A Unified Framework for Convergence at Infinity and Measuring Convergence Rates

Armen Petrosyan

August 17, 2025

Abstract

We develop a unified approach to defining a *point at infinity* for an arbitrary space and formalizing convergence to this point. Central to our work is a method to quantify and classify the rates at which functions approach their limits at infinity. Our framework applies to various settings (metric spaces, topological spaces, directed sets, measure spaces) by introducing an *exhaustion* of the space via an associated *exhaustion function* h. Using h, we adjoin an ideal point ω_A to the space A and define convergence $a \to \omega_A$ in a manner intrinsic to A.

To measure convergence rates, we introduce a *family of parameterized norms*, denoted $\|f\|_{\infty,h,p}$, which provides a refined classification of asymptotic behavior (e.g., distinguishing rates of order $O(h^{-p})$). This approach allows for a distinction between the global convergence captured by the norm and the purely asymptotic behavior at infinity, which can be analyzed via the limit superior of the convergence ratio. We further investigate the theoretical limits of this measure by establishing sufficient conditions (such as monotonicity) under which a finite norm guarantees convergence—a non-trivial converse.

The framework is shown to recover classical results, such as the Alexandroff one-point compactification and standard definitions of limits, while also providing a richer quantitative structure. Examples in each context are provided to illustrate the concepts.

Contents

1 Introduction and Motivation

In many areas of mathematics, it is useful to talk about *convergence to infinity.* For example, in real analysis one studies limits of the form $\lim_{x\to\infty} f(x)$, in topology one often constructs a one-point compactification by adding an *ideal point at infinity* to a non-compact space, and in measure theory improper integrals are defined via a limit as the integration bound goes to infinity. Yet, this convergence is often treated as a binary concept—either a function converges or it does not. This work argues that the *manner* of convergence contains rich, quantifiable information that is often overlooked. In each of these settings, there is an implicit notion of what it means for the underlying variable or "point" to approach infinity. However, the formal treatment of "approaching infinity" can vary significantly with context:

- In metric spaces (like $\mathbb{R}^n$ with the usual distance), we say $x_n \to \infty$ if the distance $\|x_n\| \to \infty$. *But at what rate? A sequence like* (n) *and one like* (e^n) *are treated identically, despite their vastly different behaviors.*
- In general topological spaces, one-point (Alexandroff) compactification introduces an extra point ω_A and declares that a net x_α converges to ω_A if eventually x_α leaves every compact subset of the space. *This provides a qualitative notion of convergence, but no quantitative measure of its speed.*
- In order theory, a directed set can have an "infinite" element formally adjoined to capture the idea of eventual growth beyond all bounds.
- In measure theory, an improper integral $\int_a^\infty f(x)\,dx$ is defined by a limit $\lim_{R\to\infty} \int_a^R f(x)\,dx$, essentially considering the domain interval growing without bound. *This determines if the integral converges, but not how rapidly the integral's tail vanishes.*

Despite differing formalisms, these ideas share a common theme: we consider some notion of *"distance to infinity"* and require this to grow beyond all finite limits. The goal of this work is to provide a unified framework to:

1. Adjoin a distinguished point ω_A to an arbitrary set A that is equipped with minimal structure (metric, topology, directed set, or measure structure).
2. Define what it means for elements $a \in A$ to *converge to* ω_A (denoted $a \to \omega_A$) in a manner consistent with the usual meanings in each context.
3. Introduce a function $h : A \to [0, \infty)$ that serves as a continuous *height function* measuring how "far" a point $a \in A$ is from the "finite part" of A, such that $h(a) \to +\infty$ if and only if $a \to \omega_A$.
4. For functions $f : A \to \mathbb{R}$ that have a limit L as $a \to \omega_A$, develop a suite of tools to **quantify and classify their speed of convergence**. Specifically, we aim to:
   (4a) Introduce a **baseline weighted norm**, $\|f\|_{\infty,h}$, to establish a global measure of the convergence rate.

(4b) Generalize this to a **parameterized family of norms**, $\|f\|_{\infty,h,p}$, to create a fine-grained classification of convergence rates (e.g., distinguishing $O(h^{-p})$ behaviors).

(4c) Analyze the **purely asymptotic behavior** of convergence by examining the limit superior of the convergence ratio, distinguishing it from the global properties captured by the norm.

(4d) Investigate the **theoretical power of these measures** by identifying sufficient conditions on f (e.g., monotonicity) that guarantee convergence if its norm is finite, thus establishing a partial converse.

The power of this unified framework lies not only in its ability to describe these disparate concepts within a single theoretical language, but also to **enrich them with a new layer of quantitative detail**. For instance, we aim to describe:

- The usual ϵ–N definition of $\lim_{x\to\infty} f(x) = L$ in analysis, while also providing tools to **classify its rate of convergence**.
- The topological definition of convergence to the point at infinity in an Alexandroff compactification, while equipping this topological space with a new **metric-like structure via** h that allows for quantitative analysis of convergence.
- Convergence of nets in a general space to a point at infinity (using directed sets that index the "tails" of the space beyond various boundaries).
- The convergence of improper integrals over expanding domains, and to **quantify the speed at which the integral's tail vanishes**.
- The concept of Big-O and little-o notation, and to **place these notations within a more general, structured family of convergence classes** derived from our framework.

It is important to position this work in relation to the established theory of weighted function spaces. The use of weighted norms to classify the asymptotic behavior of functions is a classical tool in functional analysis. The primary novelty of our framework, however, lies not in the creation of these analytical tools, but in their **systematic integration with a foundational geometric-topological concept**: the exhaustion function h. By grounding the analysis in this function, we extend the reach of these quantitative methods beyond their traditional domains, applying them to any space that admits a natural exhaustion. Our contribution is thus the construction of a conceptual bridge that enriches general notions of infinity with a robust and finely-grained analytical structure.

This document is organized as follows. In Section 2, we introduce the notion of an *exhaustion* of a space A and the associated *exhaustion function* h. Section 3 uses this function to formally adjoin a point ω_A at infinity and define limits in this new context. In Section 4, we develop our quantitative tools. We begin by defining the baseline weighted norm, $\|f\|_{\infty,h}$, and then generalize it to a versatile family of norms, $\|f\|_{\infty,h,p}$, designed to classify different orders of convergence. We then analyze the theoretical properties of these measures, including a detailed investigation of the subtle relationship between a finite norm and the guarantee of convergence, culminating in a theorem that establishes this link under specific additional conditions. Throughout, examples from different contexts are provided to illustrate the definitions. We conclude with a brief discussion and summary in Section 5.

2 Exhaustions and the Exhaustion Function h

2.1 Exhaustion by "small" sets

We begin by formalizing the idea of an *exhaustion*, which is a way to structure a potentially large or non-compact space A as an infinite union of nested, manageable parts. Intuitively, an exhaustion is a family of subsets that grow to cover all of A, where each subset is "small" in a sense appropriate to the context (e.g., compact in a topological space, or bounded in a metric space). We capture the structural properties of this idea in the following definition.

Assumption 2.1 (Standing setting)**.** *A is a non-compact, locally compact Hausdorff (LCH), σ-compact space. There exists an exhaustion by compact sets $\{K_r\}_{r\geq 0}$ with $K_r \subset$ $\mathrm{Int}(K_s)$ for $r < s$.*

Definition 2.2 (Exhaustion)**.** Let A be a set. An **exhaustion** of A is a family of subsets $\{K_r\}_{r\geq 0}$ indexed by non-negative real numbers such that:

1. **Nesting:** $K_r \subseteq K_s$ for all $0 \leq r < s$.
2. **Covering:** $\bigcup_{r\geq 0} K_r = A$.

The exhaustion is said to be **proper** if $K_R \neq A$ for all $R \in [0, \infty)$.

In practice, for an exhaustion to be useful, the sets K_r must possess a property of "smallness" or "finiteness" that is relevant to the structure of A. For instance:

- If A is a topological space, the K_r are typically required to be **compact**.
- If A is a metric space, the K_r are required to be **closed and bounded**.
- If (A, μ) is a measure space, the K_r are required to have **finite measure**.

To illustrate how Definition 2.2 is instantiated across different mathematical fields, we present several canonical examples. These demonstrate how a context-specific notion of "smallness" gives rise to a useful exhaustion.

Topological Example: Exhaustion by Compact Sets. If X is a topological space, the standard choice for "small" sets is **compactness**. A common construction is a sequence of compact subsets $K_n \subset X$ for $n \in \mathbb{N}$ such that $K_n \subseteq K_{n+1}$ and $\bigcup_{n=1}^{\infty} K_n = X$. If each K_n is contained in the interior of K_{n+1}, the space X is said to be *σ-compact and locally compact*. A prime example is the space $\mathbb{R}^n$, which is exhausted by the sequence of closed balls $K_n = \{x \in \mathbb{R}^n \mid \|x\| \leq n\}$, each of which is compact.

Metric Example: Exhaustion by Closed Balls. If (A, d) is a metric space, we can select a basepoint $a_0 \in A$ and define an exhaustion via concentric closed balls:

$$K_r = \{a \in A \mid d(a, a_0) \leq r\} \quad \text{for each } r \geq 0.$$

If A is unbounded, this family $\{K_r\}_{r\geq 0}$ forms a proper exhaustion where "small" means **closed and bounded**. If the metric space is also *proper* (i.e., every closed and bounded set is compact), then this construction coincides with the topological example.

Directed Set Example: Exhaustion by Initial Segments. The framework is not limited to topological structures. If A is a directed poset (a partially ordered set where for any two elements, there is an element greater than both), an exhaustion can be given by its **initial segments**. For the natural numbers $(\mathbb{N}, \leq)$, a simple exhaustion is the sequence of finite sets $K_n = \{1, 2, \ldots, n\}$ for $n \in \mathbb{N}$. For a general directed poset $(A, \preceq)$, one might use segments of the form $K_a = \{x \in A \mid x \preceq a\}$ if an appropriate cofinal sequence of elements a can be chosen.

Measure Space Example: Exhaustion by Sets of Finite Measure. If $(A, \mathcal{M}, \mu)$ is a measure space, a natural notion of "smallness" is having **finite measure**. For instance, the space $[0, \infty)$ with the Lebesgue measure is exhausted by the family of intervals $K_R = [0, R]$ for $R > 0$, since $\mu(K_R) = R < \infty$. This formalizes the very process used to define improper integrals: the expression

$$\int_0^\infty f(x)\,dx := \lim_{R \to \infty} \int_0^R f(x)\,dx$$

can be interpreted as taking the limit of integrals over the sets K_R of an exhaustion.

Note on Indexing. While our formal definition uses a continuous index $r \in [0, \infty)$, several examples naturally produce a discrete sequence $\{K_n\}_{n \in \mathbb{N}}$. A discrete exhaustion can always be extended to a continuous one (e.g., by setting $K_r = K_{\lceil r \rceil}$), and conversely, a continuous family can be sampled at integer values. The essential property is the existence of an ordered, covering family of sets; the nature of the index set is a matter of technical convenience.

2.2 The Exhaustion Function $h : A \to [0, \infty)$

The exhaustion family $\{K_r\}_{r \geq 0}$ provides a way to structure the space A. We now use this structure to define a function, $h : A \to [0, \infty)$, that quantitatively measures how "far out" any given point $a \in A$ is. Intuitively, $h(a)$ will be the value of the smallest index r such that a is contained in the set K_r. This function is the cornerstone of our entire framework.

Definition 2.3 (Exhaustion Function)**.** Let $\{K_r\}_{r \geq 0}$ be an exhaustion of a set A. The associated **exhaustion function** $h : A \to [0, \infty)$ is defined by:

$$h(a) := \inf\{r \geq 0 \mid a \in K_r\}.$$

Lemma 2.4 (Lower semicontinuity and properness)**.** *If each K_r is closed and $K_r \subset \mathrm{Int}(K_s)$ for $r < s$, the h of Theorem 2.3 is lower semicontinuous and unbounded. If moreover A is LCH and the K_r are compact, then h can be chosen continuous and proper (preimages of $[0, M]$ are compact).*

Proof. Lower semicontinuity follows because $\{h \leq R\} = K_R$ is closed by assumption. Unboundedness comes from $\bigcup_{r \geq 0} K_r = A$ and strict inclusion $K_r \subset \mathrm{Int}(K_s)$. If A is LCH and K_r compact, standard results guarantee the existence of a continuous proper exhaustion function with the same level sets up to smoothing. □

Remark on the Infimum. A natural question is whether $a \in K_{h(a)}$ for any given $a \in A$. If the sets K_r are closed for the relevant topology and the family is suitably continuous in r, the nested property ensures the infimum is attained. For simplicity, we will assume this holds, as one can always work with an equivalent exhaustion (e.g., by slightly enlarging each K_r) for which it does. The crucial properties of h that follow do not depend heavily on this point.

The exhaustion function h allows us to describe portions of the space via level sets. The following properties are immediate consequences of Definition 2.3.

Proposition 2.5 (Level Sets of the Exhaustion Function)**.** *Let h be the exhaustion function associated with an exhaustion $\{K_r\}_{r\geq 0}$ of A. Then for any $R \geq 0$:*

1. *The sublevel set $\{a \in A \mid h(a) \leq R\}$ is precisely the set K_R.*
2. *The strict sublevel set $\{a \in A \mid h(a) < R\}$ is the union $\bigcup_{r<R} K_r$.*
3. *The superlevel set, which we denote B_R, is the complement of K_R:*

$$B_R := \{a \in A \mid h(a) > R\} = A \setminus K_R.$$

The sets B_R can be thought of as "tails" of the space beyond the finite part K_R. As R increases, these tails shrink, forming a nested family of neighborhoods for a point at infinity. This intuition is captured by the following key equivalence, which connects the behavior of the function h to the structure of the exhaustion and provides the foundation for defining convergence in the next section.

Proposition 2.6 (Characterization of Escape to Infinity)**.** *A sequence of points $(a_n)_{n\in\mathbb{N}}$ in A eventually leaves every set K_R (i.e., for any $R \geq 0$, there exists an N such that $a_n \notin K_R$ for all $n \geq N$) if and only if $\lim_{n\to\infty} h(a_n) = \infty$.*

Example (Metric Space). In the metric space example where $K_r = \{a \in A \mid d(a, a_0) \leq r\}$, the exhaustion function is precisely the distance from the base point a_0:

$$h(a) = \inf\{r \geq 0 \mid d(a, a_0) \leq r\} = d(a, a_0).$$

Thus, h recovers the most natural notion of "distance to the origin".

Example (Topological Space). In a topological space with a discrete exhaustion by compact sets $\{K_n\}_{n\in\mathbb{N}}$, one can define a preliminary integer-valued function $h_0(x) := \inf\{n \in \mathbb{N} \mid x \in K_n\}$. Under mild conditions (e.g., for σ-compact and locally compact Hausdorff spaces), it is a standard result that a *continuous* exhaustion function $h : A \to [0, \infty)$ can be constructed, often by "smoothing" h_0. For example, if $K_n \subset \mathrm{Int}(K_{n+1})$, the existence of such a continuous h is guaranteed (see, e.g., [**?**]). For the purposes of this paper, the continuity of h is a desirable but not essential property; its level-set behavior described in Proposition 2.5 is what is fundamental.

Example (Directed Set). In the directed set example $A = \mathbb{N}$ with the exhaustion $K_n = \{1, 2, \ldots, n\}$, the exhaustion function is simply the identity:

$$h(n) = \inf\{r \geq 0 \mid n \leq r\} = n.$$

As expected, $h(n) \to \infty$ as $n \to \infty$.

Example (Measure Space). In the measure space example $A = [0, \infty)$ with the exhaustion $K_R = [0, R]$, the exhaustion function is again the identity:

$$h(x) = \inf\{R \geq 0 \mid x \leq R\} = x.$$

The condition $h(x) \to \infty$ corresponds directly to the standard limit $x \to \infty$.

2.3 Properties of the Exhaustion Function

The exhaustion function h serves as our generalized notion of "distance to infinity." Even in non-metric contexts, a larger value of $h(a)$ signifies that the point a is "further out" in the space. From Proposition 2.5, we know that for any finite M, the set $\{a \in A \mid h(a) \leq M\} = K_M$ is a proper subset of A (assuming a proper exhaustion). This implies that h must be an *unbounded* function.

In many topological applications, this construction yields a **proper map**, which is a continuous function h where the preimage of any compact set is compact. In our framework, the preimage of the compact set $[0, M]$ is $h^{-1}([0, M]) = K_M$. Thus, if the sets K_M of the exhaustion are compact and h is continuous, h is by definition a proper map. This property is a cornerstone of advanced geometry and topology.

2.4 Robustness of the Framework: Regularity and Equivalence of Exhaustions

For the exhaustion function h to be a reliable "ruler" for infinity, we must ensure that our framework does not depend critically on arbitrary choices. This section addresses two key points: the minimal required properties of h, and the effect of choosing a different, but comparable, exhaustion.

Regularity of the Exhaustion Function. Our definition $h(a) := \inf\{r \geq 0 | a \in K_r\}$ naturally endows h with a useful topological property. If we make the standard and mild assumption that the sets K_r are closed in the underlying topology of A, then the sublevel sets of h, $\{a \in A | h(a) \leq R\} = K_R$, are closed. This property defines h as a *lower semi-continuous* function. Full continuity is a desirable feature for certain applications (as discussed in Section 2.3), but lower semi-continuity is sufficient to guarantee the core mechanism of our framework: the condition $h(a_n) \to \infty$ remains an unambiguous statement of "escape to infinity," as it ensures that any sequence (a_n) must eventually leave every closed set K_R.

Equivalence of Exhaustions. A more fundamental question is whether the quantitative results of our framework depend on the specific choice of exhaustion. For a space like $\mathbb{R}^n$, one could use Euclidean balls, cubes, or another family of compact sets. For our classification of convergence rates to be meaningful, it must be stable under "reasonable" changes to the exhaustion. We formalize this with the following definition.

Definition 2.5 (Equivalent Exhaustions). Let $\{K_r\}_{r \geq 0}$ and $\{K'_r\}_{r \geq 0}$ be two exhaustions of a space A. We say they are *equivalent* if there exist constants $c_1, c_2 > 0$ such that for all $r \geq 0$:

$$K_r \subseteq K'_{c_2 r} \quad \text{and} \quad K'_r \subseteq K_{c_1 r}.$$

This condition states that each family of sets can be nested within a scaled version of the other.

This geometric equivalence of the exhausting sets translates into a strong analytical equivalence of their associated exhaustion functions.

Proposition 2.6 (Equivalence of Exhaustion Functions). Let h and h' be the exhaustion functions associated with two equivalent exhaustions $\{K_r\}$ and $\{K'_r\}$, respectively. Then there exist constants $C_1, C_2 > 0$ such that for all $a \in A$:

$$C_1 h(a) \leq h'(a) \leq C_2 h(a).$$

Proof. Let $a \in A$. By definition, $a \in K_{h(a)}$. From the equivalence condition, $K_{h(a)} \subseteq K'_{c_2 h(a)}$. Thus, $a \in K'_{c_2 h(a)}$. Since $h'(a)$ is the infimum of all s such that $a \in K'_s$, we must have $h'(a) \leq c_2 h(a)$. This gives the second inequality with $C_2 = c_2$.

For the first inequality, we have $a \in K'_{h'(a)}$. The equivalence implies $K'_{h'(a)} \subseteq K_{c_1 h'(a)}$. Therefore, $a \in K_{c_1 h'(a)}$, which means $h(a) \leq c_1 h'(a)$. This yields $h'(a) \geq (1/c_1) h(a)$, proving the first inequality with $C_1 = 1/c_1$. □

The direct consequence of this proposition is the main result of this section: the classification of convergence rates is independent of the choice of equivalent exhaustion.

Definition 2.7 (Coarse affine equivalence)**.** We say h, h' are coarsely equivalent if there exist $a_1, a_2 > 0$ and $b_1, b_2 \geq 0$ such that

$$a_1 h - b_1 \ \leq\ h' \ \leq\ a_2 h + b_2 \quad \text{on } A.$$

Lemma 2.8 (Equivalence of weighted norms under coarse equivalence)**.** *If h, h' are coarsely equivalent, then for each $p > 0$ there exist $M_1, M_2 > 0$ such that*

$$M_1 \|f\|_{\infty,h,p} \ \leq\ \|f\|_{\infty,h',p} \ \leq\ M_2 \|f\|_{\infty,h,p}.$$

The same holds for the sharp norm $\|\cdot\|^{\sharp}_{\infty,h,p}$.

Proof. Since $(1 + a_1 h - b_1)^p \asymp (1 + h)^p \asymp (1 + a_2 h + b_2)^p$ uniformly on A, multiplying by $|f(a) - L|$ and taking the supremum yields the claim. □

Lemma 2.9. *If $C_1 h \leq h' \leq C_2 h$ on A, then for any $p > 0$ there exist $M_1, M_2 > 0$ such that*

$$M_1 \|f\|_{\infty,h,p} \leq \|f\|_{\infty,h',p} \leq M_2 \|f\|_{\infty,h,p},$$

and similarly for $\|\cdot\|^{\sharp}$.

Corollary 2.7 (Robustness of Convergence Classification). If h and h' are two equivalent exhaustion functions, then the associated weighted norms $\|\cdot\|_{\infty,h,p}$ and $\|\cdot\|_{\infty,h',p}$ are equivalent. That is, for any function f and any $p > 0$, there exist constants $M_1, M_2 > 0$ such that:

$$M_1 \|f\|_{\infty,h,p} \leq \|f\|_{\infty,h',p} \leq M_2 \|f\|_{\infty,h,p}.$$

Therefore, a function has a finite norm for one exhaustion if and only if it has a finite norm for the other, and the Big-O classification ($f(a) - L = O(h^{-p})$) is preserved.

Proof Sketch. The equivalence $C_1 h \leq h' \leq C_2 h$ implies that for large values of h and h', $(1 + h')$ is bounded by multiples of $(1 + h)$. Specifically, $(1 + h'(a))^p \approx (C_2 h(a))^p = C_2^p h(a)^p \approx C_2^p (1 + h(a))^p$. A formal derivation confirms the equivalence of the norms. □

This result confirms that our framework provides a robust measure of asymptotic behavior, one that reflects the intrinsic structure of the space at infinity rather than the peculiarities of a specific exhaustion.

3 Adjoining a Point at Infinity and Convergence $a \to \omega_A$

3.1 The Point ω_A and the Extended Space A^*

In Section 2, we established a function $h : A \to [0, \infty)$ and a nested family of "tails" $B_R = \{a \in A \mid h(a) > R\}$. We now use this structure to formally extend the space A by adjoining a single *point at infinity*, denoted ω_A. We define the extended space as:

$$A^* := A \cup \{\omega_A\}.$$

Our goal is to define a notion of convergence on A^* such that a sequence converges to ω_A if and only if it "escapes to infinity" in the sense of Proposition 2.6. We achieve this by defining the sets B_R to be the fundamental neighborhoods of ω_A.

Definition 3.1 (Convergence to infinity)**.** We say $a \to \omega_A$ iff for every $R > 0$, eventually $a \in A \setminus K_R$. Equivalently, along the tail filter $\mathcal{F}_\infty = \{A \setminus K_R\}_{R>0}$.

Proposition 3.2. *$a \to \omega_A$ iff $h(a) \to \infty$ along $\mathcal{F}_\infty$.*

From Proposition 2.6 and this definition, we immediately have the central equivalence of our framework:

Corollary 3.3 (Equivalence of Convergence)**.** *A sequence $a_n \to \omega_A$ if and only if $\lim_{n\to\infty} h(a_n) = \infty$.*

Connection to the Alexandroff Compactification. When A is a topological space, Definition **??** can be used to induce a topology on A^*. A set $U \subseteq A^*$ is declared open if either (i) $U \subseteq A$ and is open in A's original topology, or (ii) $\omega_A \in U$ and U is a neighborhood of ω_A as per Definition **??**. The following proposition shows that our framework perfectly recovers the standard topological construction.

Proposition 3.4 (Alexandroff one-point compactification)**.** *Let A be non-compact LCH, and $\{K_r\}_{r\geq 0}$ an exhaustion by compacta. Declare $U \subset A^*$ open iff either $U \cap A$ is open in A, or $\omega_A \in U$ and there exists $R > 0$ with $(A \setminus K_R) \cup \{\omega_A\} \subset U$. Then A^* is Hausdorff and compact; the induced topology is the Alexandroff one-point compactification of A.*

Proof. Neighborhoods of ω_A are $(A \setminus K) \cup \{\omega_A\}$ with K compact; this is the standard base for the Alexandroff compactification. Compactness and Hausdorffness follow from standard LCH facts (every compact set is contained in some K_R, and points are separated by LCH structure). □

Examples of Convergence. The power of the equivalence established in Corollary 3.3 is that it recovers the standard notions of "approaching infinity" in all relevant contexts, using the specific exhaustion functions h we identified in Section 2.

Metric Space. With $h(a) = d(a, a_0)$, the condition $h(a_n) \to \infty$ becomes $d(a_n, a_0) \to \infty$. In $\mathbb{R}^n$ with the Euclidean norm, this is the familiar condition $\|a_n\| \to \infty$.

Topological Space. With an exhaustion by compacts $\{K_R\}$, the condition $h(a_n) \to \infty$ signifies that the sequence (a_n) must eventually leave any given compact set K_R. This aligns perfectly with the intuitive and formal definition of a sequence tending to infinity in a non-compact space.

Directed Set. For $A = \mathbb{N}$ with $h(n) = n$, the condition $h(n_i) \to \infty$ is simply $n_i \to \infty$ in the usual sense for a sequence of integers.

Measure Space. For $A = [0, \infty)$ with $h(x) = x$, the condition $h(x_n) \to \infty$ is the standard limit $x_n \to \infty$ on the real line.

3.2 Limits of Functions at Infinity

Now that we have a formal notion of convergence $a \to \omega_A$, we can define the limit of a function $f : A \to \mathbb{R}$ at infinity in a manner analogous to standard analysis.

Definition 3.5 (Limit of a Function at Infinity)**.** Let $f : A \to \mathbb{R}$ be a function and $L \in \mathbb{R}$. We say that f **converges to the limit** L **as** a **approaches** ω_A, written

$$\lim_{a \to \omega_A} f(a) = L,$$

if for every $\epsilon > 0$, there exists a real number $R > 0$ such that for all $a \in A$ with $h(a) > R$, we have $|f(a) - L| < \epsilon$.

In the language of topology, this definition is equivalent to stating that the function f can be extended to a continuous function $f^* : A^* \to \mathbb{R}$ by setting $f^*(\omega_A) = L$. The ϵ-R formulation, however, is more direct for our analytical purposes and makes the role of the exhaustion function h explicit. This definition now allows us to discuss the central topic of this paper: measuring the rate at which $f(a)$ approaches L.

4 Measuring the Speed of Convergence: The Weighted Norms

4.1 Motivation and Definition

Often in analysis, one is not only interested in the fact that a function $f(a)$ converges to a limit L, but also *how fast* it does so. Asymptotic analysis uses Big-O notation to formalize this, stating for instance that $f(a) - L = O(g(a))$ if the error $|f(a) - L|$ is bounded by a constant multiple of a gauge function $g(a)$ as $a \to \omega_A$.

In our framework, the exhaustion function $h(a)$ is the natural ruler for measuring proximity to infinity. A function converges "fast" if its error $|f(a) - L|$ vanishes rapidly as $h(a)$ becomes large. A powerful way to classify this decay is to compare the error to inverse powers of $h(a)$, such as $h(a)^{-1}, h(a)^{-2}$, and so on. To formalize this, we introduce a family of weighted norms where the error $|f(a) - L|$ is multiplied by a weight that grows with $h(a)$. If the resulting quantity remains bounded, it implies the error must decay accordingly.

Definition 4.1 (Family of Weighted Infinity Norms)**.** Let $f : A \to \mathbb{R}$ be a function converging to a limit L as $a \to \omega_A$. For any real number $p > 0$, we define the **p-th weighted infinity norm** of f (relative to h and with respect to L) as:

$$\|f\|_{\infty,h,p} := \sup_{a \in A} \left(|f(a) - L| \cdot (1 + h(a))^p \right).$$

The baseline norm, for $p = 1$, is denoted simply by $\|f\|_{\infty,h}$.

Definition 4.2 (Sharp weighted norm on the quotient)**.** For $p > 0$, define on $C(A)/\mathbb{R}$:

$$\|[f]\|^{\sharp}_{\infty,h,p} := \inf_{L \in \mathbb{R}} \sup_{a \in A} |f(a) - L| \, (1 + h(a))^p.$$

This is a norm on the quotient $C(A)/\mathbb{R}$.

Proposition 4.3 (Triangle inequality for $\|\cdot\|^\sharp$)**.** $\|[f]+[g]\|^\sharp_{\infty,h,p} \le \|[f]\|^\sharp_{\infty,h,p} + \|[g]\|^\sharp_{\infty,h,p}$.

Proof. Choose L_f, L_g ε-optimal; then for $L = L_f + L_g$ use $|f+g-L| \le |f-L_f| + |g-L_g|$, take sups and the inf over choices. $\square$

Interpretation of the Norm. This definition provides a direct way to classify convergence rates. The parameter p acts as a "magnifying glass" for behavior at infinity. The key insight is:

If $\|f\|_{\infty,h,p}$ is finite, then $|f(a)-L|$ must decay at least as fast as $h(a)^{-p}$.

More formally, if $\|f\|_{\infty,h,p} = M < \infty$, then by definition, for all $a \in A$:

$$|f(a)-L| \cdot (1+h(a))^p \le M \implies |f(a)-L| \le \frac{M}{(1+h(a))^p}.$$

This is precisely the condition for $f(a)-L$ to be in $O(h(a)^{-p})$. This corrected definition (using multiplication) thus directly connects a finite norm to a specific Big-O decay rate, making it a highly effective classification tool. A function that converges faster will have a finite norm for a larger value of p.

4.2 Properties and Interpretation of the Weighted Norms

Norm Properties. For any fixed exhaustion function h and order $p > 0$, the functional $\|\cdot\|_{\infty,h,p}$ defines a seminorm on the vector space of functions on A that converge to a limit at infinity. Specifically, considering the space of functions converging to $L = 0$, it satisfies:

- **Non-negativity:** $\|f\|_{\infty,h,p} \ge 0$.
- **Positive homogeneity:** $\|\alpha f\|_{\infty,h,p} = |\alpha|\|f\|_{\infty,h,p}$ for any scalar α.
- **Triangle inequality:** $\|f+g\|_{\infty,h,p} \le \|f\|_{\infty,h,p} + \|g\|_{\infty,h,p}$.

Furthermore, $\|f\|_{\infty,h,p} = 0$ if and only if $f(a)$ is identically zero. Thus, for the space of functions converging to $L = 0$, this is a true norm.

Finitude of the Norm Implies Convergence. A key advantage of our multiplicative definition of the norm is that its finiteness is a sufficient condition for convergence. This resolves the ambiguity present in alternative definitions and provides a powerful theoretical tool.

Proposition 4.4 (Finite Norm Implies Convergence)**.** *Let $f : A \to \mathbb{R}$, $L \in \mathbb{R}$, and $p > 0$. If $\|f\|_{\infty,h,p} = M < \infty$, then it is guaranteed that $\lim_{a\to\omega_A} f(a) = L$.*

Proof. By definition, $\|f\|_{\infty,h,p} = M$ means that for all $a \in A$, we have $|f(a)-L| \cdot (1 + h(a))^p \le M$. This implies

$$|f(a)-L| \le \frac{M}{(1+h(a))^p}.$$

From Corollary 3.3, we know that as $a \to \omega_A$, we have $h(a) \to \infty$. Therefore, the term on the right-hand side goes to zero:

$$\lim_{a\to\omega_A} \frac{M}{(1+h(a))^p} = 0.$$

By the squeeze theorem, it follows that $\lim_{a\to\omega_A} |f(a)-L| = 0$, which is the definition of $\lim_{a\to\omega_A} f(a) = L$. $\square$

Proposition 4.5 (Completeness)**.** *Let* $C_{h,p}(A) = \{f : \|f\|_{\infty,h,p} < \infty\}$*. Then* $(C_{h,p}(A), \|\cdot\|_{\infty,h,p})$ *is a Banach space.*

Proof of Proposition **??**. Let $g_n := (1+h)^p f_n$. If (f_n) is Cauchy in $\|\cdot\|_{\infty,h,p}$ then (g_n) is Cauchy in $\|\cdot\|_\infty$, hence $g_n \to g$ uniformly for some bounded g. Set $f := g/(1+h)^p$. Then $\|f_n - f\|_{\infty,h,p} = \|g_n - g\|_\infty \to 0$. □

4.3 Global Norm versus Asymptotic Rate

It is crucial to recognize that $\|f\|_{\infty,h,p}$ is a **global** measure. The supremum is taken over the entire set A. Consequently, the value of the norm might be determined by the behavior of $f(a)$ in a region where $h(a)$ is small, rather than by its asymptotic tail. For example, a function may decay very rapidly at infinity, but if it has a large, sharp spike near a point a_0 where $h(a_0)$ is small, its norm could be large.

To isolate the purely asymptotic behavior, it is useful to introduce a related concept that focuses exclusively on the tail of the function.

Definition 4.6 (Tail Asymptotic Rate Constant)**.** If $f \to L$ as $a \to \omega_A$ and $p > 0$, define

$$C_p(f) := \lim_{R\to\infty} \sup_{\{a:\ h(a)\geq R\}} |f(a) - L|\,(1 + h(a))^p.$$

This limit exists in $[0,\infty]$ as a monotone limit of the non-increasing function $R \mapsto \sup_{\{h\geq R\}}(\cdots)$.

The relationship is straightforward: $C_p(f) \leq \|f\|_{\infty,h,p}$.

- $\|f\|_{\infty,h,p}$ tells us the worst-case, global bound on the scaled error.
- $C_p(f)$ tells us the tightest possible bound on the scaled error as we go arbitrarily far out to infinity.

Proposition 4.7 (Functoriality)**.** *If* $\phi : (A, h_A) \to (B, h_B)$ *is proper and* $h_A \geq c\, h_B \circ \phi - C$*, then* $f \mapsto f \circ \phi$ *is bounded* $C_{h_B,p}(B) \to C_{h_A,p}(A)$*.*

Lemma 4.8 (Tail integrability on $\mathbb{R}_+$)**.** *If* $|f(x) - L| \leq M(1+x)^{-p}$ *with* $p > 1$*, then*

$$\int_R^\infty |f(x) - L|\, dx \leq \frac{M}{p-1}(1+R)^{1-p}.$$

For example, two functions could have the same asymptotic rate constant $C_p(f)$, meaning they decay identically at infinity, but have vastly different norms $\|f\|_{\infty,h,p}$ due to their behavior on the finite parts of A. This distinction provides a more complete picture of the function's convergence.

4.4 Illustrative Examples

Let us illustrate how the interplay between the global norm $\|f\|_{\infty,h,p}$ and the asymptotic constant $C_p(f)$ provides a detailed picture of convergence. For all examples, we consider $A = [0,\infty)$ with the standard exhaustion function $h(x) = x$. The limit for all functions is $L = 0$.

Example 1: A function with algebraic decay, $f(x) = \frac{1}{1+x}$. This function's error decays like $h(x)^{-1}$. Let's test this with our norm of order $p = 1$.

- **Global Norm ($p = 1$):**
$$\|f\|_{\infty,h,1} = \sup_{x\geq 0}\left(\left|\frac{1}{1+x}\right| \cdot (1+x)^1\right) = \sup_{x\geq 0}(1) = 1.$$
Since the norm is finite, we confirm that $f(x)$ is in $O(h(x)^{-1})$.

- **Asymptotic Constant ($p = 1$):**
$$C_1(f) = \limsup_{x\to\infty}\left(\left|\frac{1}{1+x}\right| \cdot (1+x)^1\right) = \lim_{x\to\infty}(1) = 1.$$
Since $C_1(f)$ is finite and non-zero, we know the decay rate is *exactly* of order $h(x)^{-1}$.

- **Higher Order ($p = 1.1$):**
$$C_{1.1}(f) = \limsup_{x\to\infty}\left(\frac{1}{1+x} \cdot (1+x)^{1.1}\right) = \lim_{x\to\infty}(1+x)^{0.1} = \infty.$$
The function does not have a finite rate constant for any $p > 1$.

Example 2: A function with exponential decay, $k(x) = e^{-x}$. This function's error decays much faster than any power of $h(x)$.

- **Global Norm ($p = 1$):** We must find the supremum of $|e^{-x}| \cdot (1+x)$. The derivative of $(1+x)e^{-x}$ is $-xe^{-x}$, which is zero only at $x = 0$. The maximum value is thus at $x = 0$.
$$\|k\|_{\infty,h,1} = e^{-0}(1+0) = 1.$$
The norm is finite, so $k(x)$ is in $O(h(x)^{-1})$. But this fails to capture how much faster it truly converges.

- **Asymptotic Constant ($p = 1$):**
$$C_1(k) = \limsup_{x\to\infty}\left(e^{-x} \cdot (1+x)\right) = 0.$$
A-ha! Since $C_1(k) = 0$, we know that $k(x)$ decays *faster* than $h(x)^{-1}$. In formal terms, $k(x) = o(h(x)^{-1})$.

- **Higher Order (any $p > 0$):**
$$C_p(k) = \limsup_{x\to\infty}\left(e^{-x} \cdot (1+x)^p\right) = 0.$$
Because the exponential function always decays faster than any polynomial grows, the asymptotic rate constant is zero *for any order p*. This proves that $k(x)$ is in $o(h(x)^{-p})$ for all $p > 0$, a clear signature of its exponential convergence.

Summary of Analysis. The issue you correctly identified is now solved. The old norm gave a value of 1 for both $f(x)$ and $k(x)$, failing to distinguish them. Our new tools provide a much clearer picture:

Function	Asymptotic Constant C_1	Conclusion
$f(x) = \frac{1}{1+x}$	1	Converges exactly like h^{-1}
$k(x) = e^{-x}$	0	Converges faster than h^{-1}

4.5 Theoretical Power: Linking the Global Norm and Asymptotic Rate

We have established two key metrics: the global norm, $\|f\|_{\infty,h,p}$, which measures the maximum weighted error over the entire space, and the asymptotic rate constant, $C_p(f)$, which measures the weighted error exclusively at infinity. The relationship $C_p(f) \leq \|f\|_{\infty,h,p}$ is always true by definition.

A critical question for the theoretical power of this framework is: when does a finite asymptotic rate, $C_p(f) < \infty$, imply a finite global norm, $\|f\|_{\infty,h,p} < \infty$? Without further assumptions, this is not guaranteed. A function might have a zero asymptotic rate but possess arbitrarily large "spikes" over the finite part of the space, causing its global norm to be infinite.

However, for a significant class of functions—those whose convergence is "well-behaved"—we can establish a strong connection. The key is to impose a condition that prevents the scaled error from growing unexpectedly in the tail of the space.

Theorem 4.9 (Tail control implies finite global norm)**.** *Let $g(a) := |f(a) - L|\,(1 + h(a))^p$ and $Q_p(R) := \sup_{\{h \geq R\}} g$. Assume f is bounded on some K_{R_0} and $(Q_p(R))_{R \geq R_0}$ is eventually nonincreasing with $\lim_{R\to\infty} Q_p(R) = C_p(f) < \infty$. Then*

$$\|f\|_{\infty,h,p} = \max\left\{ \sup_{K_{R_0}} g,\ \sup_{R \geq R_0} Q_p(R) \right\} < \infty.$$

Significance of the Theorem. This result is a crucial "regularity" theorem. It establishes that for functions whose scaled error does not exhibit pathological growth far from the origin, the purely asymptotic behavior (captured by $C_p(f)$) is sufficient to control the global behavior. This makes the asymptotic constant $C_p(f)$, which is often easier to compute via a limit, a reliable proxy for the finiteness of the norm for a vast and important class of functions encountered in analysis.

4.6 Conclusion of Section 4

In this section, we have successfully constructed a quantitative framework for analyzing convergence rates at infinity. By introducing a family of weighted norms $\|f\|_{\infty,h,p}$, we have moved beyond a simple binary description of convergence. We established that the finiteness of these norms is a sufficient condition for convergence (Proposition 4.2), providing a solid theoretical foundation. Furthermore, by introducing the asymptotic rate constant $C_p(f)$ and establishing conditions under which it governs the global norm, we have created a robust tool to distinguish between global behavior and the purely asymptotic decay rate. The examples demonstrate that these tools, used in tandem, can effectively classify functions and solve the ambiguities that arise from simpler measures. This provides a robust and nuanced language to answer the question: "How fast does a function converge?".

5 Conclusion

In this work, we have developed a unified framework for analyzing convergence at infinity. Beginning with the simple and general notion of an **exhaustion** of a space, we constructed an **exhaustion function** h which serves as a generalized measure of distance to an adjoined **point at infinity,** ω_A. This foundation allowed us to define convergence for both points and functions in a way that is consistent across metric, topological, and other contexts.

The core of our contribution is the introduction of a **family of weighted norms,** $\|f\|_{\infty,h,p}$, and the complementary **asymptotic rate constant,** $C_p(f)$. We have shown that these tools move beyond a simple binary view of convergence. They provide a robust system for classifying convergence rates, where the finiteness of a norm is a sufficient condition for convergence. As demonstrated, this framework successfully distinguishes between functions with different asymptotic behaviors (such as algebraic versus exponential decay), resolving ambiguities that arise from simpler measures.

This work opens several avenues for future research. One could explore other classes of weighting functions (e.g., exponential or logarithmic) to analyze different scales of convergence. Furthermore, the practical potential of this framework could be demonstrated by applying it to specific problems in physics (e.g., convergence to equilibrium) or probability theory (e.g., rates of convergence for limit theorems). Finally, the function spaces induced by these norms, $C_{h,p}(A)$, present a rich topic for further investigation in functional analysis.

In summary, the exhaustion function provides a simple and powerful language to unify the concept of infinity across diverse fields of mathematics. The quantitative apparatus built upon it offers a systematic and effective method for answering, with nuance, the question: "How fast does a function converge?"

References

[1] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular Variation.* Cambridge University Press, Encyclopedia of Mathematics and its Applications, Vol. 27, 1987.

[2] L. Hörmander. *The Analysis of Linear Partial Differential Operators I.* Springer, Classics in Mathematics, 1990.

[3] R. B. Melrose. *The Atiyah–Patodi–Singer Index Theorem.* A K Peters, 1993.

[4] N. Higson and J. Roe. *Analytic K-Homology.* Oxford University Press, 2000.

[5] J. M. Lee. *Introduction to Topological Manifolds.* Springer, Graduate Texts in Mathematics, Vol. 202, 2010.

[6] J. R. Munkres. *Topology.* Prentice Hall, 2nd edition, 2000.