Measuring Asymptotic Convergence: A Unified Framework from Isotropic Infinity to Anisotropic Ends

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Abstract

We develop a unified approach to defining a point at infinity for an arbitrary space and formalizing convergence to this point. Central to our work is a method to quantify and classify the rates at which functions approach their limits at infinity. Our framework applies to various settings (metric spaces, topological spaces, directed sets, measure spaces) by introducing an exhaustion of the space via an associated exhaustion function h. Using h, we adjoin an ideal point ω_A to the space A and define convergence $a \to \omega_A$ in a manner intrinsic to A. To measure convergence rates, we introduce a family of parameterized norms, denoted $||f||_{\infty,h,p}$ which provides a refined classification of asymptotic behavior (e.g., distinguishing rates of order $O(h^{-p})$). **Furthermore, the framework is extended to handle anisotropic spaces with multiple distinct ends by introducing a 'multi-exhaustion' formalism, allowing for a precise, directional analysis of convergence rates towards each asymptotic channel.** This approach allows for a distinction between the global convergence captured by the norm and the purely asymptotic behavior at infinity, which can be analyzed via the limit superior of the convergence ratio. We further investigate the theoretical limits of this measure by establishing sufficient conditions (such as monotonicity) under which a finite norm guarantees convergence a non-trivial converse. The framework is shown to recover classical results, such as the Alexandroff one-point compactification and standard definitions of limits, while also providing a richer quantitative structure. Examples in each context are provided to illustrate the concepts.

Contents

Symbol	Meaning
A	Hausdorff, locally compact, σ-compact space
$h : A \to [0, \infty)$	continuous proper exhaustion (Lemma 2.1)
$\varphi \in \Phi_{\mathrm{adm}}$	admissible comparison function (Def. 4.2)
$\|f\|_{\infty,h,p;L}$	global fixed-L norm with polynomial weight $(1+h)^p$ (Def. 4.47)
$\|[f]\|^{\sharp}_{\infty,h,p}$	sharp quotient norm modulo constants (Def. 4.48)
$\|f\|_{\infty,h;\varphi;L}$, $\|[f]\|^{\sharp}_{\infty,h;\varphi}$	φ-weighted global/quotient norms (Sec. 4.4)
$C^{(h)}_{\varphi}(f; L)$, $C^{(h)}_{\varphi}([f])$	φ-weighted asymptotic constants (Sec. 4.4)
U_i, h_i, φ_i	end neighborhoods, per-end exhaustions and weights (Sec. 8)
$\|[f]\|^{\sharp}_{\infty;\mathbf{h},\boldsymbol{\varphi}}$, $C^{(\mathbf{h})}_{\boldsymbol{\varphi}}([f])$	anisotropic sharp norm and asymptotic constant (Sec. 8)

Table 1: Main symbols used throughout the paper.

[scale=0.9,>=Stealth] [-Latex[length=3mm]] (-5,0)–(5,0) node[below]x; (-4.5,0) node[above]$-\infty$; (4.5,0) node[above]$+\infty$; [thick,domain=-4:4,smooth] plot(,0.6*sin(0.8*r)); at (-3,1) U_-; at (3,1) U_+; [dashed] (-1.5,-0.9) rectangle (1.5,0.9) node[pos=0.5,below=1.2cm] K_0;

Figure 1: Two ends on a cylinder $\mathbb{R} \times M$: neighborhoods $U_\pm$ and a compact core K_0.

1 Introduction and Motivation

In many areas of mathematics, it is useful to talk about *convergence to infinity.* For example, in real analysis one studies limits of the form $\lim_{x\to\infty} f(x)$, in topology one often constructs a one-point compactification by adding an *ideal point at infinity* to a non-compact space, and in measure theory improper integrals are defined via a limit as the integration bound goes to infinity. Yet, this convergence is often treated as a binary concept—either a function converges or it does not. This work argues that the *manner* of convergence contains rich, quantifiable information that is often overlooked. In each of these settings, there is an implicit notion of what it means for the underlying variable or "point" to approach infinity. However, the formal treatment of "approaching infinity" can vary significantly with context:

- In metric spaces (like $\mathbb{R}^n$ with the usual distance), we say $x_n \to \infty$ if the distance $\|x_n\| \to \infty$. *But at what rate? A sequence like* (n) *and one like* (e^n) *are treated identically, despite their vastly different behaviors.*
- In general topological spaces, one-point (Alexandroff) compactification introduces an extra point ω_A and declares that a net x_α converges to ω_A if eventually x_α leaves every compact subset of the space. *This provides a qualitative notion of convergence, but no quantitative measure of its speed.*
- In order theory, a directed set can have an "infinite" element formally adjoined to capture the idea of eventual growth beyond all bounds.
- In measure theory, an improper integral $\int_a^\infty f(x)\,dx$ is defined by a limit $\lim_{R\to\infty} \int_a^R f(x)\,dx$, essentially considering the domain interval growing without bound. *This determines if the integral converges, but not how rapidly the integral's tail vanishes.*

tikz arrows.meta

[scale=1.05,>=Stealth] [-Latex[length=3mm]] (-5,0)–(5,0) node[below]x; (0,0) circle (0.08); [below] at (0,-0.15) 0; (-4.5,0) node[above]$-\infty$; (4.5,0) node[above]$+\infty$; (-2.0,0.5) node U_{0-}; (2.0,0.5) node U_{0+}; (-3.5,-0.6) node $U_{-\infty}$; (3.5,-0.6) node $U_{+\infty}$;

Figure 2: Four ends on $\mathbb{R} \setminus \{0\}$: near $0^{\pm}$ and at $\pm\infty$.

Despite differing formalisms, these ideas share a common theme: we consider some notion of *"distance to infinity"* and require this to grow beyond all finite limits. The goal of this work is to provide a unified framework to:

1. Adjoin a distinguished point ω_A to an arbitrary set A that is equipped with minimal structure (metric, topology, directed set, or measure structure).

2. Define what it means for elements $a \in A$ to *converge to* ω_A (denoted $a \to \omega_A$) in a manner consistent with the usual meanings in each context.

3. Introduce a function $h : A \to [0, \infty)$ that serves as a continuous *height function* measuring how "far" a point $a \in A$ is from the "finite part" of A, such that $h(a) \to +\infty$ if and only if $a \to \omega_A$.

4. For functions $f : A \to \mathbb{R}$ that have a limit L as $a \to \omega_A$, develop a suite of tools to **quantify and classify their speed of convergence**. Specifically, we aim to:

   (4a) Introduce a **baseline weighted norm**, $\|f\|_{\infty,h}$, to establish a global measure of the convergence rate.

   (4b) Generalize this to a **parameterized family of norms**, $\|f\|_{\infty,h,p}$, to create a fine-grained classification of convergence rates (e.g., distinguishing $O(h^{-p})$ behaviors).

   (4c) Analyze the **purely asymptotic behavior** of convergence by examining the limit superior of the convergence ratio, distinguishing it from the global properties captured by the norm.

   (4d) Investigate the **theoretical power of these measures** by identifying sufficient conditions on f (e.g., monotonicity) that guarantee convergence if its norm is finite, thus establishing a partial converse.

The power of this unified framework lies not only in its ability to describe these disparate concepts within a single theoretical language, but also to **enrich them with a new layer of quantitative detail**. For instance, we aim to describe:

- The usual ϵ–N definition of $\lim_{x\to\infty} f(x) = L$ in analysis, while also providing tools to **classify its rate of convergence**.

- The topological definition of convergence to the point at infinity in an Alexandroff compactification, while equipping this topological space with a new **metric-like structure via** h that allows for quantitative analysis of convergence.It is this model of a single, unified point at infinity that our present framework seeks to formalize and enrich quantitatively.

- Convergence of nets in a general space to a point at infinity (using directed sets that index the "tails" of the space beyond various boundaries).

- The convergence of improper integrals over expanding domains, and to **quantify the speed at which the integral's tail vanishes**.
- The concept of Big-O and little-o notation, and to **place these notations within a more general, structured family of convergence classes** derived from our framework.

It is crucial to precisely situate our contribution within the rich and well-established theory of weighted function spaces. To be sure, the use of weighted norms to quantify asymptotic behavior is a classical tool in functional analysis. In common practice, however, the weight function is often chosen on an **ad hoc** basis, selected for its analytical convenience or to fit a specific class of problems.

The principal contribution of our approach, therefore, lies not in the use of a weight itself—a classical tool—but in the establishment of a **unifying axiomatic framework for its construction**. While disciplines such as geometric analysis routinely employ weight functions derived from the space's structure (like the distance function on a Riemannian manifold), our formalism abstracts this core idea to extend it to more general settings (topological, ordered, etc.).

Our framework thus acts as a conceptual 'Rosetta Stone': we begin with a general qualitative notion—the exhaustion of a space—to arrive at a quantitative, non-arbitrary, and finely-grained analytical structure. This provides a rigorous bridge that unifies practices that were, until now, specific to their respective domains, offering a common language to the analyst, the topologist, and the geometer.

This document is organized as follows. In Section 2, we introduce the notion of an *exhaustion* of a space A and the associated *exhaustion function* h. Section 3 uses this function to formally adjoin a point ω_A at infinity and define limits in this new context. In Section 4, we develop our quantitative tools. We begin by defining the baseline weighted norm, $\|f\|_{\infty,h}$, and then generalize it to a versatile family of norms, $\|f\|_{\infty,h,p}$, designed to classify different orders of convergence. We then analyze the theoretical properties of these measures, including a detailed investigation of the subtle relationship between a finite norm and the guarantee of convergence, culminating in a theorem that establishes this link under specific additional conditions. Throughout, examples from different contexts are provided to illustrate the definitions. We conclude with a brief discussion and summary in Section 5.

The analytical power of this framework is not merely theoretical. It is demonstrated in a companion paper A Weighted Kolmogorov Metric for Berry-Esseen Bounds under Sub-Cubic Moments, where the weighted metric is applied to the classical Berry-Esseen problem. There, we show that by focusing the measure of accuracy on the center of the distribution, our approach successfully recovers the optimal $n^{-1/2}$ convergence rate under the mild moment condition $\mathbb{E}|X|^{2+\delta} < \infty$, a setting where the standard uniform metric yields a suboptimal rate.

The numerical experiments and data analysis for this application are fully reproducible and available in my public repository. [1]

Contributions and positioning. We propose a unified framework to quantify convergence "towards infinity" in diverse settings (metric, topological, measured, or ordered) via an *exhaustion function* h and the adjunction of an ideal point ω_A. Our main contributions are:

[1] The code and data can be found at: `https://github.com/Armen0807/Mathematical-Research-Notes`

- **Axiomatic construction of the point at infinity and recovery of Alexandroff.** Starting from an exhaustion $\{K_r\}_{r\geq 0}$ we build h and define the convergence $a \to \omega_A$; the induced topology on $A^* = A \cup \{\omega_A\}$ recovers the Alexandroff one-point compactification (see Prop. 3.4).
- **Global vs. asymptotic measurement.** We introduce weighted sup norms $\|f\|_{\infty,h,p}$ and the asymptotic constant $C_\varphi(f)$, separating global uniform control from purely asymptotic rates (Sec. 4).
- **Coarse robustness.** Under *coarse affine equivalence* of exhaustion functions, the associated norms are equivalent; hence the classification $O(h^{-p})$ does not depend on a reasonable choice of exhaustion (Lem. 2.8, Cor. 2.7).
- **Large-scale functoriality.** For a proper map $\phi : (A, h_A) \to (B, h_B)$ with $h_A \geq c\, h_B \circ \phi - C$, composition $f \mapsto f \circ \phi$ defines a bounded operator $C_{h_B,p}(B) \to C_{h_A,p}(A)$ (Prop. 4.7).
- **Anisotropic extension (multi-exhaustions).** We extend h to a family (h_i) indexing disjoint *ends*, with directional norms and a global control by a central maximum plus endwise contributions (Sec. 5).

Relation to prior work. Our approach sits at the interface of:

- *Coarse geometry* and quasi-isometries, which study large-scale metric structure (Roe [1], Gromov [2]);
- *Theory of ends* (Freudenthal [3]; see also Peschke [4]) compactifying by an ideal boundary at infinity;
- *Weighted spaces and elliptic analysis on manifolds with ends* (Melrose's b-calculus [5]; Kondrat'ev [7]; Agmon [6]) using decay weights;
- *Asymptotic scales and regular variation* (Bingham–Goldie–Teugels [8]).

Novelty. Instead of postulating ad hoc weights, we derive a *canonical weight* from an abstract exhaustion and prove *coarse invariance* of the class $O(h^{-p})$. We also (i) separate *global* from *asymptotic* control, (ii) provide explicit *functoriality* under proper maps, and (iii) develop an *anisotropic* multi-exhaustion extension.

2 Exhaustions and the Exhaustion Function h

Standing assumptions for Section 2 and onward

Throughout, A denotes a Hausdorff, locally compact, σ-compact topological space. We fix an increasing exhaustion by compact sets $(K_n)_{n\in\mathbb{N}}$ with

$$K_n \subset \operatorname{int}(K_{n+1}) \quad \text{and} \quad A = \bigcup_{n\geq 0} K_n.$$

(Existence of such an exhaustion follows from local compactness and σ-compactness.) Unless stated otherwise, $p > 0$ is fixed.

Lemma 2.1 (Continuous proper exhaustion function)**.** *There exists a continuous map $h : A \to [0, \infty)$ such that:*

(i) h is proper*: for every $R < \infty$, the sublevel set $h^{-1}([0, R])$ is compact;*

(ii) $h(x) \to \infty$ iff x escapes every compact set of A (i.e. $x \to \omega_A$ in the Alexandroff sense).

Proof. Choose an exhaustion (K_n) with $K_n \subset \operatorname{int}(K_{n+1})$ and $K_{-1} := \varnothing$. By Urysohn's lemma (valid on locally compact Hausdorff spaces for closed sets with disjoint interiors), for each $n \geq 0$ there is a continuous function $\phi_n : A \to [0, 1]$ with

$$\phi_n \equiv 0 \text{ on } K_n, \qquad \phi_n \equiv 1 \text{ on } A \setminus \operatorname{int}(K_{n+1}).$$

Define

$$h(x) \;:=\; \sum_{n=0}^{\infty} n\, \phi_n(x).$$

For each fixed $x \in A$, we have $x \in K_N$ for some N, hence $\phi_n(x) = 0$ for all $n \geq N$, so the sum is finite at x; thus h is well-defined and continuous as a locally finite sum of continuous functions.

If $x \notin K_{N+1}$, then $\phi_n(x) = 1$ for all $0 \leq n \leq N$, hence $h(x) \geq \sum_{n=0}^{N} n = \frac{N(N+1)}{2} \xrightarrow[N\to\infty]{} \infty$. Therefore $h(x) \to \infty$ precisely when x escapes compact sets.

Finally, for any $R < \infty$, choose N with $\frac{N(N+1)}{2} > R$. Then $A \setminus K_{N+1} \subset \{h > R\}$, hence $h^{-1}([0, R]) \subset K_{N+1}$, which is compact. Thus h is proper. $\square$

Remark 2.2 (On the regularity of the exhaustion function). The construction of h in the preceding proof is minimal in nature. It is possible to construct a significantly more regular function whose growth is more controlled by refining the exhaustion. For instance, by subdividing each annulus $K_{n+1} \setminus \operatorname{int}(K_n)$ into a finite sub-sequence of compact sets, one could build a quasi-linear exhaustion function.

However, as Proposition 2.2 establishes, all continuous proper exhaustion functions are coarsely equivalent. Consequently, this additional regularity does not affect the asymptotic classifications of convergence rates (e.g., $O(h^{-p})$) that are the focus of this work. The simpler construction is therefore sufficient for the results that follow.

Proposition 2.3 (Coarse affine equivalence of proper exhaustions)**.** *Let $h, h' : A \to [0, \infty)$ be continuous proper functions. Then there exist constants $a, A > 0$ and $b, B \in \mathbb{R}$ such that*

$$a\, h(x) - b \;\leq\; h'(x) \;\leq\; A\, h(x) + B \qquad \text{for all } x \in A.$$

In particular, the weighted classes $O(h^{-p})$ and $O(h'^{-p})$ coincide, and the associated weighted sup norms are equivalent.

Proof sketch. Properness implies that sublevel sets $\{h \leq R\}$ and $\{h' \leq R\}$ form compact exhaustions. By σ-compactness, for each n one can find $m(n)$ with $\{h \leq n\} \subset \{h' \leq m(n)\}$ and conversely $n'(m)$ with $\{h' \leq m\} \subset \{h \leq n'(m)\}$. Monotonicity of $m(\cdot)$ and $n'(\cdot)$ then yields linear bounds $m(n) \leq An + B$ and $n'(m) \leq a^{-1}m + b$ for suitable constants, which translate into the claimed inequalities for h and h' after reparametrization of sublevel indices. Norm equivalence for weights $(1 + h)^p$ and $(1 + h')^p$ follows immediately. $\square$

Remark 2.4 (Metric case). If A is a proper metric space (A, d) (closed balls are compact), then $h(x) := d(x_0, x)$ is a continuous proper exhaustion for any fixed basepoint $x_0 \in A$. All such choices are coarsely equivalent in the sense of Prop. 2.3.

2.1 Exhaustion by "small" sets

We begin by formalizing the idea of an *exhaustion*, which is a way to structure a potentially large or non-compact space A as an infinite union of nested, manageable parts. Intuitively, an exhaustion is a family of subsets that grow to cover all of A, where each subset is "small" in a sense appropriate to the context (e.g., compact in a topological space, or bounded in a metric space). We capture the structural properties of this idea in the following definition.

Assumption 2.5 (Standing setting). *A is a non-compact, locally compact Hausdorff (LCH), σ-compact space. There exists an exhaustion by compact sets $\{K_r\}_{r\geq 0}$ with $K_r \subset \mathrm{Int}(K_s)$ for $r < s$.*

Definition 2.6 (Exhaustion). Let A be a set. An **exhaustion** of A is a family of subsets $\{K_r\}_{r\geq 0}$ indexed by non-negative real numbers such that:

1. **Nesting:** $K_r \subseteq K_s$ for all $0 \leq r < s$.
2. **Covering:** $\bigcup_{r\geq 0} K_r = A$.

The exhaustion is said to be **proper** if $K_R \neq A$ for all $R \in [0, \infty)$.

In practice, for an exhaustion to be useful, the sets K_r must possess a property of "smallness" or "finiteness" that is relevant to the structure of A. For instance:

- If A is a topological space, the K_r are typically required to be **compact**.
- If A is a metric space, the K_r are required to be **closed and bounded**.
- If (A, μ) is a measure space, the K_r are required to have **finite measure**.

To illustrate how Definition 2.6 is instantiated across different mathematical fields, we present several canonical examples. These demonstrate how a context-specific notion of "smallness" gives rise to a useful exhaustion.

Topological Example: Exhaustion by Compact Sets. If X is a topological space, the standard choice for "small" sets is **compactness**. A common construction is a sequence of compact subsets $K_n \subset X$ for $n \in \mathbb{N}$ such that $K_n \subseteq K_{n+1}$ and $\bigcup_{n=1}^{\infty} K_n = X$. If each K_n is contained in the interior of K_{n+1}, the space X is said to be *σ-compact and locally compact.* A prime example is the space $\mathbb{R}^n$, which is exhausted by the sequence of closed balls $K_n = \{x \in \mathbb{R}^n \mid \|x\| \leq n\}$, each of which is compact.

Metric Example: Exhaustion by Closed Balls. If (A, d) is a metric space, we can select a basepoint $a_0 \in A$ and define an exhaustion via concentric closed balls:

$$K_r = \{a \in A \mid d(a, a_0) \leq r\} \quad \text{for each } r \geq 0.$$

If A is unbounded, this family $\{K_r\}_{r\geq 0}$ forms a proper exhaustion where "small" means **closed and bounded**. If the metric space is also *proper* (i.e., every closed and bounded set is compact), then this construction coincides with the topological example.

Directed Set Example: Exhaustion by Initial Segments. The framework is not limited to topological structures. If A is a directed poset (a partially ordered set where for any two elements, there is an element greater than both), an exhaustion can be given by its **initial segments**. For the natural numbers $(\mathbb{N}, \leq)$, a simple exhaustion is the sequence of finite sets $K_n = \{1, 2, \ldots, n\}$ for $n \in \mathbb{N}$. For a general directed poset $(A, \preceq)$, one might use segments of the form $K_a = \{x \in A \mid x \preceq a\}$ if an appropriate cofinal sequence of elements a can be chosen.

Measure Space Example: Exhaustion by Sets of Finite Measure. If $(A, \mathcal{M}, \mu)$ is a measure space, a natural notion of "smallness" is having **finite measure**. For instance, the space $[0, \infty)$ with the Lebesgue measure is exhausted by the family of intervals $K_R = [0, R]$ for $R > 0$, since $\mu(K_R) = R < \infty$. This formalizes the very process used to define improper integrals: the expression

$$\int_0^\infty f(x)\, dx := \lim_{R\to\infty} \int_0^R f(x)\, dx$$

can be interpreted as taking the limit of integrals over the sets K_R of an exhaustion.

Note on Indexing. While our formal definition uses a continuous index $r \in [0, \infty)$, several examples naturally produce a discrete sequence $\{K_n\}_{n\in\mathbb{N}}$. A discrete exhaustion can always be extended to a continuous one (e.g., by setting $K_r = K_{\lceil r\rceil}$), and conversely, a continuous family can be sampled at integer values. The essential property is the existence of an ordered, covering family of sets; the nature of the index set is a matter of technical convenience.

2.2 The Exhaustion Function $h : A \to [0, \infty)$

The exhaustion family $\{K_r\}_{r\geq 0}$ provides a way to structure the space A. We now use this structure to define a function, $h : A \to [0, \infty)$, that quantitatively measures how "far out" any given point $a \in A$ is. Intuitively, $h(a)$ will be the value of the smallest index r such that a is contained in the set K_r. This function is the cornerstone of our entire framework.

Definition 2.7 (Exhaustion Function)**.** Let $\{K_r\}_{r\geq 0}$ be an exhaustion of a set A. The associated **exhaustion function** $h : A \to [0, \infty)$ is defined by:

$$h(a) := \inf\{r \geq 0 \mid a \in K_r\}.$$

Lemma 2.8 (Lower semicontinuity and properness)**.** *If each K_r is closed and $K_r \subset \mathrm{Int}(K_s)$ for $r < s$, the h of Theorem 2.7 is lower semicontinuous and unbounded. If moreover A is LCH and the K_r are compact, then h can be chosen continuous and proper (preimages of $[0, M]$ are compact).*

Proof. Lower semicontinuity follows because $\{h \leq R\} = K_R$ is closed by assumption. Unboundedness comes from $\bigcup_{r\geq 0} K_r = A$ and strict inclusion $K_r \subset \mathrm{Int}(K_s)$. If A is LCH and K_r compact, standard results guarantee the existence of a continuous proper exhaustion function with the same level sets up to smoothing. □

Remark on the Infimum. A natural question is whether $a \in K_{h(a)}$ for any given $a \in A$. If the sets K_r are closed for the relevant topology and the family is suitably continuous in r, the nested property ensures the infimum is attained. For simplicity, we will assume this holds, as one can always work with an equivalent exhaustion (e.g., by slightly enlarging each K_r) for which it does. The crucial properties of h that follow do not depend heavily on this point.

The exhaustion function h allows us to describe portions of the space via level sets. The following properties are immediate consequences of Definition 2.7.

Proposition 2.9 (Level Sets of the Exhaustion Function)**.** *Let h be the exhaustion function associated with an exhaustion $\{K_r\}_{r\geq 0}$ of A. Then for any $R \geq 0$:*

1. *The sublevel set* $\{a \in A \mid h(a) \leq R\}$ *is precisely the set* K_R.

2. *The strict sublevel set* $\{a \in A \mid h(a) < R\}$ *is the union* $\bigcup_{r<R} K_r$.

3. *The superlevel set, which we denote* B_R*, is the complement of* K_R*:*

$$B_R := \{a \in A \mid h(a) > R\} = A \setminus K_R.$$

The sets B_R can be thought of as "tails" of the space beyond the finite part K_R. As R increases, these tails shrink, forming a nested family of neighborhoods for a point at infinity. This intuition is captured by the following key equivalence, which connects the behavior of the function h to the structure of the exhaustion and provides the foundation for defining convergence in the next section.

Proposition 2.10 (Characterization of Escape to Infinity)**.** *A sequence of points* $(a_n)_{n\in\mathbb{N}}$ *in* A *eventually leaves every set* K_R *(i.e., for any* $R \geq 0$*, there exists an* N *such that* $a_n \notin K_R$ *for all* $n \geq N$*) if and only if* $\lim_{n\to\infty} h(a_n) = \infty$.

Example (Metric Space). In the metric space example where $K_r = \{a \in A \mid d(a, a_0) \leq r\}$, the exhaustion function is precisely the distance from the base point a_0:

$$h(a) = \inf\{r \geq 0 \mid d(a, a_0) \leq r\} = d(a, a_0).$$

Thus, h recovers the most natural notion of "distance to the origin".

Example (Topological Space). In a topological space with a discrete exhaustion by compact sets $\{K_n\}_{n\in\mathbb{N}}$, one can define a preliminary integer-valued function $h_0(x) := \inf\{n \in \mathbb{N} \mid x \in K_n\}$. Under mild conditions (e.g., for σ-compact and locally compact Hausdorff spaces), it is a standard result that a *continuous* exhaustion function $h : A \to [0, \infty)$ can be constructed, often by "smoothing" h_0. For example, if $K_n \subset \mathrm{Int}(K_{n+1})$, the existence of such a continuous h is guaranteed (see, e.g., [**?**]). For the purposes of this paper, the continuity of h is a desirable but not essential property; its level-set behavior described in Proposition 2.9 is what is fundamental.

Example (Directed Set). In the directed set example $A = \mathbb{N}$ with the exhaustion $K_n = \{1, 2, \ldots, n\}$, the exhaustion function is simply the identity:

$$h(n) = \inf\{r \geq 0 \mid n \leq r\} = n.$$

As expected, $h(n) \to \infty$ as $n \to \infty$.

Example (Measure Space). In the measure space example $A = [0, \infty)$ with the exhaustion $K_R = [0, R]$, the exhaustion function is again the identity:

$$h(x) = \inf\{R \geq 0 \mid x \leq R\} = x.$$

The condition $h(x) \to \infty$ corresponds directly to the standard limit $x \to \infty$.

2.3 Properties of the Exhaustion Function

The exhaustion function h serves as our generalized notion of "distance to infinity." Even in non-metric contexts, a larger value of $h(a)$ signifies that the point a is "further out" in the space. From Proposition 2.9, we know that for any finite M, the set $\{a \in A \mid h(a) \leq M\} = K_M$ is a proper subset of A (assuming a proper exhaustion). This implies that h must be an *unbounded* function.

In many topological applications, this construction yields a **proper map**, which is a continuous function h where the preimage of any compact set is compact. In our framework, the preimage of the compact set $[0, M]$ is $h^{-1}([0, M]) = K_M$. Thus, if the sets K_M of the exhaustion are compact and h is continuous, h is by definition a proper map. This property is a cornerstone of advanced geometry and topology.

2.4 Robustness of the Framework: Regularity and Equivalence of Exhaustions

For the exhaustion function h to be a reliable "ruler" for infinity, we must ensure that our framework does not depend critically on arbitrary choices. This section addresses two key points: the minimal required properties of h, and the effect of choosing a different, but comparable, exhaustion.

Regularity of the Exhaustion Function. Our definition $h(a) := \inf\{r \geq 0 | a \in K_r\}$ naturally endows h with a useful topological property. If we make the standard and mild assumption that the sets K_r are closed in the underlying topology of A, then the sublevel sets of h, $\{a \in A | h(a) \leq R\} = K_R$, are closed. This property defines h as a *lower semi-continuous* function. Full continuity is a desirable feature for certain applications (as discussed in Section 2.3), but lower semi-continuity is sufficient to guarantee the core mechanism of our framework: the condition $h(a_n) \to \infty$ remains an unambiguous statement of "escape to infinity," as it ensures that any sequence (a_n) must eventually leave every closed set K_R.

Equivalence of Exhaustions. A more fundamental question is whether the quantitative results of our framework depend on the specific choice of exhaustion. For a space like $\mathbb{R}^n$, one could use Euclidean balls, cubes, or another family of compact sets. For our classification of convergence rates to be meaningful, it must be stable under "reasonable" changes to the exhaustion. We formalize this with the following definition.

Definition 2.5 (Equivalent Exhaustions). Let $\{K_r\}_{r \geq 0}$ and $\{K'_r\}_{r \geq 0}$ be two exhaustions of a space A. We say they are *equivalent* if there exist constants $c_1, c_2 > 0$ such that for all $r \geq 0$:

$$K_r \subseteq K'_{c_2 r} \quad \text{and} \quad K'_r \subseteq K_{c_1 r}.$$

This condition states that each family of sets can be nested within a scaled version of the other.

This geometric equivalence of the exhausting sets translates into a strong analytical equivalence of their associated exhaustion functions.

Proposition 2.6 (Equivalence of Exhaustion Functions). Let h and h' be the exhaustion functions associated with two equivalent exhaustions $\{K_r\}$ and $\{K'_r\}$, respectively. Then there exist constants $C_1, C_2 > 0$ such that for all $a \in A$:

$$C_1 h(a) \leq h'(a) \leq C_2 h(a).$$

Proof. Let $a \in A$. By definition, $a \in K_{h(a)}$. From the equivalence condition, $K_{h(a)} \subseteq K'_{c_2 h(a)}$. Thus, $a \in K'_{c_2 h(a)}$. Since $h'(a)$ is the infimum of all s such that $a \in K'_s$, we must have $h'(a) \leq c_2 h(a)$. This gives the second inequality with $C_2 = c_2$.

For the first inequality, we have $a \in K'_{h'(a)}$. The equivalence implies $K'_{h'(a)} \subseteq K_{c_1 h'(a)}$. Therefore, $a \in K_{c_1 h'(a)}$, which means $h(a) \leq c_1 h'(a)$. This yields $h'(a) \geq (1/c_1) h(a)$, proving the first inequality with $C_1 = 1/c_1$. □

The direct consequence of this proposition is the main result of this section: the classification of convergence rates is independent of the choice of equivalent exhaustion.

Definition 2.11 (Coarse affine equivalence)**.** We say h, h' are coarsely equivalent if there exist $a_1, a_2 > 0$ and $b_1, b_2 \geq 0$ such that

$$a_1 h - b_1 \ \leq \ h' \ \leq \ a_2 h + b_2 \quad \text{on } A.$$

Lemma 2.12 (Equivalence of weighted norms under coarse equivalence)**.** *If h, h' are coarsely equivalent, then for each $p > 0$ there exist $M_1, M_2 > 0$ such that*

$$M_1 \, \|f\|_{\infty,h,p} \ \leq \ \|f\|_{\infty,h',p} \ \leq \ M_2 \, \|f\|_{\infty,h,p}.$$

The same holds for the sharp norm $\|\cdot\|^\sharp_{\infty,h,p}$.

Proof. Since $(1 + a_1 h - b_1)^p \asymp (1 + h)^p \asymp (1 + a_2 h + b_2)^p$ uniformly on A, multiplying by $|f(a) - L|$ and taking the supremum yields the claim. □

Lemma 2.13. *If $C_1 h \leq h' \leq C_2 h$ on A, then for any $p > 0$ there exist $M_1, M_2 > 0$ such that*

$$M_1 \|f\|_{\infty,h,p} \leq \|f\|_{\infty,h',p} \leq M_2 \|f\|_{\infty,h,p},$$

and similarly for $\|\cdot\|^\sharp$.

Corollary 2.7 (Robustness of Convergence Classification). If h and h' are two equivalent exhaustion functions, then the associated weighted norms $\|\cdot\|_{\infty,h,p}$ and $\|\cdot\|_{\infty,h',p}$ are equivalent. That is, for any function f and any $p > 0$, there exist constants $M_1, M_2 > 0$ such that:

$$M_1 \|f\|_{\infty,h,p} \leq \|f\|_{\infty,h',p} \leq M_2 \|f\|_{\infty,h,p}.$$

Therefore, a function has a finite norm for one exhaustion if and only if it has a finite norm for the other, and the Big-O classification ($f(a) - L = O(h^{-p})$) is preserved.

Proof Sketch. The equivalence $C_1 h \leq h' \leq C_2 h$ implies that for large values of h and h', $(1 + h')$ is bounded by multiples of $(1 + h)$. Specifically, $(1 + h'(a))^p \approx (C_2 h(a))^p = C_2^p h(a)^p \approx C_2^p (1 + h(a))^p$. A formal derivation confirms the equivalence of the norms. □

This result confirms that our framework provides a robust measure of asymptotic behavior, one that reflects the intrinsic structure of the space at infinity rather than the peculiarities of a specific exhaustion.

3 Adjoining a Point at Infinity and Convergence $a \to \omega_A$

3.1 The Point ω_A and the Extended Space A^*

In Section 2, we established a function $h : A \to [0, \infty)$ and a nested family of "tails" $B_R = \{a \in A \mid h(a) > R\}$. We now use this structure to formally extend the space A by adjoining a single *point at infinity*, denoted ω_A. We define the extended space as:

$$A^* := A \cup \{\omega_A\}.$$

Our goal is to define a notion of convergence on A^* such that a sequence converges to ω_A if and only if it "escapes to infinity" in the sense of Proposition 2.10. We achieve this by defining the sets B_R to be the fundamental neighborhoods of ω_A.

Definition 3.1 (Convergence to infinity)**.** We say $a \to \omega_A$ iff for every $R > 0$, eventually $a \in A \setminus K_R$. Equivalently, along the tail filter $\mathcal{F}_\infty = \{A \setminus K_R\}_{R>0}$.

Proposition 3.2. *$a \to \omega_A$ iff $h(a) \to \infty$ along $\mathcal{F}_\infty$.*

From Proposition 2.10 and this definition, we immediately have the central equivalence of our framework:

Corollary 3.3 (Equivalence of Convergence)**.** *A sequence $a_n \to \omega_A$ if and only if $\lim_{n\to\infty} h(a_n) = \infty$.*

Connection to the Alexandroff Compactification. When A is a topological space, Definition **??** can be used to induce a topology on A^*. A set $U \subseteq A^*$ is declared open if either (i) $U \subseteq A$ and is open in A's original topology, or (ii) $\omega_A \in U$ and U is a neighborhood of ω_A as per Definition **??**. The following proposition shows that our framework perfectly recovers the standard topological construction.

Proposition 3.4 (Alexandroff one-point compactification)**.** *Let A be non-compact LCH, and $\{K_r\}_{r\geq 0}$ an exhaustion by compacta. Declare $U \subset A^*$ open iff either $U \cap A$ is open in A, or $\omega_A \in U$ and there exists $R > 0$ with $(A \setminus K_R) \cup \{\omega_A\} \subset U$. Then A^* is Hausdorff and compact; the induced topology is the Alexandroff one-point compactification of A.*

Proof. Neighborhoods of ω_A are $(A \setminus K) \cup \{\omega_A\}$ with K compact; this is the standard base for the Alexandroff compactification. Compactness and Hausdorffness follow from standard LCH facts (every compact set is contained in some K_R, and points are separated by LCH structure). ☐

Theorem 3.5 (Universal property of the h-compactification)**.** *Let A be LCH with compact exhaustion $\{K_R\}$ and $A^* = A \cup \{\omega_A\}$ endowed with the Alexandroff topology as in Proposition 3.4. For every compact Hausdorff space Y and every continuous map $F : A \to Y$ such that $F(\{h > R\}) \subset U$ for some neighborhood U independent of R (i.e., F is constant at infinity along the tail filter), there exists a unique continuous $\widetilde{F} : A^* \to Y$ with $\widetilde{F}|_A = F$ and $\widetilde{F}(\omega_A) = \lim_{h\to\infty} F$.*

Proof. Neighborhoods of ω_A are $(A \setminus K) \cup \{\omega_A\}$ with K compact (Prop. 3.4). The tail assumption guarantees that F is Cauchy/constant along this filter and hence has a unique limit in compact Y; define $\widetilde{F}(\omega_A)$ to be this limit. Continuity at ω_A is exactly the tail condition; uniqueness is clear. ☐

Examples of Convergence. The power of the equivalence established in Corollary 3.3 is that it recovers the standard notions of "approaching infinity" in all relevant contexts, using the specific exhaustion functions h we identified in Section 2.

Metric Space. With $h(a) = d(a, a_0)$, the condition $h(a_n) \to \infty$ becomes $d(a_n, a_0) \to \infty$. In $\mathbb{R}^n$ with the Euclidean norm, this is the familiar condition $\|a_n\| \to \infty$.

Topological Space. With an exhaustion by compacts $\{K_R\}$, the condition $h(a_n) \to \infty$ signifies that the sequence (a_n) must eventually leave any given compact set K_R. This aligns perfectly with the intuitive and formal definition of a sequence tending to infinity in a non-compact space.

Directed Set. For $A = \mathbb{N}$ with $h(n) = n$, the condition $h(n_i) \to \infty$ is simply $n_i \to \infty$ in the usual sense for a sequence of integers.

Measure Space. For $A = [0, \infty)$ with $h(x) = x$, the condition $h(x_n) \to \infty$ is the standard limit $x_n \to \infty$ on the real line.

3.2 Limits of Functions at Infinity

Now that we have a formal notion of convergence $a \to \omega_A$, we can define the limit of a function $f : A \to \mathbb{R}$ at infinity in a manner analogous to standard analysis.

Definition 3.6 (Limit of a Function at Infinity)**.** Let $f : A \to \mathbb{R}$ be a function and $L \in \mathbb{R}$. We say that f **converges to the limit** L **as** a **approaches** ω_A, written

$$\lim_{a \to \omega_A} f(a) = L,$$

if for every $\epsilon > 0$, there exists a real number $R > 0$ such that for all $a \in A$ with $h(a) > R$, we have $|f(a) - L| < \epsilon$.

In the language of topology, this definition is equivalent to stating that the function f can be extended to a continuous function $f^* : A^* \to \mathbb{R}$ by setting $f^*(\omega_A) = L$. The ϵ-R formulation, however, is more direct for our analytical purposes and makes the role of the exhaustion function h explicit. This definition now allows us to discuss the central topic of this paper: measuring the rate at which $f(a)$ approaches L.

4 Measuring the Speed of Convergence: The Weighted Norms

4.1 Motivation and Definition

Often in analysis, one is not only interested in the fact that a function $f(a)$ converges to a limit L, but also *how fast* it does so. To formalize this, we need to compare the error $|f(a) - L|$ to a "gauge" function that vanishes at infinity. Our framework provides a natural candidate for this gauge: the inverse of a function of $h(a)$.

To capture the widest possible spectrum of asymptotic behaviors, from slow logarithmic decay to rapid exponential decay, we introduce the concept of a **scale of comparison functions**, Φ. This will be a set of functions $\phi : [0, \infty) \to (0, \infty)$ which are typically chosen to be continuous, strictly increasing, and such that $\lim_{s \to \infty} \phi(s) = \infty$. Each function $\phi \in \Phi$ defines a specific rate of convergence to be tested.

4.2 Admissible comparison functions and basic notation

We work with a class Φ_{adm} of *admissible comparison functions* $\varphi : [0,\infty) \to [1,\infty)$ satisfying:

(A1) **Monotonicity & normalization:** φ is continuous, nondecreasing, and $\varphi(0) = 1$.

(A2) **Submultiplicativity up to a constant:** There exists $K \geq 1$ such that

$$\varphi(r+s) \;\leq\; K\,\varphi(r)\,\varphi(s) \qquad \forall r,s \geq 0.$$

(This implies a "doubling" type control $\varphi(2t) \leq K\,\varphi(t)^2$.)

Typical examples (all in Φ_{adm}):

$$\varphi_p(t) = (1+t)^p \ (p \geq 0), \qquad \varphi_{p,q}(t) = (1+t)^p\big(1+\log(1+t)\big)^q, \qquad \varphi_\alpha(t) = e^{\alpha t} \ (\alpha \geq 0).$$

We write $\varphi_1 \preceq \varphi_2$ if there exist $R < \infty$ and $C \geq 1$ with $\varphi_1(t) \leq C\,\varphi_2(t)$ for all $t \geq R$ ("eventual domination"), and $\varphi_1 \simeq \varphi_2$ if both $\varphi_1 \preceq \varphi_2$ and $\varphi_2 \preceq \varphi_1$ hold.

4.3 Orlicz sharp norms via Young functions

Let $\Phi : [0,\infty) \to [0,\infty)$ be a Young function (convex, increasing, $\Phi(0) = 0$) satisfying the Δ_2-condition at infinity. Let h be a proper exhaustion and set $W(x) := 1 + h(x)$.

Definition 4.1 (Luxemburg sharp norm)**.** For $[f] \in C(A)/\mathbb{R}$ define

$$\|[f]\|^{\sharp}_{\Phi,h} := \inf\Big\{\lambda > 0 : \inf_{c\in\mathbb{R}} \sup_{x\in A} \Phi\Big(\frac{|f(x)-c|\,W(x)}{\lambda}\Big) \ \leq 1\Big\}.$$

Theorem 4.2 (Basic properties)**.** *If Φ satisfies Δ_2 at infinity, then $\|\cdot\|^{\sharp}_{\Phi,h}$ is a norm and $\big(C(A)/\mathbb{R}, \|\cdot\|^{\sharp}_{\Phi,h}\big)$ is complete. If Φ_1 and Φ_2 satisfy $\Phi_1(t) \leq C\,\Phi_2(Dt)$ for all sufficiently large t, then there exists $K \geq 1$ such that*

$$\|[f]\|^{\sharp}_{\Phi_1,h} \;\leq\; K\,\|[f]\|^{\sharp}_{\Phi_2,h} \qquad \textit{for all } [f].$$

If h' is coarsely affine equivalent to h, then the corresponding Orlicz sharp norms are equivalent.

Corollary 4.3 (Polynomial case)**.** *For $\Phi_q(t) = t^q$ with $q \geq 1$ and $p > 0$, the Luxemburg sharp norm associated with Φ_q and weight $W(x) = 1 + h(x)$ is (up to equivalence of norms) the polynomially weighted sharp norm with $\varphi_p(t) = (1+t)^p$, upon identifying exponents in the usual manner.*

4.4 φ-weighted global and asymptotic functionals

Let $h : A \to [0,\infty)$ be a continuous proper exhaustion (Lemma 2.1). For $L \in \mathbb{R}$ and $\varphi \in \Phi_{\mathrm{adm}}$, define the *global* fixed-L norm

$$\|f\|_{\infty,h;\varphi;L} \;:=\; \sup_{a\in A} \varphi\big(h(a)\big)\,|f(a)-L| \;\in [0,\infty],$$

and the *sharp* quotient norm on $C(A)/\mathbb{R}$:

$$\|[f]\|^{\sharp}_{\infty,h;\varphi} \;:=\; \inf_{c\in\mathbb{R}} \sup_{a\in A} \varphi\big(h(a)\big)\,|f(a)-c|.$$

(These extend Definitions 4.47 and 4.48 obtained for the special case $\varphi(t) = (1+t)^p$.)

The *asymptotic* version uses a limsup along $h \to \infty$:

$$C^{(h)}_{\varphi}(f;L) \;:=\; \limsup_{h(a)\to\infty} \varphi\big(h(a)\big)\,|f(a)-L|, \qquad C^{(h)}_{\varphi}([f]) \;:=\; \inf_{c\in\mathbb{R}} C^{(h)}_{\varphi}(f;c).$$

When the choice of h is clear, we simply write $C_\varphi(f;L)$ and $C_\varphi([f])$.

4.5 Banach lattice structure

Fix a proper exhaustion h and an admissible φ; write $W = \varphi \circ h$.

Proposition 4.4. *On $C(A)/\mathbb{R}$ define $|[f]| := [\,|f|\,]$, $[f] \vee [g] := [\max(f,g)\,]$, and $[f] \wedge [g] := [\min(f,g)\,]$. Then:*

1. $\|\,|[f]|\,\|^{\sharp}_{\infty,h;\varphi} = \|[f]\|^{\sharp}_{\infty,h;\varphi}$ *and* $0 \le [f] \le [g]$ *implies* $\|[f]\|^{\sharp}_{\infty,h;\varphi} \le \|[g]\|^{\sharp}_{\infty,h;\varphi}$*;*
2. $\big(C^{\sharp}_{h;\varphi}(A), \|\cdot\|^{\sharp}_{\infty,h;\varphi}\big)$ *is a Banach lattice.*

Proof. For (1), $W\,||f| - c| \le W\,|f - c|$ for suitable $c \ge 0$, and monotonicity is immediate from $|f - c| \le |g - c|$ whenever $0 \le f \le g$. Completeness is already established, and stability under $\vee, \wedge$ follows from continuity of these operations and uniform weighted control on compacts together with the patching principle. □

4.6 Minimizers for the sharp norm

Fix $\varphi \in \Phi_{\mathrm{adm}}$ and a proper h.

Proposition 4.5 (Existence of a sharp minimizer)**.** *For every $f \in C(A)$, the function $J(c) := \sup_{a\in A}\ \varphi(h(a))\,|f(a) - c|$ is convex, lower semicontinuous, and attains its minimum at some $c_* \in \mathbb{R}$. Hence*

$$\|[f]\|^{\sharp}_{\infty,h;\varphi} = J(c_*).$$

Proof. For each a, $c \mapsto \varphi(h(a))|f(a) - c|$ is convex; the supremum of convex l.s.c. functions is convex and l.s.c. Properness of h and continuity of f imply that for large $|c|$, $J(c) \ge |c| - \sup_a \varphi(h)|f| \to \infty$; thus J is coercive and attains its minimum. □

Lemma 4.6 (Two-sided contact condition)**.** *Let c_* be a minimizer. Then for every $\varepsilon > 0$ there exist $a_+, a_- \in A$ such that*

$$\varphi(h(a_\pm))\,|f(a_\pm) - c_*| \ge \|[f]\|^{\sharp}_{\infty,h;\varphi} - \varepsilon \quad \textit{and} \quad \operatorname{sign}\big(f(a_+) - c_*\big) = -\operatorname{sign}\big(f(a_-) - c_*\big).$$

Idea. If all near-maximizers lie on one side of c_*, shifting c_* slightly toward that side strictly decreases J, contradicting minimality. □

Remark 4.7 (Uniqueness up to a flat plateau). The set of minimizers is a closed interval on which J is constant. If the set of near-maximizers has non-empty interior in both signs, the minimizer is unique.

4.7 Baire-generic uniqueness of the sharp minimizer

Let $(\mathcal{B}, \|\cdot\|^{\sharp})$ be $C(A)/\mathbb{R}$ with the sharp norm.

Theorem 4.8 (Generic uniqueness)**.** *The set*

$$\mathcal{U} := \{[f] \in \mathcal{B} :\ J_f(c) = \sup W|f - c| \ \textit{has a unique minimizer}\}$$

is residual (comeager) in $\mathcal{B}$.

Sketch. For $k \in \mathbb{N}$ let E_k be the set of $[f]$ for which there exist $c_1 < c_2$ with $J_f(c_i) \le \inf J_f + 1/k$. Each E_k is closed and nowhere dense (perturb f by a tiny bump supported on a compact core to break ties using the two-sided contact Lemma 4.2). Then $\mathcal{U} = \mathcal{B} \setminus \bigcup_k E_k$ is residual. □

4.8 Lipschitz functional calculus

Let $W := \varphi \circ h \geq 1$ and denote by $\|[f]\|^\sharp := \|[f]\|^\sharp_{\infty,h;\varphi}$.

Proposition 4.9 (Composition with Lipschitz maps)**.** *Let $\psi : \mathbb{R} \to \mathbb{R}$ be Lipschitz with* $\mathrm{Lip}(\psi) = L$*. Then for all $[f] \in C(A)/\mathbb{R}$,*

$$\|[\psi \circ f]\|^\sharp \;\leq\; L\,\|[f]\|^\sharp.$$

In particular, the decay class $\{[f] : C^{(h)}_\varphi([f]) = 0\}$ is stable under Lipschitz compositions.

Proof. For any $c \in \mathbb{R}$ choose $d = \psi(c)$. Then $W\,|\psi(f) - d| \leq L\,W\,|f - c|$. Take sup over A and inf over c. □

Corollary 4.10 (Multipliers)**.** *If $g \in C(A)$ is bounded and $[f] \in C(A)/\mathbb{R}$, then $\|[g\,f]\|^\sharp \leq \|g\|_\infty\,\|[f]\|^\sharp$. If, moreover, $g \to g_\infty$ along $h \to \infty$, then $[g\,f]$ and $[g_\infty f]$ have the same decay class.*

4.9 Real interpolation between polynomially weighted sharp spaces

Fix a proper exhaustion h and set $X_p := \big(C^\sharp_{h;\varphi_p}(A),\ \|\cdot\|^\sharp_{\infty,h;\varphi_p}\big)$ with $\varphi_p(t) = (1+t)^p$. For $p_0, p_1 \geq 0$ and $\theta \in (0,1)$ define $p_\theta := \theta p_0 + (1-\theta)p_1$.

Theorem 4.11 (Real K-method)**.** *For the compatible couple (X_{p_0}, X_{p_1}) one has the continuous embeddings*

$$X_{p_\theta} \;\hookrightarrow\; (X_{p_0}, X_{p_1})_{\theta,\infty} \;\hookrightarrow\; X_{p_\theta},$$

hence $(X_{p_0}, X_{p_1})_{\theta,\infty}$ and X_{p_θ} are isomorphic as normed spaces (with equivalence constants depending only on p_0, p_1, θ).

Idea. Let $W_j = (1+h)^{p_j}$ and $W_\theta = W_0^\theta W_1^{1-\theta} = (1+h)^{p_\theta}$. For the left embedding, split $[f] = [g] + [h]$ by a threshold on $\{W_0/W_1 \gtrless \tau\}$ so that $K(\tau, [f]) \lesssim \|[f]\|_{W_\theta}$; take $\sup_{\tau>0} \tau^{-\theta} K(\tau, [f])$. For the right embedding, given a near-optimal decomposition at scale τ, bound $W_\theta|f - c| \leq W_0|g - c|^\theta\, W_1|h - c|^{1-\theta}$ and use the pointwise max-splitting as in your patching principle to control the global sharp norm. □

4.10 Stability of minimizers

Let $J_f(c) := \sup_a \varphi(h(a))\,|f(a) - c|$ and denote by $\mathrm{Argmin}\,J_f$ the set of sharp minimizers.

Proposition 4.12 (Upper semicontinuity of minimizers)**.** *If $\|[f_n] - [f]\|^\sharp_{\infty,h;\varphi} \to 0$ and $c_n \in \mathrm{Argmin}\,J_{f_n}$, then every cluster point of (c_n) lies in $\mathrm{Argmin}\,J_f$.*

Proof. For any c, $|J_{f_n}(c) - J_f(c)| \leq \|[f_n] - [f]\|^\sharp_{\infty,h;\varphi}$. With c_n minimizing J_{f_n}, take lim sup and compare to J_f. □

Corollary 4.13 (Continuity under uniqueness)**.** *If each J_{f_n} and J_f have unique minimizers c_n and c_*, then $c_n \to c_*$ whenever $\|[f_n] - [f]\|^\sharp_{\infty,h;\varphi} \to 0$.*

Remark 4.14 (Quantitative two-sided contact). Your Lemma 4.2 provides a near-maximizer on each sign at c_*. Under a uniform two-sided contact gap on compacts, one can obtain local Lipschitz dependence of c_* on f.

4.11 Uniqueness criterion for the sharp minimizer

Let $J_f(c) = \sup_a W(a)\,|f(a) - c|$ with $W = \varphi \circ h$.

Proposition 4.15 (Alternation-type criterion)**.** *Assume there exist points $a_+, a_- \in A$ such that*

$$W(a_\pm)\,|f(a_\pm) - c_*| = \|[f]\|^\sharp_{\infty,h;\varphi}, \qquad \operatorname{sign}\big(f(a_+) - c_*\big) = -\operatorname{sign}\big(f(a_-) - c_*\big),$$

and, moreover, the contact set $\mathcal{C}_* = \{a : W(a)\,|f(a) - c_*| = \|[f]\|^\sharp\}$ contains a neighborhood of $a_\pm$ only on those two branches (no flat plateau). Then c_* is the unique minimizer of J_f.

Idea. If $c \neq c_*$, shifting c toward the sign of the dominant contact increases the opposite contact value past the sup at c_*. Absence of a flat plateau prevents equality for $c \neq c_*$. □

Remark 4.16 (Practical check). On a compact core K, uniqueness holds if $\max_K W|f - c_*|$ and $\min_K W(f - c_*)$ are attained at unique points with opposite signs, and tails are strictly below the global sup (by $C^{(h)}_\varphi([f]) < \|[f]\|^\sharp$).

4.12 Embeddings and interpolation on the Φ-scale

Proposition 4.17 (Tail embeddings for asymptotic constants)**.** *Let $\varphi_1, \varphi_2 \in \Phi_{\mathrm{adm}}$ with $\varphi_1 \preceq \varphi_2$. Then for all $L \in \mathbb{R}$ and $f \in C(A)$,*

$$C^{(h)}_{\varphi_1}(f; L) \;\le\; C\; C^{(h)}_{\varphi_2}(f; L),$$

for some constant C depending only on the domination constants in $\varphi_1 \preceq \varphi_2$. Consequently, $C_{\varphi_2}(f; L) = 0 \;\Rightarrow\; C_{\varphi_1}(f; L) = 0$.

Proof. Choose R and C with $\varphi_1(t) \le C\varphi_2(t)$ for all $t \ge R$. In the limsup defining $C^{(h)}_{\varphi_1}(f; L)$ one may restrict to points with $h(a) \ge R$, giving the inequality. □

Proposition 4.18 (Global embeddings modulo a local patch)**.** *Let $\varphi_1 \preceq \varphi_2$. Fix $R < \infty$ such that $\varphi_1(t) \le C\varphi_2(t)$ for all $t \ge R$. Then for all $[f] \in C(A)/\mathbb{R}$,*

$$\|[f]\|^\sharp_{\infty,h;\varphi_1} \;\le\; \max\Big\{ \sup_{h \le R} \operatorname{osc}_{[f]}(a),\; C\,\|[f]\|^\sharp_{\infty,h;\varphi_2} \Big\},$$

where $\operatorname{osc}_{[f]}(a) := \inf_{c \in \mathbb{R}} |f(a) - c|$ denotes the pointwise oscillation modulo constants. In particular, the two global norms are equivalent up to a local *finite patch on $\{h \le R\}$.*

Sketch. Pick c near-optimal for $\|[f]\|^\sharp_{\infty,h;\varphi_2}$. Split the supremum over $\{h \le R\}$ and $\{h > R\}$ and use $\varphi_1 \le C\varphi_2$ on the tail. The local term is finite as a supremum over a compact set $\{h \le R\}$. □

Proposition 4.19 (Interpolation on the Φ-scale)**.** *Let $\varphi, \varphi_1, \varphi_2 \in \Phi_{\mathrm{adm}}$ and $\theta \in [0, 1]$. Assume that for some $K \ge 1$ we have*

$$\varphi(t) \;\le\; K\,\varphi_1(t)^\theta\,\varphi_2(t)^{1-\theta} \qquad \forall t \ge 0.$$

Then for all L and $f \in C(A)$,

$$C^{(h)}_\varphi(f; L) \;\le\; K\,\Big(C^{(h)}_{\varphi_1}(f; L)\Big)^\theta\,\Big(C^{(h)}_{\varphi_2}(f; L)\Big)^{1-\theta}.$$

Proof. Apply the inequality inside the limsup and use the elementary bound $\limsup(xy) \le (\limsup x)\,(\limsup y)$ with nonnegative sequences. □

4.13 Monotonicity and log-convexity in p

Fix a proper exhaustion h and $\varphi_p(t) = (1+t)^p$ for $p \geq 0$.

Proposition 4.20 (Monotonicity). *For fixed $[f] \in C(A)/\mathbb{R}$, the map $p \mapsto \|[f]\|^{\sharp}_{\infty,h;\varphi_p}$ is nondecreasing on $[0,\infty)$.*

Proof. If $p_1 \leq p_2$, then $\varphi_{p_1} \leq \varphi_{p_2}$ pointwise, hence the sharp norm with φ_{p_1} is dominated by that with φ_{p_2}. ☐

Theorem 4.21 (Log-convexity). *For $\theta \in [0,1]$ and $p_\theta = \theta p_1 + (1-\theta)p_2$,*

$$\|[f]\|^{\sharp}_{\infty,h;\varphi_{p_\theta}} \;\leq\; \big(\|[f]\|^{\sharp}_{\infty,h;\varphi_{p_1}}\big)^{\theta}\big(\|[f]\|^{\sharp}_{\infty,h;\varphi_{p_2}}\big)^{1-\theta}.$$

Proof. Use your interpolation on the Φ-scale (Sec. 4.5): $(1+t)^{p_\theta} = (1+t)^{\theta p_1}(1+t)^{(1-\theta)p_2}$ and apply the multiplicative interpolation inequality to the tail, then patch globally as in Sec. 4.6 if needed. ☐

4.14 Global vs. asymptotic: a patching principle and examples

We record a precise relation between the *global* (sharp) norm and the *asymptotic* constant. Fix a continuous proper exhaustion h and $\varphi \in \Phi_{\mathrm{adm}}$.

Definition 4.22 (Local and tail functionals). For $R \geq 0$ and a class $[f] \in C(A)/\mathbb{R}$, define

$$\|[f]\|_{\mathrm{loc},R;\varphi} := \inf_{c\in\mathbb{R}} \sup_{\{a\,:\,h(a)\leq R\}} \varphi\big(h(a)\big)\,|f(a)-c| \in [0,\infty),$$

and the *tail sup* functional

$$T^{(h)}_{R;\varphi}([f]) := \inf_{c\in\mathbb{R}} \sup_{\{a\,:\,h(a)>R\}} \varphi\big(h(a)\big)\,|f(a)-c| \in [0,\infty].$$

We keep $C^{(h)}_{\varphi}([f]) = \inf_c \limsup_{h(a)\to\infty} \varphi(h(a))\,|f(a)-c|$ for the asymptotic constant.

Lemma 4.23 (Patching principle). *For every $R \geq 0$ and $[f] \in C(A)/\mathbb{R}$,*

$$\max\big\{\, \|[f]\|_{\mathrm{loc},R;\varphi},\; C^{(h)}_{\varphi}([f]) \,\big\} \;\leq\; \|[f]\|^{\sharp}_{\infty,h;\varphi} \;\leq\; \max\big\{\, \|[f]\|_{\mathrm{loc},R;\varphi},\; T^{(h)}_{R;\varphi}([f]) \,\big\}.$$

Moreover, $T^{(h)}_{R;\varphi}([f]) \downarrow C^{(h)}_{\varphi}([f])$ as $R \to \infty$.

Proof. For any $c \in \mathbb{R}$, split the global supremum over $\{h \leq R\}$ and $\{h > R\}$; then take the infimum in c to obtain the upper bound. For the lower bound, observe that restricting the supremum to either part can only decrease the value, and the limsup is always bounded by the global sup. The monotone convergence $T_R \downarrow C_\varphi$ is standard: $R \mapsto T_R$ is decreasing and its limit equals the limsup envelope of the tail. ☐

Corollary 4.24 (Finiteness criterion). *For $[f] \in C(A)/\mathbb{R}$ the following are equivalent:*

1. *$\|[f]\|^{\sharp}_{\infty,h;\varphi} < \infty$;*
2. *$C^{(h)}_{\varphi}([f]) < \infty$ and $\|[f]\|_{\mathrm{loc},R;\varphi} < \infty$ for some (equivalently, for all) $R \geq 0$.*

In particular, for continuous f the local term is always finite on any compact $\{h \leq R\}$, so the global finiteness is equivalent to finiteness of the asymptotic constant.

Proof. Immediate from Lemma 4.23, using compactness of $\{h \le R\}$ and continuity of f for the local bound. □

Proposition 4.25 (Quantitative patching with a near-optimal tail constant)**.** *Fix $R \ge 0$ and $\varepsilon > 0$. Choose c_ε with $\sup_{\{h>R\}} \varphi(h)\,|f - c_\varepsilon| \le T^{(h)}_{R;\varphi}([f]) + \varepsilon$. Then*

$$\|[f]\|^{\sharp}_{\infty,h;\varphi} \;\le\; \max\Big\{ \sup_{\{h\le R\}} \varphi(h)\,|f - c_\varepsilon|\;,\; T^{(h)}_{R;\varphi}([f]) + \varepsilon \Big\}.$$

Proof. Evaluate the global supremum at $c = c_\varepsilon$ and split into local/tail parts. □

Two simple examples (global $\neq$ asymptotic). Let $A = \mathbb{R}$, $h(x) = |x|$, and $\varphi(t) = (1+t)^p$ with $p > 0$.

- *Asymptotically perfect but globally nonzero.* Let f be a smooth bump with $f(x) = \sin x$ on $[-1, 1]$ and $f(x) = 0$ for $|x| \ge 2$. Then $C^{(h)}_\varphi([f]) = 0$ (since $f \to 0$ at infinity), while $\|[f]\|^{\sharp}_{\infty,h;\varphi} \ge \inf_c \sup_{|x|\le 2} |f(x) - c| > 0$ by non-constancy on the compact core.
- *Same asymptotic rate, different global size.* Let $g(x) = f(x) + \eta\,\chi_{[-2,2]}(x)$ with $\eta \neq 0$ and a smooth cutoff $\chi \equiv 1$ on $[-1, 1]$, supported in $[-2, 2]$. Then $C^{(h)}_\varphi([g]) = C^{(h)}_\varphi([f]) = 0$, but $\|[g]\|^{\sharp}_{\infty,h;\varphi}$ can be strictly larger due to the amplified local oscillation.

Remark 4.26 (Operational takeaway). For estimates that only depend on the *decay class* (whether $C_\varphi = 0$), asymptotic control suffices. For uniform quantitative bounds, combine a compact *local patch* $\|[f]\|_{\mathrm{loc},R;\varphi}$ with an *asymptotic tail* C_φ via Lemma 4.23 or Proposition 4.25.

4.15 A weighted Schur test for sharp norms

Let (A, μ) be a Radon measure space and $W = \varphi \circ h \ge 1$. Consider the integral operator

$$(Tf)(x) \;:=\; \int_A K(x, y)\, f(y)\, d\mu(y), \qquad K \ge 0.$$

Theorem 4.27 (Weighted Schur test)**.** *Assume there exist $C_1, C_2 < \infty$ such that*

$$\sup_{x\in A} \int_A K(x, y)\, \frac{W(x)}{W(y)}\, d\mu(y) \;\le\; C_1, \qquad \sup_{y\in A} \int_A K(x, y)\, d\mu(x) \;\le\; C_2.$$

Then T defines a bounded map on $C^{\sharp}_{h;\varphi}(A)$ with

$$\|[Tf]\|^{\sharp}_{\infty,h;\varphi} \;\le\; C_1\, \|[f]\|^{\sharp}_{\infty,h;\varphi}.$$

If K is positivity preserving and $\int_A K(x, y)\, d\mu(x) = 1$ for all y, then $C^{(h)}_\varphi([Tf]) \le C_1\, C^{(h)}_\varphi([f])$.

Proof. For any $c \in \mathbb{R}$ and $x \in A$, $W(x)\,|(Tf)(x) - c| \le \int K(x, y)\, \frac{W(x)}{W(y)}\, W(y)\, |f(y) - c|\, d\mu(y)$. Take sup in x, then inf in c to get the bound by C_1. The tail estimate follows by the same inequality restricted to $\{h > R\}$ and letting $R \to \infty$. □

4.16 Isometry with C_0 on the decay class

Let $\mathcal{X} := \{[f] \in C(A)/\mathbb{R} : \; C_\varphi^{(h)}([f]) = 0\}$ and $W = \varphi \circ h$.

Theorem 4.28. *Choose for each $[f] \in \mathcal{X}$ a sharp minimizer $c_*([f])$. The map*

$$\mathcal{T} : \; \mathcal{X} \;\longrightarrow\; C_0(A), \qquad \mathcal{T}([f]) \;:=\; W(\cdot)\,\big(f(\cdot) - c_*([f])\big)$$

is well-defined up to equality in $C_0(A)$, linear, and isometric:

$$\|\mathcal{T}([f])\|_{C_0} \;=\; \|[f]\|^\sharp_{\infty,h;\varphi}.$$

Consequently, $\mathcal{X}$ is separable (as $C_0(A)$ is separable when A is σ-compact LCH).

Sketch. Decay means $W(f - c_*) \to 0$ along $h \to \infty$, hence $\mathcal{T}([f]) \in C_0(A)$ and $\|\cdot\|_{C_0}$ equals the sharp norm by definition of c_*. If c_* is not unique, the image differs by a function vanishing at infinity (same norm), so $\mathcal{T}$ is well-defined in C_0. Linearity follows from the choice of minimizers via a standard ε-argument. $\square$

4.17 Regularization by inf-convolution on proper metric spaces

Assume (A, d) is a proper metric space. For $\lambda > 0$, define the (Moreau) inf-convolution

$$\mathsf{M}_\lambda f(x) \;:=\; \inf_{y \in A} \Big\{ \, f(y) \;+\; \frac{d(x,y)^2}{2\lambda} \, \Big\}.$$

Proposition 4.29 (Preservation of decay class & convergence)**.** *If $[f] \in C(A)/\mathbb{R}$ satisfies $C_\varphi^{(h)}([f]) = 0$, then $[\mathsf{M}_\lambda f] \in C(A)/\mathbb{R}$ satisfies $C_\varphi^{(h)}([\mathsf{M}_\lambda f]) = 0$ for all $\lambda > 0$, and*

$$\lim_{\lambda \downarrow 0} \; \|[\mathsf{M}_\lambda f] - [f]\|^\sharp_{\infty,h;\varphi} \;=\; 0.$$

Idea. M_λ is 1-Lipschitz in the sup-norm and preserves monotone limits on compacts. Use the weighted Dini theorem (Sec. 4.19) for convergence local+tête, and the inequality $W(x)\,|\mathsf{M}_\lambda f(x) - c| \le \sup_y W(y)\,|f(y) - c|$ (since the inf over y does not increase the weighted sup). $\square$

4.18 Exact patch formula and a projective-limit viewpoint

Fix a proper exhaustion h and $\varphi \in \Phi_{\mathrm{adm}}$. Recall the local/tail functionals $\|[f]\|_{\mathrm{loc},R;\varphi}$ and $T^{(h)}_{R;\varphi}([f])$ from Section 4.6.

Proposition 4.30 (Exact patch formula)**.** *For every $[f] \in C(A)/\mathbb{R}$,*

$$\boxed{\|[f]\|^\sharp_{\infty,h;\varphi} \;=\; \inf_{R \ge 0} \; \max\Big\{ \, \|[f]\|_{\mathrm{loc},R;\varphi} \; , \; T^{(h)}_{R;\varphi}([f]) \, \Big\}.}$$

Equivalently, with $C_\varphi^{(h)}$ from Section 4.6,

$$\|[f]\|^\sharp_{\infty,h;\varphi} \;=\; \inf_{R \ge 0} \; \max\Big\{ \, \|[f]\|_{\mathrm{loc},R;\varphi} \; , \; T^{(h)}_{R;\varphi}([f]) \, \Big\} \;=\; \inf_{R \ge 0} \; \max\Big\{ \, \|[f]\|_{\mathrm{loc},R;\varphi} \; , \; C^{(h)}_{\varphi}([f]) \, \Big\}.$$

Proof. Upper bound: Lemma 4.8 gives $\|[f]\|^\sharp \le \max\{\|[f]\|_{\mathrm{loc},R}, T_R\}$ for every R; take the infimum over R. Lower bound: let c_* be a sharp minimizer (Prop. 4.1); then for each R,

$$\|[f]\|^\sharp = \sup_A \varphi(h)|f - c_*| = \max\Big\{ \sup_{\{h \le R\}} \varphi(h)|f - c_*|, \; \sup_{\{h > R\}} \varphi(h)|f - c_*| \Big\} \;\ge\; \max\{\|[f]\|_{\mathrm{loc},R}, T_R\}.$$

Take the infimum in R. The variant with $C_\varphi^{(h)}$ follows since $T_R \downarrow C_\varphi^{(h)}$ (Lemma 4.8). $\square$

Remark 4.31 (Projective-limit viewpoint). The seminorms $p_R([f]) := \max\{\|[f]\|_{\mathrm{loc},R;\varphi}, T^{(h)}_{R;\varphi}([f])\}$ decrease in R and $\|[f]\|^\sharp = \inf_R p_R([f])$. Thus $C^\sharp_{h;\varphi}(A)$ est la limite projective des Banach "tronqués" $\big(C(\{h \le R\})/\mathbb{R}\big) \times$ (tail) soudés par Proposition 4.30.

4.19 A weighted Dini-type convergence theorem

Fix a proper h and $\varphi \in \Phi_{\mathrm{adm}}$. Let (f_n) be continuous functions on A and $f \in C(A)$.

Theorem 4.32 (Weighted Dini)**.** *Assume:*

(i) For every $R < \infty$, $f_n \to f$ uniformly *on the compact set $\{h \le R\}$ (e.g. $f_n \downarrow f$ or $f_n \uparrow f$ on $\{h \le R\}$ so that classical Dini applies);*

(ii) $C^{(h)}_\varphi\big([f_n - f]\big) \to 0$ as $n \to \infty$.

Then $\|[f_n] - [f]\|^\sharp_{\infty,h;\varphi} \to 0$.

Proof. By Lemma 4.23 (Section 4.6), for any R,

$$\|[f_n] - [f]\|^\sharp_{\infty,h;\varphi} \le \max\Big\{ \sup_{\{h\le R\}} \varphi(h)\ \mathrm{osc}_{[f_n-f]}\ ,\ T^{(h)}_{R;\varphi}\big([f_n - f]\big) \Big\}.$$

The first term goes to 0 by (i), the second by (ii) letting $R \to \infty$. □

4.20 Density of compactly supported perturbations

Fix $\varphi \in \Phi_{\mathrm{adm}}$ and a proper h.

Proposition 4.33 (Truncation density)**.** *For every $[f] \in C(A)/\mathbb{R}$ with $\|[f]\|^\sharp_{\infty,h;\varphi} < \infty$ and every $\varepsilon > 0$, there exist $g \in C(A)$ and $c \in \mathbb{R}$ such that:*

1. *$g = f$ on a compact set K;*
2. *g is constant c on $A \setminus K$;*
3. *$\|[f] - [g]\|^\sharp_{\infty,h;\varphi} < \varepsilon$.*

Idea. Choisir R avec $T^{(h)}_{R;\varphi}([f]) < \varepsilon$ (Sec. 4.6). Prendre c ε-optimal sur la queue et lisser la transition sur un col autour de $\{h = R\}$ via une partition de l'unité; la contribution de queue reste $< \varepsilon$, le cœur est inchangé. □

4.21 Stability under coarse change of exhaustion

We record a coarse invariance for the asymptotic constants that is sufficient for applications.

Definition 4.34 (Power-dilation control)**.** An admissible $\varphi \in \Phi_{\mathrm{adm}}$ has *power-dilation control* if for each $\lambda \ge 1$ there exist $\kappa(\lambda) \ge 1$ and $C(\lambda) \ge 1$ with

$$\varphi(\lambda t) \ \le\ C(\lambda)\, \varphi(t)^{\kappa(\lambda)} \qquad \forall t \ge 0.$$

All polynomial, log-polynomial, and exponential examples above satisfy this property.

Theorem 4.35 (Coarse stability of asymptotic decay)**.** *Let $h, h' : A \to [0, \infty)$ be continuous proper exhaustions with*

$$a\,h - b \;\le\; h' \;\le\; A\,h + B \qquad \textit{for some } a, A > 0,\ b, B \in \mathbb{R}.$$

If $\varphi \in \Phi_{\mathrm{adm}}$ has power-dilation control, then there exist positive constants C_1, C_2 and exponents $\kappa_1, \kappa_2 \ge 1$ (depending only on a, A, b, B and φ) such that for all f and L,

$$C_\varphi^{(h')}(f;L) \;\le\; C_1 \left(C_\varphi^{(h)}(f;L)\right)^{\kappa_1}, \qquad C_\varphi^{(h)}(f;L) \;\le\; C_2 \left(C_\varphi^{(h')}(f;L)\right)^{\kappa_2}.$$

In particular, $C_\varphi^{(h)}(f;L) = 0$ iff $C_\varphi^{(h')}(f;L) = 0$; the decay class $\{f :\ C_\varphi^{(h)}(f;L) = 0\}$ *is independent of the chosen proper exhaustion.*

Proof sketch. From $h' \le Ah + B$, monotonicity and (A2) give $\varphi(h') \le K\,\varphi(B)\,\varphi(Ah)$. By power-dilation control, $\varphi(Ah) \le C(A)\,\varphi(h)^{\kappa(A)}$. Passing to limsups yields the first inequality with constants $C_1 := K\,\varphi(B)\,C(A)$ and $\kappa_1 := \kappa(A)$. The other side uses $h \le a^{-1}h' + a^{-1}b$ similarly. □

Corollary 4.36 (Exact equivalence for polynomial weights)**.** *For $\varphi(t) = (1+t)^p$ with fixed $p > 0$ and any two proper exhaustions h, h', there exists $C \ge 1$ such that*

$$C^{-1}\,C_\varphi^{(h)}(f;L) \;\le\; C_\varphi^{(h')}(f;L) \;\le\; C\,C_\varphi^{(h)}(f;L)$$

for all f, L. Hence the asymptotic polynomially weighted constant is coarsely invariant *up to a multiplicative factor.*

Remark 4.37 (Quick usage guide). If your analysis only needs a decay class (i.e. whether $C_\varphi = 0$), use Theorem 4.35 directly. If you need quantitative comparison between two choices of φ, first reduce via $\preceq$ or interpolation (Propositions 4.17–4.19); for global norms, patch a fixed compact region as in Proposition 4.18.

Definition 4.38 (Generalized Weighted Norm)**.** Let ϕ be a comparison function. For a function $f : A \to \mathbb{R}$ with limit L at infinity, we define the weighted norm with respect to ϕ as:

$$||f||_{\infty,h,\phi} := \sup_{a\in A} \left(|f(a) - L| \cdot \phi(h(a))\right)$$

A finite norm $||f||_{\infty,h,\phi} = M < \infty$ implies that $|f(a) - L| \le M/\phi(h(a))$, which is precisely the condition for $f(a) - L$ to be in $O(1/\phi(h(a)))$.

The Algebraic Scale. A particularly important and common choice is the **algebraic scale**, corresponding to functions with polynomial decay. This scale is given by the family of functions $\phi_p(s) := (1+s)^p$ for $p > 0$. For this specific choice, we denote the norm more simply:

$$||f||_{\infty,h,p} := ||f||_{\infty,h,\phi_p} = \sup_{a\in A} \left(|f(a) - L| \cdot (1 + h(a))^p\right)$$

This family of norms, indexed by p, acts as a "magnifying glass" for behavior at infinity, allowing us to classify decay rates such as $O(h^{-1})$, $O(h^{-2})$, etc.

Definition 4.39 (Sharp weighted norm on the quotient)**.** For $p > 0$, define on $C(A)/\mathbb{R}$:

$$\|[f]\|^\sharp_{\infty,h,p} := \inf_{L\in\mathbb{R}}\ \sup_{a\in A}\ |f(a) - L|\,(1 + h(a))^p.$$

This is a norm on the quotient $C(A)/\mathbb{R}$.

Proposition 4.40 (Triangle inequality for $\|\cdot\|^\sharp$)**.** $\|[f] + [g]\|^\sharp_{\infty,h,p} \le \|[f]\|^\sharp_{\infty,h,p} + \|[g]\|^\sharp_{\infty,h,p}$.

Proof. Choose L_f, L_g ε-optimal; then for $L = L_f + L_g$ use $|f + g - L| \le |f - L_f| + |g - L_g|$, take sups and the inf over choices. □

Interpretation of the Norm. This definition provides a direct way to classify convergence rates. The parameter p acts as a "magnifying glass" for behavior at infinity. The key insight is:

If $\|f\|_{\infty,h,p}$ is finite, then $|f(a) - L|$ must decay at least as fast as $h(a)^{-p}$.

More formally, if $\|f\|_{\infty,h,p} = M < \infty$, then by definition, for all $a \in A$:

$$|f(a) - L| \cdot (1 + h(a))^p \leq M \implies |f(a) - L| \leq \frac{M}{(1 + h(a))^p}.$$

This is precisely the condition for $f(a) - L$ to be in $O(h(a)^{-p})$. This corrected definition (using multiplication) thus directly connects a finite norm to a specific Big-O decay rate, making it a highly effective classification tool. A function that converges faster will have a finite norm for a larger value of p.

4.22 Continuity of the norm under parameter changes

Let $\varphi_p(t) = (1 + t)^p$ and assume h, h' satisfy $ah - b \leq h' \leq Ah + B$.

Proposition 4.41 (Continuity in p)**.** *Fix $[f]$ with $\|[f]\|^{\sharp}_{\infty,h;\varphi_{p_0}} < \infty$. Then $p \mapsto \|[f]\|^{\sharp}_{\infty,h;\varphi_p}$ is continuous at p_0. Moreover, by Theorem 4.21 it is log-convex on $[0, \infty)$.*

Sketch. Use uniform control on compact cores and the dominated tail bound $\varphi_p(h) \leq \varphi_{p_0+\delta}(h)$ for $|p - p_0| \leq \delta$, then apply the weighted Dini theorem (Thm. 4.32). □

Proposition 4.42 (Uniform continuity under coarse change of h)**.** *If h' is coarsely affine equivalent to h, then for each fixed $p > 0$ there exists $C \geq 1$ (depending only on a, A, b, B, p) with*

$$C^{-1} \|[f]\|^{\sharp}_{\infty,h;\varphi_p} \;\leq\; \|[f]\|^{\sharp}_{\infty,h';\varphi_p} \;\leq\; C \|[f]\|^{\sharp}_{\infty,h;\varphi_p}.$$

Thus $h \mapsto \|[f]\|^{\sharp}_{\infty,h;\varphi_p}$ is continuous along coarse-equivalent families.

4.23 Joint continuity in (p, h) along coarse-equivalent families

Let $p \in [0, \infty)$ and let $(h_\tau)_{\tau \in T}$ be a family of proper exhaustions on A. Assume that there exist constants $a, A > 0$ and $b, B \in \mathbb{R}$ such that for all τ in a neighborhood of τ_0,

$$a\, h_{\tau_0} - b \;\leq\; h_\tau \;\leq\; A\, h_{\tau_0} + B.$$

Write $\varphi_p(t) = (1 + t)^p$.

Theorem 4.43 (Joint continuity of the sharp norm)**.** *For every $[f] \in C(A)/\mathbb{R}$, the map*

$$(p, \tau) \;\longmapsto\; \|[f]\|^{\sharp}_{\infty,h_\tau;\varphi_p}$$

is continuous at each (p_0, τ_0).

Sketch. Continuity in p at fixed h_{τ_0} follows from monotonicity and log-convexity in p, together with a weighted Dini argument on compact sublevel sets and tail control by the patching principle. The coarse-affine equivalence gives uniform norm equivalences between $(1 + h_\tau)^p$ and $(1 + h_{\tau_0})^p$ for τ near τ_0, which transfers the continuity to (p, τ) jointly. □

4.24 Coarse isomorphism for weighted sharp spaces

Theorem 4.44 (Coarse affine equivalence ⇒ normed-space isomorphism)**.** *Let $h, h' : A \to [0, \infty)$ be continuous proper exhaustions with*

$$a\,h - b \;\le\; h' \;\le\; A\,h + B \qquad (a, A > 0,\; b, B \in \mathbb{R}).$$

Let $\varphi \in \Phi_{\mathrm{adm}}$ have power-dilation control (Def. 4.34). Then the identity map on $C(A)/\mathbb{R}$ induces a linear topological isomorphism

$$\mathrm{Id} : \; \big(C^{\sharp}_{h;\varphi}(A),\; \|\cdot\|^{\sharp}_{\infty,h;\varphi}\big) \;\longrightarrow\; \big(C^{\sharp}_{h';\varphi}(A),\; \|\cdot\|^{\sharp}_{\infty,h';\varphi}\big),$$

with two-sided bounds

$$\|[f]\|^{\sharp}_{\infty,h';\varphi} \;\le\; C_1 \left(\|[f]\|^{\sharp}_{\infty,h;\varphi}\right)^{\kappa_1}, \qquad \|[f]\|^{\sharp}_{\infty,h;\varphi} \;\le\; C_2 \left(\|[f]\|^{\sharp}_{\infty,h';\varphi}\right)^{\kappa_2},$$

for suitable constants $C_1, C_2 \ge 1$ and exponents $\kappa_1, \kappa_2 \ge 1$ depending only on a, A, b, B and φ.

Corollary 4.45 (Polynomial weights: linear equivalence)**.** *If $\varphi(t) = (1 + t)^p$ with $p > 0$, there exists $C \ge 1$ such that*

$$C^{-1}\,\|[f]\|^{\sharp}_{\infty,h;\varphi} \;\le\; \|[f]\|^{\sharp}_{\infty,h';\varphi} \;\le\; C\,\|[f]\|^{\sharp}_{\infty,h;\varphi} \quad \textit{for all } [f].$$

Hence the Banach spaces $C^{\sharp}_{h;p}(A)$ and $C^{\sharp}_{h';p}(A)$ are (linearly) isomorphic.

Idea. Combine Theorem 4.35 (for asymptotic constants) with the patching Lemma 4.23 to control the global sharp norms; absorb the compact-core terms. For polynomial weights, power-dilation exponents equal 1, giving linear two-sided bounds. □

4.25 Properties and Interpretation of the Weighted Norms

Norm Properties. For any fixed exhaustion function h and order $p > 0$, the functional $\|\cdot\|_{\infty,h,p}$ defines a seminorm on the vector space of functions on A that converge to a limit at infinity. Specifically, considering the space of functions converging to $L = 0$, it satisfies:

- **Non-negativity:** $\|f\|_{\infty,h,p} \ge 0$.
- **Positive homogeneity:** $\|\alpha f\|_{\infty,h,p} = |\alpha|\|f\|_{\infty,h,p}$ for any scalar α.
- **Triangle inequality:** $\|f + g\|_{\infty,h,p} \le \|f\|_{\infty,h,p} + \|g\|_{\infty,h,p}$.

Furthermore, $\|f\|_{\infty,h,p} = 0$ if and only if $f(a)$ is identically zero. Thus, for the space of functions converging to $L = 0$, this is a true norm.

Finitude of the Norm Implies Convergence. A key advantage of our multiplicative definition of the norm is that its finiteness is a sufficient condition for convergence. This resolves the ambiguity present in alternative definitions and provides a powerful theoretical tool.

Proposition 4.46 (Finite Norm Implies Convergence)**.** *Let $f : A \to \mathbb{R}$, $L \in \mathbb{R}$, and $p > 0$. If $\|f\|_{\infty,h,p} = M < \infty$, then it is guaranteed that* $\lim_{a\to\omega_A} f(a) = L$.

Proof. By definition, $\|f\|_{\infty,h,p} = M$ means that for all $a \in A$, we have $|f(a)-L|\cdot(1+h(a))^p \leq M$. This implies

$$|f(a) - L| \leq \frac{M}{(1 + h(a))^p}.$$

From Corollary 3.3, we know that as $a \to \omega_A$, we have $h(a) \to \infty$. Therefore, the term on the right-hand side goes to zero:

$$\lim_{a\to\omega_A} \frac{M}{(1 + h(a))^p} = 0.$$

By the squeeze theorem, it follows that $\lim_{a\to\omega_A} |f(a) - L| = 0$, which is the definition of $\lim_{a\to\omega_A} f(a) = L$. ☐

Definition 4.47 (Weighted space with fixed constant)**.** For fixed $L \in \mathbb{R}$ and $p > 0$, set

$$\|f\|_{\infty,h,p;L} \;:=\; \sup_{a\in A} (1 + h(a))^p \, |f(a) - L| \quad (\in [0, \infty]),$$

and define

$$C^L_{h,p}(A) \;:=\; \Big\{ f \in C(A) \;:\; \|f\|_{\infty,h,p;L} < \infty \Big\}.$$

Definition 4.48 (*Sharp* norm on the quotient)**.** Consider the quotient $C(A)/\mathbb{R}$ of continuous functions modulo constants. For $[f] \in C(A)/\mathbb{R}$, set

$$\|[f]\|^{\sharp}_{\infty,h,p} \;:=\; \inf_{c\in\mathbb{R}} \sup_{a\in A} (1 + h(a))^p \, |f(a) - c|.$$

This defines a norm on $C(A)/\mathbb{R}$ (see Remark 4.49). We write

$$C^{\sharp}_{h,p}(A) \;:=\; \big(C(A)/\mathbb{R},\, \|\cdot\|^{\sharp}_{\infty,h,p}\big).$$

Remark 4.49. (i) The value above is independent of the representative of $[f]$ since replacing f by $f + c_0$ shifts the infimum $c \mapsto c + c_0$. (ii) Non-degeneracy: if $\|[f]\|^{\sharp}_{\infty,h,p} = 0$, there exist c_n with $\sup_a (1 + h(a))^p |f(a) - c_n| \to 0$. As $(1 + h)^p \geq 1$, we also have $\sup_a |f(a) - c_n| \to 0$, hence f is constant and $[f] = 0$. (iii) Homogeneity and the triangle inequality are immediate.

Theorem 4.50 (Completeness: fixed-L and quotient)**.** *Let A be a topological space, $h : A \to [0, \infty)$ continuous, and $p > 0$. Then:*

1. *For every $L \in \mathbb{R}$, the normed space $\big(C^L_{h,p}(A),\, \|\cdot\|_{\infty,h,p;L}\big)$ is complete.*
2. *The quotient normed space $\big(C^{\sharp}_{h,p}(A),\, \|\cdot\|^{\sharp}_{\infty,h,p}\big)$ is complete.*

Proof. (1) Let $(f_n)_n$ be Cauchy in $\|\cdot\|_{\infty,h,p;L}$. Then $g_n := (1+h)^p(f_n - L)$ is Cauchy in the sup-norm, hence converges uniformly to some continuous $g \in C_b(A)$. Set $f := L+(1+h)^{-p}g$. Then

$$\|f_n - f\|_{\infty,h,p;L} = \sup_a \big|(1 + h)^p(f_n - L) - (1 + h)^p(f - L)\big| = \|g_n - g\|_\infty \to 0,$$

so $C^L_{h,p}(A)$ is complete.

(2) Let $([f_n])_n$ be Cauchy in $\|\cdot\|^{\sharp}_{\infty,h,p}$. For each n, choose a representative f_n and $c_n \in \mathbb{R}$ such that

$$\|(1 + h)^p(f_n - c_n)\|_\infty \;\leq\; \|[f_n]\|^{\sharp}_{\infty,h,p} + \tfrac{1}{2^n}.$$

Then $g_n := (1+h)^p(f_n - c_n)$ is Cauchy in the sup-norm, hence converges uniformly to some continuous g. Fix any $c_* \in \mathbb{R}$ and set $f := c_* + (1+h)^{-p}g$, with class $[f] \in C(A)/\mathbb{R}$. For all n,

$$\|[f_n] - [f]\|^{\sharp}_{\infty,h,p} \;\le\; \sup_a \bigl|(1+h)^p\bigl((f_n - c_n) - (f - c_*)\bigr)\bigr| \;=\; \|g_n - g\|_\infty \xrightarrow[n\to\infty]{} 0.$$

Thus $([f_n])_n$ converges to $[f]$ and $C^{\sharp}_{h,p}(A)$ is complete. □

Corollary 4.51 (Fixed-L vs. quotient viewpoints)**.** *Fix $L_0 \in \mathbb{R}$. The linear map*

$$\Psi:\; C^{L_0}_{h,p}(A) \longrightarrow C^{\sharp}_{h,p}(A), \qquad \Psi(f) \;:=\; [\,f\,],$$

is continuous with closed graph and dense image. In particular, completeness of one framework implies completeness of the other (Theorem 4.50).

Sketch. For any f, $\|[f]\|^{\sharp}_{\infty,h,p} \le \|f\|_{\infty,h,p;L_0}$ by taking $c = L_0$ in the infimum. Density follows by normalizing each class $[f]$ with nearly optimal choices of c (up to ε). □

4.26 Global Norm versus Asymptotic Rate

It is crucial to recognize that $\|f\|_{\infty,h,p}$ is a **global** measure. The supremum is taken over the entire set A. Consequently, the value of the norm might be determined by the behavior of $f(a)$ in a region where $h(a)$ is small, rather than by its asymptotic tail. For example, a function may decay very rapidly at infinity, but if it has a large, sharp spike near a point a_0 where $h(a_0)$ is small, its norm could be large.

To isolate the purely asymptotic behavior, it is useful to introduce a related concept that focuses exclusively on the tail of the function.

Definition 4.52 (Generalized Asymptotic Rate Constant)**.** Let $f \to L$ as $a \to \omega_A$ and let ϕ be a comparison function. We define the asymptotic rate constant as:

$$C_\phi(f) := \limsup_{a\to\omega_A} \left(|f(a) - L| \cdot \phi(h(a))\right)$$

This value in $[0,\infty]$ precisely captures the tightest asymptotic bound on the scaled error. A value of $C_\phi(f) = 0$ indicates that f converges faster than $1/\phi(h)$, i.e., $f(a) - L = o(1/\phi(h(a)))$.

The relationship is straightforward: $C_p(f) \le \|f\|_{\infty,h,p}$.

- $\|f\|_{\infty,h,p}$ tells us the worst-case, global bound on the scaled error.
- $C_p(f)$ tells us the tightest possible bound on the scaled error as we go arbitrarily far out to infinity.

Proposition 4.53 (Functoriality)**.** *If $\phi : (A, h_A) \to (B, h_B)$ is proper and $h_A \ge c\, h_B \circ \phi - C$, then $f \mapsto f \circ \phi$ is bounded $C_{h_B,p}(B) \to C_{h_A,p}(A)$.*

Lemma 4.54 (Tail integrability on $\mathbb{R}_+$)**.** *If $|f(x) - L| \le M(1+x)^{-p}$ with $p > 1$, then*

$$\int_R^\infty |f(x) - L|\, dx \le \frac{M}{p-1}(1+R)^{1-p}.$$

For example, two functions could have the same asymptotic rate constant $C_p(f)$, meaning they decay identically at infinity, but have vastly different norms $\|f\|_{\infty,h,p}$ due to their behavior on the finite parts of A. This distinction provides a more complete picture of the function's convergence.

4.27 Illustrative Examples: The Power of Comparison Scales

To showcase the power of the generalized framework, we analyze functions with different asymptotic behaviors. We consider $A = [0, \infty)$ with $h(x) = x$ and limit $L = 0$. Let us define three distinct scales of comparison:

- **The Algebraic Scale:** $\Phi_{\text{alg}} = \{\phi_p(s) = (1+s)^p \mid p > 0\}$
- **The Exponential Scale:** $\Phi_{\text{exp}} = \{\psi_c(s) = e^{cs} \mid c > 0\}$
- **The Log-Polynomial Scale:** $\Phi_{\text{log}} = \{\chi_p(s) = (\ln(s+2))^p \mid p > 0\}$

Example 1: Algebraic Decay, $f(x) = \frac{1}{(1+x)^2}$
This function is designed to have a clear algebraic decay.

- Using $\phi_2(s) = (1+s)^2 \in \Phi_{\text{alg}}$: $C_{\phi_2}(f) = \lim_{x\to\infty} \frac{1}{(1+x)^2} \cdot (1+x)^2 = 1$. The rate is exactly of order $h(x)^{-2}$.
- Using $\phi_3(s) = (1+s)^3 \in \Phi_{\text{alg}}$: $C_{\phi_3}(f) = \lim_{x\to\infty} \frac{1}{(1+x)^2} \cdot (1+x)^3 = \infty$. It does not converge as fast as $h(x)^{-3}$.
- Using any $\psi_c \in \Phi_{\text{exp}}$: $C_{\psi_c}(f) = \lim_{x\to\infty} \frac{1}{(1+x)^2} \cdot e^{cx} = \infty$. The function is not of exponential decay.

Example 2: Exponential Decay, $k(x) = e^{-x}$
This is the kind of function our original framework could not fully classify.

- Using any $\phi_p \in \Phi_{\text{alg}}$: $C_{\phi_p}(k) = \lim_{x\to\infty} e^{-x} \cdot (1+x)^p = 0$. This tells us $k(x) = o(h(x)^{-p})$ for all $p > 0$, confirming it decays faster than any algebraic rate. But how fast?
- Using the exponential scale Φ_{exp}: Let's test against $\psi_c(s) = e^{cs}$. The asymptotic constant is $C_{\psi_c}(k) = \lim_{x\to\infty} e^{-x} \cdot e^{cx} = \lim_{x\to\infty} e^{(c-1)x}$. This limit gives a sharp trichotomy:
$$C_{\psi_c}(k) = \begin{cases} 0 & \text{if } c < 1 \\ 1 & \text{if } c = 1 \\ \infty & \text{if } c > 1 \end{cases}$$
Conclusion: We have pinpointed the exact rate of convergence. The function decays precisely like $e^{-h(x)}$. The framework can now distinguish e^{-x} from, say, e^{-2x} (for which the constant would be finite for $c \leq 2$).

Example 3: Slow Logarithmic Decay, $g(x) = \frac{1}{\ln(x+2)}$
This function converges very slowly to zero.

- Using the algebraic scale Φ_{alg} with any $p > 0$: $C_{\phi_p}(g) = \lim_{x\to\infty} \frac{1}{\ln(x+2)} \cdot (1+x)^p = \infty$. The function converges slower than any power law.
- Using the log-polynomial scale Φ_{log} with $\chi_1(s) = \ln(s+2)$: $C_{\chi_1}(g) = \lim_{x\to\infty} \frac{1}{\ln(x+2)} \cdot \ln(x+2) = 1$. **Conclusion:** The framework successfully identified the precise, slow logarithmic rate of convergence, a task impossible with the previous fixed scale.

4.28 Theoretical Power: Linking the Global Norm and Asymptotic Rate

We have established two key metrics: the global norm, $\|f\|_{\infty,h,p}$, which measures the maximum weighted error over the entire space, and the asymptotic rate constant, $C_p(f)$, which measures the weighted error exclusively at infinity. The relationship $C_p(f) \le \|f\|_{\infty,h,p}$ is always true by definition.

A critical question for the theoretical power of this framework is: when does a finite asymptotic rate, $C_p(f) < \infty$, imply a finite global norm, $\|f\|_{\infty,h,p} < \infty$? Without further assumptions, this is not guaranteed. A function might have a zero asymptotic rate but possess arbitrarily large "spikes" over the finite part of the space, causing its global norm to be infinite.

However, for a significant class of functions—those whose convergence is "well-behaved"—we can establish a strong connection. The key is to impose a condition that prevents the scaled error from growing unexpectedly in the tail of the space.

Theorem 4.55 (Tail control implies finite global norm). *Let $g(a) := |f(a) - L|\,(1 + h(a))^p$ and $Q_p(R) := \sup_{\{h \ge R\}} g$. Assume f is bounded on some K_{R_0} and $(Q_p(R))_{R \ge R_0}$ is eventually nonincreasing with $\lim_{R\to\infty} Q_p(R) = C_p(f) < \infty$. Then*

$$\|f\|_{\infty,h,p} = \max\Big\{ \sup_{K_{R_0}} g,\ \sup_{R \ge R_0} Q_p(R) \Big\} < \infty.$$

Significance of the Theorem. This result is a crucial "regularity" theorem. It establishes that for functions whose scaled error does not exhibit pathological growth far from the origin, the purely asymptotic behavior (captured by $C_p(f)$) is sufficient to control the global behavior. This makes the asymptotic constant $C_p(f)$, which is often easier to compute via a limit, a reliable proxy for the finiteness of the norm for a vast and important class of functions encountered in analysis.

4.29 A weighted Schur test for sharp norms

Let (A, μ) be a Radon measure space and $W = \varphi \circ h \ge 1$. Consider the integral operator

$$(Tf)(x) \ := \ \int_A K(x,y)\, f(y)\, d\mu(y), \qquad K \ge 0.$$

Theorem 4.56 (Weighted Schur test). *Assume there exist $C_1, C_2 < \infty$ such that*

$$\sup_{x\in A} \int_A K(x,y)\, \frac{W(x)}{W(y)}\, d\mu(y) \ \le \ C_1, \qquad \sup_{y \in A} \int_A K(x,y)\, d\mu(x) \ \le \ C_2.$$

Then T defines a bounded map on $C^\sharp_{h;\varphi}(A)$ with

$$\|[Tf]\|^\sharp_{\infty,h;\varphi} \ \le \ C_1\, \|[f]\|^\sharp_{\infty,h;\varphi}.$$

If K is positivity preserving and $\int_A K(x,y)\, d\mu(x) = 1$ for all y, then $C^{(h)}_\varphi([Tf]) \le C_1\, C^{(h)}_\varphi([f])$.

Proof. For any $c \in \mathbb{R}$ and $x \in A$, $W(x)\,|(Tf)(x) - c| \le \int K(x,y)\, \frac{W(x)}{W(y)}\, W(y)\, |f(y) - c|\, d\mu(y)$. Take sup in x, then inf in c to get the bound by C_1. The tail estimate follows by the same inequality restricted to $\{h > R\}$ and letting $R \to \infty$. ∎

4.30 Conclusion of Section 4

In this section, we have successfully constructed a quantitative framework for analyzing convergence rates at infinity. By introducing a family of weighted norms $\|f\|_{\infty,h,p}$, we have moved beyond a simple binary description of convergence. We established that the finiteness of these norms is a sufficient condition for convergence (Proposition 4.2), providing a solid theoretical foundation. Furthermore, by introducing the asymptotic rate constant $C_p(f)$ and establishing conditions under which it governs the global norm, we have created a robust tool to distinguish between global behavior and the purely asymptotic decay rate. The examples demonstrate that these tools, used in tandem, can effectively classify functions and solve the ambiguities that arise from simpler measures. This provides a robust and nuanced language to answer the question: "How fast does a function converge?".

5 Duality & compactness

5.1 Riesz representation and full duality on the tail-tight subspace

Set $W(a) := \varphi(h(a)) \geq 1$.

Weighted zero-mass measures. Let $\mathcal{M}_0^{W^{-1}}(A)$ be the space of finite signed regular Borel measures ν on A such that

$$\nu(A) = 0 \qquad \text{and} \qquad \|\nu\|_{W^{-1}} := \int_A W(a)^{-1}\, d|\nu|(a) < \infty.$$

Proposition 5.1 (Canonical embedding). *For $\nu \in \mathcal{M}_0^{W^{-1}}(A)$, the functional*

$$\Lambda_\nu([f]) := \int_A f\, d\nu$$

is well-defined on $C(A)/\mathbb{R}$ and continuous on $\big(C^\sharp_{h;\varphi}(A), \|\cdot\|^\sharp_{\infty,h;\varphi}\big)$, with

$$\big|\Lambda_\nu([f])\big| \;\leq\; \|\nu\|_{W^{-1}}\; \|[f]\|^\sharp_{\infty,h;\varphi}.$$

Hence $\nu \mapsto \Lambda_\nu$ is an isometric embedding of $\big(\mathcal{M}_0^{W^{-1}}(A), \|\cdot\|_{W^{-1}}\big)$ into the dual space.

Proof. Independence of the representative uses $\nu(A) = 0$. For any $c \in \mathbb{R}$, $\big|\int (f - c)\, d\nu\big| \leq \int |f - c|\, d|\nu| \leq \|[f]\|^\sharp_{\infty,h;\varphi} \int W^{-1}\, d|\nu|$. □

Theorem 5.2 (Partial representation). *Let Λ be a bounded linear functional on $C^\sharp_{h;\varphi}(A)$ that is* tight on tails, *i.e.*

$$\lim_{R\to\infty} \sup_{\substack{[f] \\ \|[f]\|^\sharp_{\infty,h;\varphi} \leq 1}} \inf_{c\in\mathbb{R}} \sup_{\{h>R\}} \varphi(h)\, |f - c| \;=\; 0.$$

Then there exists $\nu \in \mathcal{M}_0^{W^{-1}}(A)$ such that $\Lambda = \Lambda_\nu$.

Sketch. By Hahn–Banach on compact sublevel sets $\{h \leq R\}$ one obtains representing measures of zero total mass; the tail-tightness and the bound $\int W^{-1}\, d|\nu| < \infty$ allow passage to the limit as $R \to \infty$, yielding $\nu \in \mathcal{M}_0^{W^{-1}}(A)$ with $\Lambda = \Lambda_\nu$. □

Decay subspace. Let

$$\mathcal{X} \;:=\; \big\{[f] \in C(A)/\mathbb{R} : \; C_\varphi^{(h)}([f]) = 0\big\}$$

equipped with $\|\cdot\|^\sharp_{\infty,h;\varphi}$.

Theorem 5.3 (Full duality on the tail-tight subspace)**.** *The map $\nu \mapsto \Lambda_\nu$ induces an isometric isomorphism*

$$\Big(\mathcal{M}_0^{W^{-1}}(A),\; \|\nu\|_{W^{-1}}\Big) \;\cong\; \big(\mathcal{X}\big)^*.$$

Consequently, for $[f] \in \mathcal{X}$,

$$\|[f]\|^\sharp_{\infty,h;\varphi} \;=\; \sup\Big\{ \Big|\int f\, d\nu\Big| : \; \nu \in \mathcal{M}_0^{W^{-1}}(A),\; \|\nu\|_{W^{-1}} \le 1\Big\}.$$

Sketch. Proposition 5.1 gives an isometric embedding. On $\mathcal{X}$, tails vanish by definition, and the patching principle (Lemma 4.23) together with the truncation-density result (Proposition 4.33) reduces the dual problem to the compact case, where zero-mass Radon measures represent all continuous linear functionals. A limiting argument with uniform W^{-1}-integrability yields the claimed identification. □

Corollary 5.4 (Optimality certificate at a sharp minimizer)**.** *Let $[f] \in \mathcal{X}$ and let c_* be a sharp minimizer. There exists $\nu_* \in \mathcal{M}_0^{W^{-1}}(A)$ with $\|\nu_*\|_{W^{-1}} = 1$ such that*

$$\int (f - c_*)\, d\nu_* \;=\; \|[f]\|^\sharp_{\infty,h;\varphi}, \qquad \operatorname{supp} \nu_* \;\subset\; \Big\{a : \; W(a)\,|f(a) - c_*| = \|[f]\|^\sharp_{\infty,h;\varphi}\Big\}.$$

One may choose $\nu_* = \nu_+ - \nu_-$ with $\nu_\pm \ge 0$ supported on the positive/negative contact sets.

5.2 Subdifferential and weak-* compactness in the weighted dual

Set $W := \varphi \circ h \ge 1$ and let

$$\mathcal{M}_0^{W^{-1}}(A) \;:=\; \Big\{\nu \text{ finite signed Radon on } A : \; \nu(A) = 0, \;\; \|\nu\|_{W^{-1}} := \int_A W^{-1}\, d|\nu| < \infty\Big\}.$$

Definition 5.5 (Contact set at a sharp minimizer)**.** For $[f] \in C(A)/\mathbb{R}$ and a sharp minimizer c_*, define the contact set

$$\mathcal{C}_* := \Big\{a \in A : \; W(a)\,|f(a) - c_*| = \|[f]\|^\sharp_{\infty,h;\varphi}\Big\},$$

and its signed parts $\mathcal{C}_*^\pm := \{a \in \mathcal{C}_* : \; \pm(f(a) - c_*) \ge 0\}$.

Theorem 5.6 (Subdifferential description)**.** *The subdifferential of $\|\cdot\|^\sharp_{\infty,h;\varphi}$ at $[f]$ is the weak-* compact convex set*

$$\partial\|[f]\|^\sharp = \Big\{\; \nu \in \mathcal{M}_0^{W^{-1}}(A) : \; \|\nu\|_{W^{-1}} \le 1, \; \operatorname{supp} \nu \subset \mathcal{C}_*, \; \nu(\mathcal{C}_*^+) = -\nu(\mathcal{C}_*^-) = \tfrac{1}{2} \;\Big\}.$$

Consequently, for any direction $[g] \in C(A)/\mathbb{R}$ the one-sided Gâteaux derivatives exist and satisfy

$$D^\pm\|[f]\|^\sharp([g]) \;=\; \max_{\nu \in \partial\|[f]\|^\sharp} \; \pm\int g\, d\nu.$$

If $\partial\|[f]\|^\sharp = \{\nu_*\}$ is a singleton, then the norm is Gâteaux differentiable at $[f]$ and $D\|[f]\|^\sharp([g]) = \int g\, d\nu_*$.

Sketch. The dual pairing $\langle f, \nu\rangle = \int f\, d\nu$ with $\|\nu\|_{W^{-1}} \le 1$ describes the polar of the unit ball, cf. Proposition 5.1. Extremal measures are supported on the contact set and split with equal total variation on the positive/negative branches to annihilate constants (use existence of sharp minimizers and two-sided contact, cf. Proposition 4.1 and Lemma 4.2). The formulae for directional derivatives follow from convex analysis via the subdifferential. ☐

Proposition 5.7 (Weak-* compactness in the weighted dual)**.** *The closed unit ball*

$$\mathbb{B} := \left\{\nu \in \mathcal{M}_0^{W^{-1}}(A) : \ \|\nu\|_{W^{-1}} \le 1, \ \nu(A) = 0\right\}$$

is sequentially weak- compact with respect to the topology induced by* $C_0(A)$*: every sequence in* $\mathbb{B}$ *admits a subsequence converging against* $C_0(A)$*.*

Sketch. The integrability of W^{-1} controls mass on the tails: for each $\varepsilon > 0$ choose R with $\int_{\{h>R\}} W^{-1}\, d|\nu| \le \varepsilon$ uniformly over $\nu \in \mathbb{B}$. On the compact $\{h \le R\}$, Prokhorov/Banach–Alaoglu gives weak-* compactness of restrictions. A diagonal argument in R and the tail control yield the claim. ☐

Corollary 5.8 (Existence of dual extremals)**.** *Let* $[f]$ *belong to the tail-tight subspace* $\{[u] : C_\varphi^{(h)}([u]) = 0\}$*. Then*

$$\|[f]\|^\sharp_{\infty,h;\varphi} = \max\left\{ \left|\int f\, d\nu\right| : \ \nu \in \mathcal{M}_0^{W^{-1}}(A), \ \|\nu\|_{W^{-1}} \le 1\right\}.$$

Proof. By Theorem 5.3 (Section 5.1) the supremum is over the weak-* compact unit ball of $\mathcal{M}_0^{W^{-1}}(A)$; Proposition 5.7 ensures existence of a maximizing measure. ☐

5.3 Compactness: a weighted Arzelà–Ascoli theorem and compact embeddings

Fix a proper exhaustion h and $\varphi \in \Phi_{\mathrm{adm}}$. Recall the tail functional $T^{(h)}_{R;\varphi}$ from Lemma 4.23.

Definition 5.9 (Uniform local equicontinuity)**.** A family $\mathcal{F} \subset C(A)/\mathbb{R}$ is *uniformly locally equicontinuous* if, for every compact $K \subset A$ and every $\varepsilon > 0$, there exists a neighborhood base $\mathcal{U}$ on K such that for all $[f] \in \mathcal{F}$ and all $x, y \in K$ contained in a common $U \in \mathcal{U}$,

$$|f(x) - f(y)| < \varepsilon$$

for some representative f of $[f]$.

Theorem 5.10 (Weighted Arzelà–Ascoli)**.** *Let* $\mathcal{F} \subset C(A)/\mathbb{R}$ *satisfy:*

1. ***Uniform sharp boundedness:*** $\sup_{[f]\in\mathcal{F}} \|[f]\|^\sharp_{\infty,h;\varphi} < \infty$*;*
2. ***Uniform local equicontinuity*** *on every compact (Def. 5.9);*
3. ***Tight tails:*** $\lim_{R\to\infty} \sup_{[f]\in\mathcal{F}} T^{(h)}_{R;\varphi}([f]) = 0.$

Then $\mathcal{F}$ *is relatively compact in* $\left(C(A)/\mathbb{R}, \ \|\cdot\|^\sharp_{\infty,h;\varphi}\right)$.

Sketch. Given $\varepsilon > 0$, pick R such that $\sup_{[f]\in\mathcal{F}} T^{(h)}_{R;\varphi}([f]) < \varepsilon$. On the compact set $\{h \le R\}$, (1)+(2) yield precompactness in the uniform topology modulo constants (classical Arzelà–Ascoli). Using Lemma 4.23, the tail contribution is $< \varepsilon$ uniformly in $[f] \in \mathcal{F}$, hence one extracts a subsequence that is Cauchy in the sharp norm. ☐

Theorem 5.11 (Compact inclusion between weighted sharp spaces). *Let $\varphi_2, \varphi_1 \in \Phi_{\mathrm{adm}}$ with $\varphi_1 \preceq \varphi_2$ and*

$$\lim_{t\to\infty} \frac{\varphi_1(t)}{\varphi_2(t)} = 0.$$

Then the identity map

$$\iota: \ \big(C^\sharp_{h;\varphi_2}(A),\ \|\cdot\|^\sharp_{\infty,h;\varphi_2}\big) \ \longrightarrow\ \big(C^\sharp_{h;\varphi_1}(A),\ \|\cdot\|^\sharp_{\infty,h;\varphi_1}\big)$$

is compact on every subset that is bounded in the φ_2-sharp norm and uniformly locally equicontinuous on compacts.

Sketch. Let $\mathcal{F}$ be bounded in $\|\cdot\|^\sharp_{\infty,h;\varphi_2}$ and uniformly locally equicontinuous. For $\varepsilon > 0$, choose R so that $\sup_{[f]\in\mathcal{F}} \ \sup_{\{h>R\}} \frac{\varphi_1(h)}{\varphi_2(h)} < \varepsilon$. Then for each $[f] \in \mathcal{F}$,

$$\|[f]\|^\sharp_{\infty,h;\varphi_1} \ \le\ \max\Big\{ \sup_{\{h\le R\}} \varphi_1(h)\ \mathrm{osc}([f])\ ,\ \varepsilon \sup_{\{h>R\}} \varphi_2(h)\ \mathrm{osc}([f]) \ \Big\}.$$

By Theorem 5.10, $\mathcal{F}$ is relatively compact on the core $\{h \le R\}$; the tail term is uniformly $\le \varepsilon$ times the φ_2-sharp bound. Hence any sequence in $\mathcal{F}$ admits a subsequence converging in the φ_1-sharp norm. □

Corollary 5.12 (Polynomial case). *If $\varphi_2(t) = (1+t)^{p_2}$ and $\varphi_1(t) = (1+t)^{p_1}$ with $p_2 > p_1 \ge 0$, then the identity $C^\sharp_{h;p_2}(A) \to C^\sharp_{h;p_1}(A)$ is compact on sets that are bounded in $\|\cdot\|^\sharp_{\infty,h;\varphi_2}$ and uniformly locally equicontinuous on compacts.*

6 Measured extensions and L^q embeddings

Let $(A, \mathcal{B}, \mu)$ be a Radon measure space on A and h a continuous proper exhaustion.

Definition 6.1 (Weighted L^q spaces). For $\varphi \in \Phi_{\mathrm{adm}}$ and $q \in [1,\infty)$, define

$$\|f\|_{L^q(h;\varphi;L)} := \Big(\int_A \big(\varphi(h(a))\,|f(a)-L|\big)^q \, d\mu(a)\Big)^{1/q}.$$

Write $L^q_{h;\varphi}(A)$ for the set of f with finite norm, and analogously on the quotient by constants using the infimum over $L \in \mathbb{R}$.

Definition 6.2 (Volume growth index). Set $V(R) := \mu(\{h \le R\})$. We say that μ has polynomial volume growth of order $\gamma \ge 0$ if $V(R) \le C(1+R)^\gamma$ for all $R \ge 0$.

Theorem 6.3 (Sup-to-L^q embedding under volume growth). *Assume $V(R) \le C(1+R)^\gamma$ and $\varphi(t) = (1+t)^p$ with $p > 0$. If $pq > \gamma$, then for every $L \in \mathbb{R}$ there exists $C_{p,q,\gamma}$ such that*

$$\|f\|_{L^q(h;\varphi;L)} \ \le\ C_{p,q,\gamma}\ \|f\|_{\infty,h;\varphi;L} \qquad \text{for all } f.$$

The same holds on the quotient: $\|[f]\|_{L^q(h;\varphi)} \le C\,\|[f]\|^\sharp_{\infty,h;\varphi}$.

Proof. Decompose into layers $A_R := \{R < h \le R+1\}$ and use $\int_{A_R}(1+h)^{pq}|f-L|^q \le \|f\|^q_{\infty,h;\varphi;L}\,\mu(A_R)\,(1+R)^{pq}$. Sum over $R \in \mathbb{N}$ and apply $\mu(A_R) \le V(R+1) - V(R) \lesssim (1+R)^{\gamma-1}$. Converges iff $pq > \gamma$. □

Remark 6.4. For general $\varphi \in \Phi_{\mathrm{adm}}$, replace $(1+R)^{pq}$ by $\varphi(R)^q$ and assume a summability $\sum_R \varphi(R)^q\big(V(R+1)-V(R)\big) < \infty$.

7 Functoriality under coarse-affine maps

Definition 7.1 (Category of exhaustions)**.** Let Exh be the category whose objects are pairs (A, h) with A a Hausdorff, locally compact, σ-compact space and $h : A \to [0, \infty)$ a continuous proper exhaustion.

A morphism $\phi : (A, h_A) \to (B, h_B)$ is a *proper* map $\phi : A \to B$ such that there exist constants $A_0 \geq 1$ and $B_0 \in \mathbb{R}$ with

$$h_A(x) \;\leq\; A_0\, h_B(\phi(x)) + B_0 \qquad \forall x \in A. \tag{$\star$}$$

Composition is the usual composition of maps. Identities are the identity maps.

Remark 7.2 (On the direction of control). Condition $(\star)$ is a *forward* coarse-affine control ensuring that "going forward" through ϕ does not make h grow too fast. If in addition there are $a_0 \geq 1$, $b_0 \in \mathbb{R}$ with $h_B(\phi(x)) \leq a_0\, h_A(x) + b_0$, then ϕ is a coarse bi-affine equivalence and induces norm equivalences (Cor. 7.5).

Definition 7.3 (Pullback)**.** Given $\phi : (A, h_A) \to (B, h_B)$ in Exh and $f \in C(B)$, define the pullback $\phi^* f := f \circ \phi \in C(A)$. This descends to classes $[f] \in C(B)/\mathbb{R}$ via $\phi^*[f] := [f \circ \phi] \in C(A)/\mathbb{R}$.

Theorem 7.4 (Bounded pullback for admissible weights)**.** *Let $\phi : (A, h_A) \to (B, h_B)$ be a morphism in* Exh *satisfying $(\star)$. Let $\varphi \in \Phi_{\mathrm{adm}}$ (Def. 4.2) and assume φ has power-dilation control (Def. 4.34). Then there exist constants $C \geq 1$ and $\kappa \geq 1$, depending only on A_0, B_0 and φ, such that for all $[f] \in C(B)/\mathbb{R}$,*

$$\|\phi^*[f]\|^{\sharp}_{\infty,h_A;\varphi} \;\leq\; C\; \max\Big\{ \sup_{\{h_B \leq R\}} \mathrm{osc}_{[f]},\; \big(\|[f]\|^{\sharp}_{\infty,h_B;\varphi}\big)^{\kappa} \Big\}$$

for some finite R depending only on A_0, B_0. In particular, if one cares only about the asymptotic class*, then*

$$C^{(h_A)}_{\varphi}(\phi^*[f]) \;\leq\; C'\; \big(C^{(h_B)}_{\varphi}([f])\big)^{\kappa}.$$

Proof sketch. From $(\star)$ we have $\varphi(h_A) \leq \varphi(A_0 h_B \circ \phi + B_0) \leq K\, \varphi(B_0)\, \varphi(A_0 h_B \circ \phi)$; power-dilation gives $\varphi(A_0 t) \leq C(A_0)\, \varphi(t)^{\kappa(A_0)}$. Evaluate the sharp norm with a near-optimal constant for $[f]$ on B, split into $\{h_B \leq R\}$ and $\{h_B > R\}$ as in Lemma 4.23, and absorb constants. ☐

Corollary 7.5 (Polynomial weights)**.** *Let $\varphi(t) = (1 + t)^p$ with $p > 0$ and ϕ as above. Then there exists $C = C(A_0, B_0, p) \geq 1$ such that for all $[f] \in C(B)/\mathbb{R}$,*

$$\|\phi^*[f]\|^{\sharp}_{\infty,h_A;\varphi} \;\leq\; C\; \Big(\sup_{\{h_B \leq R\}} \mathrm{osc}_{[f]} \;+\; \|[f]\|^{\sharp}_{\infty,h_B;\varphi} \Big),$$

and $C^{(h_A)}_{\varphi}(\phi^*[f]) \leq C\, C^{(h_B)}_{\varphi}([f])$. If, moreover, there is a reverse affine bound $h_B \circ \phi \leq a_0 h_A + b_0$, then ϕ^* is an isomorphism between the corresponding decay classes and the global norms are equivalent up to constants.

Proposition 7.6 (Contravariant functor)**.** *For any fixed admissible φ, the assignment*

$$(A, h) \;\longmapsto\; \big(C^{\sharp}_{h;\varphi}(A),\; \|\cdot\|^{\sharp}_{\infty,h;\varphi}\big), \qquad \phi \;\longmapsto\; \phi^*$$

defines a contravariant functor from Exh *to the category of normed spaces and bounded linear maps. If $\varphi(t) = (1 + t)^p$, the target can be taken as Banach spaces by Theorem 4.50.*

8 Anisotropic infinity: multi-exhaustions and ends

Definition 8.1 (Ends via complements of compacts). Let A be Hausdorff, locally compact, σ-compact. Fix a compact exhaustion (K_n) with $K_n \subset \mathrm{int}(K_{n+1})$ and $\bigcup_n K_n = A$. An *end* of A is a choice of a connected component E_n of $A \setminus K_n$ for each n, with $E_{n+1} \subset E_n$. Two such choices are identified if they agree for all large n. The (possibly infinite) set of ends is denoted $\mathsf{Ends}(A)$.

Definition 8.2 (Multi-exhaustion). Choose a finite index set $I = \{1, \dots, m\}$ of ends, open pairwise disjoint neighborhoods $(U_i)_{i \in I}$ with U_i eventually containing the corresponding end, and a compact *central core* $K_0 \subset A$ with $A = K_0 \cup \bigcup_{i=1}^m U_i$ and $K_0 \cap U_i = \partial U_i$. A *multi-exhaustion* is a family of continuous proper functions $h_i : U_i \to [0, \infty)$, one on each U_i. We extend them to A by setting $h_i \equiv 0$ on $A \setminus U_i$.

Remark 8.3. Typical examples: (a) $A = \mathbb{R} \times M$ (two ends $\pm\infty$), take $U_\pm = \{\pm x > 1\}$, $h_\pm(x, y) = |x|$; (b) $A = \mathbb{R} \setminus \{0\}$ (four ends at $\pm\infty$ and $0^\pm$), take $U_{0^\pm} = \{0 < \pm x < 1\}$ with $h_{0^\pm}(x) = \log(1/|x|)$, and $U_{\pm\infty} = \{\pm x > 1\}$ with $h_{\pm\infty}(x) = |x|$.

Definition 8.4 (Anisotropic weights). For each end $i \in I$, fix $\varphi_i \in \Phi_{\mathrm{adm}}$. Define the *block weight* on A by

$$W(x) \;:=\; \max\Big\{1,\; \max_{i \in I}\; \varphi_i\big(h_i(x)\big)\Big\}.$$

Definition 8.5 (Anisotropic sharp norm and asymptotic constants). For $[f] \in C(A)/\mathbb{R}$ define

$$\|[f]\|^\sharp_{\infty;\mathbf{h},\boldsymbol{\varphi}} := \inf_{c \in \mathbb{R}} \sup_{x \in A} W(x)\,|f(x) - c|, \qquad C^{(\mathbf{h})}_{\boldsymbol{\varphi}}([f]) := \inf_{c \in \mathbb{R}} \max_{i \in I} \limsup_{\substack{x \to \text{end } i \\ x \in U_i}} \varphi_i\big(h_i(x)\big)\,|f(x) - c|.$$

When all $\varphi_i(t) = (1 + t)^{p_i}$, we simply write the exponents $\mathbf{p} = (p_i)_{i \in I}$.

Lemma 8.6 (Gluing: core + per-end tails). *Fix a compact core K_0 as in Def. 8.2. Then for all $[f] \in C(A)/\mathbb{R}$,*

$$\|[f]\|^\sharp_{\infty;\mathbf{h},\boldsymbol{\varphi}} \;\simeq\; \max\Big\{\; \inf_c \sup_{x \in K_0} |f(x) - c| \;,\; \max_{i \in I} \inf_c \sup_{x \in U_i} \varphi_i\big(h_i(x)\big)\,|f(x) - c| \;\Big\},$$

with equivalence constants depending only on the choice of (U_i) and a finite overlap buffer around ∂U_i. Moreover,

$$C^{(\mathbf{h})}_{\boldsymbol{\varphi}}([f]) \;=\; \max_{i \in I}\; C^{(h_i)}_{\varphi_i}([f|_{U_i}]).$$

Proof sketch. Pick a partition of unity $\{\eta_0, \eta_1, \dots, \eta_m\}$ with η_0 supported in K_0 and η_i supported in U_i. Evaluate the sharp norm with a near-optimal constant c and split into the core and the ends. Conversely, take near-optimal constants per region and use that max of weights controls the global weight. The asymptotic identity follows by restricting to U_i and taking limsups along the corresponding end. □

8.1 Exact sequence for endwise limits

Let $(U_i, h_i)_{i=1}^m$ be a multi-exhaustion and write $L_i(f)$ for the limit of f at end i when it exists.

Definition 8.7 (Vanishing tail class). $\mathrm{X}_{\mathrm{van}} := \{[f] \in C(A)/\mathbb{R} :\; C^{(h_i)}_{\varphi_i}([f|_{U_i}]) = 0 \text{ for each } i\}$.

Theorem 8.8 (Exact sequence). *Define $\mathcal{Q} := \{(x_1, \ldots, x_m) \in \mathbb{R}^m : \sum_{i=1}^m x_i = 0\}$ with the norm $\|(x_i)\| := \max_i |x_i|$. There is a short exact sequence of normed spaces*

$$0 \longrightarrow \mathcal{X}_{\mathrm{van}} \xrightarrow{\iota} C^\sharp_{\mathbf{h};\boldsymbol{\varphi}}(A) \xrightarrow{\Pi} \mathcal{Q} \longrightarrow 0,$$

where $\Pi([f]) = (L_1(f) - \bar{L}, \ldots, L_m(f) - \bar{L})$ with $\bar{L} := \frac{1}{m}\sum_i L_i(f)$, and ι is the inclusion. The map Π is well-defined, linear, continuous, and surjective; its kernel is exactly $\mathcal{X}_{\mathrm{van}}$. Moreover, there exists a continuous linear right inverse $S : \mathcal{Q} \to C^\sharp_{\mathbf{h};\boldsymbol{\varphi}}(A)$ constructed via a partition of unity supported in the U_i.

Idea. Well-definedness uses the anisotropic gluing (core + queues) and the fact that constants annihilate $\mathcal{Q}$. Kernel= $\mathcal{X}_{\mathrm{van}}$ by definition of the tail constants. Surjectivity: pick a bump η_i supported in U_i with $\sum_i \eta_i \equiv 1$ on $A \setminus K_0$ and set $S((x_i)) := \sum_i x_i\eta_i$; the sharp norm is controlled by the local patch on K_0 and the bounded overlaps near ∂U_i. □

8.2 Product spaces

Let (A, h_A) and (B, h_B) be spaces with proper exhaustions.

Proposition 8.9 (Product exhaustions and equivalence). *On $A \times B$, the functions*

$$h_{\max}(a, b) := \max\{h_A(a), h_B(b)\}, \qquad h_+(a, b) := h_A(a) + h_B(b)$$

are continuous proper exhaustions and are coarsely affine equivalent: $h_{\max} \le h_+ \le 2\, h_{\max}$. Consequently, for any admissible φ with power-dilation control, the corresponding sharp spaces for $h_{\max}$ and h_+ are isomorphic (Cor. 4.45 for polynomial weights).

Corollary 8.10 (Tensor-type bounds for decoupled functions). *Let $f(a, b) = f_A(a) + f_B(b)$ with $f_A \in C(A)$, $f_B \in C(B)$, and take $\varphi(t) = (1 + t)^p$. Then*

$$\|[f]\|^\sharp_{\infty,h_{\max};\varphi} \;\le\; C_p\Big(\|[f_A]\|^\sharp_{\infty,h_A;\varphi} \;+\; \|[f_B]\|^\sharp_{\infty,h_B;\varphi} \Big).$$

Sketch. Use $h_{\max} \simeq h_+$ and $(1 + h_A + h_B)^p \lesssim (1 + h_A)^p + (1 + h_B)^p$, then the triangle inequality for the sharp norm. □

Proposition 8.11 (Endwise coarse invariance). *Suppose for each $i \in I$ we have another exhaustion h'_i on U_i with $a_i h_i - b_i \le h'_i \le A_i h_i + B_i$. If each φ_i has power-dilation control (Def. 4.34), then*

$$C^{(\mathbf{h})}_{\boldsymbol{\varphi}}([f]) = 0 \quad\Longleftrightarrow\quad C^{(\mathbf{h}')}_{\boldsymbol{\varphi}}([f]) = 0,$$

and the corresponding sharp norms are equivalent up to multiplicative constants depending only on $\{a_i, A_i, b_i, B_i, \varphi_i\}$ and the finite overlap near the core.

Proposition 8.12 (Anisotropic pullbacks). *Let $\phi : (A, \{U_i, h_i\}) \to (B, \{V_j, k_j\})$ be a proper map such that for each i there exists $j = \tau(i)$ with $\phi(U_i) \subset V_j$ and $h_i \le A_i\, k_{\tau(i)} \circ \phi + B_i$. Let $\varphi_j \in \Phi_{\mathrm{adm}}$ have power-dilation control and set $\varphi_i := \varphi_{\tau(i)}$. Then the pullback $\phi^* : C^\sharp_{\mathbf{k};\boldsymbol{\varphi}}(B) \to C^\sharp_{\mathbf{h};\boldsymbol{\varphi}}(A)$ is bounded, and*

$$C^{(\mathbf{h})}_{\boldsymbol{\varphi}}(\phi^*[f]) \;\le\; C\, \max_{i\in I}\Big(C^{(k_{\tau(i)})}_{\varphi_{\tau(i)}}([f]) \Big)^{\kappa}$$

for constants C, κ depending only on $\{A_i, B_i\}$ and the φ_j.

Example 1: Cylinder $\mathbb{R} \times M$. Take $U_\pm = \{\pm x > 1\}$, $h_\pm = |x|$, and $\varphi_\pm(t) = (1+t)^{p_\pm}$. Then

$$C_{\boldsymbol{\varphi}}^{(\mathbf{h})}([f]) = \max\left\{ \limsup_{x\to+\infty} (1+|x|)^{p_+} \operatorname{osc}_{[f]}(x,\cdot),\ \limsup_{x\to-\infty} (1+|x|)^{p_-} \operatorname{osc}_{[f]}(x,\cdot) \right\}.$$

Example 2: $\mathbb{R} \setminus \{0\}$ with four ends. Let $U_{\pm\infty} = \{\pm x > 1\}$, $h_{\pm\infty} = |x|$; and $U_{0^\pm} = \{0 < \pm x < 1\}$ with $h_{0^\pm}(x) = \log(1/|x|)$. Choosing $(p_{\pm\infty}, p_{0^\pm})$, anisotropic decay couples polynomial rates at infinity with logarithmic rates near $0^\pm$.

8.3 Infinitely many ends

Let $\mathsf{Ends}(A)$ be the (Freudenthal) end space of A (a compact, totally disconnected space in many cases). Assume a family $\{(U_e, h_e)\}_{e\in E}$ indexed by a (possibly infinite) closed subset $E \subset \mathsf{Ends}(A)$ such that: (i) $\{U_e\}_{e\in E}$ is a locally finite cover of $A \setminus K_0$, pairwise disjoint; (ii) each $h_e : U_e \to [0,\infty)$ is a continuous proper exhaustion on its channel; (iii) $\{\varphi_e\}_{e\in E} \subset \Phi_{\mathrm{adm}}$ with uniform admissibility constants on compact subsets of E.

Define $W(x) := \max\{1,\ \sup_{e\in E}\ \varphi_e(h_e(x))\}$ (finite by local finiteness) and

$$\|[f]\|^\sharp_{\infty;\mathbf{h},\boldsymbol{\varphi}} := \inf_{c\in\mathbb{R}} \sup_{x\in A} W(x)\,|f(x) - c|.$$

Theorem 8.13 (Well-posedness and endwise gluing)**.** *Under the above hypotheses, $\|\cdot\|^\sharp_{\infty;\mathbf{h},\boldsymbol{\varphi}}$ is a norm on $C(A)/\mathbb{R}$ and the gluing Lemma 7.6 extends verbatim with $i \in I$ replaced by $e \in E$ and "$\max_{i\in I}$" replaced by "$\sup_{e\in E}$". Moreover, if all per-end limits $L_e(f)$ exist, the map*

$$\Pi_\infty :\ [f] \longmapsto \big(L_e(f) - \bar{L}\big)_{e\in E} \in C(E)/\mathbb{R}$$

(with $\bar{L}$ any barycenter) is well-defined, linear and continuous, with kernel equal to the vanishing-tail class $\{[f] : \sup_e C^{(h_e)}_{\varphi_e}([f|_{U_e}]) = 0\}$.

8.4 A bounded projection onto the vanishing-tail class

Let Π and S be as in Theorem 8.8. Define

$$\mathsf{P} \ := \ \mathrm{Id} \ - \ S \circ \Pi \ : \ C^\sharp_{\mathbf{h};\boldsymbol{\varphi}}(A) \ \longrightarrow \ C^\sharp_{\mathbf{h};\boldsymbol{\varphi}}(A).$$

Proposition 8.14 (Bounded projection)**.** *The operator P is a bounded linear projection with*

$$\operatorname{Ran}\mathsf{P} = \mathcal{X}_{\mathrm{van}} \quad \textit{and} \quad \ker\mathsf{P} = \operatorname{Ran} S \cong \mathcal{Q}.$$

In particular,

$$C^\sharp_{\mathbf{h};\boldsymbol{\varphi}}(A) \ \cong \ \mathcal{X}_{\mathrm{van}} \ \oplus \ \mathcal{Q}$$

as a topological direct sum.

Proof. Since $\Pi \circ S = \mathrm{Id}_{\mathcal{Q}}$, $(S\circ\Pi)^2 = S \circ \Pi$ and hence $\mathsf{P}^2 = \mathsf{P}$. Boundedness follows from the construction of S via a partition of unity supported in the end neighborhoods and the uniform control near the interfaces with the compact core. $\square$

9 An Extension for Anisotropic Spaces: The Multi-Exhaustion Framework

The framework developed thus far excels for spaces whose infinity can be modeled by a single point. We now address the challenge of spaces with multiple distinct "ends" or anisotropic geometries. The solution is to generalize the single exhaustion function h to a collection of functions $\{h_i\}$, each characterizing a specific asymptotic direction.

Assumption 9.1 (Decomposition at Infinity). *Let A be a space. We assume there exists a compact subset $K_0 \subset A$ such that its complement can be written as a finite, disjoint union of open, non-compact sets U_i:*

$$A \setminus K_0 = \bigsqcup_{i=1}^{n} U_i$$

*Each set U_i is called an **end** or an **asymptotic channel** of the space A.*

Remark 9.2 (On the Existence and Structure of Ends). The assumption of a decomposition into a *finite, disjoint union* of ends is a powerful simplification that covers many important cases (e.g., cylinders, cones). It is crucial, however, to recognize its limitations. The topological theory of ends reveals far richer asymptotic structures, including spaces with infinitely many ends, or spaces whose set of ends forms a complex topological space, such as a Cantor set.

A future extension of this work could involve indexing the family of exhaustion functions $\{h_i\}_{i\in I}$ not by a finite set, but by a topological space I modeling the boundary at infinity of A, such as the Martin boundary or the Thurston compactification. Our current framework should thus be seen as the foundation for the analysis of the most 'regular' asymptotic geometries.

Furthermore, each local exhaustion function h_i is not chosen arbitrarily. It should be constructed systematically from a **local exhaustion** of the channel U_i by a nested family of compact sets $\{K_{r,i}\}_{r\geq 0}$, in direct analogy with the construction of the global function h in Section 2. That is, $h_i(a) := \inf\{r \geq 0 \mid a \in K_{r,i}\}$. This ensures that our directional measures of convergence are just as fundamentally grounded in the space's structure as our original global measure. A further refinement for analytical applications would involve ensuring the family of functions $\{h_i\}$ can be chosen to be smooth across the boundaries of their respective domains.

Definition 9.3 (Multi-Exhaustion Functions). For each end U_i, we define a local **exhaustion function** $h_i : U_i \to [0, \infty)$ such that for any $R > 0$, the sublevel set $\{a \in U_i \mid h_i(a) \leq R\}$ is a compact subset of A. We can extend each h_i to all of A by setting it to 0 (or $-\infty$) outside of $U_i \cup K_0$.

Definition 9.4 (Convergence to a Specific End). We denote the point at infinity associated with the end U_i as ω_i. We say a sequence (a_k) converges to ω_i, written $a_k \to \omega_i$, if for any $R > 0$, the sequence is eventually in U_i and $h_i(a_k) > R$. This is equivalent to $\lim_{k\to\infty} h_i(a_k) = \infty$.

Definition 9.5 (Directional Limits and Norms). Let $f : A \to \mathbb{R}$ be a function.

1. The **limit of f at the end** ω_i is the value $L_i \in \mathbb{R}$ such that $\lim_{a\to\omega_i} f(a) = L_i$.

2. The **convergence rate of f towards** ω_i is measured by the **directional weighted norm**:

$$||f||^{(i)}_{\infty,h_i,\phi} := \sup_{a\in U_i} \left(|f(a) - L_i| \cdot \phi(h_i(a))\right)$$

Definition 9.6 (Global Norm on Anisotropic Spaces)**.** To provide a single measure for the overall behavior of a function f on A, we define the **global multi-exhaustion norm** as the maximum of its behavior on the central part and on each end:

$$||f||_{\text{global}} := \max\left(\sup_{a\in K_0}|f(a)|, ||f||^{(1)}_{\infty,h_1,\phi}, \dots, ||f||^{(n)}_{\infty,h_n,\phi}\right)$$

where it is assumed that f converges to a limit L_i at each end ω_i (if an end has no limit, the corresponding directional norm is infinite). A function has a finite global norm only if it is bounded on the central part and converges at a specified rate at **all** ends of the space.

9.1 Illustrative Examples of the Multi-Exhaustion Framework

To demonstrate the power and necessity of this extended framework, we analyze two spaces whose structure at infinity is not reducible to a single point.

9.1.1 Example 1: Convergence on an Infinite Strip

Let $A = \{z = x + iy \in \mathbb{C} \mid 0 < y < 1\}$ be an infinite strip. This space has two intuitive ends: one to the right ($x \to +\infty$, denoted ω_R) and one to the left ($x \to -\infty$, denoted ω_L).

- **Decomposition:** Let $K_0 = \{z \in A \mid -1 \le x \le 1\}$. We define the two ends as $U_R = \{z \in A \mid x > 1\}$ and $U_L = \{z \in A \mid x < -1\}$.
- **Exhaustion Functions:** The natural exhaustion functions are $h_R(z) := x$ for $z \in U_R$, and $h_L(z) := -x$ for $z \in U_L$.
- **Function Analysis:** Consider the function $f(z) = e^{-z} = e^{-x}e^{-iy}$.
  - **At the right end (ω_R):** As $a \to \omega_R$, we have $h_R(a) = x \to \infty$. The limit is $L_R = \lim_{x\to+\infty} e^{-x}e^{-iy} = 0$. The convergence is exponential, meaning the norm $||f||^{(R)}_{\infty,h_R,\psi_c}$ is finite for any $c \le 1$ in the exponential scale $\psi_c(s) = e^{cs}$.
  - **At the left end (ω_L):** As $a \to \omega_L$, we have $h_L(a) = -x \to \infty$. The term $|f(z)| = e^{-x}$ explodes. Therefore, the function does not converge at the left end.

The multi-exhaustion framework allows us to formally distinguish the behavior at each end, a task impossible with the original isotropic framework.

9.1.2 Example 2: A Space with Four Ends, $\mathbb{R} \setminus \{0\}$

Consider the space $A = \mathbb{R} \setminus \{0\}$. This space has four ends: $\omega_{+\infty}$, $\omega_{-\infty}$, ω_{0+}, and ω_{0-}. Let $K_0 = [-2, -1/2] \cup [1/2, 2]$. This defines four channels: $U_{+\infty} = (2, \infty)$, $U_{-\infty} = (-\infty, -2)$, $U_{0+} = (0, 1/2)$, and $U_{0-} = (-1/2, 0)$.

The natural exhaustion functions are: $h_{+\infty}(x) = x$, $h_{-\infty}(x) = -x$, $h_{0+}(x) = 1/x$, and $h_{0-}(x) = -1/x$.

Now, consider the function $f(x) = \frac{\sin(x)}{x^2} + e^x$.

- **At $\omega_{+\infty}$ ($h_{+\infty} \to \infty$):** $f(x)$ explodes due to e^x. The limit is infinite.
- **At $\omega_{-\infty}$ ($h_{-\infty} \to \infty$):** $f(x) \to 0$. The convergence rate is precisely of order $h_{-\infty}(x)^{-2}$ (i.e., $O((-x)^{-2})$) because e^x vanishes.
- **At ω_{0+} ($h_{0+} \to \infty$):** $f(x) \approx \frac{x}{x^2} = \frac{1}{x}$. The limit is infinite.

- **At ω_{0-} ($h_{0-} \to \infty$):** $f(x) \approx \frac{1}{x}$. The limit is infinite.

Our framework allows a precise characterization: the function converges towards the end $\omega_{-\infty}$ only, with an algebraic rate of $h_{-\infty}^{-2}$, and diverges at the three other ends.

10 Conclusion

In this work, we have developed a unified framework for analyzing convergence at infinity. Beginning with the simple and general notion of an **exhaustion** of a space, we constructed an **exhaustion function** h which serves as a generalized measure of distance to an adjoined **point at infinity,** ω_A. This foundation allowed us to define convergence for both points and functions in a way that is consistent across metric, topological, and other contexts.

The core of our contribution is the introduction of a **family of weighted norms,** $\|f\|_{\infty,h,p}$, and the complementary **asymptotic rate constant,** $C_p(f)$. We have shown that these tools move beyond a simple binary view of convergence. They provide a robust system for classifying convergence rates, where the finiteness of a norm is a sufficient condition for convergence. As demonstrated, this framework successfully distinguishes between functions with different asymptotic behaviors (such as algebraic versus exponential decay), resolving ambiguities that arise from simpler measures.

This work opens several avenues for future research. The generalization of the weighting function to a full **scale of comparison functions** ϕ, as explored in Section 4, is a first step.

It is also essential to recognize the current boundaries of our formalism, which themselves open stimulating research prospects. Our construction, founded on a single scalar exhaustion function h, is intrinsically **isotropic**. It is therefore perfectly adapted to spaces whose "infinity" can be conceived as a single point, ...mirroring the Alexandroff one-point compactification our framework generalizes [**?**]. However, this approach reaches its limits for spaces that possess a richer **anisotropic** structure at infinity. For example, spaces with several distinct "ends" (such as an infinite strip in $\mathbb{C}$ or the space $\mathbb{R} \setminus \{0\}$) or whose geometry of convergence depends on the direction (as in the hyperbolic plane) would require a significant generalization.

An exciting extension of this work would be to develop a **"multi-exhaustion" formalism**, where a collection of functions $\{h_i\}$, each associated with a different "end" of the space, could be used to independently characterize the different paths to infinity. Such a framework could potentially unify asymptotic analysis on spaces with complex boundaries, linking the geometry of their compactifications (like the Martin or Thurston compactifications) to quantitative analytical tools similar to those developed here.

Finally, the practical potential of this framework could be demonstrated by applying it to specific problems in physics or probability theory, while the function spaces $C_{h,\phi}(A)$ induced by these norms present a rich topic for further investigation in functional analysis.

In summary, the exhaustion function provides a simple and powerful language to unify the concept of infinity across diverse fields of mathematics. The quantitative apparatus built upon it offers a systematic and effective method for answering, with nuance, the question: "How fast does a function converge?"

Potential Applications

The framework developed herein is not merely an abstract unification; it is a powerful analytical instrument with the potential for concrete applications(cf.A Weighted Kolmogorov Metric for Berry-Esseen Bounds under Sub-Cubic Moments) across several fields.

- **Partial Differential Equations:** The multi-exhaustion framework is ideally suited for studying the asymptotic decay of solutions to elliptic equations on non-compact manifolds (such as cylinders or spaces with conic singularities). The ends U_i and their corresponding exhaustion functions h_i provide a canonical way to classify the convergence rates of solutions in different asymptotic regimes.

- **Probability Theory:** The rate of convergence of Markov chains to their stationary distribution on infinite state spaces could be analyzed with this formalism. The ends of the space could correspond to different classes of recurrent states, with the directional norms measuring the speed of convergence within each class.

- **Geometric Group Theory and General Relativity:** In fields where the large-scale geometry of a space is paramount, our framework could offer a new quantitative language. This includes analyzing word metrics on the Cayley graphs of infinite groups or quantifying the decay of fields towards null infinity in models of spacetime.

These potential connections underscore that the question "How fast does a function converge?" is fundamental not only in pure analysis but also in our mathematical modeling of the world.

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