\pi}[\int_{-\pi}^{0} 0 \cos nx \, dx + \int_0^{\pi} 1 \cos nx \, dx]$$

$$= 0$$

$$\text{Aging } b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} 0 \sin nx \, dx + \int_{0}^{\pi} 1 \sin nx \, dx \right.$$

$$= \frac{1}{\pi} \left[\frac{-conx}{n} \right]_{0}^{\pi}$$

$$\frac{1}{n\pi} [1 \text{-} \cos n\pi]$$

$$= \begin{cases} 0 & \text{for even } n \\ \dfrac{2}{n\pi} & \text{for odd } n \end{cases}$$

Again we know that

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Putting the values of a_0, a_n and b_n in above equation and we get below expression

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} 0 \cos nx + \sum_{n=1(odd)}^{\infty} \frac{2}{n\pi} \sin nx = \frac{1}{2} + 0 + \sum_{n=1(odd)}^{\infty} \frac{2}{n\pi} \sin nx$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \left(\frac{sinx}{1} + \frac{sin3x}{3} + \frac{sin5x}{5} + \frac{sin7x}{7} + \text{---------------------------------------} + \infty \right)$$

The above this equation represent mathematical equation for example 01

1.2 Conversion of Trigonometric Form Fourier series in to Complex Form Fourier series for Continuous Time Periodic Signal

The article (1.1) represents trigonometrically form Fourier series but many books are explained the Fourier series in complex form because it is easily represent and analysis the different type of signals. It is remember that the complex form not complicated form but also simple form compare to others form such as trigonometric form, polar form etc. So we are convert trigonometrically form Fourier series in to a complex form Fourier series. However the trigonometrically form Fourier series and complex form Fourier series is same case but only representation form is different. Fourier series for periodic function as given by

$$f(\theta)=a_0+\sum_{n=1}^{\infty}(a_n \cos\theta + b_n \sin\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta \text{ ---------(8)}$$

$$\text{Or} \quad f(\theta)=a_0+\sum_{n=1}^{\infty} a_n(e^{jn\theta}+e^{-jn\theta})/2 - j\sum_{n=1}^{\infty} b_n(e^{jn\theta}-e^{-jn\theta})/2 \text{ ------------------------- (9)}$$

[By using Appendix -(A)]

$$\text{Or} \quad f(\theta)=a_0+\sum_{n=1}^{\infty} e^{jn\theta}(a_n-jb_n)/2 + \sum_{n=1}^{\infty} e^{-jn\theta}(a_n+jb_n)/2 \text{ ------------------------------- (10)}$$

$$\text{Or} \quad f(\theta)= I_0 + I_1 + I_2 \text{ --- (11)}$$

Where $I_0 = a_0$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta \quad \textbf{[Using Summery 1.1(A), page-5]}$$

$$= \lim_{n\to 0}\frac{1}{2\pi}\{\int_0^{2\pi} f(\theta)\cos n\theta d\theta - j\int_0^{2\pi} f(\theta)\sin n\theta d\theta \}$$

$$= \frac{1}{2}\lim_{n\to 0}\frac{1}{\pi}\{\int_0^{2\pi} f(\theta)\cos n\theta d\theta - j\int_0^{2\pi} f(\theta)\sin n\theta d\theta \}$$

$$= \frac{1}{2}\sum_{n=0}^{0}\frac{1}{\pi}\{\int_0^{2\pi} f(\theta)\cos n\theta d\theta - j\int_0^{2\pi} f(\theta)\sin n\theta d\theta \}$$

$$= \frac{1}{2}\sum_{n=0}^{0} e^{jn\theta}\frac{1}{\pi}\{\int_0^{2\pi} f(\theta)\cos n\theta d\theta - j\int_0^{2\pi} f(\theta)\sin n\theta d\theta \}$$

$$= \frac{1}{2}\sum_{n=0}^{0} e^{jn\theta} \{a_n - jb_n\} \qquad \textbf{[Using Summery 1.1(A), page-5]}$$

Or $\quad I_0 = \sum_{n=0}^{0} e^{jn\theta} \frac{\{a_n - jb_n\}}{2}$

$\quad I_1 = \sum_{n=1}^{\infty} e^{jn\theta} \frac{\{a_n - jb_n\}}{2}$

$\quad I_2 = \sum_{n=1}^{\infty} e^{-jn\theta} \frac{\{a_n + jb_n\}}{2}$

$$= \sum_{n=1}^{\infty} e^{-jn\theta} \frac{1}{2} \{\frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta + j\frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta d\theta \}$$

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[Using Summery 1.1(A), page-5]

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$$= \sum_{n=-1}^{-\infty} e^{jn\theta} \frac{1}{2} \{\frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(-n\theta)d\theta + j\frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(-n\theta)d\theta \}$$

$$= \sum_{n=-1}^{-\infty} e^{jn\theta} \frac{1}{2} \{\frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)d\theta + j\frac{1}{\pi}\int_0^{2\pi} -f(\theta)\sin(n\theta)d\theta \}$$

$$= \sum_{n=-1}^{-\infty} e^{jn\theta} \frac{1}{2} \{\frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)d\theta - j\frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)d\theta \}$$

$$= \sum_{n=-1}^{-\infty} e^{jn\theta} \frac{1}{2} \{a_n - jb_n\} \quad \textbf{[Using Summery 1.1(A), page-5]}$$

$I_2 = \sum_{n=-1}^{-\infty} e^{jn\theta} \frac{\{a_n - jb_n\}}{2}$

Now, the value of I_0, I_1 and I_2 put in equation (11) and we get following expression

Or $\quad f(\theta) = \sum_{n=0}^{0} e^{jn\theta}(a_n-jb_n)/2 + \sum_{n=1}^{\infty} e^{jn\theta}(a_n-b_n)/2 + \sum_{n=-1}^{-\infty} e^{jn\theta}(a_n-jb_n)/2$

Or $\quad f(\theta) = \sum_{n=-\infty}^{-\infty} e^{jn\theta}(a_n-jb_n)/2$

Or $\quad f(\theta) = \sum_{n=-\infty}^{\infty} C_n e^{jn\theta}$ -- (12)

Or $\quad f(t) = \sum_{-\infty}^{\infty} C_n e^{jn\omega_0 t}$

Or $\quad f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn2\pi F_0 t} \qquad [\omega_0 = 2\pi F_0]$ -- (13)

Or $f(t) = \sum_{k=-\infty}^{\infty} C_k\, e^{jk2\pi F0t}$ [just n replace by k]------------------------------------- (14)

Where $C_n = (a_n - jb_n)/2$

$C_n = 1/2\{1/\pi \int_0^{2\pi} f(\theta)\cos n\theta\, d\theta - j1/\pi \int_0^{2\pi} f(\theta)\sin n\theta\, d\theta\}$

$C_n = 1/2\pi \int_0^{2\pi} f(\theta)\, e^{-jn\theta}\, d\theta$ --- (15)

$C_n = 1/T \int_0^T f(t)\, e^{-jn2\pi F0t}\, dt$ [$w_0 = 2\pi F_0$ and $F_0 = 1/T$] ---------------------------- (16)

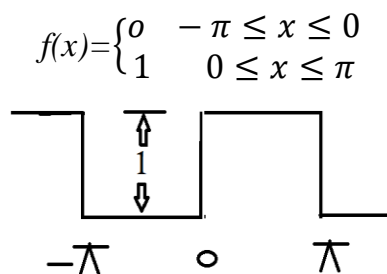$C_k = 1/T \int_0^T f(t)\, e^{-jk2\pi F0t}\, dt$ [just n replace by k] ------------------------------- (17)

Summary 1.1(B): Fourier series for Continuous Time Periodic Signal

$$f(\theta) = \sum_{n=-\infty}^{\infty} C_n\, e^{jn\theta} \quad and \quad C_n = 1/2\pi \int_0^{2\pi} f(\theta)\, e^{-jn\theta}\, d\theta$$

$$Or\ f(t) = \sum_{n=-\infty}^{\infty} C_n\, e^{jn2\pi F0t} \quad and \quad C_n = 1/T\int_0^T f(t)\, e^{-jn2\pi F0t}\, dt$$

$$Or\, f(t) = \sum_{k=-\infty}^{\infty} C_k\, e^{jk2\pi F0t} \quad and \quad C_k = 1/T\int_0^T f(t)\, e^{-jk2\pi F0t}\, dt$$

Example-02: Find the mathematical equation for given signal using Fourier series as complex form.

$$f(x) = \begin{cases} o & -\pi \le x \le 0 \\ 1 & 0 \le x \le \pi \end{cases}$$

Solution:

We know that complex form Fourier series for continuous time periodic signal is given by following expression

$$f(x) = \sum_{n=-\infty}^{n=\infty} C_n e^{inx} \quad [x=\theta] \quad \textbf{[See summary 1.1(B), page-9]}$$

Where $C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$

$\Rightarrow C_n = \frac{1}{2\pi} [\int_{-\pi}^{0} f(x) e^{-inx}\, dx + \int_{0}^{\pi} f(x) e^{-inx}\, dx]$

$\Rightarrow C_n = \frac{1}{2\pi} [\int_{-\pi}^{0} 0\, e^{-inx}\, dx + \int_{0}^{\pi} 1\, e^{-inx}\, dx \quad (*)$

$$C_n = -\frac{1}{2in\pi}(e^{in\pi} - 1)$$

If n is odd then $C_n = \frac{1}{in\pi}$

If n is even then $C_n = 0 \quad [n \neq 0]$

(*) If n=0 then $C_n = \frac{1}{2\pi} \int_{0}^{\pi} 1\, dx = \frac{1}{2}$

Putting the value of C_n in equation (12) and we get

$$\Rightarrow f(x) = \frac{1}{2} + \frac{1}{i\pi}(\frac{e^{ix}}{1} + \frac{e^{i3x}}{3} + \frac{e^{i5x}}{5} + \ldots\ldots\ldots)$$

$$+ \frac{1}{i\pi}(\frac{e^{-ix}}{-1} + \frac{e^{-i3x}}{-3} + \frac{e^{-i5x}}{-5} + \ldots..)$$

$$\Rightarrow f(x) = \frac{1}{2} + \frac{2}{\pi}[\frac{1}{1}(\frac{e^{ix}-e^{-ix}}{2i}) + \frac{1}{3}(\frac{e^{3ix}-e^{-3ix}}{2i}) +$$

$$\frac{1}{5}(\frac{e^{5ix}-e^{-5ix}}{2i}) + \ldots\ldots\ldots\ldots\ldots]$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi}(\frac{sinx}{1} + \frac{sin3x}{3} + \frac{sin5x}{5} + \frac{sin7x}{7} + \text{---------------------------------})$$

The above this equation represent mathematical equation for example-02

NOTE: Example-01and Example-02 are same problem. Example-01 solved by using of trigonometrical form Fourier series and Example-02 solved by using of complex form Fourier series but we get same result. So it is clear that the trigonometrically form Fourier series and complex form Fourier series is same case but only representation form is different.