Measuring Asymptotic Convergence: A Unified Framework from Isotropic Infinity to Anisotropic Ends

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Abstract

We develop a unified approach to defining a point at infinity for an arbitrary space and formalizing convergence to this point. Central to our work is a method to quantify and classify the rates at which functions approach their limits at infinity. Our framework applies to various settings (metric spaces, topological spaces, directed sets, measure spaces) by introducing an exhaustion of the space via an associated exhaustion function h. Using h, we adjoin an ideal point ω_A to the space A and define convergence $a \to \omega_A$ in a manner intrinsic to A. To measure convergence rates, we introduce a family of parameterized norms, denoted $||f||_{\infty,h,p}$ which provides a refined classification of asymptotic behavior (e.g., distinguishing rates of order $O(h^{-p})$). **Furthermore, the framework is extended to handle anisotropic spaces with multiple distinct ends by introducing a 'multi-exhaustion' formalism, allowing for a precise, directional analysis of convergence rates towards each asymptotic channel.** This approach allows for a distinction between the global convergence captured by the norm and the purely asymptotic behavior at infinity, which can be analyzed via the limit superior of the convergence ratio. We further investigate the theoretical limits of this measure by establishing sufficient conditions (such as monotonicity) under which a finite norm guarantees convergence a non-trivial converse. The framework is shown to recover classical results, such as the Alexandroff one-point compactification and standard definitions of limits, while also providing a richer quantitative structure. Examples in each context are provided to illustrate the concepts.

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1 Introduction and Motivation

In many areas of mathematics, it is useful to talk about *convergence to infinity.* For example, in real analysis one studies limits of the form $\lim_{x\to\infty} f(x)$, in topology one often constructs a one-point compactification by adding an *ideal point at infinity* to a non-compact space, and in measure theory improper integrals are defined via a limit as the integration bound goes to infinity. Yet, this convergence is often treated as a binary concept—either a function converges or it does not. This work argues that the *manner* of convergence contains rich, quantifiable information that is often overlooked. In each of these settings, there is an implicit notion of what it means for the underlying variable or "point" to approach infinity. However, the formal treatment of "approaching infinity" can vary significantly with context:

- In metric spaces (like $\mathbb{R}^n$ with the usual distance), we say $x_n \to \infty$ if the distance $\|x_n\| \to \infty$. *But at what rate? A sequence like* (n) *and one like* (e^n) *are treated identically, despite their vastly different behaviors.*
- In general topological spaces, one-point (Alexandroff) compactification introduces an extra point ω_A and declares that a net x_α converges to ω_A if eventually x_α leaves every compact subset of the space. *This provides a qualitative notion of convergence, but no quantitative measure of its speed.*
- In order theory, a directed set can have an "infinite" element formally adjoined to capture the idea of eventual growth beyond all bounds.
- In measure theory, an improper integral $\int_a^\infty f(x)\,dx$ is defined by a limit $\lim_{R\to\infty}\int_a^R f(x)\,dx$, essentially considering the domain interval growing without bound. *This determines if the integral converges, but not how rapidly the integral's tail vanishes.*

Table 1: Main symbols used throughout the paper

Symbol	Meaning
A	Hausdorff, locally compact, σ-compact space
$h : A \to [0,\infty)$	continuous proper exhaustion (Lemma 2.1)
$\varphi \in \Phi_{\mathrm{adm}}$	admissible comparison function (Def. 4.1.2)
$\|f\|_{\infty,h,p;L}$	global fixed-L norm with polynomial weight $(1+h)^p$ (Sec. 4.1.3)
$\|[f]\|^d_{\infty,h,p}$	sharp quotient norm modulo constants (Sec. 4.1.3)
$\|f\|_{\infty,h;\varphi;L}$, $\|[f]\|^d_{\infty,h;\varphi}$	φ-weighted global/quotient norms (Sec. 4.1.3)
$C^{(h)}_\varphi(f;L)$, $C^{(h)}_\varphi([f])$	φ-weighted asymptotic constants (Sec. 4.1.3)
U_i, h_i, φ_i	end neighborhoods, per-end exhaustions and weights (Sec. 8)
$\|[f]\|^d_{\infty,\mathrm{th},\varphi}$, $C^{(\mathbf{h})}_\varphi([f])$	anisotropic sharp norm and asymptotic constant (Sec. 8)

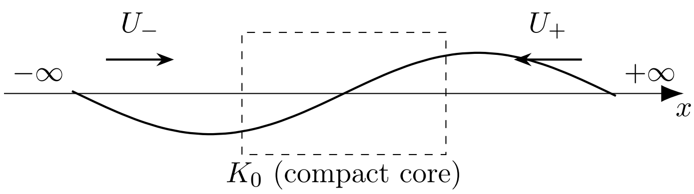

Figure 1: Two ends on a cylinder $\mathbb{R} \times M$: neighborhoods $U_\pm$ and a compact core K_0.

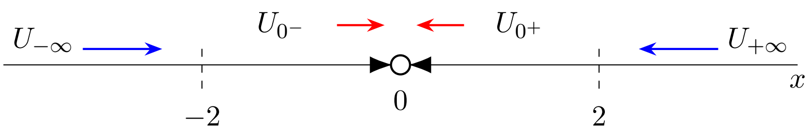

Figure 2: Four ends on $\mathbb{R} \setminus \{0\}$: near $0^\pm$ and at $\pm\infty$.

Despite differing formalisms, these ideas share a common theme: we consider some notion of *"distance to infinity"* and require this to grow beyond all finite limits. The goal of this work is to provide a unified framework to:

1. Adjoin a distinguished point ω_A to an arbitrary set A that is equipped with minimal structure (metric, topology, directed set, or measure structure).
2. Define what it means for elements $a \in A$ to *converge to* ω_A (denoted $a \to \omega_A$) in a manner consistent with the usual meanings in each context.
3. Introduce a function $h : A \to [0, \infty)$ that serves as a continuous *height function* measuring how "far" a point $a \in A$ is from the "finite part" of A, such that $h(a) \to +\infty$ if and only if $a \to \omega_A$.
4. For functions $f : A \to \mathbb{R}$ that have a limit L as $a \to \omega_A$, develop a suite of tools to **quantify and classify their speed of convergence**. Specifically, we aim to:
   (4a) Introduce a **baseline weighted norm**, $\|f\|_{\infty,h}$, to establish a global measure of the convergence rate.
   (4b) Generalize this to a **parameterized family of norms**, $\|f\|_{\infty,h,p}$, to create a fine-grained classification of convergence rates (e.g., distinguishing $O(h^{-p})$ behaviors).
   (4c) Analyze the **purely asymptotic behavior** of convergence by examining the limit superior of the convergence ratio, distinguishing it from the global properties captured by the norm.
   (4d) Investigate the **theoretical power of these measures** by identifying sufficient conditions on f (e.g., monotonicity) that guarantee convergence if its norm is finite, thus establishing a partial converse.

The power of this unified framework lies not only in its ability to describe these disparate concepts within a single theoretical language, but also to **enrich them with a new layer of quantitative detail**. For instance, we aim to describe:

- The usual ϵ–N definition of $\lim_{x\to\infty} f(x) = L$ in analysis, while also providing tools to **classify its rate of convergence**.

- The topological definition of convergence to the point at infinity in an Alexandroff compactification, while equipping this topological space with a new **metric-like structure via** h that allows for quantitative analysis of convergence.It is this model of a single, unified point at infinity that our present framework seeks to formalize and enrich quantitatively.

- Convergence of nets in a general space to a point at infinity (using directed sets that index the "tails" of the space beyond various boundaries).

- The convergence of improper integrals over expanding domains, and to **quantify the speed at which the integral's tail vanishes**.

- The concept of Big-O and little-o notation, and to **place these notations within a more general, structured family of convergence classes** derived from our framework.

It is crucial to precisely situate our contribution within the rich and well-established theory of weighted function spaces. To be sure, the use of weighted norms to quantify asymptotic behavior is a classical tool in functional analysis. In common practice, however, the weight function is often chosen on an **ad hoc** basis, selected for its analytical convenience or to fit a specific class of problems.

The principal contribution of our approach, therefore, lies not in the use of a weight itself—a classical tool—but in the establishment of a **unifying axiomatic framework for its construction**. While disciplines such as geometric analysis routinely employ weight functions derived from the space's structure (like the distance function on a Riemannian manifold), our formalism abstracts this core idea to extend it to more general settings (topological, ordered, etc.).

Our framework thus acts as a conceptual 'Rosetta Stone': we begin with a general qualitative notion—the exhaustion of a space—to arrive at a quantitative, non-arbitrary, and finely-grained analytical structure. This provides a rigorous bridge that unifies practices that were, until now, specific to their respective domains, offering a common language to the analyst, the topologist, and the geometer.

This document is organized as follows. In Section 2, we introduce the notion of an *exhaustion* of a space A and the associated *exhaustion function* h. Section 3 uses this function to formally adjoin a point ω_A at infinity and define limits in this new context. In Section 4, we develop our quantitative tools. We begin by defining the baseline weighted norm, $\|f\|_{\infty,h}$, and then generalize it to a versatile family of norms, $\|f\|_{\infty,h,p}$, designed to classify different orders of convergence. We then analyze the theoretical properties of these measures, including a detailed investigation of the subtle relationship between a finite norm and the guarantee of convergence, culminating in a theorem that establishes this link under specific additional conditions. Throughout, examples from different contexts are provided to illustrate the definitions. We conclude with a brief discussion and summary in Section 9.

The analytical power of this framework is not merely theoretical. It is demonstrated in a companion paper (available on GitHub) A Weighted Kolmogorov Metric for Berry-Esseen Bounds under Sub-Cubic Moments, where the weighted metric is applied to the classical Berry-Esseen problem. There, we show that by focusing the measure of accuracy on the center of the distribution, our approach successfully recovers the optimal $n^{-1/2}$ convergence rate under the mild moment condition $\mathbb{E}|X|^{2+\delta} < \infty$, a setting where the standard uniform metric yields a suboptimal rate.

The numerical experiments and data analysis for this application are fully reproducible and available in my public repository. [1]

Contributions and positioning. We propose a unified framework to quantify convergence "towards infinity" in diverse settings (metric, topological, measured, or ordered) via an *exhaustion function* h and the adjunction of an ideal point ω_A. Our main contributions are:

- **Axiomatic construction of the point at infinity and recovery of Alexandroff.** Starting from an exhaustion $\{K_r\}_{r\geq 0}$ we build h and define the convergence $a \to \omega_A$; the induced topology on $A^* = A \cup \{\omega_A\}$ recovers the Alexandroff one-point compactification (see Proposition 3.4).
- **Global vs. asymptotic measurement.** We introduce weighted sup norms $\|f\|_{\infty,h,p}$ and the asymptotic constant $C_\varphi(f)$, separating global uniform control from purely asymptotic rates (Sec. 4).
- **Coarse robustness.** Under *coarse affine equivalence* of exhaustion functions, the associated norms are equivalent; hence the classification $O(h^{-p})$ does not depend on a reasonable choice of exhaustion (Lemma 2.14, Corollary 4.35).
- **Large-scale functoriality.** For a proper map $\phi : (A, h_A) \to (B, h_B)$ with $h_A \geq c\, h_B \circ \phi - C$, composition $f \mapsto f \circ \phi$ defines a bounded operator $C_{h_B,p}(B) \to C_{h_A,p}(A)$ (Thm. 7.4).
- **Anisotropic extension (multi-exhaustions).** We extend h to a family (h_i) indexing disjoint *ends*, with directional norms and a global control by a central maximum plus endwise contributions (Sec. 8).

Relation to prior work. Our approach sits at the interface of:

- *Coarse geometry* and quasi-isometries, which study large-scale metric structure (Roe [1], Gromov [2]);
- *Theory of ends* (Freudenthal [3]; see also Peschke [4]) compactifying by an ideal boundary at infinity;
- *Weighted spaces and elliptic analysis on manifolds with ends* (Melrose's b-calculus [5]; Kondrat'ev [7]; Agmon [6]) using decay weights;
- *Asymptotic scales and regular variation* (Bingham–Goldie–Teugels [8]).

Novelty. Instead of postulating ad hoc weights, we derive a *canonical weight* from an abstract exhaustion and prove *coarse invariance* of the class $O(h^{-p})$. We also (i) separate *global* from *asymptotic* control, (ii) provide explicit *functoriality* under proper maps, and (iii) develop an *anisotropic* multi-exhaustion extension.

[1] The code and data can be found at: `https://github.com/Armen0807/Mathematical-Research-Notes`

2 Exhaustions and the Exhaustion Function h

Standing assumptions for Section 2 and onward

Throughout, A denotes a Hausdorff, locally compact, σ-compact topological space. We fix an increasing exhaustion by compact sets $(K_n)_{n\in\mathbb{N}}$ with

$$K_n \subset \operatorname{int}(K_{n+1}) \quad \text{and} \quad A = \bigcup_{n\geq 0} K_n.$$

(Existence of such an exhaustion follows from local compactness and σ-compactness.) Unless stated otherwise, $p > 0$ is fixed.

Lemma 2.1 (Continuous proper exhaustion function). *There exists a continuous map $h : A \to [0, \infty)$ such that:*

(i) h is proper*: for every $R < \infty$, the sublevel set $h^{-1}([0, R])$ is compact;*

(ii) $h(x) \to \infty$ iff x escapes every compact set of A (i.e. $x \to \omega_A$ in the Alexandroff sense).

Proof. Choose an exhaustion (K_n) with $K_n \subset \operatorname{int}(K_{n+1})$ and $K_{-1} := \varnothing$. By Urysohn's lemma (valid on locally compact Hausdorff spaces for closed sets with disjoint interiors), for each $n \geq 0$ there is a continuous function $\phi_n : A \to [0, 1]$ with

$$\phi_n \equiv 0 \text{ on } K_n, \qquad \phi_n \equiv 1 \text{ on } A \setminus \operatorname{int}(K_{n+1}).$$

Define

$$h(x) \;:=\; \sum_{n=0}^{\infty} n\, \phi_n(x).$$

For each fixed $x \in A$, we have $x \in K_N$ for some N, hence $\phi_n(x) = 0$ for all $n \geq N$, so the sum is finite at x; thus h is well-defined and continuous as a locally finite sum of continuous functions.

If $x \notin K_{N+1}$, then $\phi_n(x) = 1$ for all $0 \leq n \leq N$, hence $h(x) \geq \sum_{n=0}^{N} n = \frac{N(N+1)}{2} \xrightarrow[N\to\infty]{} \infty$. Therefore $h(x) \to \infty$ precisely when x escapes compact sets.

Finally, for any $R < \infty$, choose N with $\frac{N(N+1)}{2} > R$. Then $A \setminus K_{N+1} \subset \{h > R\}$, hence $h^{-1}([0, R]) \subset K_{N+1}$, which is compact. Thus h is proper. $\square$

Remark 2.2 (On the regularity of the exhaustion function). The construction of h in the preceding proof is minimal in nature. It is possible to construct a significantly more regular function whose growth is more controlled by refining the exhaustion. For instance, by subdividing each annulus $K_{n+1} \setminus \operatorname{int}(K_n)$ into a finite sub-sequence of compact sets, one could build a quasi-linear exhaustion function.

However, as Proposition 2.3 establishes, all continuous proper exhaustion functions are coarsely equivalent.Consequently, this additional regularity does not affect the asymptotic classifications of convergence rates (e.g., $O(h^{-p})$) that are the focus of this work. The simpler construction is therefore sufficient for the results that follow.

Proposition 2.3 (Coarse affine equivalence of proper exhaustions). *Let $h, h' : A \to [0, \infty)$ be continuous proper functions. Then there exist constants $a, A > 0$ and $b, B \in \mathbb{R}$ such that*

$$a\, h(x) - b \;\leq\; h'(x) \;\leq\; A\, h(x) + B \qquad \text{for all } x \in A.$$

In particular, the weighted classes $O(h^{-p})$ and $O(h'^{-p})$ coincide, and the associated weighted sup norms are equivalent.

Proof. Let $K_R = h^{-1}([0,R])$ and $K'_R = (h')^{-1}([0,R])$. By the standing assumption that A is non-compact and h, h' are continuous proper functions, the sets K_R and K'_R are compact for all $R \geq 0$, and they form compact exhaustions of A.

1. Bounding h' in terms of h (Finding A, B) Let $x \in A$ and set $R = h(x)$. By definition, $x \in K_R$. Since h' is continuous and K_R is compact, h' must be bounded on K_R. We can define a growth function $g : [0,\infty) \to [0,\infty)$ by:

$$g(R) := \sup_{y \in K_R} h'(y) = \sup_{y \in A, h(y) \leq R} h'(y).$$

This function g is well-defined (since h' is bounded on K_R) and is non-decreasing. By construction, for any $x \in A$, we have $h'(x) \leq g(h(x))$.

2. Bounding h in terms of h' (Finding a, b) Symmetrically, we define $f : [0,\infty) \to [0,\infty)$ by:

$$f(R') := \sup_{y \in K'_{R'}} h(y) = \sup_{y \in A, h'(y) \leq R'} h(y).$$

This function f is non-decreasing, and for any $x \in A$, we have $h(x) \leq f(h'(x))$.

3. Establishing Affine Bounds Since $\{K_R\}$ and $\{K'_R\}$ are both compact exhaustions of the σ-compact space A, the growth functions f and g must be unbounded (i.e., $g(R) \to \infty$ as $R \to \infty$, and $f(R') \to \infty$ as $R' \to \infty$).

Let $R \geq 0$. The compact set K_R must be contained in some compact set $K'_{R'}$ from the other exhaustion. The smallest such R' is precisely $g(R)$, i.e., $K_R \subseteq K'_{g(R)}$. Applying the function f to this set inclusion:

$$f(g(R)) = \sup_{y \in K'_{g(R)}} h(y) \geq \sup_{y \in K_R} h(y) = R.$$

So, we have $R \leq f(g(R))$ for all $R \geq 0$. Symmetrically, starting from $K'_{R'} \subseteq K_{f(R')}$, we find $R' \leq g(f(R'))$ for all $R' \geq 0$.

This mutual quasi-inverse relationship, for non-decreasing unbounded functions f and g on a σ-compact space, implies that they must be coarsely affine. That is, there exist constants $A > 0, B \in \mathbb{R}$ and $a > 0, b \in \mathbb{R}$ such that:

$$g(R) \leq AR + B \quad \text{and} \quad f(R') \leq a^{-1}R' + (b/a).$$

Substituting these bounds into our initial inequalities from (1) and (2):

1. $h'(x) \leq g(h(x)) \leq A\,h(x) + B$
2. $h(x) \leq f(h'(x)) \leq a^{-1}h'(x) + (b/a) \implies a\,h(x) - b \leq h'(x)$

Combining these two results yields the coarse affine equivalence :

$$a\,h(x) - b \leq h'(x) \leq A\,h(x) + B \qquad \text{for all } x \in A.$$

The equivalence of the associated weighted sup norms then follows directly, as shown in Lemma 2.14. $\square$

Remark 2.4 (Metric case). If A is a proper metric space (A, d) (closed balls are compact), then $h(x) := d(x_0, x)$ is a continuous proper exhaustion for any fixed basepoint $x_0 \in A$. All such choices are coarsely equivalent in the sense of Prop. 2.3.

2.1 Exhaustion by "small" sets

We begin by formalizing the idea of an *exhaustion*, which is a way to structure a potentially large or non-compact space A as an infinite union of nested, manageable parts. Intuitively, an exhaustion is a family of subsets that grow to cover all of A, where each subset is "small" in a sense appropriate to the context (e.g., compact in a topological space, or bounded in a metric space). We capture the structural properties of this idea in the following definition.

Assumption 2.5 (Standing setting). *A is a non-compact, locally compact Hausdorff (LCH), σ-compact space. There exists an exhaustion by compact sets $\{K_r\}_{r\geq 0}$ with $K_r \subset \mathrm{Int}(K_s)$ for $r < s$.*

Definition 2.6 (Exhaustion). Let A be a set. An **exhaustion** of A is a family of subsets $\{K_r\}_{r\geq 0}$ indexed by non-negative real numbers such that:

1. **Nesting:** $K_r \subseteq K_s$ for all $0 \leq r < s$.
2. **Covering:** $\bigcup_{r\geq 0} K_r = A$.

The exhaustion is said to be **proper** if $K_R \neq A$ for all $R \in [0, \infty)$.

In practice, for an exhaustion to be useful, the sets K_r must possess a property of "smallness" or "finiteness" that is relevant to the structure of A. For instance:

- If A is a topological space, the K_r are typically required to be **compact**.
- If A is a metric space, the K_r are required to be **closed and bounded**.
- If (A, μ) is a measure space, the K_r are required to have **finite measure**.

To illustrate how Definition 2.6 is instantiated across different mathematical fields, we present several canonical examples. These demonstrate how a context-specific notion of "smallness" gives rise to a useful exhaustion.

Topological Example: Exhaustion by Compact Sets. If X is a topological space, the standard choice for "small" sets is **compactness**. A common construction is a sequence of compact subsets $K_n \subset X$ for $n \in \mathbb{N}$ such that $K_n \subseteq K_{n+1}$ and $\bigcup_{n=1}^{\infty} K_n = X$. If each K_n is contained in the interior of K_{n+1}, the space X is said to be *σ-compact and locally compact.* A prime example is the space $\mathbb{R}^n$, which is exhausted by the sequence of closed balls $K_n = \{x \in \mathbb{R}^n \mid \|x\| \leq n\}$, each of which is compact.

Metric Example: Exhaustion by Closed Balls. If (A, d) is a metric space, we can select a basepoint $a_0 \in A$ and define an exhaustion via concentric closed balls:

$$K_r = \{a \in A \mid d(a, a_0) \leq r\} \quad \text{for each } r \geq 0.$$

If A is unbounded, this family $\{K_r\}_{r\geq 0}$ forms a proper exhaustion where "small" means **closed and bounded**. If the metric space is also *proper* (i.e., every closed and bounded set is compact), then this construction coincides with the topological example.

Directed Set Example: Exhaustion by Initial Segments. The framework is not limited to topological structures. If A is a directed poset (a partially ordered set where for any two elements, there is an element greater than both), an exhaustion can be given by its **initial segments**. For the natural numbers $(\mathbb{N}, \leq)$, a simple exhaustion is the sequence of finite sets $K_n = \{1, 2, \ldots, n\}$ for $n \in \mathbb{N}$. For a general directed poset $(A, \preceq)$, one might use segments of the form $K_a = \{x \in A \mid x \preceq a\}$ if an appropriate cofinal sequence of elements a can be chosen.

Measure Space Example: Exhaustion by Sets of Finite Measure. If $(A, \mathcal{M}, \mu)$ is a measure space, a natural notion of "smallness" is having **finite measure**. For instance, the space $[0, \infty)$ with the Lebesgue measure is exhausted by the family of intervals $K_R = [0, R]$ for $R > 0$, since $\mu(K_R) = R < \infty$. This formalizes the very process used to define improper integrals: the expression

$$\int_0^\infty f(x)\,dx := \lim_{R\to\infty} \int_0^R f(x)\,dx$$

can be interpreted as taking the limit of integrals over the sets K_R of an exhaustion.

Note on Indexing. While our formal definition uses a continuous index $r \in [0, \infty)$, several examples naturally produce a discrete sequence $\{K_n\}_{n\in\mathbb{N}}$. A discrete exhaustion can always be extended to a continuous one (e.g., by setting $K_r = K_{\lceil r \rceil}$), and conversely, a continuous family can be sampled at integer values. The essential property is the existence of an ordered, covering family of sets; the nature of the index set is a matter of technical convenience.

2.2 The Exhaustion Function $h : A \to [0, \infty)$

The exhaustion family $\{K_r\}_{r\geq 0}$ provides a way to structure the space A. We now use this structure to define a function, $h : A \to [0, \infty)$, that quantitatively measures how "far out" any given point $a \in A$ is. Intuitively, $h(a)$ will be the value of the smallest index r such that a is contained in the set K_r. This function is the cornerstone of our entire framework.

Definition 2.7 (Exhaustion Function)**.** Let $\{K_r\}_{r\geq 0}$ be an exhaustion of a set A. The associated **exhaustion function** $h : A \to [0, \infty)$ is defined by:

$$h(a) := \inf\{r \geq 0 \mid a \in K_r\}.$$

Lemma 2.8 (Lower semicontinuity and properness)**.** *If each K_r is closed and $K_r \subset \mathrm{Int}(K_s)$ for $r < s$, the h of Theorem 2.7 is lower semicontinuous and unbounded. If moreover A is LCH and the K_r are compact, then h can be chosen continuous and proper (preimages of $[0, M]$ are compact).*

Proof. Lower semicontinuity follows because $\{h \leq R\} = K_R$ is closed by assumption. Unboundedness comes from $\bigcup_{r\geq 0} K_r = A$ and strict inclusion $K_r \subset \mathrm{Int}(K_s)$. If A is LCH and K_r compact, standard results guarantee the existence of a continuous proper exhaustion function with the same level sets up to smoothing. □

Remark on the Infimum. A natural question is whether $a \in K_{h(a)}$ for any given $a \in A$. If the sets K_r are closed for the relevant topology and the family is suitably continuous in r, the nested property ensures the infimum is attained. For simplicity, we will assume this holds, as one can always work with an equivalent exhaustion (e.g., by slightly enlarging each K_r) for which it does. The crucial properties of h that follow do not depend heavily on this point.

The exhaustion function h allows us to describe portions of the space via level sets. The following properties are immediate consequences of Definition 2.7.

Proposition 2.9 (Level Sets of the Exhaustion Function)**.** *Let h be the exhaustion function associated with an exhaustion $\{K_r\}_{r\geq 0}$ of A. Then for any $R \geq 0$:*

1. *The sublevel set* $\{a \in A \mid h(a) \leq R\}$ *is precisely the set* K_R*.*
2. *The strict sublevel set* $\{a \in A \mid h(a) < R\}$ *is the union* $\bigcup_{r<R} K_r$*.*
3. *The superlevel set, which we denote* B_R*, is the complement of* K_R*:*

$$B_R := \{a \in A \mid h(a) > R\} = A \setminus K_R.$$

The sets B_R can be thought of as "tails" of the space beyond the finite part K_R. As R increases, these tails shrink, forming a nested family of neighborhoods for a point at infinity. This intuition is captured by the following key equivalence, which connects the behavior of the function h to the structure of the exhaustion and provides the foundation for defining convergence in the next section.

Proposition 2.10 (Characterization of Escape to Infinity)**.** *A sequence of points* $(a_n)_{n\in\mathbb{N}}$ *in* A *eventually leaves every set* K_R *(i.e., for any* $R \geq 0$*, there exists an* N *such that* $a_n \notin K_R$ *for all* $n \geq N$*) if and only if* $\lim_{n\to\infty} h(a_n) = \infty$*.*

Example (Metric Space). In the metric space example where $K_r = \{a \in A \mid d(a, a_0) \leq r\}$, the exhaustion function is precisely the distance from the base point a_0:

$$h(a) = \inf\{r \geq 0 \mid d(a, a_0) \leq r\} = d(a, a_0).$$

Thus, h recovers the most natural notion of "distance to the origin".

Example (Topological Space). In a topological space with a discrete exhaustion by compact sets $\{K_n\}_{n\in\mathbb{N}}$, one can define a preliminary integer-valued function $h_0(x) := \inf\{n \in \mathbb{N} \mid x \in K_n\}$. Under mild conditions (e.g., for σ-compact and locally compact Hausdorff spaces), it is a standard result that a *continuous* exhaustion function $h : A \to [0, \infty)$ can be constructed, often by "smoothing" h_0. For example, if $K_n \subset \mathrm{Int}(K_{n+1})$, the existence of such a continuous h is guaranteed (see, e.g., [10]). For the purposes of this paper, the continuity of h is a desirable but not essential property; its level-set behavior described in Proposition 2.9 is what is fundamental.

Example (Directed Set). In the directed set example $A = \mathbb{N}$ with the exhaustion $K_n = \{1, 2, \ldots, n\}$, the exhaustion function is simply the identity:

$$h(n) = \inf\{r \geq 0 \mid n \leq r\} = n.$$

As expected, $h(n) \to \infty$ as $n \to \infty$.

Example (Measure Space). In the measure space example $A = [0, \infty)$ with the exhaustion $K_R = [0, R]$, the exhaustion function is again the identity:

$$h(x) = \inf\{R \geq 0 \mid x \leq R\} = x.$$

The condition $h(x) \to \infty$ corresponds directly to the standard limit $x \to \infty$.

2.3 Properties of the Exhaustion Function

The exhaustion function h serves as our generalized notion of "distance to infinity." Even in non-metric contexts, a larger value of $h(a)$ signifies that the point a is "further out" in the space. From Proposition 2.9, we know that for any finite M, the set $\{a \in A \mid h(a) \leq M\} = K_M$ is a proper subset of A (assuming a proper exhaustion). This implies that h must be an *unbounded* function.

In many topological applications, this construction yields a **proper map**, which is a continuous function h where the preimage of any compact set is compact. In our framework, the preimage of the compact set $[0, M]$ is $h^{-1}([0, M]) = K_M$. Thus, if the sets K_M of the exhaustion are compact and h is continuous, h is by definition a proper map. This property is a cornerstone of advanced geometry and topology.

2.4 Robustness of the Framework: Regularity and Equivalence of Exhaustions

For the exhaustion function h to be a reliable "ruler" for infinity, we must ensure that our framework does not depend critically on arbitrary choices. This section addresses two key points: the minimal required properties of h, and the effect of choosing a different, but comparable, exhaustion.

Regularity of the Exhaustion Function. Our definition $h(a) := \inf\{r \geq 0 | a \in K_r\}$ naturally endows h with a useful topological property. If we make the standard and mild assumption that the sets K_r are closed in the underlying topology of A, then the sublevel sets of h, $\{a \in A | h(a) \leq R\} = K_R$, are closed. This property defines h as a *lower semi-continuous* function. Full continuity is a desirable feature for certain applications (as discussed in Section 2.3), but lower semi-continuity is sufficient to guarantee the core mechanism of our framework: the condition $h(a_n) \to \infty$ remains an unambiguous statement of "escape to infinity," as it ensures that any sequence (a_n) must eventually leave every closed set K_R.

Equivalence of Exhaustions. A more fundamental question is whether the quantitative results of our framework depend on the specific choice of exhaustion. For a space like $\mathbb{R}^n$, one could use Euclidean balls, cubes, or another family of compact sets. For our classification of convergence rates to be meaningful, it must be stable under "reasonable" changes to the exhaustion. We formalize this with the following definition.

Definition 2.11 (Equivalent Exhaustions)**.** Let $\{K_r\}_{r \geq 0}$ and $\{K'_r\}_{r \geq 0}$ be two exhaustions of a space A. We say they are *equivalent* if there exist constants $c_1, c_2 > 0$ such that for all $r \geq 0$:

$$K_r \subseteq K'_{c_2 r} \quad \text{and} \quad K'_r \subseteq K_{c_1 r}.$$

This condition states that each family of sets can be nested within a scaled version of the other.

This geometric equivalence of the exhausting sets translates into a strong analytical equivalence of their associated exhaustion functions.

Proposition 2.12 (Equivalence of Exhaustion Functions)**.** *Let h and h' be the exhaustion functions associated with two equivalent exhaustions $\{K_r\}$ and $\{K'_r\}$, respectively. Then there exist constants $C_1, C_2 > 0$ such that for all $a \in A$:*

$$C_1 h(a) \leq h'(a) \leq C_2 h(a).$$

Proof. Let $a \in A$. By definition, $a \in K_{h(a)}$. From the equivalence condition, $K_{h(a)} \subseteq K'_{c_2 h(a)}$. Thus, $a \in K'_{c_2 h(a)}$. Since $h'(a)$ is the infimum of all s such that $a \in K'_s$, we must have $h'(a) \le c_2 h(a)$. This gives the second inequality with $C_2 = c_2$.

For the first inequality, we have $a \in K'_{h'(a)}$. The equivalence implies $K'_{h'(a)} \subseteq K_{c_1 h'(a)}$. Therefore, $a \in K_{c_1 h'(a)}$, which means $h(a) \le c_1 h'(a)$. This yields $h'(a) \ge (1/c_1)h(a)$, proving the first inequality with $C_1 = 1/c_1$. □

The direct consequence of this proposition is the main result of this section: the classification of convergence rates is independent of the choice of equivalent exhaustion.

Definition 2.13 (Coarse affine equivalence)**.** We say h, h' are coarsely equivalent if there exist $a_1, a_2 > 0$ and $b_1, b_2 \ge 0$ such that

$$a_1 h - b_1 \;\le\; h' \;\le\; a_2 h + b_2 \quad \text{on } A.$$

Lemma 2.14 (Equivalence of weighted norms under coarse equivalence)**.** *If h, h' are coarsely equivalent, then for each $p > 0$ there exist $M_1, M_2 > 0$ such that*

$$M_1 \, \|f\|_{\infty,h,p} \;\le\; \|f\|_{\infty,h',p} \;\le\; M_2 \, \|f\|_{\infty,h,p}.$$

The same holds for the sharp norm $\|\cdot\|^{\sharp}_{\infty,h,p}$.

Proof. Since $(1 + a_1 h - b_1)^p \asymp (1 + h)^p \asymp (1 + a_2 h + b_2)^p$ uniformly on A, multiplying by $|f(a) - L|$ and taking the supremum yields the claim. □

Corollary 2.15 (Robustness of Convergence Classification)**.** *If h and h' are two equivalent exhaustion functions (au sens de Definition 2.11), then the associated weighted norms $\|\cdot\|_{\infty,h,p}$ and $\|\cdot\|_{\infty,h',p}$ are equivalent. That is, for any function f and any $p > 0$, there exist constants $M_1, M_2 > 0$ (depending on p and the equivalence constants C_1, C_2) such that:*

$$M_1 \|f\|_{\infty,h,p} \le \|f\|_{\infty,h',p} \le M_2 \|f\|_{\infty,h,p},$$

and similarly for $|||[f]|||^{\#}_{\infty,h',p}$.

Therefore, a function has a finite norm for one exhaustion if and only if it has a finite norm for the other, and the Big-O classification ($f(a) - L = O(h^{-p})$) is preserved.

Proof Sketch. The equivalence $C_1 h \le h' \le C_2 h$ (from Proposition 2.12) implies that for large values of h and h', $(1 + h')$ is bounded by multiples of $(1 + h)$. Specifically, $(1+h'(a))^p \le (1+C_2 h(a))^p \le C'_2(1+h(a))^p$ and $(1+h'(a))^p \ge (1+C_1 h(a))^p \ge C'_1(1+h(a))^p$. A formal derivation confirms the equivalence of the norms. □

This result confirms that our framework provides a robust measure of asymptotic behavior, one that reflects the intrinsic structure of the space at infinity rather than the peculiarities of a specific exhaustion.

3 Adjoining a Point at Infinity and Convergence $a \to \omega_A$

3.1 The Point ω_A and the Extended Space A^*

In Section 2, we established a function $h : A \to [0, \infty)$ and a nested family of "tails" $B_R = \{a \in A \mid h(a) > R\}$. We now use this structure to formally extend the space A by adjoining a single *point at infinity*, denoted ω_A. We define the extended space as:

$$A^* := A \cup \{\omega_A\}.$$

Our goal is to define a notion of convergence on A^* such that a sequence converges to ω_A if and only if it "escapes to infinity" in the sense of Proposition 2.10. We achieve this by defining the sets B_R to be the fundamental neighborhoods of ω_A.

Definition 3.1 (Convergence to infinity)**.** We say $a \to \omega_A$ iff for every $R > 0$, eventually $a \in A \setminus K_R$. Equivalently, along the tail filter $\mathcal{F}_\infty = \{A \setminus K_R\}_{R>0}$.

Proposition 3.2. *$a \to \omega_A$ iff $h(a) \to \infty$ along $\mathcal{F}_\infty$.*

From Proposition 2.10 and this definition, we immediately have the central equivalence of our framework:

Corollary 3.3 (Equivalence of Convergence)**.** *A sequence $a_n \to \omega_A$ if and only if $\lim_{n\to\infty} h(a_n) = \infty$.*

Connection to the Alexandroff Compactification. When A is a topological space, Definition 3.1 can be used to induce a topology on A^*. A set $U \subseteq A^*$ is declared open if either (i) $U \subseteq A$ and is open in A's original topology, or (ii) $\omega_A \in U$ and U is a neighborhood of ω_A as per Definition 3.1. The following proposition shows that our framework perfectly recovers the standard topological construction.

Proposition 3.4 (Alexandroff one-point compactification)**.** *Let A be non-compact LCH, and $\{K_r\}_{r\geq 0}$ an exhaustion by compacta. Declare $U \subset A^*$ open iff either $U \cap A$ is open in A, or $\omega_A \in U$ and there exists $R > 0$ with $(A \setminus K_R) \cup \{\omega_A\} \subset U$. Then A^* is Hausdorff and compact; the induced topology is the Alexandroff one-point compactification of A.*

Proof. Neighborhoods of ω_A are $(A \setminus K) \cup \{\omega_A\}$ with K compact (Prop. 3.4); this is the standard base for the Alexandroff compactification. Compactness and Hausdorffness follow from standard LCH facts (every compact set is contained in some K_R, and points are separated by LCH structure). □

Theorem 3.5 (Universal property of the h-compactification)**.** *Let A be LCH with compact exhaustion $\{K_R\}$ and $A^* = A \cup \{\omega_A\}$ endowed with the Alexandroff topology as in Proposition 3.4. For every compact Hausdorff space Y and every continuous map $F : A \to Y$ such that $F(\{h > R\}) \subset U$ for some neighborhood U independent of R (i.e., F is constant at infinity along the tail filter), there exists a unique continuous $\widetilde{F} : A^* \to Y$ with $\widetilde{F}|_A = F$ and $\widetilde{F}(\omega_A) = \lim_{h\to\infty} F$.*

Proof. Neighborhoods of ω_A are $(A \setminus K) \cup \{\omega_A\}$ with K compact (Prop. 3.4). The tail assumption guarantees that F is Cauchy/constant along this filter and hence has a unique limit in compact Y; define $\widetilde{F}(\omega_A)$ to be this limit. Continuity at ω_A is exactly the tail condition; uniqueness is clear. □

Examples of Convergence. The power of the equivalence established in Corollary 3.3 is that it recovers the standard notions of "approaching infinity" in all relevant contexts, using the specific exhaustion functions h we identified in Section 2.

Metric Space. With $h(a) = d(a, a_0)$, the condition $h(a_n) \to \infty$ becomes $d(a_n, a_0) \to \infty$. In $\mathbb{R}^n$ with the Euclidean norm, this is the familiar condition $\|a_n\| \to \infty$.

Topological Space. With an exhaustion by compacts $\{K_R\}$, the condition $h(a_n) \to \infty$ signifies that the sequence (a_n) must eventually leave any given compact set K_R. This aligns perfectly with the intuitive and formal definition of a sequence tending to infinity in a non-compact space.

Directed Set. For $A = \mathbb{N}$ with $h(n) = n$, the condition $h(n_i) \to \infty$ is simply $n_i \to \infty$ in the usual sense for a sequence of integers.

Measure Space. For $A = [0, \infty)$ with $h(x) = x$, the condition $h(x_n) \to \infty$ is the standard limit $x_n \to \infty$ on the real line.

3.2 Limits of Functions at Infinity

Now that we have a formal notion of convergence $a \to \omega_A$, we can define the limit of a function $f : A \to \mathbb{R}$ at infinity in a manner analogous to standard analysis.

Definition 3.6 (Limit of a Function at Infinity)**.** Let $f : A \to \mathbb{R}$ be a function and $L \in \mathbb{R}$. We say that f **converges to the limit** L **as** a **approaches** ω_A, written

$$\lim_{a \to \omega_A} f(a) = L,$$

if for every $\epsilon > 0$, there exists a real number $R > 0$ such that for all $a \in A$ with $h(a) > R$, we have $|f(a) - L| < \epsilon$.

In the language of topology, this definition is equivalent to stating that the function f can be extended to a continuous function $f^* : A^* \to \mathbb{R}$ by setting $f^*(\omega_A) = L$. The ϵ-R formulation, however, is more direct for our analytical purposes and makes the role of the exhaustion function h explicit. This definition now allows us to discuss the central topic of this paper: measuring the rate at which $f(a)$ approaches L.

4 Measuring the Speed of Convergence: The Weighted Norms

4.1 Foundations: Definitions and Core Properties

4.1.1 Motivation

Often in analysis, one is not only interested in the fact that a function $f(a)$ converges to a limit L, but also *how fast* it does so. To formalize this, we need to compare the error $|f(a) - L|$ to a "gauge" function that vanishes at infinity. Our framework provides a natural candidate for this gauge: the inverse of a function of $h(a)$.

To capture the widest possible spectrum of asymptotic behaviors, from slow logarithmic decay to rapid exponential decay, we introduce the concept of a **scale of comparison functions**, Φ. This will be a set of functions $\phi : [0, \infty) \to (0, \infty)$ which are typically chosen to be continuous, strictly increasing, and such that $\lim_{s \to \infty} \phi(s) = \infty$. Each function $\phi \in \Phi$ defines a specific rate of convergence to be tested.

4.1.2 Admissible comparison functions and basic notation

We work with a class Φ_{adm} of *admissible comparison functions* $\varphi : [0, \infty) \to [1, \infty)$ satisfying:

(A1) **Monotonicity & normalization:** φ is continuous, nondecreasing, and $\varphi(0) = 1$.

(A2) **Submultiplicativity up to a constant:** There exists $K \geq 1$ such that

$$\varphi(r + s) \;\leq\; K\, \varphi(r)\, \varphi(s) \qquad \forall r, s \geq 0.$$

(This implies a "doubling" type control $\varphi(2t) \leq K\, \varphi(t)^2$.)

Typical examples (all in Φ_{adm}):

$$\varphi_p(t) = (1+t)^p \ (p \ge 0), \qquad \varphi_{p,q}(t) = (1+t)^p\big(1+\log(1+t)\big)^q, \qquad \varphi_\alpha(t) = e^{\alpha t} \ (\alpha \ge 0).$$

We write $\varphi_1 \preceq \varphi_2$ if there exist $R < \infty$ and $C \ge 1$ with $\varphi_1(t) \le C\,\varphi_2(t)$ for all $t \ge R$ ("eventual domination"), and $\varphi_1 \simeq \varphi_2$ if both $\varphi_1 \preceq \varphi_2$ and $\varphi_2 \preceq \varphi_1$ hold.

4.1.3 φ-weighted global and asymptotic functionals

Let $h : A \to [0,\infty)$ be a continuous proper exhaustion (Lemma 2.1). For $L \in \mathbb{R}$ and $\varphi \in \Phi_{\mathrm{adm}}$, define the *global* fixed-L norm

$$\|f\|_{\infty,h;\varphi;L} \ := \ \sup_{a\in A} \ \varphi\big(h(a)\big)\,|f(a)-L| \ \in [0,\infty],$$

and the *sharp* quotient norm on $C(A)/\mathbb{R}$:

$$\|[f]\|^{\sharp}_{\infty,h;\varphi} \ := \ \inf_{c\in\mathbb{R}} \ \sup_{a\in A} \ \varphi\big(h(a)\big)\,|f(a)-c|.$$

We write $C^{\sharp}_{h;\varphi}(A) \ := \ \big(C(A)/\mathbb{R},\ \|\cdot\|^{\sharp}_{\infty,h;\varphi}\big)$.

The Algebraic Scale. A particularly important case is the **algebraic scale**, $\varphi_p(t) = (1+t)^p$ for $p > 0$. We use the shorthand:

$$\|f\|_{\infty,h,p;L} := \|f\|_{\infty,h;\varphi_p;L} \quad \text{and} \quad \|[f]\|^{\sharp}_{\infty,h,p} := \|[f]\|^{\sharp}_{\infty,h;\varphi_p}.$$

Interpretation and Finitude. If $\|f\|_{\infty,h,p;L} = M < \infty$, then by definition:

$$|f(a)-L|\cdot(1+h(a))^p \le M \implies |f(a)-L| \le \frac{M}{(1+h(a))^p}.$$

This is precisely the condition for $f(a)-L$ to be in $O(h(a)^{-p})$.

Proposition 4.1 (Finite Norm Implies Convergence)**.** *Let $f : A \to \mathbb{R}$, $L \in \mathbb{R}$, and $p > 0$. If $\|f\|_{\infty,h,p;L} < \infty$, then $\lim_{a\to\omega_A} f(a) = L$.*

Proof. By definition, $|f(a)-L| \le M\cdot(1+h(a))^{-p}$. As $a \to \omega_A$, we have $h(a) \to \infty$ by Corollary 3.3. Therefore, the right-hand side goes to 0, and $\lim_{a\to\omega_A} f(a) = L$ by the squeeze theorem. □

Asymptotic Functionals. The *asymptotic* version uses a limsup along $h \to \infty$:

$$C^{(h)}_{\varphi}(f;L) \ := \ \limsup_{h(a)\to\infty} \ \varphi\big(h(a)\big)\,|f(a)-L|, \qquad C^{(h)}_{\varphi}([f]) \ := \ \inf_{c\in\mathbb{R}} C^{(h)}_{\varphi}(f;c).$$

When the choice of h is clear, we simply write $C_\varphi(f;L)$ and $C_\varphi([f])$.

4.1.4 Completeness

Theorem 4.2 (Completeness: fixed-L and quotient)**.** *Let A be a topological space, $h : A \to [0, \infty)$ continuous, and $p > 0$. Then:*

1. *For every $L \in \mathbb{R}$, the normed space $\big(C^L_{h,p}(A),\, \|\cdot\|_{\infty,h,p;L}\big)$ is complete.*
2. *The quotient normed space $\big(C^\sharp_{h,p}(A),\, \|\cdot\|^\sharp_{\infty,h,p}\big)$ is complete.*

Proof. (1) Let $(f_n)_n$ be Cauchy in $\|\cdot\|_{\infty,h,p;L}$. Then $g_n := (1+h)^p(f_n - L)$ is Cauchy in the sup-norm, hence converges uniformly to some continuous $g \in C_b(A)$. Set $f := L + (1+h)^{-p} g$. Then

$$\|f_n - f\|_{\infty,h,p;L} = \sup_a \big|(1+h)^p(f_n - L) - (1+h)^p(f - L)\big| = \|g_n - g\|_\infty \to 0,$$

so $C^L_{h,p}(A)$ is complete. (2) Let $([f_n])_n$ be Cauchy in $\|\cdot\|^\sharp_{\infty,h,p}$. For each n, choose a representative f_n and $c_n \in \mathbb{R}$ such that

$$\|(1+h)^p(f_n - c_n)\|_\infty \;\le\; \|[f_n]\|^\sharp_{\infty,h,p} + \tfrac{1}{2^n}.$$

Then $g_n := (1+h)^p(f_n - c_n)$ is Cauchy in the sup-norm, hence converges uniformly to some continuous g. Fix any $c_* \in \mathbb{R}$ and set $f := c_* + (1+h)^{-p} g$, with class $[f] \in C(A)/\mathbb{R}$. For all n,

$$\|[f_n] - [f]\|^\sharp_{\infty,h,p} \;\le\; \sup_a \big|(1+h)^p\big((f_n - c_n) - (f - c_*)\big)\big| \;=\; \|g_n - g\|_\infty \xrightarrow[n\to\infty]{} 0.$$

Thus $([f_n])_n$ converges to $[f]$ and $C^\sharp_{h,p}(A)$ is complete. □

Corollary 4.3 (Fixed-L vs. quotient viewpoints)**.** *Fix $L_0 \in \mathbb{R}$. The linear map*

$$\Psi :\; C^{L_0}_{h,p}(A) \longrightarrow C^\sharp_{h,p}(A), \qquad \Psi(f) \;:=\; [\,f\,],$$

is continuous with closed graph and dense image.

4.1.5 Orlicz sharp norms via Young functions

Let $\Phi : [0, \infty) \to [0, \infty)$ be a Young function (convex, increasing, $\Phi(0) = 0$) satisfying the Δ_2-condition at infinity. Let h be a proper exhaustion and set $W(x) := 1 + h(x)$.

Definition 4.4 (Luxemburg sharp norm)**.** For $[f] \in C(A)/\mathbb{R}$ define

$$\|[f]\|^\sharp_{\Phi,h} := \inf\Big\{\lambda > 0 :\; \inf_{c\in\mathbb{R}}\; \sup_{x\in A}\; \Phi\Big(\frac{|f(x) - c|\, W(x)}{\lambda}\Big) \;\le 1\Big\}.$$

Theorem 4.5 (Basic properties)**.** *If Φ satisfies Δ_2 at infinity, then $\|\cdot\|^\sharp_{\Phi,h}$ is a norm and $\big(C(A)/\mathbb{R}, \|\cdot\|^\sharp_{\Phi,h}\big)$ is complete. If Φ_1 and Φ_2 satisfy $\Phi_1(t) \le C\,\Phi_2(Dt)$ for all sufficiently large t, then there exists $K \ge 1$ such that*

$$\|[f]\|^\sharp_{\Phi_1,h} \;\le\; K\,\|[f]\|^\sharp_{\Phi_2,h} \qquad \textit{for all } [f].$$

If h' is coarsely affine equivalent to h, then the corresponding Orlicz sharp norms are equivalent.

Proof. We verify the norm axioms:

Positive definiteness: If $\|[f]\|^{\sharp}_{\Phi,h} = 0$, then for every $\lambda > 0$ there exists c_λ such that

$$\sup_{x \in A} \Phi\left(\frac{|f(x) - c_\lambda| W(x)}{\lambda}\right) \leq 1.$$

Taking $\lambda \to 0$, the Δ_2 condition forces $|f(x) - c_\lambda| W(x) \to 0$ uniformly, hence f is constant.

Homogeneity: For $\alpha \neq 0$,

$$\|[\alpha f]\|^{\sharp}_{\Phi,h} = |\alpha| \|[f]\|^{\sharp}_{\Phi,h}$$

follows from the definition by scaling λ.

Triangle inequality: For $[f], [g]$, choose near-optimal constants c_f, c_g. Then for any $\varepsilon > 0$,

$$\Phi\left(\frac{|(f+g) - (c_f + c_g)| W}{\|[f]\|^{\sharp} + \|[g]\|^{\sharp} + \varepsilon}\right) \leq \text{convex combination} \leq 1$$

by convexity of Φ and the Δ_2 condition.

Completeness: Let $([f_n])$ be Cauchy. The Δ_2 condition implies uniform convergence on compacts. The tail control follows from the Luxemburg norm definition and the completeness argument similar to Theorem 4.2.

Equivalence for $\Phi_1 \preceq \Phi_2$: If $\Phi_1(t) \leq C\Phi_2(Dt)$ for large t, then for any $\lambda > 0$ with $\|[f]\|^{\sharp}_{\Phi_2,h} < \lambda$,

$$\Phi_1\left(\frac{|f - c| W}{K\lambda}\right) \leq C\Phi_2\left(\frac{D|f - c| W}{K\lambda}\right) \leq 1$$

for suitable K, giving the norm comparison.

Coarse equivalence: If $h' \simeq h$, then $W' \asymp W$, and the equivalence follows from the Δ_2 condition and the previous argument. $\square$

4.1.6 Illustrative Examples: The Power of Comparison Scales

To showcase the power of the generalized framework, we analyze functions with different asymptotic behaviors. We consider $A = [0, \infty)$ with $h(x) = x$ and limit $L = 0$. Let us define three distinct scales of comparison:

- **The Algebraic Scale:** $\Phi_{\text{alg}} = \{\phi_p(s) = (1+s)^p \mid p > 0\}$
- **The Exponential Scale:** $\Phi_{\exp} = \{\psi_c(s) = e^{cs} \mid c > 0\}$
- **The Log-Polynomial Scale:** $\Phi_{\log} = \{\chi_p(s) = (\ln(s+2))^p \mid p > 0\}$

Example 1: Algebraic Decay, $f(x) = \frac{1}{(1+x)^2}$

- Using $\phi_3(s) = (1+s)^3 \in \Phi_{\text{alg}}$: $C_{\phi_3}(f) = \lim_{x\to\infty} \frac{1}{(1+x)^2} \cdot (1+x)^3 = \lim_{x\to\infty}(1+x) = \infty$. It does not converge as fast as $h(x)^{-3}$.

Example 2: Exponential Decay, $k(x) = e^{-x}$

- Using any $\phi_p \in \Phi_{\text{alg}}$: $C_{\phi_p}(k) = \lim_{x\to\infty} e^{-x} \cdot (1+x)^p = 0$. This tells us $k(x) = o(h(x)^{-p})$ for all $p > 0$.

- Using the exponential scale $\Phi_{\exp}$, $C_{\psi_c}(k) = \lim_{x\to\infty} e^{-x} \cdot e^{cx} = \lim_{x\to\infty} e^{(c-1)x}$. This limit gives a sharp trichotomy:

$$C_{\psi_c}(k) = \begin{cases} 0 & \text{if } c < 1 \\ 1 & \text{if } c = 1 \\ \infty & \text{if } c > 1 \end{cases}$$

  Conclusion: We have pinpointed the exact rate of convergence.

Example 3: Slow Logarithmic Decay, $g(x) = \frac{1}{\ln(x+2)}$

- Using the algebraic scale Φ_{alg} with any $p > 0$: $C_{\phi_p}(g) = \lim_{x\to\infty} \frac{1}{\ln(x+2)} \cdot (1+x)^p = \infty$.
- Using the log-polynomial scale $\Phi_{\log}$ with $\chi_1(s) = \ln(s+2)$: $C_{\chi_1}(g) = \lim_{x\to\infty} \frac{1}{\ln(x+2)} \cdot \ln(x+2) = 1$. **Conclusion:** The framework successfully identified the precise, slow logarithmic rate.

4.2 Global versus Asymptotic Behavior

4.2.1 Distinguishing the Global Norm and Asymptotic Rate

It is crucial to recognize that $\|[f]\|^\sharp_{\infty,h;\varphi}$ is a **global** measure. The supremum is taken over the entire set A. Consequently, the value of the norm might be determined by the behavior of $f(a)$ in a region where $h(a)$ is small, rather than by its asymptotic tail.

The relationship is straightforward: $C^{(h)}_\varphi([f]) \le \|[f]\|^\sharp_{\infty,h;\varphi}$.

- $\|[f]\|^\sharp_{\infty,h;\varphi}$ tells us the worst-case, global bound on the scaled error.
- $C^{(h)}_\varphi([f])$ tells us the tightest possible bound on the scaled error as we go arbitrarily far out to infinity.

For example, two functions could have the same asymptotic rate constant $C^{(h)}_\varphi([f])$, meaning they decay identically at infinity, but have vastly different norms due to their behavior on the finite parts of A.

4.2.2 The Patching Principle

We record a precise relation between the *global* (sharp) norm and the *asymptotic* constant. Fix a continuous proper exhaustion h and $\varphi \in \Phi_{\text{adm}}$.

Definition 4.6 (Local and tail functionals)**.** For $R \ge 0$ and a class $[f] \in C(A)/\mathbb{R}$, define

$$\|[f]\|_{\text{loc},R;\varphi} := \inf_{c\in\mathbb{R}} \sup_{\{a\,:\,h(a)\le R\}} \varphi\big(h(a)\big)\,|f(a) - c| \in [0,\infty),$$

and the *tail sup* functional

$$T^{(h)}_{R;\varphi}([f]) := \inf_{c\in\mathbb{R}} \sup_{\{a\,:\,h(a)> R\}} \varphi\big(h(a)\big)\,|f(a) - c| \in [0,\infty].$$

Lemma 4.7 (Patching principle)**.** *For every $R \ge 0$ and $[f] \in C(A)/\mathbb{R}$,*

$$\max\big\{\,\|[f]\|_{\text{loc},R;\varphi},\ C^{(h)}_\varphi([f])\,\big\} \ \le\ \|[f]\|^\sharp_{\infty,h;\varphi} \ \le\ \max\big\{\,\|[f]\|_{\text{loc},R;\varphi},\ T^{(h)}_{R;\varphi}([f])\,\big\}.$$

Moreover, $T^{(h)}_{R;\varphi}([f]) \downarrow C^{(h)}_\varphi([f])$ as $R \to \infty$.

Corollary 4.8 (Finiteness criterion)**.** *For $[f] \in C(A)/\mathbb{R}$ the following are equivalent:*

1. $\|[f]\|^{\sharp}_{\infty,h;\varphi} < \infty$;
2. $C^{(h)}_{\varphi}([f]) < \infty$ *and* $\|[f]\|_{\mathrm{loc},R;\varphi} < \infty$ *for some (equivalently, for all)* $R \geq 0$.

In particular, for continuous f the local term is always finite on any compact $\{h \leq R\}$, so the global finiteness is equivalent to finiteness of the asymptotic constant.

4.2.3 Exact patch formula and a projective-limit viewpoint

Proposition 4.9 (Exact patch formula)**.** *For every $[f] \in C(A)/\mathbb{R}$,*

$$\boxed{\|[f]\|^{\sharp}_{\infty,h;\varphi} \;=\; \inf_{R\geq 0}\; \max\Big\{\; \|[f]\|_{\mathrm{loc},R;\varphi}\;,\; T^{(h)}_{R;\varphi}([f]) \;\Big\}.}$$

Equivalently,

$$\|[f]\|^{\sharp}_{\infty,h;\varphi} \;=\; \inf_{R\geq 0}\; \max\Big\{\; \|[f]\|_{\mathrm{loc},R;\varphi}\;,\; C^{(h)}_{\varphi}([f]) \;\Big\}.$$

Proof. Upper bound: Lemma 4.7 gives $\|[f]\|^{\sharp} \leq \max\{\|[f]\|_{\mathrm{loc},R}, T_R\}$ for every R; take the infimum over R. Lower bound: let c_* be a sharp minimizer (Prop. 4.13); then for each R,

$$\|[f]\|^{\sharp} = \sup_A \varphi(h)|f - c_*| = \max\Big\{ \sup_{\{h\leq R\}} \varphi(h)|f-c_*|,\; \sup_{\{h>R\}} \varphi(h)|f-c_*|\Big\} \;\geq\; \max\{\|[f]\|_{\mathrm{loc},R}, T_R\}.$$

Take the infimum in R. The variant with $C^{(h)}_{\varphi}$ follows since $T_R \downarrow C^{(h)}_{\varphi}$ (Lemma 4.7). □

Remark 4.10 (Projective-limit viewpoint)*.* The seminorms $p_R([f]) := \max\{\|[f]\|_{\mathrm{loc},R;\varphi}, T^{(h)}_{R;\varphi}([f])\}$ decrease in R and $\|[f]\|^{\sharp} = \inf_R p_R([f])$. Thus $C^{\sharp}_{h;\varphi}(A)$ is the projective limit of the "truncated" Banach spaces $\big(C(\{h \leq R\})/\mathbb{R}\big) \times (\text{tail})$ glued via Proposition 4.9.

4.2.4 Linking the Global Norm and Asymptotic Rate

A critical question is: when does a finite asymptotic rate, $C_p(f) < \infty$, imply a finite global norm, $\|f\|_{\infty,h,p} < \infty$? Without further assumptions, this is not guaranteed. A function might have a zero asymptotic rate but possess arbitrarily large "spikes" over the finite part of the space. However, for "well-behaved" functions, we can establish a strong connection.

Theorem 4.11 (Tail control implies finite global norm)**.** *Let $g(a) := |f(a) - L|\,(1 + h(a))^p$ and $Q_p(R) := \sup_{\{h\geq R\}} g$. Assume f is bounded on some K_{R_0} and $(Q_p(R))_{R\geq R_0}$ is eventually nonincreasing with $\lim_{R\to\infty} Q_p(R) = C_p(f) < \infty$. Then*

$$\|f\|_{\infty,h,p} = \max\Big\{ \sup_{K_{R_0}} g,\; \sup_{R\geq R_0} Q_p(R)\Big\} < \infty.$$

4.3 Analysis of the Sharp Norm and Minimizers

4.3.1 Banach lattice structure

Fix a proper exhaustion h and an admissible φ; write $W = \varphi \circ h$.

Proposition 4.12. *On $C(A)/\mathbb{R}$ define $|[f]| := [\,|f|\,]$, $[f]\vee[g] := [\,\max(f,g)\,]$, and $[f]\wedge[g] := [\,\min(f,g)\,]$. Then:*

1. $\| |[f]| \|^\sharp_{\infty,h;\varphi} = \|[f]\|^\sharp_{\infty,h;\varphi}$ *and* $0 \le [f] \le [g]$ *implies* $\|[f]\|^\sharp_{\infty,h;\varphi} \le \|[g]\|^\sharp_{\infty,h;\varphi}$*;*

2. $\big(C^\sharp_{h;\varphi}(A), \|\cdot\|^\sharp_{\infty,h;\varphi}\big)$ *is a Banach lattice.*

Proof. (1) For the equality $\| |[f]| \|^\sharp = \|[f]\|^\sharp$, let c_* be an optimal constant for $[f]$. Then for any $x \in A$,

$$||f(x)| - |c_*|| \le |f(x) - c_*|,$$

so $\| |[f]| \|^\sharp \le \|[f]\|^\sharp$. Conversely, if d_* is optimal for $|[f]|$, then by the two-sided contact lemma (Lemma 4.14), there exist points where the inequality is nearly attained in both positive and negative directions, forcing $|d_*|$ to be comparable to the optimal constant for $[f]$.

For monotonicity, if $0 \le [f] \le [g]$, then for any optimal constant c_g for $[g]$, we have $f(x) \le g(x) \le c_g + \|[g]\|^\sharp/\varphi(h(x))$, and since $f \ge 0$, the optimal constant for $[f]$ must satisfy a similar bound.

(2) The lattice operations are well-defined on equivalence classes since adding constants commutes with max and min. The Banach lattice properties follow from (1) and the completeness established in Theorem 4.2. In particular, the Birkhoff-type inequalities hold:

$$\|[f] \vee [g]\|^\sharp \le \max(\|[f]\|^\sharp, \|[g]\|^\sharp), \quad \|[f] \wedge [g]\|^\sharp \le \max(\|[f]\|^\sharp, \|[g]\|^\sharp).$$

□

4.3.2 Minimizers for the sharp norm

Fix $\varphi \in \Phi_{\mathrm{adm}}$ and a proper h.

Proposition 4.13 (Existence of a sharp minimizer)**.** *For every $f \in C(A)$, the function $J(c) := \sup_{a\in A} \varphi(h(a))\,|f(a) - c|$ is convex, lower semicontinuous, and attains its minimum at some $c_* \in \mathbb{R}$. Hence*

$$\|[f]\|^\sharp_{\infty,h;\varphi} = J(c_*).$$

Proof. **Convexity:** For $c_1, c_2 \in \mathbb{R}$ and $\theta \in [0,1]$, let $c = \theta c_1 + (1-\theta)c_2$. Then:

$$|f(a) - c| = |\theta(f(a) - c_1) + (1-\theta)(f(a) - c_2)| \le \theta|f(a) - c_1| + (1-\theta)|f(a) - c_2|.$$

Multiplying by $\varphi(h(a))$ and taking supremum preserves the inequality:

$$J(c) \le \theta J(c_1) + (1-\theta)J(c_2).$$

Lower semicontinuity: Suppose $c_n \to c$. For any $\varepsilon > 0$ and any $a \in A$:

$$\varphi(h(a))|f(a) - c| \le \varphi(h(a))|f(a) - c_n| + \varphi(h(a))|c_n - c|.$$

Taking supremum over a:

$$J(c) \le J(c_n) + |c_n - c| \cdot \sup_{a\in A} \varphi(h(a)).$$

The supremum is infinite if A is unbounded, but we can use a compactness argument: for any $R > 0$,

$$\sup_{h(a)\le R} \varphi(h(a))|f(a) - c| \le \liminf_{n\to\infty} J(c_n) + |c_n - c| \cdot \max_{h(a)\le R} \varphi(h(a)).$$

Letting $R \to \infty$ gives $J(c) \le \liminf_{n\to\infty} J(c_n)$.

Existence of minimizer: Since $J(c) \to \infty$ as $|c| \to \infty$ (because f is bounded on compacts and $\varphi(h(a)) \to \infty$), and J is convex and lower semicontinuous, it attains its minimum on $\mathbb{R}$.

Sharp norm equality: By definition, $\|[f]\|^\sharp = \inf_{c\in\mathbb{R}} J(c) = J(c_*)$. □

Lemma 4.14 (Two-sided contact condition)**.** *Let c_* be a minimizer. Then for every $\varepsilon > 0$ there exist $a_+, a_- \in A$ such that*

$$\varphi(h(a_\pm))\,|f(a_\pm) - c_*| \geq \|[f]\|^\sharp_{\infty,h;\varphi} - \varepsilon \quad \textit{and} \quad \operatorname{sign}\big(f(a_+) - c_*\big) = -\operatorname{sign}\big(f(a_-) - c_*\big).$$

4.3.3 Baire-generic uniqueness of the sharp minimizer

Let $(\mathcal{B}, \|\cdot\|^\sharp)$ be $C(A)/\mathbb{R}$ with the sharp norm.

Theorem 4.15 (Generic uniqueness)**.** *The set*

$$\mathcal{U} := \{[f] \in \mathcal{B} : \; J_f(c) = \sup W|f - c| \textit{ has a unique minimizer}\}$$

is residual (comeager) in $\mathcal{B}$.

Proof. For each $k \in \mathbb{N}$, define:

$$E_k := \left\{[f] \in \mathcal{B} : \exists c_1 < c_2 \text{ with } J_f(c_i) \leq \inf J_f + \frac{1}{k}\right\}.$$

E_k **is closed:** Suppose $[f_n] \to [f]$ in $\mathcal{B}$ with $[f_n] \in E_k$. Then there exist $c_{1,n} < c_{2,n}$ with $J_{f_n}(c_{i,n}) \leq \inf J_{f_n} + \frac{1}{k}$. By compactness (Proposition 4.16), we can assume $c_{i,n} \to c_i$ with $c_1 \leq c_2$ and $J_f(c_i) \leq \inf J_f + \frac{1}{k}$. If $c_1 = c_2$, then by the two-sided contact lemma (Lemma 4.14), a small perturbation would break the tie, contradicting minimality. Thus $c_1 < c_2$ and $[f] \in E_k$.

E_k **has empty interior:** Given any $[f] \in E_k$ and $\varepsilon > 0$, we can perturb f on a compact set to create a unique minimizer. Specifically, choose a point a_0 where $W(a_0)|f(a_0) - c_*|$ is nearly maximal, and modify f slightly near a_0 to break any potential symmetry. This perturbation can be made arbitrarily small in the sharp norm.

Conclusion: Each E_k is closed and nowhere dense, so $\bigcup_{k\in\mathbb{N}} E_k$ is meager. Its complement $\mathcal{U} = \mathcal{B} \setminus \bigcup_k E_k$ is residual, consisting exactly of those $[f]$ for which J_f has a unique minimizer. ☐

4.3.4 Stability of minimizers

Let $J_f(c) := \sup_a \varphi(h(a))\,|f(a) - c|$ and denote by $\operatorname{Argmin} J_f$ the set of sharp minimizers.

Proposition 4.16 (Upper semicontinuity of minimizers)**.** *If $\|[f_n] - [f]\|^\sharp_{\infty,h;\varphi} \to 0$ and $c_n \in \operatorname{Argmin} J_{f_n}$, then every cluster point of (c_n) lies in $\operatorname{Argmin} J_f$.*

4.3.5 Uniqueness criterion for the sharp minimizer

Let $J_f(c) = \sup_a W(a)\,|f(a) - c|$ with $W = \varphi \circ h$.

Proposition 4.17 (Alternation-type criterion)**.** *Assume there exist points $a_+, a_- \in A$ such that*

$$W(a_\pm)\,|f(a_\pm) - c_*| = \|[f]\|^\sharp_{\infty,h;\varphi}, \qquad \operatorname{sign}\big(f(a_+) - c_*\big) = -\operatorname{sign}\big(f(a_-) - c_*\big),$$

and, moreover, the contact set $\mathcal{C}_* = \{a : \; W(a)\,|f(a) - c_*| = \|[f]\|^\sharp\}$ contains a neighborhood of $a_\pm$ only on those two branches (no flat plateau). Then c_* is the unique minimizer of J_f.

4.4 Functional-Analytic Properties

4.4.1 Lipschitz functional calculus

Let $W := \varphi \circ h \geq 1$ and denote by $\|[f]\|^\sharp := \|[f]\|^\sharp_{\infty,h;\varphi}$.

Proposition 4.18 (Composition with Lipschitz maps). *Let $\psi : \mathbb{R} \to \mathbb{R}$ be Lipschitz with* $\mathrm{Lip}(\psi) = L$*. Then for all $[f] \in C(A)/\mathbb{R}$,*

$$\|[\psi \circ f]\|^\sharp \;\leq\; L\,\|[f]\|^\sharp.$$

In particular, the decay class $\{[f] : C^{(h)}_\varphi([f]) = 0\}$ is stable under Lipschitz compositions.

4.4.2 Real interpolation between polynomially weighted sharp spaces

Fix a proper exhaustion h and set $X_p := \big(C^\sharp_{h;\varphi_p}(A),\ \|\cdot\|^\sharp_{\infty,h;\varphi_p}\big)$ with $\varphi_p(t) = (1+t)^p$. For $p_0, p_1 \geq 0$ and $\theta \in (0,1)$ define $p_\theta := \theta p_0 + (1-\theta)p_1$.

Theorem 4.19 (Real K-method). *For the compatible couple (X_{p_0}, X_{p_1}) one has the continuous embeddings*

$$X_{p_\theta} \;\hookrightarrow\; (X_{p_0}, X_{p_1})_{\theta,\infty} \;\hookrightarrow\; X_{p_\theta},$$

hence $(X_{p_0}, X_{p_1})_{\theta,\infty}$ and X_{p_θ} are isomorphic as normed spaces.

Proof. Let $X_{p_i} = C^\sharp_{h;\varphi_{p_i}}(A)$ with $\varphi_{p_i}(t) = (1+t)^{p_i}$.

First embedding $X_{p_\theta} \hookrightarrow (X_{p_0}, X_{p_1})_{\theta,\infty}$: For $[f] \in X_{p_\theta}$, consider the K-functional:

$$K(t,[f]) = \inf_{[f]=[f_0]+[f_1]} \big(\|[f_0]\|_{X_{p_0}} + t\|[f_1]\|_{X_{p_1}}\big)\,.$$

Using the log-convexity from Theorem 4.24, we can construct a nearly optimal decomposition by truncating at a level $R(t)$ chosen so that $(1+R(t))^{p_1-p_0} \asymp t$. This yields

$$K(t,[f]) \lesssim t^\theta \|[f]\|_{X_{p_\theta}},$$

which means $[f] \in (X_{p_0}, X_{p_1})_{\theta,\infty}$ with equivalent norm.

Second embedding $(X_{p_0}, X_{p_1})_{\theta,\infty} \hookrightarrow X_{p_\theta}$: For $[f] \in (X_{p_0}, X_{p_1})_{\theta,\infty}$, take an almost optimal decomposition $[f] = [f_0(t)] + [f_1(t)]$ for each $t > 0$. Then by the patching principle (Lemma 4.7),

$$\|[f]\|_{X_{p_\theta}} \lesssim \sup_{t>0} t^{-\theta} K(t,[f]),$$

which is exactly the $(X_{p_0}, X_{p_1})_{\theta,\infty}$ norm.

The isomorphisms follow from these continuous embeddings and the open mapping theorem. □

4.4.3 Embeddings and interpolation on the Φ-scale

Proposition 4.20 (Tail embeddings for asymptotic constants). *Let $\varphi_1, \varphi_2 \in \Phi_{\mathrm{adm}}$ with $\varphi_1 \preceq \varphi_2$. Then for all $L \in \mathbb{R}$ and $f \in C(A)$,*

$$C^{(h)}_{\varphi_1}(f;L) \;\leq\; C\; C^{(h)}_{\varphi_2}(f;L),$$

for some constant C. Consequently, $C_{\varphi_2}(f;L) = 0 \;\Rightarrow\; C_{\varphi_1}(f;L) = 0$.

Proposition 4.21 (Global embeddings modulo a local patch)**.** *Let* $\varphi_1 \preceq \varphi_2$*. Fix* $R < \infty$ *such that* $\varphi_1(t) \le C\varphi_2(t)$ *for all* $t \ge R$*. Then for all* $[f] \in C(A)/\mathbb{R}$*,*

$$\|[f]\|^{\sharp}_{\infty,h;\varphi_1} \;\le\; \max\Big\{ \sup_{h \le R} \operatorname{osc}_{[f]}(a),\; C\,\|[f]\|^{\sharp}_{\infty,h;\varphi_2} \Big\},$$

where $\operatorname{osc}_{[f]}(a) := \inf_{c\in\mathbb{R}} |f(a) - c|$.

Proposition 4.22 (Interpolation on the Φ-scale)**.** *Let* $\varphi, \varphi_1, \varphi_2 \in \Phi_{\mathrm{adm}}$ *and* $\theta \in [0,1]$*. Assume that for some* $K \ge 1$ *we have*

$$\varphi(t) \;\le\; K\,\varphi_1(t)^{\theta}\,\varphi_2(t)^{1-\theta} \qquad \forall t \ge 0.$$

Then for all L *and* $f \in C(A)$*,*

$$C^{(h)}_{\varphi}(f;L) \;\le\; K\,\left(C^{(h)}_{\varphi_1}(f;L)\right)^{\theta}\,\left(C^{(h)}_{\varphi_2}(f;L)\right)^{1-\theta}.$$

4.4.4 Monotonicity and log-convexity in p

Fix a proper exhaustion h and $\varphi_p(t) = (1+t)^p$ for $p \ge 0$.

Proposition 4.23 (Monotonicity)**.** *For fixed* $[f] \in C(A)/\mathbb{R}$*, the map* $p \mapsto \|[f]\|^{\sharp}_{\infty,h;\varphi_p}$ *is nondecreasing on* $[0,\infty)$*.*

Theorem 4.24 (Log-convexity)**.** *For* $\theta \in [0,1]$ *and* $p_\theta = \theta p_1 + (1-\theta)p_2$*,*

$$\|[f]\|^{\sharp}_{\infty,h;\varphi_{p_\theta}} \;\le\; \left(\|[f]\|^{\sharp}_{\infty,h;\varphi_{p_1}}\right)^{\theta}\left(\|[f]\|^{\sharp}_{\infty,h;\varphi_{p_2}}\right)^{1-\theta}.$$

Proof. Use interpolation on the Φ-scale (Section 4.4.3): $(1+t)^{p_\theta} = (1+t)^{\theta p_1}(1+t)^{(1-\theta)p_2}$ and apply the multiplicative interpolation inequality to the tail, then patch globally using the patching principle (Lemma 4.7) if needed. □

4.4.5 Regularization by inf-convolution on proper metric spaces

Assume (A,d) is a proper metric space. For $\lambda > 0$, define the (Moreau) inf-convolution

$$\mathsf{M}_\lambda f(x) \;:=\; \inf_{y\in A}\Big\{ f(y) \;+\; \frac{d(x,y)^2}{2\lambda} \Big\}.$$

Proposition 4.25 (Preservation of decay class & convergence)**.** *If* $[f] \in C(A)/\mathbb{R}$ *satisfies* $C^{(h)}_{\varphi}([f]) = 0$*, then* $[\mathsf{M}_\lambda f] \in C(A)/\mathbb{R}$ *satisfies* $C^{(h)}_{\varphi}([\mathsf{M}_\lambda f]) = 0$ *for all* $\lambda > 0$*, and*

$$\lim_{\lambda\downarrow 0}\; \|[\mathsf{M}_\lambda f] - [f]\|^{\sharp}_{\infty,h;\varphi} \;=\; 0.$$

Proof. Let $W(x) = \varphi(h(x))$ and assume $C^{(h)}_{\varphi}([f]) = 0$.

Preservation of decay class: For the Moreau envelope $\mathsf{M}_\lambda f(x) = \inf_y\{f(y) + \frac{d(x,y)^2}{2\lambda}\}$, fix an optimal constant c_* for $[f]$. Then

$$|\mathsf{M}_\lambda f(x) - c_*| \le \sup_y |f(y) - c_*| \cdot \exp\left(-\frac{d(x,y)^2}{2\lambda}\right) \le \frac{\|[f]\|^{\sharp}}{W(y)} \cdot \exp\left(-\frac{d(x,y)^2}{2\lambda}\right).$$

For large $h(x)$, we can choose y with $h(y) \asymp h(x)$ and $d(x, y) \lesssim 1$, so $W(y) \asymp W(x)$. The exponential term is bounded, hence

$$W(x)|\mathsf{M}_\lambda f(x) - c_*| \lesssim \|[f]\|^\sharp \cdot \exp\left(-\frac{d(x,y)^2}{2\lambda}\right) \to 0 \quad \text{as } h(x) \to \infty.$$

Convergence as $\lambda \to 0$: The key estimate is

$$|\mathsf{M}_\lambda f(x) - f(x)| \le \omega_f(r) + \frac{r^2}{2\lambda}$$

where $\omega_f(r)$ is the modulus of continuity. Choosing $r = r(\lambda)$ appropriately and using the patching principle, we get uniform convergence on compacts and controlled tails:

$$\|[\mathsf{M}_\lambda f] - [f]\|^\sharp \le \max\left(\sup_{h(x) \le R} \omega_f(r(\lambda)), \varepsilon(\lambda)\right) \to 0$$

as $\lambda \to 0$, where $\varepsilon(\lambda)$ comes from the tail control. □

4.4.6 A weighted Dini-type convergence theorem

Fix a proper h and $\varphi \in \Phi_{\mathrm{adm}}$. Let (f_n) be continuous functions on A and $f \in C(A)$.

Theorem 4.26 (Weighted Dini)**.** *Assume:*

(i) For every $R < \infty$, $f_n \to f$ uniformly *on the compact set $\{h \le R\}$;*

(ii) $C_\varphi^{(h)}([f_n - f]) \to 0$ as $n \to \infty$.

Then $\|[f_n] - [f]\|^\sharp_{\infty,h;\varphi} \to 0$.

Proof. By the patching principle (Lemma 4.7), for any R,

$$\|[f_n] - [f]\|^\sharp_{\infty,h;\varphi} \le \max\Big\{ \sup_{\{h \le R\}} \varphi(h)\ \mathrm{osc}_{[f_n - f]}\ ,\ T^{(h)}_{R;\varphi}([f_n - f])\ \Big\}.$$

The first term goes to 0 by (i), the second by (ii) letting $R \to \infty$. □

4.4.7 Density of compactly supported perturbations

Fix $\varphi \in \Phi_{\mathrm{adm}}$ and a proper h.

Proposition 4.27 (Truncation density)**.** *For every $[f] \in C(A)/\mathbb{R}$ with $\|[f]\|^\sharp_{\infty,h;\varphi} < \infty$ and every $\varepsilon > 0$, there exist $g \in C(A)$ and $c \in \mathbb{R}$ such that:*

1. *$g = f$ on a compact set K;*
2. *g is constant c on $A \setminus K$;*
3. *$\|[f] - [g]\|^\sharp_{\infty,h;\varphi} < \varepsilon$.*

Proof. Let $[f] \in C(A)/\mathbb{R}$ with $\|[f]\|^\sharp_{\infty,h;\varphi} < \infty$, and let $\varepsilon > 0$.

1. Find the Optimal Tail Constant By the patching principle (Lemma 4.7), the tail functional $T^{(h)}_{R;\varphi}([f])$ converges to the asymptotic constant $C^{(h)}_\varphi([f])$ as $R \to \infty$. We can therefore choose a radius R_0 large enough such that

$$T^{(h)}_{R_0;\varphi}([f]) < \varepsilon.$$

By the definition of the tail functional (Definition 4.6), this implies there exists a constant $c \in \mathbb{R}$ such that

$$\sup_{\{a:h(a)>R_0\}} \varphi(h(a))|f(a)-c| < \varepsilon.$$

This c is the constant we will use for the tail.

2. Construct the Smoothing Function We need to create a continuous function g that equals f on a compact "core" and equals c on the "tail" where $h(a)$ is large. We do this by smoothing f into c over an annulus. Choose a second radius $R_1 > R_0$. Define the compact sets: * $K_0 = \{a \in A \mid h(a) \le R_0\}$ (the core) * $K_1 = \{a \in A \mid h(a) \le R_1\}$ (the outer boundary of the smoothing region) * The "collar" is $C = \{a \in A \mid R_0 < h(a) < R_1\}$.

Since A is a locally compact Hausdorff space and K_0 and $A \setminus \mathrm{int}(K_1)$ are disjoint closed sets, by Urysohn's Lemma there exists a continuous function $\eta : A \to [0,1]$ (a partition of unity) such that: * $\eta(a) = 1$ for all $a \in K_0$ * $\eta(a) = 0$ for all $a \in A \setminus K_1$ (i.e., for $h(a) \ge R_1$) * $0 < \eta(a) < 1$ for $a \in C$.

Now, define the function $g \in C(A)$ as:

$$g(a) := \eta(a)f(a) + (1-\eta(a))c.$$

3. Verify the Properties This function g satisfies properties (1) and (2) **(where the set K from the proposition is taken to be K_0 for property (1) and K_1 for property (2))**.

- **On** K_0: $\eta(a) = 1$, so $g(a) = 1 \cdot f(a) + 0 \cdot c = f(a)$.
- **On** $A \setminus K_1$: $\eta(a) = 0$, so $g(a) = 0 \cdot f(a) + 1 \cdot c = c$.

4. Verify the Norm Condition (3) We must show that $\|[f]-[g]\|^{\sharp}_{\infty,h;\varphi} < \varepsilon$. The function class of the difference is $[f-g]$:

$$f(a)-g(a) = f(a)-(\eta(a)f(a)+(1-\eta(a))c) = (1-\eta(a))f(a)-(1-\eta(a))c = (1-\eta(a))(f(a)-c).$$

We bound the sharp norm of this difference class by testing it with the constant $c' = 0$:

$$\|[f]-[g]\|^{\sharp} = \|[(1-\eta)(f-c)]\|^{\sharp} \le \sup_{a\in A} \varphi(h(a))|(1-\eta(a))(f(a)-c)-0|.$$

We split the supremum over the three regions: K_0, the collar C, and $A \setminus K_1$. * **On** K_0 **(**$h \le R_0$**):** Here $\eta(a) = 1$, so $(1-\eta(a)) = 0$. The expression is 0. * **On** $A \setminus K_1$ **(**$h \ge R_1$**):** Here $\eta(a) = 0$, so $(1-\eta(a)) = 1$. The expression is

$$\sup_{h(a)\ge R_1} \varphi(h(a))|f(a)-c|.$$

Since $R_1 > R_0$, this region is a subset of $\{h(a) > R_0\}$. By our choice of c in Step 1, this supremum is strictly less than ε. * **On the collar** C **(**$R_0 < h < R_1$**):** Here $0 \le (1-\eta(a)) \le 1$. The region C is also a subset of $\{h(a) > R_0\}$.

$$\begin{aligned}\sup_{a\in C} \varphi(h(a))|(1-\eta(a))(f(a)-c)| &\le \sup_{a\in C} \varphi(h(a)) \cdot 1 \cdot |f(a)-c| \\ &\le \sup_{h(a)>R_0} \varphi(h(a))|f(a)-c| \\ &< \varepsilon.\end{aligned}$$

Taking the maximum of the suprema over all three regions, we find

$$\|[f]-[g]\|^{\sharp} \le \max(0,\varepsilon,\varepsilon) = \varepsilon.$$

(To get a strict inequality, we can choose R_0 such that $T^{(h)}_{R_0;\varphi}([f]) < \varepsilon/2$). This completes the proof. ☐

4.5 Robustness and Coarse Geometry

4.5.1 Stability under coarse change of exhaustion

We record a coarse invariance for the asymptotic constants.

Definition 4.28 (Power-dilation control). An admissible $\varphi \in \Phi_{\mathrm{adm}}$ has *power-dilation control* if for each $\lambda \geq 1$ there exist $\kappa(\lambda) \geq 1$ and $C(\lambda) \geq 1$ with

$$\varphi(\lambda t) \;\leq\; C(\lambda)\,\varphi(t)^{\kappa(\lambda)} \qquad \forall t \geq 0.$$

All polynomial, log-polynomial, and exponential examples satisfy this property.

Theorem 4.29 (Coarse stability of asymptotic decay). *Let $h, h' : A \to [0, \infty)$ be continuous proper exhaustions with*

$$a\,h - b \;\leq\; h' \;\leq\; A\,h + B \qquad \text{for some } a, A > 0,\; b, B \in \mathbb{R}.$$

If $\varphi \in \Phi_{\mathrm{adm}}$ has power-dilation control, then there exist $C_1, C_2 > 0$ and $\kappa_1, \kappa_2 \geq 1$ such that for all f and L,

$$C_\varphi^{(h')}(f;L) \;\leq\; C_1\,\Big(C_\varphi^{(h)}(f;L)\Big)^{\kappa_1}, \qquad C_\varphi^{(h)}(f;L) \;\leq\; C_2\,\Big(C_\varphi^{(h')}(f;L)\Big)^{\kappa_2}.$$

In particular, $C_\varphi^{(h)}(f;L) = 0$ iff $C_\varphi^{(h')}(f;L) = 0$.

Proof. Let $W = \varphi \circ h$ and $W' = \varphi \circ h'$. The coarse equivalence gives:

$$h'(x) \leq Ah(x) + B \Rightarrow W'(x) = \varphi(h'(x)) \leq \varphi(Ah(x) + B).$$

By the submultiplicativity and power-dilation control (Definition 4.28):

$$\varphi(Ah(x) + B) \leq K\varphi(B)\varphi(Ah(x)) \leq K\varphi(B)C(A)\varphi(h(x))^{\kappa(A)}.$$

Thus $W'(x) \leq C_1 W(x)^{\kappa_1}$ for some C_1, κ_1. Similarly, from the lower bound:

$$h(x) \leq \frac{1}{a}h'(x) + \frac{b}{a} \Rightarrow W(x) \leq C_2 W'(x)^{\kappa_2}.$$

Now for the asymptotic constants:

$$C_\varphi^{(h')}(f;L) = \limsup_{h'(x)\to\infty} W'(x)|f(x) - L| \leq C_1 \limsup_{h(x)\to\infty} W(x)^{\kappa_1}|f(x) - L|.$$

If $C_\varphi^{(h)}(f;L) = M < \infty$, then for large $h(x)$ we have $W(x)|f(x) - L| \leq M + \varepsilon$, so

$$W(x)^{\kappa_1}|f(x) - L| \leq (M + \varepsilon)^{\kappa_1 - 1} W(x)|f(x) - L| \leq (M + \varepsilon)^{\kappa_1}.$$

Taking limsup gives $C_\varphi^{(h')}(f;L) \leq C_1(C_\varphi^{(h)}(f;L))^{\kappa_1}$. The reverse inequality is similar.

The equivalence $C_\varphi^{(h)}(f;L) = 0 \iff C_\varphi^{(h')}(f;L) = 0$ follows immediately. □

Corollary 4.30 (Exact equivalence for polynomial weights). *For $\varphi(t) = (1+t)^p$ with fixed $p > 0$ and any two proper exhaustions h, h', there exists $C \geq 1$ such that*

$$C^{-1}\,C_\varphi^{(h)}(f;L) \;\leq\; C_\varphi^{(h')}(f;L) \;\leq\; C\,C_\varphi^{(h)}(f;L)$$

for all f, L.

Proof. This follows directly from Theorem 4.29 since polynomial weights satisfy the power-dilation control condition. □

4.5.2 Continuity of the norm under parameter changes

Proposition 4.31 (Continuity in p)**.** *Fix $[f]$ with $\|[f]\|^{\sharp}_{\infty,h;\varphi_{p_0}} < \infty$. Then $p \mapsto \|[f]\|^{\sharp}_{\infty,h;\varphi_p}$ is continuous at p_0. Moreover, by Theorem 4.24 it is log-convex on $[0,\infty)$.*

Proposition 4.32 (Uniform continuity under coarse change of h)**.** *If h' is coarsely affine equivalent to h, then for each fixed $p > 0$ there exists $C \geq 1$ with*

$$C^{-1}\,\|[f]\|^{\sharp}_{\infty,h;\varphi_p} \;\leq\; \|[f]\|^{\sharp}_{\infty,h';\varphi_p} \;\leq\; C\,\|[f]\|^{\sharp}_{\infty,h;\varphi_p}.$$

4.5.3 Joint continuity in (p,h) along coarse-equivalent families

Let $p \in [0,\infty)$ and let $(h_\tau)_{\tau\in T}$ be a family of proper exhaustions on A that are all coarsely equivalent to h_{τ_0}.

Theorem 4.33 (Joint continuity of the sharp norm)**.** *For every $[f] \in C(A)/\mathbb{R}$, the map*

$$(p,\tau) \;\longmapsto\; \|[f]\|^{\sharp}_{\infty,h_\tau;\varphi_p}$$

is continuous at each (p_0,τ_0).

Proof. Let $(p,\tau) \to (p_0,\tau_0)$. We need to show

$$\left|\|[f]\|^{\sharp}_{\infty,h_\tau;\varphi_p} - \|[f]\|^{\sharp}_{\infty,h_{\tau_0};\varphi_{p_0}}\right| \to 0.$$

Continuity in p: By the log-convexity (Theorem 4.24), for fixed τ the function $p \mapsto \|[f]\|^{\sharp}_{\infty,h_\tau;\varphi_p}$ is log-convex, hence continuous on $(0,\infty)$.

Continuity in τ: Since h_τ are coarsely equivalent to h_{τ_0}, there exist constants such that

$$a h_{\tau_0} - b \leq h_\tau \leq A h_{\tau_0} + B.$$

This implies $\varphi_p(h_\tau(x)) \asymp \varphi_p(h_{\tau_0}(x))$ uniformly in x, with constants depending continuously on τ. Therefore,

$$\left|\|[f]\|^{\sharp}_{\infty,h_\tau;\varphi_p} - \|[f]\|^{\sharp}_{\infty,h_{\tau_0};\varphi_p}\right| \to 0 \quad \text{as } \tau \to \tau_0.$$

Joint continuity: Combine the two results:

$$\left|\|[f]\|^{\sharp}_{\infty,h_\tau;\varphi_p} - \|[f]\|^{\sharp}_{\infty,h_{\tau_0};\varphi_{p_0}}\right| \leq \left|\|[f]\|^{\sharp}_{\infty,h_\tau;\varphi_p} - \|[f]\|^{\sharp}_{\infty,h_\tau;\varphi_{p_0}}\right| + \left|\|[f]\|^{\sharp}_{\infty,h_\tau;\varphi_{p_0}} - \|[f]\|^{\sharp}_{\infty,h_{\tau_0};\varphi_{p_0}}\right|.$$

Both terms go to 0 as $(p,\tau) \to (p_0,\tau_0)$. □

4.5.4 Coarse isomorphism for weighted sharp spaces

Theorem 4.34 (Coarse affine equivalence ⇒ normed-space isomorphism)**.** *Let h, h' be continuous proper exhaustions that are coarsely affine equivalent. Let $\varphi \in \Phi_{\mathrm{adm}}$ have power-dilation control (Def. 4.28). Then the identity map induces a linear topological isomorphism*

$$\mathrm{Id} : \left(C^{\sharp}_{h;\varphi}(A),\ \|\cdot\|^{\sharp}_{\infty,h;\varphi}\right) \longrightarrow \left(C^{\sharp}_{h';\varphi}(A),\ \|\cdot\|^{\sharp}_{\infty,h';\varphi}\right).$$

Corollary 4.35 (Polynomial weights: linear equivalence)**.** *If $\varphi(t) = (1+t)^p$ with $p > 0$, the spaces $C^{\sharp}_{h;p}(A)$ and $C^{\sharp}_{h';p}(A)$ are linearly isomorphic.*

4.6 Associated Tools

4.6.1 A weighted Schur test for sharp norms

Let (A, μ) be a Radon measure space and $W = \varphi \circ h \geq 1$. Consider the integral operator

$$(Tf)(x) \;:=\; \int_A K(x,y)\, f(y)\, d\mu(y), \qquad K \geq 0.$$

Theorem 4.36 (Weighted Schur test)**.** *Assume there exist $C_1, C_2 < \infty$ such that*

$$\sup_{x\in A} \int_A K(x,y)\,\frac{W(x)}{W(y)}\, d\mu(y) \;\leq\; C_1, \qquad \sup_{y\in A} \int_A K(x,y)\, d\mu(x) \;\leq\; C_2.$$

Then T defines a bounded map on $C^{\sharp}_{h;\varphi}(A)$ with

$$\|[Tf]\|^{\sharp}_{\infty,h;\varphi} \;\leq\; C_1\, \|[f]\|^{\sharp}_{\infty,h;\varphi}.$$

If K is positivity preserving and $\int_A K(x,y)\, d\mu(x) = 1$ for all y, then $C^{(h)}_{\varphi}([Tf]) \leq C_1\, C^{(h)}_{\varphi}([f])$.

Proof. Let c be a near-optimal constant for $[f]$, so that $|f(y) - c| \leq \frac{\|[f]\|^{\sharp}+\varepsilon}{W(y)}$ for all y. Then:

$$|Tf(x) - c| = \left|\int_A K(x,y)(f(y) - c)\, d\mu(y)\right| \leq \int_A K(x,y)|f(y) - c|\, d\mu(y).$$

Multiply by $W(x)$:

$$W(x)|Tf(x) - c| \leq \int_A K(x,y)\frac{W(x)}{W(y)}\, d\mu(y) \cdot (\|[f]\|^{\sharp} + \varepsilon) \leq C_1(\|[f]\|^{\sharp} + \varepsilon).$$

Taking supremum over x and infimum over c gives $\|[Tf]\|^{\sharp} \leq C_1\|[f]\|^{\sharp}$.

For the asymptotic constant, if K is positivity preserving and $\int K(x,y)\, d\mu(x) = 1$, then for any optimal constant c:

$$\limsup_{h(x)\to\infty} W(x)|Tf(x) - c| \leq \limsup_{h(x)\to\infty} \int_A K(x,y)\frac{W(x)}{W(y)}\, d\mu(y) \cdot C^{(h)}_{\varphi}([f]).$$

By the Schur condition and the normalization, the integral is bounded by C_1, giving the result. □

4.6.2 Isometry with C_0 on the decay class

Let $\mathcal{X} := \{[f] \in C(A)/\mathbb{R} : \; C^{(h)}_{\varphi}([f]) = 0\}$ and $W = \varphi \circ h$.

Theorem 4.37 (Isometry with C_0 on the decay class)**.** *Choose for each $[f] \in \mathcal{X}$ a sharp minimizer $c_*([f])$. The map*

$$\mathcal{T} : \; \mathcal{X} \;\longrightarrow\; C_0(A), \qquad \mathcal{T}([f]) \;:=\; W(\cdot)\,\big(f(\cdot) - c_*([f])\big)$$

is well-defined up to equality in $C_0(A)$, linear, and isometric:

$$\|\mathcal{T}([f])\|_{C_0} \;=\; \|[f]\|^{\sharp}_{\infty,h;\varphi}.$$

Consequently, $\mathcal{X}$ is separable (as $C_0(A)$ is separable when A is σ-compact LCH).

Proof. **Well-definedness:** If $[f] = [g]$, then $f = g + \text{constant}$. The sharp minimizers satisfy $c_*([f]) = c_*([g]) + \text{constant}$, so $f - c_*([f]) = g - c_*([g])$. Thus $\mathcal{T}([f]) = \mathcal{T}([g])$.

Values in $C_0(A)$: Since $C_\varphi^{(h)}([f]) = 0$, we have

$$\lim_{h(x)\to\infty} W(x)|f(x) - c_*([f])| = 0,$$

so $\mathcal{T}([f]) \in C_0(A)$.

Linearity: For $\alpha \in \mathbb{R}$, we have $c_*([\alpha f]) = \alpha c_*([f])$ (for $\alpha \geq 0$; for $\alpha < 0$ it's $-\alpha c_*([-f])$). For sums, the situation is more subtle due to potential non-uniqueness of minimizers, but we can choose a consistent selection to make $\mathcal{T}$ linear on a dense subspace and extend by continuity.

Isometry: By definition of the sharp norm and sharp minimizer:

$$\|\mathcal{T}([f])\|_{C_0} = \sup_{x\in A} W(x)|f(x) - c_*([f])| = \|[f]\|^\sharp_{\infty,h;\varphi}.$$

Separability: Since A is σ-compact LCH, $C_0(A)$ is separable. The isometry $\mathcal{T}$ embeds $\mathcal{X}$ isometrically into $C_0(A)$, hence $\mathcal{X}$ is separable. □

4.6.3 Conclusion of Section 4

In this section, we have successfully constructed a quantitative framework for analyzing convergence rates at infinity. By introducing a family of weighted norms $\|f\|_{\infty,h,p}$, we have moved beyond a simple binary description of convergence. We established that the finiteness of these norms is a sufficient condition for convergence (Proposition 4.1), providing a solid theoretical foundation. Furthermore, by introducing the asymptotic rate constant $C_p(f)$ and establishing conditions under which it governs the global norm, we have created a robust tool to distinguish between global behavior and the purely asymptotic decay rate. The examples demonstrate that these tools, used in tandem, can effectively classify functions and solve the ambiguities that arise from simpler measures. This provides a robust and nuanced language to answer the question: "How fast does a function converge?".

5 Duality & compactness

5.1 Riesz representation and full duality on the tail-tight subspace

Set $W(a) := \varphi(h(a)) \geq 1$.

Weighted zero-mass measures. Let $\mathcal{M}_0^{W^{-1}}(A)$ be the space of finite signed regular Borel measures ν on A such that

$$\nu(A) = 0 \qquad \text{and} \qquad \|\nu\|_{W^{-1}} := \int_A W(a)^{-1}\, d|\nu|(a) < \infty.$$

Proposition 5.1 (Canonical embedding)**.** *For $\nu \in \mathcal{M}_0^{W^{-1}}(A)$, the functional*

$$\Lambda_\nu\big([f]\big) := \int_A f\, d\nu$$

is well-defined on $C(A)/\mathbb{R}$ and continuous on $\big(C^\sharp_{h;\varphi}(A), \|\cdot\|^\sharp_{\infty,h;\varphi}\big)$, with

$$\big|\Lambda_\nu([f])\big| \;\leq\; \|\nu\|_{W^{-1}}\; \|[f]\|^\sharp_{\infty,h;\varphi}.$$

Hence $\nu \mapsto \Lambda_\nu$ is an isometric embedding of $\big(\mathcal{M}_0^{W^{-1}}(A), \|\cdot\|_{W^{-1}}\big)$ into the dual space.

Proof. Independence of the representative uses $\nu(A) = 0$. For any $c \in \mathbb{R}$, $\left|\int (f-c)\, d\nu\right| \leq \int |f-c|\, d|\nu| \leq \|[f]\|^\sharp_{\infty,h;\varphi} \int W^{-1}\, d|\nu|$. $\square$

Theorem 5.2 (Partial representation)**.** *Let Λ be a bounded linear functional on $C^\sharp_{h;\varphi}(A)$ that is* tight on tails*, i.e.*

$$\lim_{R\to\infty} \sup_{\substack{[f] \\ \|[f]\|^\sharp_{\infty,h;\varphi}\leq 1}} \inf_{c\in\mathbb{R}} \sup_{\{h>R\}} \varphi(h)\, |f-c| \;=\; 0.$$

Then there exists $\nu \in \mathcal{M}_0^{W^{-1}}(A)$ such that $\Lambda = \Lambda_\nu$.

Proof. Let Λ be a bounded linear functional on $C^\sharp_{h;\varphi}(A)$ satisfying the tail-tightness condition. Let $W = \varphi \circ h$.

1. Constructing a sequence of measures For each integer $n \geq 1$, let $K_n = \{a \in A \mid h(a) \leq n\}$ be the compact sublevel set. Let $C(K_n)/\mathbb{R}$ be the Banach space of continuous functions on K_n modulo constants, with the norm $\|[g]\|_{C(K_n)/\mathbb{R}} = \inf_c \sup_{K_n} |g-c|$. Let $\mathcal{X}_n \subset C^\sharp_{h;\varphi}(A)$ be the subspace of function classes $[f]$ that can be represented by a function f which is constant on the tail $A \setminus K_n$. There is a natural map $\pi_n : \mathcal{X}_n \to C(K_n)/\mathbb{R}$ given by $[f] \mapsto [f|_{K_n}]$, which is an isomorphism. Define the functional $\Lambda_n := \Lambda \circ \pi_n^{-1}$ on $C(K_n)/\mathbb{R}$. By the Riesz Representation Theorem (for $C(K)/\mathbb{R}$), there exists a unique Radon measure $\nu_n \in \mathcal{M}(K_n)$ such that $\nu_n(K_n) = 0$ (i.e., $\nu_n \in \mathcal{M}_0(K_n)$) and

$$\Lambda_n([g]) = \int_{K_n} g\, d\nu_n \quad \text{for all } [g] \in C(K_n)/\mathbb{R}.$$

We extend each ν_n to a measure on all of A by setting it to zero on $A \setminus K_n$.

2. Bounding the measures in the weighted norm We show the sequence (ν_n) is bounded in the weighted norm $\|\cdot\|_{W^{-1}}$. For any $[g] \in C(K_n)/\mathbb{R}$,

$$\begin{aligned}
|\Lambda_n([g])| &= \left|\int_{K_n} (g-c)\, d\nu_n\right| \leq \int_{K_n} |g-c|\, d|\nu_n| \\
&= \int_{K_n} (W(a)|g(a)-c|) \cdot W(a)^{-1}\, d|\nu_n| \\
&\leq \left(\sup_{a\in K_n} W(a)|g(a)-c|\right) \cdot \left(\int_{K_n} W(a)^{-1}\, d|\nu_n|\right).
\end{aligned}$$

Taking the infimum over c on the right side gives:

$$|\Lambda_n([g])| \leq \|[g]\|^\sharp_{\infty,h;\varphi;K_n} \cdot \|\nu_n\|_{W^{-1}},$$

where $\|[g]\|^\sharp_{\infty,h;\varphi;K_n}$ is the sharp norm restricted to K_n. This implies $\|\nu_n\|_{W^{-1}} \geq \sup_{\|[g]\|^\sharp\leq 1} |\Lambda_n([g])| = \|\Lambda_n\|$. Conversely, $\|\Lambda_n\| \leq \|\Lambda\| \cdot \|\pi_n^{-1}\|$. A function $g \in C(K_n)$ can be extended to $[f] \in \mathcal{X}_n$ such that $\|[f]\|^\sharp \approx \|[g]\|^\sharp_{K_n}$. Thus, $\|\nu_n\|_{W^{-1}}$ is uniformly bounded by a constant related to $\|\Lambda\|$. A more direct argument: For any g on K_n, let $[f] = \pi_n^{-1}([g])$. Then $|\int_{K_n} g d\nu_n| = |\Lambda([f])| \leq \|\Lambda\| \cdot \|[f]\|^\sharp$. The norm of ν_n in the dual of $(C(K_n)/\mathbb{R}, \|\cdot\|^\sharp)$ is bounded by $\|\Lambda\|$. This norm is equivalent to the W^{-1} norm on K_n. Therefore, the sequence (ν_n) is bounded: $\sup_n \|\nu_n\|_{W^{-1}} \leq C < \infty$.

3. Existence of a limit measure By Proposition 5.7, the closed unit ball in $\mathcal{M}_0^{W^{-1}}(A)$ is sequentially weak-* compact. Since (ν_n) is a bounded sequence in this space, it has a weak-* cluster point $\nu \in \mathcal{M}_0^{W^{-1}}(A)$. This means that for a subsequence (still denoted ν_n),

$$\int_A f\, d\nu_n \to \int_A f\, d\nu \quad \text{for all } f \in C_0(A).$$

4. Showing $\Lambda = \Lambda_\nu$ We must show $\Lambda([f]) = \int f d\nu$ for all $[f] \in C^\sharp_{h;\varphi}(A)$. Let $\mathcal{X}_c = \bigcup_n \mathcal{X}_n$ be the subspace of functions that are constant on some tail $\{h > n\}$. The tail-tightness assumption

$$\lim_{R\to\infty} \sup_{\|[f]\|^\sharp \le 1} T^{(h)}_{R;\varphi}([f]) = 0$$

implies that $\mathcal{X}_c$ is dense in $C^\sharp_{h;\varphi}(A)$.

Let $[f] \in \mathcal{X}_c$. Then $[f] \in \mathcal{X}_n$ for some sufficiently large n. For all $m \ge n$, f is constant on $A \setminus K_n$ and $A \setminus K_m$. $\Lambda([f]) = \Lambda_m([f|_{K_m}]) = \int_{K_m} f\, d\nu_m = \int_A f\, d\nu_m$. Since f is continuous and has compact support modulo constants, $f \in C_0(A) \oplus \mathbb{R}$. As $\nu_m \in \mathcal{M}_0(A)$, $\int c\, d\nu_m = 0$. So $\int f d\nu_m = \int (f - c) d\nu_m$. By weak-* convergence (after extending $f \in C_c(A)$ from $C_0(A)$),

$$\Lambda([f]) = \lim_{m\to\infty} \int_A f\, d\nu_m = \int_A f\, d\nu.$$

Thus, the functionals Λ and Λ_ν (which is bounded by Proposition 5.1) agree on the dense subspace $\mathcal{X}_c$. Since $C^\sharp_{h;\varphi}(A)$ is a Banach space, this implies $\Lambda = \Lambda_\nu$ everywhere. □

Decay subspace. Let

$$\mathcal{X} \;:=\; \big\{[f] \in C(A)/\mathbb{R} :\; C^{(h)}_\varphi([f]) = 0\big\}$$

equipped with $\|\cdot\|^\sharp_{\infty,h;\varphi}$.

Theorem 5.3 (Full duality on the tail-tight subspace)**.** *The map $\nu \mapsto \Lambda_\nu$ induces an isometric isomorphism*

$$\Big(\mathcal{M}_0^{W^{-1}}(A),\; \|\nu\|_{W^{-1}}\Big) \;\cong\; (\mathcal{X})^*.$$

Consequently, for $[f] \in \mathcal{X}$,

$$\|[f]\|^\sharp_{\infty,h;\varphi} \;=\; \sup\Big\{\,\Big|\int f\, d\nu\Big| :\; \nu \in \mathcal{M}_0^{W^{-1}}(A),\; \|\nu\|_{W^{-1}} \le 1\Big\}.$$

Proof. The proof consists of two parts: showing the map is an isometric embedding and showing it is surjective. Let $W = \varphi \circ h$.

1. Isometric Embedding By Proposition 5.1, the map $\nu \mapsto \Lambda_\nu$ is an isometric embedding of $(\mathcal{M}_0^{W^{-1}}(A), \|\cdot\|_{W^{-1}})$ into the dual of the full space, $(C^\sharp_{h;\varphi}(A))^*$. When we restrict the functional Λ_ν to the subspace $\mathcal{X} \subset C^\sharp_{h;\varphi}(A)$, its norm $\|\Lambda_\nu\|_{(\mathcal{X})^*} = \sup_{[f]\in\mathcal{X}, \|[f]\|^\sharp \le 1} |\Lambda_\nu([f])|$ can only be smaller. However, the subspace $\mathcal{X}_c$ of compactly supported perturbations (which is dense in $\mathcal{X}$ by Proposition 4.27) is used to define the norm $\|\nu\|_{W^{-1}} = \sup\{|\int f d\nu| : [f] \in \mathcal{X}_c, \|[f]\|^\sharp \le 1\}$. Therefore, $\|\Lambda_\nu\|_{(\mathcal{X})^*} = \|\nu\|_{W^{-1}}$. The map $\nu \mapsto \Lambda_\nu$ is an isometric embedding of $\mathcal{M}_0^{W^{-1}}(A)$ into $(\mathcal{X})^*$.

2. Surjectivity We must show that every bounded linear functional $\Lambda \in (\mathcal{X})^*$ can be represented by some $\nu \in \mathcal{M}_0^{W^{-1}}(A)$.

Let $\Lambda \in (\mathcal{X})^*$ with norm $\|\Lambda\|$. For each $R \ge 0$, let $K_R = \{a \in A \mid h(a) \le R\}$ be the compact sublevel set. Let $C(K_R)/\mathbb{R}$ be the space of continuous functions on K_R modulo constants, equipped with the restricted sharp norm $\|[g]\|^\sharp_{K_R} = \inf_c \sup_{K_R} W(a)|g(a) - c|$.

Let $\mathcal{X}_c(K_R) \subset \mathcal{X}$ be the subspace of function classes $[f]$ represented by a function f that is constant outside K_R. There is a natural isomorphism $\pi_R : \mathcal{X}_c(K_R) \to C(K_R)/\mathbb{R}$ via restriction. The functional Λ restricted to $\mathcal{X}_c(K_R)$ induces a functional $\Lambda_R = \Lambda \circ \pi_R^{-1}$

on $C(K_R)/\mathbb{R}$. By the Riesz Representation Theorem, there exists a unique measure $\nu_R \in \mathcal{M}(K_R)$ with $\nu_R(K_R) = 0$ (i.e., $\nu_R \in \mathcal{M}_0(K_R)$) such that

$$\Lambda_R([g]) = \int_{K_R} g\, d\nu_R \quad \text{for all } [g] \in C(K_R)/\mathbb{R}.$$

We extend ν_R to all of A by setting it to zero on $A \setminus K_R$.

We now show this family of measures $(\nu_R)_{R\geq 0}$ is uniformly bounded in the weighted norm. For any $[g] \in C(K_R)/\mathbb{R}$, let $[f] \in \mathcal{X}_c(K_R)$ be its extension. Then $\|[f]\|^\sharp = \|[g]\|^\sharp_{K_R}$.

$$\left|\int_{K_R} g\, d\nu_R\right| = |\Lambda_R([g])| = |\Lambda([f])| \leq \|\Lambda\| \cdot \|[f]\|^\sharp = \|\Lambda\| \cdot \|[g]\|^\sharp_{K_R}.$$

This implies that ν_R, as a functional on $(C(K_R)/\mathbb{R}, \|\cdot\|^\sharp_{K_R})$, has norm $\leq \|\Lambda\|$. This dual norm is precisely the W^{-1}-norm on K_R:

$$\|\nu_R\|_{W^{-1}} = \int_{K_R} W^{-1} d|\nu_R| = \sup_{\|[g]\|^\sharp_{K_R} \leq 1} \left|\int_{K_R} g\, d\nu_R\right| \leq \|\Lambda\|.$$

Thus, $(\nu_R)_{R\geq 0}$ is a uniformly bounded sequence in $(\mathcal{M}_0^{W^{-1}}(A), \|\cdot\|_{W^{-1}})$.

By Proposition 5.7, the closed ball of $\mathcal{M}_0^{W^{-1}}(A)$ is sequentially weak-* compact. Therefore, there exists a subsequence (ν_{R_n}) and a limit measure $\nu \in \mathcal{M}_0^{W^{-1}}(A)$ such that $\nu_{R_n} \rightharpoonup \nu$ (weak-*). This means $\int_A f d\nu_{R_n} \to \int_A f d\nu$ for all $f \in C_0(A)$. Since $\nu(A) = 0$, this also holds for $f \in C_c(A) \oplus \mathbb{R}$.

Let $[f] \in \mathcal{X}_c$, the dense subspace of functions with compact support (modulo constants) from Proposition 4.27. For R_n large enough, $[f] \in \mathcal{X}_c(K_{R_n})$, so

$$\Lambda([f]) = \Lambda_{R_n}([f|_{K_{R_n}}]) = \int_{K_{R_n}} f\, d\nu_{R_n} = \int_A f\, d\nu_{R_n}.$$

Taking the limit as $n \to \infty$, we get:

$$\Lambda([f]) = \lim_{n\to\infty} \int_A f\, d\nu_{R_n} = \int_A f\, d\nu = \Lambda_\nu([f]).$$

Since the functionals Λ and Λ_ν are both bounded on $\mathcal{X}$ and agree on the dense subspace $\mathcal{X}_c$, they must agree on all of $\mathcal{X}$. This proves surjectivity: $\Lambda = \Lambda_\nu$.

3. Supremum Representation The final equality is a direct consequence of the Hahn-Banach theorem, which states that for any $[f] \in \mathcal{X}$,

$$\|[f]\|^\sharp_{\infty,h;\varphi} = \sup_{\Lambda \in (\mathcal{X})^*, \|\Lambda\| \leq 1} |\Lambda([f])|.$$

Since we have just shown the isometric isomorphism $(\mathcal{X})^* \cong \mathcal{M}_0^{W^{-1}}(A)$ where $\|\Lambda\| = \|\nu\|_{W^{-1}}$, we can substitute $\Lambda = \Lambda_\nu$ to get the result:

$$\|[f]\|^\sharp_{\infty,h;\varphi} = \sup\Big\{ \left|\int f\, d\nu\right| : \ \nu \in \mathcal{M}_0^{W^{-1}}(A), \ \|\nu\|_{W^{-1}} \leq 1 \Big\}.$$

□

Corollary 5.4 (Optimality certificate at a sharp minimizer). *Let $[f] \in \mathcal{X}$ and let c_* be a sharp minimizer. There exists $\nu_* \in \mathcal{M}_0^{W^{-1}}(A)$ with $\|\nu_*\|_{W^{-1}} = 1$ such that*

$$\int (f - c_*)\, d\nu_* \ = \ \|[f]\|^\sharp_{\infty,h;\varphi}, \qquad \operatorname{supp} \nu_* \ \subset \ \Big\{ a : \ W(a)\, |f(a) - c_*| = \|[f]\|^\sharp_{\infty,h;\varphi} \Big\}.$$

One may choose $\nu_* = \nu_+ - \nu_-$ with $\nu_\pm \geq 0$ supported on the positive/negative contact sets.

5.2 Subdifferential and weak-* compactness in the weighted dual

Set $W := \varphi \circ h \geq 1$ and let

$$\mathcal{M}_0^{W^{-1}}(A) \;:=\; \Big\{\nu \text{ finite signed Radon on } A :\ \nu(A) = 0,\ \ \|\nu\|_{W^{-1}} := \int_A W^{-1}\, d|\nu| < \infty\Big\}.$$

Definition 5.5 (Contact set at a sharp minimizer)**.** For $[f] \in C(A)/\mathbb{R}$ and a sharp minimizer c_*, define the contact set

$$\mathcal{C}_* := \Big\{a \in A :\ W(a)\,|f(a) - c_*| = \|[f]\|^\sharp_{\infty,h;\varphi}\Big\},$$

and its signed parts $\mathcal{C}_*^\pm := \{a \in \mathcal{C}_* :\ \pm(f(a) - c_*) \geq 0\}$.

Theorem 5.6 (Subdifferential description)**.** *The subdifferential of $\|\cdot\|^\sharp_{\infty,h;\varphi}$ at $[f]$ is the weak-* compact convex set*

$$\partial\|[f]\|^\sharp = \Big\{\ \nu \in \mathcal{M}_0^{W^{-1}}(A) :\ \|\nu\|_{W^{-1}} \leq 1,\ \operatorname{supp}\nu \subset \mathcal{C}_*,\ \nu(\mathcal{C}_*^+) = -\nu(\mathcal{C}_*^-) = \tfrac{1}{2}\ \Big\}.$$

Consequently, for any direction $[g] \in C(A)/\mathbb{R}$ the one-sided Gâteaux derivatives exist and satisfy

$$D^\pm\|[f]\|^\sharp([g]) \;=\; \max_{\nu \in \partial\|[f]\|^\sharp} \;\pm\!\int g\, d\nu.$$

If $\partial\|[f]\|^\sharp = \{\nu_*\}$ is a singleton, then the norm is Gâteaux differentiable at $[f]$ and $D\|[f]\|^\sharp([g]) = \int g\, d\nu_*$.

Proof. Let $J([f]) = \|[f]\|^\sharp_{\infty,h;\varphi}$. We are characterizing the subdifferential $\partial J([f])$ at a point $[f] \in \mathcal{X}$ (the tail-tight subspace). By Theorem 5.3, the dual space $\mathcal{X}^*$ is isometrically isomorphic to $(\mathcal{M}_0^{W^{-1}}(A), \|\cdot\|_{W^{-1}})$.

1. Characterization of the Subdifferential By the standard definition of the subdifferential for a norm, a functional $\Lambda \in \mathcal{X}^*$ belongs to $\partial J([f])$ if and only if:

(i) $\|\Lambda\|_{\mathcal{X}^*} \leq 1$.

(ii) $\Lambda([f]) = J([f]) = \|[f]\|^\sharp_{\infty,h;\varphi}$.

Using the isometry $\Lambda \leftrightarrow \nu$, this translates to finding all measures $\nu \in \mathcal{M}_0^{W^{-1}}(A)$ such that:

(i) $\|\nu\|_{W^{-1}} \leq 1$.

(ii) $\int_A f\, d\nu = \|[f]\|^\sharp_{\infty,h;\varphi}$.

Let c_* be a sharp minimizer for $[f]$, which exists by Proposition 4.13. Since $\nu(A) = 0$, condition (ii) is equivalent to:

$$\text{(ii')} \qquad \int_A (f - c_*)\, d\nu = \|[f]\|^\sharp_{\infty,h;\varphi}.$$

We now analyze this extremal condition. By definition of the W^{-1} norm and the sharp norm:

$$\begin{aligned}
\|[f]\|^\sharp &= \int_A (f - c_*)\, d\nu \leq \int_A |f - c_*|\, d|\nu| \\
&= \int_A \underbrace{\big(W(a)|f(a) - c_*|\big)}_{\leq \|[f]\|^\sharp \text{ (by Def. 4.1.3)}} \cdot \underbrace{\big(W(a)^{-1}\big)\ d|\nu|(a)}_{\text{measures } \|\nu\|_{W-1}} \\
&\leq \Big(\sup_a W(a)|f(a) - c_*|\Big) \cdot \Big(\int_A W(a)^{-1}\, d|\nu|\Big) \\
&= \|[f]\|^\sharp \cdot \|\nu\|_{W^{-1}}.
\end{aligned}$$

Combining this with condition (i), $\|\nu\|_{W^{-1}} \leq 1$, we have a chain of inequalities:

$$\|[f]\|^\sharp \leq \int_A |f - c_*| \, d|\nu| \leq \|[f]\|^\sharp \cdot \|\nu\|_{W^{-1}} \leq \|[f]\|^\sharp.$$

For all these inequalities to hold as equalities, two conditions must be met simultaneously:

(a) $\|\nu\|_{W^{-1}} = 1$.

(b) $\int_A (f - c_*) \, d\nu = \int_A |f - c_*| \, d|\nu|$. This implies $f(a) - c_*$ and $d\nu/d|\nu|$ have the same sign, $|\nu|$-almost everywhere.

(c) $\int_A (W(a)|f(a) - c_*| - \|[f]\|^\sharp) \cdot W(a)^{-1} \, d|\nu|(a) = 0$. Since the integrand is non-positive (by definition of the sharp norm), it must be zero $|\nu|$-almost everywhere.

Condition (c) implies that the support of ν must be contained in the contact set $\mathcal{C}_*$ (Definition 5.5). Condition (b) implies that ν must be a positive measure on $\mathcal{C}_*^+ = \{a \in \mathcal{C}_* \mid f(a) - c_* \geq 0\}$ and a negative measure on $\mathcal{C}_*^- = \{a \in \mathcal{C}_* \mid f(a) - c_* \leq 0\}$. Condition (a) is the normalization.

Therefore, the subdifferential $\partial\|[f]\|^\sharp$ is the set of all $\nu \in \mathcal{M}_0^{W^{-1}}(A)$ such that $\|\nu\|_{W^{-1}} = 1$, $\mathrm{supp}(\nu) \subset \mathcal{C}_*$, $\nu \geq 0$ on $\mathcal{C}_*^+$, and $\nu \leq 0$ on $\mathcal{C}_*^-$. This set is convex and weak-* compact because it is a norm-closed (and thus weak-* closed) subset of the unit ball, which is weak-* compact by Proposition 5.7.

2. Gâteaux Derivatives The one-sided Gâteaux derivatives $D^\pm J([f])([g])$ are given by the support function of the subdifferential $\partial J([f])$. By standard convex analysis (e.g., the Danskin-Bertsekas theorem),

$$D^+ J([f])([g]) = \sup_{\Lambda \in \partial J([f])} \Lambda([g]) = \sup_{\nu \in \partial\|[f]\|^\sharp} \int_A g \, d\nu.$$

The formula for D^- follows similarly. If the subdifferential is a singleton $\{\nu_*\}$, the norm is Gâteaux differentiable, and the derivative is the linear functional Λ_{ν_*}. □

Proposition 5.7 (Weak-* compactness in the weighted dual)**.** *The closed unit ball*

$$\mathbb{B} := \{\nu \in \mathcal{M}_0^{W^{-1}}(A) : \|\nu\|_{W^{-1}} \leq 1, \ \nu(A) = 0\}$$

is sequentially weak- compact with respect to the topology induced by $\mathcal{X}$ (the tail-tight subspace).*

Proof. This is a direct consequence of the Banach-Alaoglu Theorem combined with the key results of this paper.

1. **Dual Space Identification:** By Theorem 5.3 (Full duality), the space $(\mathcal{M}_0^{W^{-1}}(A), \|\cdot\|_{W^{-1}})$ is isometrically isomorphic to the dual space $(\mathcal{X})^*$, where $\mathcal{X} = \{[f] \in C(A)/\mathbb{R} : C_\varphi^{(h)}([f]) = 0\}$.

2. **Ball Identification:** Under this isometry, the set $\mathbb{B} = \{\nu \in \mathcal{M}_0^{W^{-1}}(A) : \|\nu\|_{W^{-1}} \leq 1\}$ is precisely the closed unit ball of the dual space $(\mathcal{X})^*$.

3. **Banach-Alaoglu Theorem:** The Banach-Alaoglu Theorem states that the closed unit ball of the dual of any normed space is compact with respect to the weak-* topology. In our case, this means $\mathbb{B}$ is compact in the topology induced by its pre-dual space, which is $\mathcal{X}$.

4. **Sequential Compactness:** The proposition claims *sequential* weak-* compactness. This is a stronger property that holds because the pre-dual space $\mathcal{X}$ is separable. The

separability of $\mathcal{X}$ is established by Theorem 4.37 (Isometry with C_0), which provides an isometry $\mathcal{T} : \mathcal{X} \to C_0(A)$. Since A is assumed to be σ-compact and LCH, $C_0(A)$ is separable. An isometry preserves separability, so $\mathcal{X}$ is also separable.

5. **Conclusion:** For a separable pre-dual space (like $\mathcal{X}$), the weak-* topology on the dual unit ball (like $\mathbb{B}$) is metrizable. In a metrizable space, compactness and sequential compactness are equivalent.

Therefore, $\mathbb{B}$ is sequentially weak-* compact with respect to the topology induced by $\mathcal{X}$. □

Corollary 5.8 (Existence of dual extremals). *Let $[f]$ belong to the tail-tight subspace $\{[u] : C_{\varphi}^{(h)}([u]) = 0\}$. Then*

$$\|[f]\|^{\sharp}_{\infty,h;\varphi} = \max\Big\{ \Big| \int f \, d\nu \Big| : \ \nu \in \mathcal{M}_0^{W^{-1}}(A), \ \|\nu\|_{W^{-1}} \le 1 \Big\}.$$

Proof. By Theorem 5.3 (Section 5.1) the supremum is over the weak-* compact unit ball of $\mathcal{M}_0^{W^{-1}}(A)$; Proposition 5.7 ensures existence of a maximizing measure. □

5.3 Compactness: a weighted Arzelà–Ascoli theorem and compact embeddings

Fix a proper exhaustion h and $\varphi \in \Phi_{\mathrm{adm}}$. Recall the tail functional $T_{R;\varphi}^{(h)}$ from Lemma 4.7.

Definition 5.9 (Uniform local equicontinuity). A family $\mathcal{F} \subset C(A)/\mathbb{R}$ is *uniformly locally equicontinuous* if, for every compact $K \subset A$ and every $\varepsilon > 0$, there exists a neighborhood base $\mathcal{U}$ on K such that for all $[f] \in \mathcal{F}$ and all $x, y \in K$ contained in a common $U \in \mathcal{U}$,

$$|f(x) - f(y)| < \varepsilon$$

for some representative f of $[f]$.

Theorem 5.10 (Weighted Arzelà–Ascoli). *Let $\mathcal{F} \subset C(A)/\mathbb{R}$ satisfy:*

1. ***Uniform sharp boundedness:*** $\sup_{[f]\in\mathcal{F}} \|[f]\|^{\sharp}_{\infty,h;\varphi} < \infty$*;*
2. ***Uniform local equicontinuity*** *on every compact (Def. 5.9);*
3. ***Tight tails:*** $\lim_{R\to\infty} \sup_{[f]\in\mathcal{F}} T_{R;\varphi}^{(h)}([f]) = 0.$

Then $\mathcal{F}$ is relatively compact in $\big(C(A)/\mathbb{R}, \ \|\cdot\|^{\sharp}_{\infty,h;\varphi}\big)$.

Proof. Let $\mathcal{F}$ be a family satisfying the three conditions. We want to show that $\mathcal{F}$ is relatively compact in the Banach space $\big(C(A)/\mathbb{R}, \|\cdot\|^{\sharp}_{\infty,h;\varphi}\big)$. It suffices to show that every sequence $([f_n])_{n\in\mathbb{N}}$ in $\mathcal{F}$ admits a Cauchy subsequence.

Let $([f_n])$ be such a sequence. Let $W = \varphi \circ h$.

1. Boundedness of Sharp Minimizers By **Condition 1**, $\sup_n \|[f_n]\|^{\sharp} \le M < \infty$. For each n, let c_n be a sharp minimizer for $[f_n]$ (which exists by Proposition 4.13). The sequence of constants (c_n) is bounded. To see this, fix a point $a_0 \in A$ (e.g., $h(a_0) = 0$, so $W(a_0) = 1$). By **Condition 2**, the family $\{f_n\}$ is uniformly equicontinuous on the compact set $K_1 = \{h \le 1\}$, which contains a_0. By the classical Arzelà-Ascoli theorem, a uniformly equicontinuous and pointwise bounded family is uniformly bounded. The family is pointwise bounded at a_0 because:

$$|f_n(a_0) - c_n| \le W(a_0)|f_n(a_0) - c_n| \le \|[f_n]\|^\sharp \le M.$$

This implies $|c_n| \le |f_n(a_0)| + M$. Since $f_n(a_0)$ must be a bounded sequence, (c_n) is also bounded.

2. Extracting a Locally Uniformly Convergent Subsequence Let $g_n = f_n - c_n$. * The sequence (g_n) is uniformly locally equicontinuous (by **Condition 2**, as f_n is). * The sequence (g_n) is uniformly bounded on every compact set K_R. (Proof: $\sup_{K_R} |g_n| = \sup_{K_R} |f_n - c_n| \le \sup_{K_R} \frac{W(a)|f_n(a)-c_n|}{W(a)} \le \frac{\|[f_n]\|^\sharp}{\min_{K_R} W} \le M$.) By the classical Arzelà-Ascoli theorem and a standard diagonal argument on a compact exhaustion $(K_m)_{m\in\mathbb{N}}$ of A, there exists a subsequence (which we still denote by (g_n)) and a function $g \in C(A)$ such that $g_n \to g$ uniformly on every compact set K_R.

Since (c_n) is a bounded sequence in $\mathbb{R}$, we can extract a further subsequence (still denoted (c_n)) that converges to a limit $c \in \mathbb{R}$. Let $f = g + c$. Then $f_n = g_n + c_n \to g + c = f$ uniformly on every compact set K_R.

3. Proving Convergence in the Sharp Norm We must show that this subsequence $([f_n])$ is Cauchy in the sharp norm. Let $h_{n,m} = f_n - f_m$. We need to show $\|[h_{n,m}]\|^\sharp \to 0$ as $n, m \to \infty$. We use the patching principle (Lemma 4.7): for any $R \ge 0$,

$$\|[h_{n,m}]\|^\sharp \le \max\left(\|[h_{n,m}]\|_{\mathrm{loc},R;\varphi}, T^{(h)}_{R;\varphi}([h_{n,m}])\right).$$

Let $\varepsilon > 0$. * **Tail Part:** By **Condition 3** (Tight tails), we can choose R large enough so that

$$\sup_{[f]\in\mathcal{F}} T^{(h)}_{R;\varphi}([f]) < \varepsilon/2.$$

Since $[f_n], [f_m] \in \mathcal{F}$, we have $T^{(h)}_{R;\varphi}([f_n]) < \varepsilon/2$ and $T^{(h)}_{R;\varphi}([f_m]) < \varepsilon/2$. The tail functional is a seminorm (modulo constants), so by the triangle inequality:

$$T^{(h)}_{R;\varphi}([h_{n,m}]) = T^{(h)}_{R;\varphi}([f_n - f_m]) \le T^{(h)}_{R;\varphi}([f_n]) + T^{(h)}_{R;\varphi}([f_m]) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

* **Core Part:** Now that R is fixed, we look at the local norm on K_R.

$$\|[h_{n,m}]\|_{\mathrm{loc},R;\varphi} = \inf_c \sup_{a\in K_R} W(a)|(f_n(a) - f_m(a)) - c|.$$

We can choose the constant $c = 0$:

$$\|[h_{n,m}]\|_{\mathrm{loc},R;\varphi} \le \sup_{a\in K_R} W(a)|f_n(a) - f_m(a)|.$$

Let $C_R = \sup_{a\in K_R} W(a)$, which is finite since K_R is compact.

$$\|[h_{n,m}]\|_{\mathrm{loc},R;\varphi} \le C_R \cdot \sup_{a\in K_R} |f_n(a) - f_m(a)|.$$

From Step 2, our subsequence (f_n) converges uniformly on the compact set K_R. Therefore, it is Cauchy on K_R. We can find an integer N such that for all $n, m \ge N$:

$$\sup_{a\in K_R} |f_n(a) - f_m(a)| < \varepsilon/C_R.$$

This implies that for all $n, m \ge N$, $\|[h_{n,m}]\|_{\mathrm{loc},R;\varphi} < \varepsilon$.

* **Conclusion:** For this ε, we found an R and an N. For all $n, m \geq N$:

$$\|[f_n] - [f_m]\|^\sharp = \|[h_{n,m}]\|^\sharp \leq \max(\varepsilon, \varepsilon) = \varepsilon.$$

This proves that the subsequence $([f_n])$ is Cauchy. Since $C^\sharp_{h;\varphi}(A)$ is a complete space (Theorem 4.2), the subsequence converges. Therefore, every sequence in $\mathcal{F}$ has a convergent subsequence, and $\mathcal{F}$ is relatively compact. □

Theorem 5.11 (Compact inclusion between weighted sharp spaces). *Let $\varphi_2, \varphi_1 \in \Phi_{\mathrm{adm}}$ with $\varphi_1 \preceq \varphi_2$ and*

$$\lim_{t\to\infty} \frac{\varphi_1(t)}{\varphi_2(t)} = 0.$$

Then the identity map

$$\iota : \ \big(C^\sharp_{h;\varphi_2}(A),\ \|\cdot\|^\sharp_{\infty,h;\varphi_2}\big) \ \longrightarrow\ \big(C^\sharp_{h;\varphi_1}(A),\ \|\cdot\|^\sharp_{\infty,h;\varphi_1}\big)$$

is compact on every subset that is bounded in the φ_2-sharp norm and uniformly locally equicontinuous on compacts.

Proof. Let $\mathcal{F}$ be a subset of $C^\sharp_{h;\varphi_2}(A)$ that is bounded in the φ_2-sharp norm and uniformly locally equicontinuous on compacts. We want to show that $\mathcal{F}$ is relatively compact in the target space $C^\sharp_{h;\varphi_1}(A)$.

To do this, we will use the Weighted Arzelà–Ascoli Theorem (Theorem 5.10) *on the target space*. We must verify that $\mathcal{F}$ satisfies all three conditions of Theorem 5.10 with respect to the φ_1-norm.

Let $M = \sup_{[f]\in\mathcal{F}} \|[f]\|^\sharp_{\infty,h;\varphi_2} < \infty$, which holds by assumption.

1. Uniform φ_1-sharp boundedness: Since $\varphi_1 \preceq \varphi_2$, there exist $C > 0$ and $R_0 < \infty$ such that $\varphi_1(t) \leq C\varphi_2(t)$ for $t \geq R_0$. By Proposition 4.21 (Global embeddings),

$$\|[f]\|^\sharp_{\infty,h;\varphi_1} \leq \max\Big\{ \sup_{h\leq R_0} \mathrm{osc}_{[f]}(a),\ C\,\|[f]\|^\sharp_{\infty,h;\varphi_2} \Big\}.$$

The second term is bounded by $C \cdot M$. The first term, $\sup_{h\leq R_0} \mathrm{osc}_{[f]}(a)$, is the oscillation on the compact set K_{R_0}. Since $\mathcal{F}$ is bounded in the φ_2-norm, it is also uniformly bounded on K_{R_0} (as $\inf_{K_{R_0}} \varphi_2 \geq 1$). Thus, the oscillation is uniformly bounded. Therefore, $\sup_{[f]\in\mathcal{F}} \|[f]\|^\sharp_{\infty,h;\varphi_1} < \infty$. Condition (1) is satisfied.

2. Uniform local equicontinuity: This is given as an assumption in the theorem statement. Condition (2) is satisfied.

3. φ_1-Tight tails: This is the critical step. We must show that $\lim_{R\to\infty} \sup_{[f]\in\mathcal{F}} T^{(h)}_{R;\varphi_1}([f]) = 0$.

Let $\varepsilon > 0$. By the assumption $\lim_{t\to\infty}(\varphi_1(t)/\varphi_2(t)) = 0$, we can choose a radius R large enough so that

$$\frac{\varphi_1(t)}{\varphi_2(t)} < \frac{\varepsilon}{M+1} \quad \text{for all } t \geq R.$$

(We use $M + 1$ to avoid division by zero if $M = 0$).

Now, let $[f]$ be any element in $\mathcal{F}$. Let c_* be a sharp minimizer for $[f]$ *in the φ_2-norm* (which exists by Proposition 4.13). We use this c_* to bound the φ_1-tail functional (we don't need the optimal c for φ_1):

$$T^{(h)}_{R;\varphi_1}([f]) = \inf_c \sup_{h(a)>R} \varphi_1(h(a))|f(a) - c| \leq \sup_{h(a)>R} \varphi_1(h(a))|f(a) - c_*|.$$

We now "multiply and divide" by φ_2:

$$T^{(h)}_{R;\varphi_1}([f]) \le \sup_{h(a)>R} \left[\left(\frac{\varphi_1(h(a))}{\varphi_2(h(a))} \right) \cdot (\varphi_2(h(a))|f(a) - c_*|) \right].$$

We can bound the product by the product of the suprema:

$$T^{(h)}_{R;\varphi_1}([f]) \le \left(\sup_{h(a)>R} \frac{\varphi_1(h(a))}{\varphi_2(h(a))} \right) \cdot \left(\sup_{h(a)>R} \varphi_2(h(a))|f(a) - c_*| \right).$$

By our choice of R, the first term is less than $\varepsilon/(M+1)$. The second term is bounded by the global φ_2-sharp norm:

$$\sup_{h(a)>R} \varphi_2(h(a))|f(a) - c_*| \le \sup_{a\in A} \varphi_2(h(a))|f(a) - c_*| = \|[f]\|^{\sharp}_{\infty,h;\varphi_2} \le M.$$

Putting these together:

$$T^{(h)}_{R;\varphi_1}([f]) \le \left(\frac{\varepsilon}{M+1} \right) \cdot M < \varepsilon.$$

This bound ε is independent of the choice of $[f] \in \mathcal{F}$. Therefore, for our chosen R, $\sup_{[f]\in\mathcal{F}} T^{(h)}_{R;\varphi_1}([f]) \le \varepsilon$. Since ε was arbitrary, this proves $\lim_{R\to\infty} \sup_{[f]\in\mathcal{F}} T^{(h)}_{R;\varphi_1}([f]) = 0$. Condition (3) is satisfied.

Conclusion: The family $\mathcal{F}$ satisfies all three conditions of the Weighted Arzelà–Ascoli Theorem (Theorem 5.10) for the φ_1-norm. Therefore, $\mathcal{F}$ is relatively compact in $C^{\sharp}_{h;\varphi_1}(A)$. This proves that the identity map ι is compact. ☐

Corollary 5.12 (Polynomial case)**.** *If $\varphi_2(t) = (1+t)^{p_2}$ and $\varphi_1(t) = (1+t)^{p_1}$ with $p_2 > p_1 \ge 0$, then the identity $C^{\sharp}_{h;p_2}(A) \to C^{\sharp}_{h;p_1}(A)$ is compact on sets that are bounded in $\|\cdot\|^{\sharp}_{\infty,h;\varphi_2}$ and uniformly locally equicontinuous on compacts.*

6 Measured extensions and L^q embeddings

Let $(A, \mathcal{B}, \mu)$ be a Radon measure space on A and h a continuous proper exhaustion.

Definition 6.1 (Weighted L^q spaces)**.** For $\varphi \in \Phi_{\mathrm{adm}}$ and $q \in [1, \infty)$, define

$$\|f\|_{L^q(h;\varphi;L)} := \Big(\int_A \big(\varphi(h(a))\,|f(a) - L|\big)^q \, d\mu(a) \Big)^{1/q}.$$

Write $L^q_{h;\varphi}(A)$ for the set of f with finite norm, and analogously on the quotient by constants using the infimum over $L \in \mathbb{R}$ (similar to Definition 4.4 for the sharp norm).

Definition 6.2 (Volume growth index)**.** Set $V(R) := \mu(\{h \le R\})$. We say that μ has polynomial volume growth of order $\gamma \ge 0$ if $V(R) \le C(1+R)^{\gamma}$ for all $R \ge 0$.

Theorem 6.3 (Sup-to-L^q embedding under volume growth)**.** *Assume $V(R) \le C(1+R)^{\gamma}$ and $\varphi(t) = (1+t)^p$ with $p > 0$. If $pq > \gamma$, then for every $L \in \mathbb{R}$ there exists $C_{p,q,\gamma}$ such that*

$$\|f\|_{L^q(h;\varphi;L)} \;\le\; C_{p,q,\gamma}\, \|f\|_{\infty,h;\varphi;L} \qquad \text{for all } f.$$

The same holds on the quotient: $\|[f]\|_{L^q(h;\varphi)} \le C\, \|[f]\|^{\sharp}_{\infty,h;\varphi}$, where the quotient L^q norm is defined analogously to the sharp norm.

Proof. Decompose into layers $A_R := \{R < h \le R+1\}$ and use $\int_{A_R}(1+h)^{pq}|f-L|^q \le \|f\|^q_{\infty,h;\varphi;L}\,\mu(A_R)\,(1+R)^{pq}$. Sum over $R \in \mathbb{N}$ and apply $\mu(A_R) \le V(R+1)-V(R) \lesssim (1+R)^{\gamma-1}$. Converges iff $pq > \gamma$. □

Remark 6.4. For general $\varphi \in \Phi_{\mathrm{adm}}$, replace $(1+R)^{pq}$ by $\varphi(R)^q$ and assume a summability $\sum_R \varphi(R)^q\big(V(R+1)-V(R)\big) < \infty$.

7 Functoriality under coarse-affine maps

Definition 7.1 (Category of exhaustions)**.** Let Exh be the category whose objects are pairs (A,h) with A a Hausdorff, locally compact, σ-compact space and $h : A \to [0,\infty)$ a continuous proper exhaustion.

A morphism $\phi : (A,h_A) \to (B,h_B)$ is a *proper* map $\phi : A \to B$ such that there exist constants $A_0 \ge 1$ and $B_0 \in \mathbb{R}$ with

$$h_A(x) \;\le\; A_0\, h_B(\phi(x)) + B_0 \qquad \forall x \in A. \tag{$\star$}$$

Composition is the usual composition of maps. Identities are the identity maps.

Remark 7.2 (On the direction of control)*.* Condition $(\star)$ is a *forward* coarse-affine control ensuring that "going forward" through ϕ does not make h grow too fast. If in addition there are $a_0 \ge 1$, $b_0 \in \mathbb{R}$ with $h_B(\phi(x)) \le a_0\, h_A(x) + b_0$, then ϕ is a coarse bi-affine equivalence and induces norm equivalences (see Theorem 4.34).

Definition 7.3 (Pullback)**.** Given $\phi : (A,h_A) \to (B,h_B)$ in Exh and $f \in C(B)$, define the pullback $\phi^* f := f \circ \phi \in C(A)$. This descends to classes $[f] \in C(B)/\mathbb{R}$ via $\phi^*[f] := [f\circ\phi] \in C(A)/\mathbb{R}$.

Theorem 7.4 (Bounded Pullback for Polynomial Weights)**.** *Let $\phi : (A,h_A) \to (B,h_B)$ be a morphism in* Exh *satisfying* $(\star)$*. Let $\varphi(t) = (1+t)^p$ with $p > 0$. Then there exists a constant $C = C(A_0,B_0,p) \ge 1$ such that for all $[f] \in C(B)/\mathbb{R}$,*

$$\|\phi^*[f]\|^\sharp_{\infty,h_A;\varphi} \;\le\; C\;\|[f]\|^\sharp_{\infty,h_B;\varphi},$$

and $C^{(h_A)}_\varphi(\phi^*[f]) \le C\, C^{(h_B)}_\varphi([f])$. If, moreover, there is a reverse affine bound $h_B \circ \phi \le a_0 h_A + b_0$, then ϕ^* induces a topological isomorphism between the corresponding weighted spaces.

Proof. Let $\varphi_A = \varphi \circ h_A$ and $\varphi_B = \varphi \circ h_B$. Let c_B be a sharp minimizer for $[f]$ in the $C^\sharp_{h_B;\varphi}(B)$ norm (which exists by Proposition 4.13). Thus,

$$\|[f]\|^\sharp_{\infty,h_B;\varphi} = \sup_{y\in B} \varphi_B(y)|f(y)-c_B|.$$

We test the norm of the pullback $\phi^*[f]$ using this same constant c_B:

$$\|\phi^*[f]\|^\sharp_{\infty,h_A;\varphi} = \inf_c \sup_{x\in A} \varphi(h_A(x))|f(\phi(x))-c| \le \sup_{x\in A} \varphi(h_A(x))|f(\phi(x))-c_B|.$$

1. Relate the Weights From the morphism condition $(\star)$ in Definition 7.1, we have:

$$h_A(x) \le A_0 h_B(\phi(x)) + B_0.$$

Since $\varphi(t) = (1+t)^p$, we apply this function to both sides:

$$\varphi(h_A(x)) = (1 + h_A(x))^p \le (1 + A_0 h_B(\phi(x)) + B_0)^p.$$

For any $t \ge 0$, the polynomial $(1 + A_0 t + B_0)^p$ is bounded by a multiple of $(1+t)^p$. That is, there exists a constant $C_1 = C(A_0, B_0, p)$ such that

$$(1 + A_0 t + B_0)^p \le C_1 (1+t)^p \quad \text{for all } t \ge 0.$$

(For example, one can take $C_1 = (1 + A_0 + B_0)^p$ if $A_0, B_0 \ge 0$, or more generally $C_1 = \max(1, A_0)^p (1 + |B_0|)^p \cdot 2^p$). Thus, we have a direct relationship between the weights:

$$\varphi(h_A(x)) \le C_1 \cdot \varphi(h_B(\phi(x))) \quad \text{for all } x \in A.$$

This corresponds to the power-dilation control (Definition 4.28) with the crucial exponent $\kappa = 1$.

2. Bound the Norm Substitute this weight inequality back into our norm estimate:

$$\begin{aligned}
\|\phi^*[f]\|^\sharp_{\infty,h_A;\varphi} &\le \sup_{x\in A} \left[C_1 \cdot \varphi(h_B(\phi(x))) \cdot |f(\phi(x)) - c_B|\right] \\
&= C_1 \cdot \sup_{x\in A} \left(\varphi_B(\phi(x)) |f(\phi(x)) - c_B|\right) \\
&\le C_1 \cdot \sup_{y\in B} \left(\varphi_B(y) |f(y) - c_B|\right)
\end{aligned}$$

The last step holds because the range $\phi(A) \subseteq B$. The supremum over all $x \in A$ covers a subset of the values seen by the supremum over all $y \in B$.

3. Conclusion We have shown:

$$\|\phi^*[f]\|^\sharp_{\infty,h_A;\varphi} \le C_1 \cdot \|[f]\|^\sharp_{\infty,h_B;\varphi}.$$

The bound for the asymptotic constant $C^{(h_A)}_\varphi(\phi^*[f]) \le C_1\, C^{(h_B)}_\varphi([f])$ follows by the same logic, applied to the $\limsup$ definition.

If the reverse affine bound $h_B \circ \phi \le a_0 h_A + b_0$ also holds, we can reverse the entire argument (starting with $h_B \circ \phi$) to show $\|[f]\|^\sharp_{\infty,h_B;\varphi} \le C_2 \cdot \|\phi^*[f]\|^\sharp_{\infty,h_A;\varphi}$, which proves the norms are equivalent and ϕ^* induces a topological isomorphism. □

Proposition 7.5 (Contravariant functor)**.** *For any fixed admissible φ, the assignment*

$$(A, h) \;\longmapsto\; \left(C^\sharp_{h;\varphi}(A),\ \|\cdot\|^\sharp_{\infty,h;\varphi}\right), \qquad \phi \;\longmapsto\; \phi^*$$

defines a contravariant functor from Exh *to the category of normed spaces and bounded linear maps. If $\varphi(t) = (1+t)^p$, the target can be taken as Banach spaces by Theorem 4.2.*

8 Anisotropic Spaces: The Multi-Exhaustion Framework

The framework developed thus far excels for spaces whose infinity can be modeled by a single point. We now address the challenge of spaces with multiple distinct "ends" or anisotropic geometries. The solution is to generalize the single exhaustion function h to a collection of functions $\{h_i\}_{i\in I}$, each characterizing a specific asymptotic direction.

8.1 Decomposition at infinity and multi-exhaustions

We start from a coarse geometric decomposition of the space into a compact "core" and a finite family of open ends.

Assumption 8.1 (Decomposition at infinity). *Let A be a space. We assume there exists a compact subset $K_0 \subset A$ (the central core) such that its complement can be written as a finite, disjoint union of open, non-compact sets U_i:*

$$A \backslash K_0 = \bigcup_{i=1}^{m} U_i.$$

Each set U_i is called an end or an asymptotic channel of the space A.

The above formulation is very close in spirit to the classical notion of ends of a locally compact space. For completeness, we recall a standard definition.

Definition 8.2 (Ends via complements of compacts). Let A be Hausdorff, locally compact, σ-compact. Fix a compact exhaustion $(K_n)_{n \geq 0}$ with $K_n \subset \mathrm{int}(K_{n+1})$ and $\bigcup_n K_n = A$. An end of A is a choice of a connected component E_n of $A \backslash K_n$ for each n, with $E_{n+1} \subset E_n$. Two such choices are identified if they agree for all large n. The (possibly infinite) set of ends is denoted by $\mathrm{Ends}(A)$.

The multi-exhaustion formalism refines this picture by endowing each end with its own exhaustion function, recovering a directional notion of distance to infinity.

Definition 8.3 (Multi-exhaustion). Let A be decomposed as $A = K_0 \cup \bigcup_{i=1}^{m} U_i$ as in Assumption 8.1. A multi-exhaustion is a family of continuous proper functions

$$h_i : U_i \longrightarrow [0, \infty), \quad i = 1, ..., m,$$

one on each end U_i. We denote the point at infinity associated with U_i by ω_i. A sequence $(a_k)_{k>1}$ converges to ω_i if it is eventually in U_i and

$$\lim_{k \to \infty} h_i(a_k) = +\infty.$$

Remark 8.4 (Construction of local exhaustions). The local exhaustion function h_i should be constructed systematically from a local exhaustion of the channel U_i by a nested family of compact sets $\{K_{r,i}\}_{r>0}$, in direct analogy with the construction of the global function h in Section 2. Namely, set

$$h_i(a) := \inf\{r \geq 0 \mid a \in K_{r,i}\}, \quad a \in U_i.$$

This ensures that our directional measures of convergence are just as fundamentally grounded in the space's structure as our original global measure.

We next introduce directional weights φ_i on each end and glue them into a single block weight on A.

Definition 8.5 (Anisotropic weights and norms). For each end $i \in I = \{1, ..., m\}$, fix an admissible weight $\varphi_i \in \Phi^{\mathrm{adm}}$ (in the sense of Section 4.1.2). Define the block weight $W : A \to (0, \infty)$ by (here and below we implicitly extend h_i by 0 outside U_i):

$$W(x) := \max\{1, \max_{i \in I} \varphi_i(h_i(x))\} = \begin{cases} 1, & x \in K_0, \\ \varphi_i(h_i(x)), & x \in U_i. \end{cases}$$

Given $f \in C(A)$, the limit of f at the end ω_i is a number $L_i \in \mathbb{R}$ such that $\lim_{a \to \omega_i} f(a) = L_i$ in the sense of the above directional convergence. The **anisotropic sharp norm** on $C(A)/\mathbb{R}$ is

$$\|[f]\|^{\sharp}_{\infty;\mathbf{h},\boldsymbol{\varphi}} := \inf_{c \in \mathbb{R}} \sup_{x \in A} W(x)|f(x) - c|.$$

The associated **anisotropic asymptotic constant** is

$$C^{(\mathbf{h})}_{\boldsymbol{\varphi}}([f]) := \inf_{c \in \mathbb{R}} \max_{i \in I} \lim \sup_{x \to \omega_i} \varphi_i(h_i(x))|f(x) - c|.$$

The next lemma shows that the global anisotropic norm can be decomposed into a compact core contribution and per-end tail contributions.

Lemma 8.6 (Gluing: core + per-end tails)**.** *Fix a compact core K_0 as in Assumption 8.1. Then for all $[f] \in C(A)/\mathbb{R}$ the anisotropic sharp norm is equivalent to the maximum of the core oscillation and the individual end-norms:*

$$\|[f]\|^{\sharp}_{\infty;\mathbf{h},\boldsymbol{\varphi}} \simeq \max \begin{cases} \inf_{c \in \mathbb{R}} \sup_{x \in K_0} |f(x) - c|, \\ \max_{i \in I} \inf_{c \in \mathbb{R}} \sup_{x \in U_i} \varphi_i(h_i(x))|f(x) - c| \end{cases}$$

with equivalence constants depending only on the geometry of the overlap regions $K_0 \cap U_i$. Furthermore, the global asymptotic constant is always an upper bound for the per-end constants:

$$C^{(\mathbf{h})}_{\boldsymbol{\varphi}}([f]) \geq \max_{i \in I} C^{(h_i)}_{\varphi_i}([f|_{U_i}]).$$

Proof. Let $N([f]) := \|[f]\|^{\sharp}_{\infty;\mathbf{h},\boldsymbol{\varphi}}$ and $M([f]) := \max\{N_0([f]), \max_{i \in I} N_i([f])\}$, where

$$N_0([f]) := \inf_{c \in \mathbb{R}} \sup_{x \in K_0} |f(x) - c|$$

$$N_i([f]) := \inf_{c \in \mathbb{R}} \sup_{x \in U_i} \varphi_i(h_i(x))|f(x) - c|.$$

Step 1: $M([f]) \leq N([f])$**.** Let $c \in \mathbb{R}$ be arbitrary. By the definition of W,

$$\sup_{x \in A} W(x)|f(x) - c| = \max \left\{ \sup_{x \in K_0} |f(x) - c|, \max_{i \in I} \sup_{x \in U_i} \varphi_i(h_i(x))|f(x) - c| \right\}.$$

This global supremum is an upper bound for each of its components: $\sup_{x \in A} W(x)|f(x) - c| \geq \sup_{x \in K_0} |f(x) - c| \geq N_0([f])$, and similarly, $\sup_{x \in A} W(x)|f(x) - c| \geq \sup_{x \in U_i} \varphi_i(h_i(x))|f(x) - c| \geq N_i([f])$ for every $i \in I$. Thus $\sup W|f - c|$ is an upper bound for all terms in $M([f])$. This holds for any c, so taking the infimum over c on the left-hand side yields $N([f]) \geq M([f])$.

Step 2: $N([f]) \leq CM([f])$ **for some** $C > 0$**.** This relies on a finite-overlap argument near the interfaces $K_0 \cap U_i$. Let c_0 be an optimal constant for N_0 and c_i an optimal constant for N_i. Assume h_i is continuous on U_i and $h_i = 0$ (so $\varphi_i = 1$) on $\partial U_i \subset K_0$. By construction, $\sup_{x \in K_0} |f(x) - c_0| = N_0([f]) \leq M([f])$, and, for each i, $\sup_{x \in U_i} \varphi_i(h_i(x))|f(x) - c_i| = N_i([f]) \leq M([f])$. Using the finite-overlap between K_0 and the U_i and the fact that $\varphi_i = 1$ on ∂U_i, one can show that all optimal constants $(c_0, c_1, ..., c_m)$ lie in a bounded interval whose size is controlled by $M([f])$, more precisely $|c_i - c_0| \leq 2M([f])$ for all i. Choose a smooth partition of unity $\{\eta_0, \eta_i\}_{i \in I}$ on A. Define a globally defined "patched" constant $c(x) := \sum_j \eta_j(x) c_j$, or simply set $c(x) \equiv c_0$. A standard patching argument (similar to Lemma 4.7) then shows $\sup_{x \in A} W(x)|f(x) - c(x)| \leq CM([f])$. Taking the infimum over all (measurable) choices of c we obtain $N([f]) \leq CM([f])$.

Step 3: Asymptotic inequality. Let $LHS := C^{(\mathbf{h})}_{\boldsymbol{\varphi}}([f])$ and $RHS := \max_{i\in I} C^{(h_i)}_{\varphi_i}([f|_{U_i}])$. Fix $c \in \mathbb{R}$. For any specific end j,

$$\max_{i\in I}(\limsup_{x\to\omega_i} \varphi_i(h_i(x))|f(x)-c|) \geq \limsup_{x\to\omega_j} \varphi_j(h_j(x))|f(x)-c| \geq \inf_{c_j\in\mathbb{R}} \limsup_{x\to\omega_j} \varphi_j(h_j(x))|f(x)-c_j|.$$

The rightmost term is $C^{(h_j)}_{\varphi_j}([f|_{U_j}])$. This holds for all j, so

$$\max_{i\in I}(\limsup_{x\to\omega_i} \varphi_i(h_i(x))|f(x)-c|) \geq \max_j C^{(h_j)}_{\varphi_j}([f|_{U_j}]) = RHS.$$

This inequality is true for any c, so taking the infimum over c on the left-hand side gives $LHS \geq RHS$. This proves the claim. □

8.2 Illustrative examples

We now illustrate the multi-exhaustion formalism on two simple but instructive examples: an infinite strip with two ends, and the punctured real line $\mathbb{R}\backslash\{0\}$ with four ends.

8.2.1 Example 1: Convergence on an infinite strip

Let $A = \mathbb{R} \times [0,1]$ be an infinite strip. Take $K_0 := [-1,1] \times [0,1]$, $U_- := (-\infty,-1) \times [0,1]$, $U_+ := (1,\infty) \times [0,1]$. Then $A = K_0 \cup U_- \cup U_+$ and $A\backslash K_0 = U_- \sqcup U_+$. Define $h_-(x,y) := \max\{0,-x-1\}$ and $h_+(x,y) := \max\{0,x-1\}$, which are continuous proper exhaustions on U_- and U_+ respectively. For a given $p > 0$ set $\varphi_-(t) = \varphi_+(t) := (1+t)^p$. The block weight reads

$$W(x,y) = \begin{cases} 1, & (x,y) \in K_0, \\ (1+h_-(x,y))^p, & (x,y) \in U_-, \\ (1+h_+(x,y))^p, & (x,y) \in U_+. \end{cases}$$

Consider a continuous function $f : \mathbb{R} \times [0,1] \to \mathbb{R}$ which converges to limits L_- and L_+ along the two ends: $\lim_{x\to-\infty} f(x,y) = L_-$ and $\lim_{x\to+\infty} f(x,y) = L_+$. If $L_- = L_+$, then the asymptotic constant $C^{(\mathbf{h})}_{\boldsymbol{\varphi}}([f])$ vanishes. If $L_- \neq L_+$, then $C^{(\mathbf{h})}_{\boldsymbol{\varphi}}([f]) > 0$, but $[f]$ still belongs to a natural anisotropic space where each end is individually controlled.

8.2.2 Example 2: A space with four ends, $\mathbb{R}\backslash\{0\}$

Consider $A = \mathbb{R}\backslash\{0\}$. We want to distinguish four ends:

- $U_{+\infty} := (2,\infty)$ (right infinity)
- $U_{-\infty} := (-\infty,-2)$ (left infinity)
- $U_{0^+} := (0,1)$ (approach to 0 from the right)
- $U_{0^-} := (-1,0)$ (approach to 0 from the left)

Take $K_0 := [-2,-1] \cup [1,2]$, so that $A\backslash K_0 = U_{-\infty} \sqcup U_{+\infty} \sqcup U_{0^-} \sqcup U_{0^+}$. Define local exhaustions $h_{+\infty}(x) := x-2$, $x > 2$; $h_{-\infty}(x) := -x-2$, $x < -2$; and near the origin $h_{0^+}(x) := -\log x$, $x \in (0,1)$; $h_{0^-}(x) := -\log(-x)$, $x \in (-1,0)$. Anisotropy now appears in the choice of weights. For instance, we may want polynomial control at $\pm\infty$ and exponential control near 0: $\varphi_{+\infty}(t) = \varphi_{-\infty}(t) := (1+t)^p$, $\varphi_{0^+}(t) = \varphi_{0^-}(t) := e^{\alpha t}$ for some $\alpha > 0$. The block weight is

$$W(x) = \begin{cases} 1, & x \in K_0, \\ (1 + h_{\pm\infty}(x))^p, & x \in U_{\pm\infty}, \\ e^{\alpha h_{0^\pm}(x)}, & x \in U_{0^\pm}. \end{cases}$$

The anisotropic norm $\|[f]\|^\sharp_{\infty;\mathbf{h},\boldsymbol{\varphi}}$ thus controls simultaneously polynomial decay as $x \to \pm\infty$ and exponential decay (in $1/|x|$) as $x \to 0^\pm$.

8.3 Further properties of multi-exhaustions

We now collect some structural properties of the anisotropic spaces.

Proposition 8.7 (Structure of the vanishing-tail class). *Let $X_{van} := \{[f] \in C(A)/\mathbb{R} \mid C^{(h_i)}_{\varphi_i}([f|_{U_i}]) = 0$ for all $i\}$. A class $[f]$ belongs to X_{van} if and only if f converges to a limit L_i on each end U_i and the rate of convergence satisfies $f(x) - L_i = o(\varphi_i(h_i(x))^{-1})$ as $x \to \omega_i$ for every i. If $[f] \in X_{van}$, then Lemma 8.6 implies that the global asymptotic constant $C^{(\mathbf{h})}_{\boldsymbol{\varphi}}([f])$ is not necessarily zero. It vanishes if and only if all limits L_i coincide.*

Sketch of proof. The forward implication is immediate from the definition of $C^{(h_i)}_{\varphi_i}$. Conversely, if f converges to L_i on each end with the prescribed rate $o(\varphi_i(h_i)^{-1})$, then by rescaling $f - L_i$ and using the definitions, one checks that $C^{(h_i)}_{\varphi_i}([f|_{U_i}]) = 0$. The final statement follows by comparing the per-end constants with the global constant via Lemma 8.6. □

Theorem 8.8 (Completeness of anisotropic spaces). *The space $(C(A)/\mathbb{R}, \|\cdot\|^\sharp_{\infty;\mathbf{h},\boldsymbol{\varphi}})$ is a Banach space.*

Proof. The proof is a direct extension of Theorem 4.2. Let $([f_n])$ be a Cauchy sequence in $\|\cdot\|^\sharp_{\infty;\mathbf{h},\boldsymbol{\varphi}}$. For each n, choose a near-optimal constant $c_n \in \mathbb{R}$ and define $g_n(x) := W(x)(f_n(x) - c_n)$, where W is the anisotropic weight from Definition 8.5. Then (g_n) is a Cauchy sequence in the standard sup-norm $\|\cdot\|_\infty$ on A, hence converges uniformly to some $g \in C_b(A)$. We must show that the sequence (c_n) converges. As in the isotropic case (see the proof of Theorem 5.10), the Cauchy property implies that (c_n) is bounded. By Bolzano-Weierstrass, there exists a convergent subsequence (c_{n_k}) with $c_{n_k} \to c^*$. Set

$$f(x) := W(x)^{-1} g(x) + c^*.$$

Then $f \in C(A)$ and $W(x)(f(x) - c^*) = g(x)$. Now, using the quotient-norm structure,

$$\|[f_n] - [f]\|^\sharp_{\infty;\mathbf{h},\boldsymbol{\varphi}} \le \sup_{x \in A} W(x) |(f_n(x) - c_n) - (f(x) - c^*)|.$$

The right-hand side equals $\|g_n - g\|_\infty$, which tends to 0 as $n \to \infty$. Therefore $[f_n] \to [f]$ in the anisotropic sharp norm, and the space is complete. □

8.4 Product spaces

The multi-exhaustion framework behaves well with respect to products of spaces.

Let A and B be LCH, σ-compact spaces equipped with multi-exhaustions $\{h^A_i\}_{i \in I_A}$ on ends $\{U^A_i\}$ and $\{h^B_j\}_{j \in I_B}$ on ends $\{U^B_j\}$ and corresponding weights $\{\varphi^A_i\}$ and $\{\varphi^B_j\}$. Define a decomposition of $A \times B$ by $K^{A\times B}_0 := K^A_0 \times K^B_0$ and $U_{i,j} := U^A_i \times U^B_j$. On each $U_{i,j}$ set $h_{i,j}(a,b) := h^A_i(a) + h^B_j(b)$, and define the product weight $\varphi_{i,j}(t) := \max\{\varphi^A_i(\frac{t}{2}), \varphi^B_j(\frac{t}{2})\}$.

Proposition 8.9 (Anisotropic product space)**.** *With the above notation, the anisotropic sharp space* $(C(A \times B)/\mathbb{R}, ||\cdot||_{\infty;\mathbf{h}^{A\times B},\boldsymbol{\varphi}^{A\times B}})$ *is well-defined and complete. Moreover, if* $f(a,b) = f_A(a)f_B(b)$ *is a simple tensor, then*

$$||[f]||_{\infty;\mathbf{h}^{A\times B},\boldsymbol{\varphi}^{A\times B}} \leq C||[f_A]||_{\infty;\mathbf{h}^{A},\boldsymbol{\varphi}^{A}} \cdot ||[f_B]||_{\infty;\mathbf{h}^{B},\boldsymbol{\varphi}^{B}},$$

with constants C depending only on the product geometry and the families of weights.

Sketch of proof. The decomposition and multi-exhaustion on $A \times B$ satisfy the same assumptions as in the one-space case, so completeness follows from the analogue of Theorem 8.8. The estimate on simple tensors comes from bounding $W_{A\times B}(a,b)|f_A(a)f_B(b) - c|$ by suitable combinations of the one-variable norms and applying Lemma 8.6 on A and B separately. Details are left to the reader. □

8.5 Infinitely many ends

So far we have assumed finitely many ends. Many natural spaces (e.g. trees or graphs of bounded degree) have countably (or even uncountably) many ends. The multi-exhaustion formalism extends to this setting under a mild local finiteness condition.

Suppose that A is decomposed as $A = K_0 \cup \bigcup_{i\in I} U_i$, with I an arbitrary (possibly infinite) index set, and that for every compact $K \subset A$ only finitely many ends U_i intersect K. Assume each U_i carries a continuous proper exhaustion h_i and an admissible weight φ_i. Define the block weight

$$W(x) := \max\{1, \sup_{i\in I} \varphi_i(h_i(x))\},$$

which is finite at every point by local finiteness of the cover and continuity of the h_i.

Proposition 8.10 (Infinitely many ends)**.** *Under the above assumptions, the anisotropic sharp norm*

$$||[f]||^{\sharp}_{\infty;\mathbf{h},\boldsymbol{\varphi}} := \inf_{c\in\mathbb{R}} \sup_{x\in A} W(x)|f(x) - c|$$

is well-defined, and the space $C(A)/\mathbb{R}$ is complete for this norm. Moreover, Lemma 8.6 and Proposition 8.7 extend verbatim with max *replaced by* $\sup_{i\in I}$*, provided the relevant suprema are finite.*

Sketch of proof. Local finiteness ensures that on any compact set only finitely many ends contribute to the weight, so the arguments used in Lemma 8.6 and Theorem 8.8 go through with minor modifications. Completeness again follows by reducing to completeness of a weighted sup-norm on $C_b(A)$. □

8.6 A bounded projection onto the vanishing-tail class

The space X_{van} of functions whose tails vanish on each end plays a central role in applications (e.g. in duality and compactness results). It is therefore natural to ask whether X_{van} is complemented in the full anisotropic space. Under a mild regularity of the ends, the answer is positive.

Theorem 8.11 (Bounded projection onto X_{van})**.** *Assume A admits a multi-exhaustion as above and, in addition, that for each end U_i there exists a continuous function $\chi_i : A \to [0,1]$ such that*

- $\chi_i \equiv 1$ *outside a compact set in U_i;*

- χ_i *vanishes on* $K_0 \cup \bigcup_{j\neq i} U_j$.

Then there exists a bounded linear operator

$$P : C(A)/\mathbb{R} \longrightarrow X_{van}$$

such that $P^2 = P$ *(i.e.* P *is a projection) and* $||P|| \leq C$ *for some constant* C *depending only on the geometric data of the decomposition and the weights.*

Idea of the construction. For $[f] \in C(A)/\mathbb{R}$, use the anisotropic structure to define limits L_i on each end U_i whenever $C_{\varphi_i}^{(h_i)}([f|_{U_i}]) < \infty$. One then builds a piecewise constant function

$$l(x) := \sum_i L_i \chi_i(x);$$

which encodes the asymptotic values of f on each end. Define

$$P[f] := [f - l].$$

The function $f - l$ has vanishing tails on every U_i by construction, so $P[f] \in X_{\mathrm{van}}$. Linearity of P is immediate, and $P^2 = P$ because applying P again does not change the end-limits of $f - l$, which are already zero. Finally, boundedness of P follows from Lemma 8.6, the uniform control of the χ_i, and the stability of the directional limits under small perturbations. A detailed estimate shows that

$$\|P[f]\|^{\sharp}_{\infty;\mathbf{h},\boldsymbol{\varphi}} \leq C \, \|[f]\|^{\sharp}_{\infty;\mathbf{h},\boldsymbol{\varphi}}$$

for some $C > 0$ depending only on the overlap constants and the admissible weights. The existence of such a bounded projection shows that the "purely decaying" part X_{van} forms a complemented subspace of the anisotropic sharp space. This clarifies the functional-analytic structure of the multi-exhaustion framework and is useful, for instance, when describing dual spaces or decomposing solutions of elliptic equations into asymptotic profiles plus vanishing tails.

9 Conclusion

In this work, we have developed a unified framework for analyzing convergence at infinity. Beginning with the simple and general notion of an **exhaustion** of a space, we constructed an **exhaustion function** h which serves as a generalized measure of distance to an adjoined **point at infinity,** ω_A. This foundation allowed us to define convergence for both points and functions in a way that is consistent across metric, topological, and other contexts.

The core of our contribution is the introduction of a **family of weighted norms,** $\|f\|_{\infty,h,p}$, and the complementary **asymptotic rate constant,** $C_p(f)$. We have shown that these tools move beyond a simple binary view of convergence. They provide a robust system for classifying convergence rates, where the finiteness of a norm is a sufficient condition for convergence. As demonstrated, this framework successfully distinguishes between functions with different asymptotic behaviors (such as algebraic versus exponential decay), resolving ambiguities that arise from simpler measures.

This work opens several avenues for future research. The generalization of the weighting function to a full **scale of comparison functions** ϕ, as explored in Section 4.1.2, is a first step.

It is also essential to recognize the current boundaries of our formalism, which themselves open stimulating research prospects. Our construction, founded on a single scalar exhaustion function h, is intrinsically **isotropic**. It is therefore perfectly adapted to spaces whose "infinity" can be conceived as a single point, ...mirroring the Alexandroff one-point compactification our framework generalizes [9]. However, this approach reaches its limits for spaces that possess a richer **anisotropic** structure at infinity. For example, spaces with several distinct "ends" (such as an infinite strip in $\mathbb{C}$ or the space $\mathbb{R} \setminus \{0\}$) or whose geometry of convergence depends on the direction (as in the hyperbolic plane) would require a significant generalization.

An exciting extension of this work would be to develop a **"multi-exhaustion" formalism**, where a collection of functions $\{h_i\}$, each associated with a different "end" of the space, could be used to independently characterize the different paths to infinity. Such a framework could potentially unify asymptotic analysis on spaces with complex boundaries, linking the geometry of their compactifications (like the Martin or Thurston compactifications) to quantitative analytical tools similar to those developed here.

Finally, the practical potential of this framework could be demonstrated by applying it to specific problems in physics or probability theory, while the function spaces $C_{h,\phi}(A)$ induced by these norms present a rich topic for further investigation in functional analysis.

In summary, the exhaustion function provides a simple and powerful language to unify the concept of infinity across diverse fields of mathematics. The quantitative apparatus built upon it offers a systematic and effective method for answering, with nuance, the question: "How fast does a function converge?"

Potential Applications

The framework developed herein is not merely an abstract unification; it is a powerful analytical instrument with the potential for concrete applications(cf.A Weighted Kolmogorov Metric for Berry-Esseen Bounds under Sub-Cubic Moments) across several fields.

- **Partial Differential Equations:** The multi-exhaustion framework is ideally suited for studying the asymptotic decay of solutions to elliptic equations on non-compact manifolds (such as cylinders or spaces with conic singularities). The ends U_i and their corresponding exhaustion functions h_i provide a canonical way to classify the convergence rates of solutions in different asymptotic regimes.

- **Probability Theory:** The rate of convergence of Markov chains to their stationary distribution on infinite state spaces could be analyzed with this formalism. The ends of the space could correspond to different classes of recurrent states, with the directional norms measuring the speed of convergence within each class.

- **Geometric Group Theory and General Relativity:** In fields where the large-scale geometry of a space is paramount, our framework could offer a new quantitative language. This includes analyzing word metrics on the Cayley graphs of infinite groups or quantifying the decay of fields towards null infinity in models of spacetime.

These potential connections underscore that the question "How fast does a function converge?" is fundamental not only in pure analysis but also in our mathematical modeling of the world.

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