Convolution

1. Assume x(n) = u(n)-u(n-10) and h(n)= an u(n), with a=0.5.

a) Find the theoretical closed-form expression for y(n) = x(n)*h(n)

b) Plot h(n)

c) Plot x(n) and y(n).

answer) $y[n] = \sum_{k=-\infty}^{k=+\infty} x[k]h[n-k]$ or $y[n] = \sum_{k=-\infty}^{k=+\infty} h[k]x[n-k]$

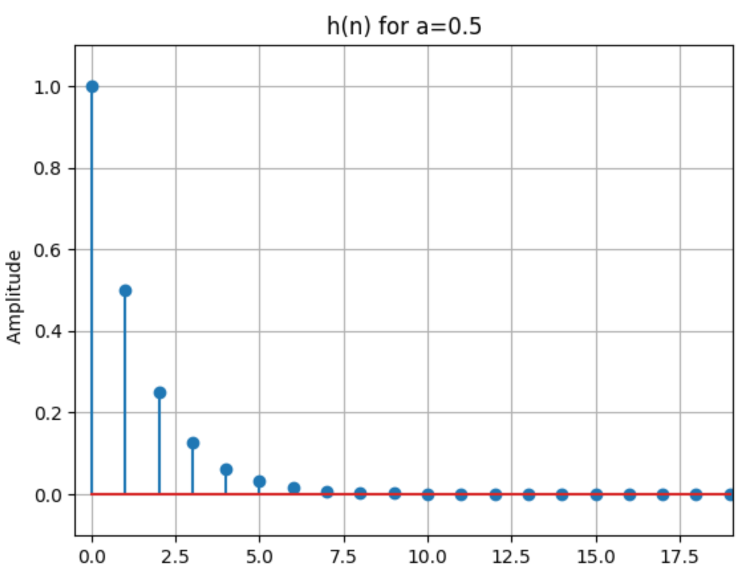
based on eq1 : $y[n] = \sum_{k=-\infty}^{k=+\infty} \propto^k u[k]x[n-k] = \sum_{k=0}^{k=n} \propto^k$ (Partial overlap is starting from 0 to n=N-1=9)

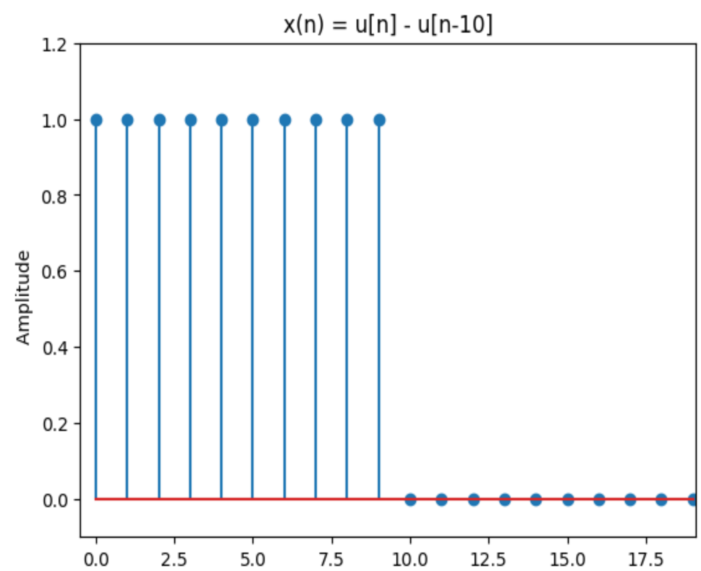
for $n \leq N-1$ (n≤ 9), the sequence is finite

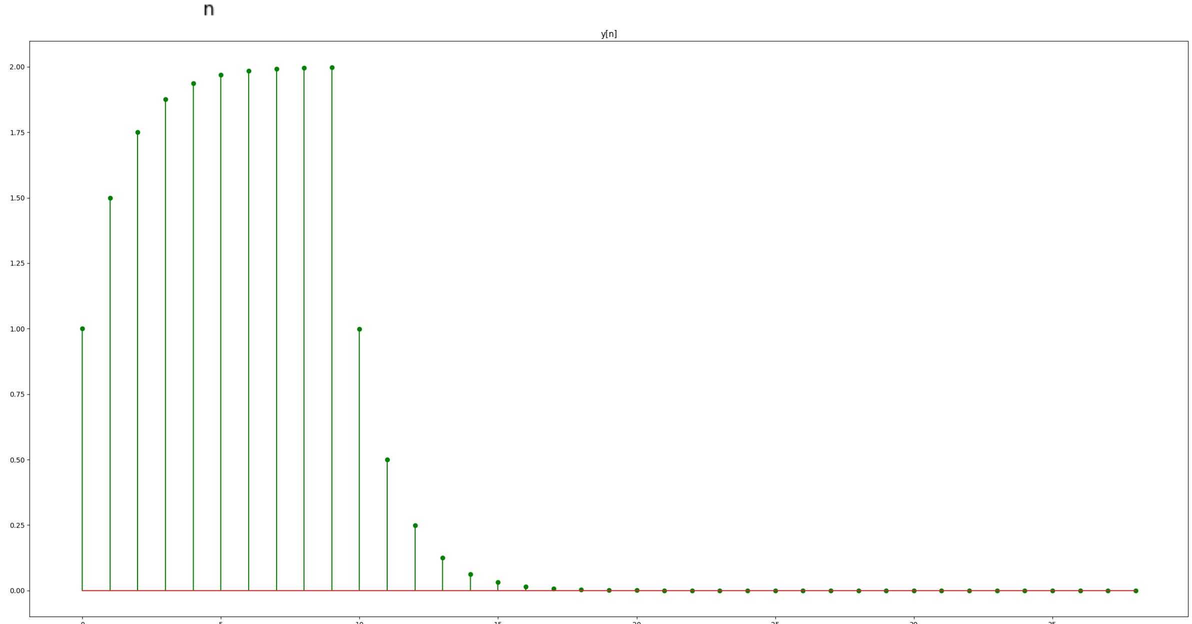
$$y[n] = \sum_{k=0}^{k=n} \propto^k = \frac{(\propto)^0 - (\propto)^{n+1}}{1-\propto} = \frac{1-\propto^{n+1}}{1-\propto} \quad \rightarrow \quad 0 \leq n \leq 9$$

for $n \geq 10$

$$y[n] = \sum_{k=9}^{k=n} \propto^k = \frac{(\propto)^{n-9} - (\propto)^{n+1}}{1-\propto} = \propto^{n-11} \left(\frac{1-\propto^{10}}{1-\propto}\right) \quad \rightarrow \quad n \geq 10$$







Difference Equation

2. Let y(n) - 0.4y(n-1) = x(n) and assume y(n) = 0 when n<0.
   a) Find the impulse response.
   b) Using x(n) = δ(n) + sin(0.08πn) u(n), find and plot y(n). Hint: use recursive (iterative) computation to solve the difference equation instead of the impulse response.
   c) What can you expect when n is close to zero?
   d) What can you expect when n is very large?

a)

$x[n] = \delta[n]$ and $y[n] = h[n]$

rewrite the problem equation :

$h[n] - 0.4h[n-1] = \delta[n] \rightarrow h[n] = \delta[n] + 0.4h[n-1]$ we know $h[-1] = h[-2] = \ldots = 0$

$n = 0 \quad h[0] = \underbrace{\delta[0]}_{1} + 0.4\, \underbrace{h[0-1]}_{0} \approx 1$

$n = 1 \quad h[1] = \underbrace{\delta[1]}_{0} + 0.4\, \underbrace{h[0]}_{1} \approx 0.4$

$n = 2 \quad h[2] = \underbrace{\delta[2]}_{0} + 0.4\, \underbrace{h[1]}_{0.4} \approx 0.16$

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b) $y[n] = 0.4\, y[n-1] + \underbrace{\delta[n] + \sin[0.8\pi n]\, u[n]}_{x[n]}$

n=0 $\quad y[0] = 0.4\, \underbrace{y[-1]}_{0} + \underbrace{\delta[0]}_{1} + \underbrace{\sin[0.8\pi 0]\, u[0]}_{0} \approx 1$

n=1 $\quad y[1] = 0.4\, \underbrace{y[0]}_{0.4} + \underbrace{\delta[1]}_{0} + \underbrace{\sin[0.8\pi]}_{0.24}\, \underbrace{u[1]}_{1} \approx 0.64$

n=2 $\quad y[1] = 0.4\, \underbrace{y[0]}_{0.64} + \underbrace{\delta[1]}_{0} + \underbrace{\sin[0.8\pi]}_{0.48}\, \underbrace{u[1]}_{1} \approx 0.736$

$y[n] = \{\ \ldots,\ 1,\ 0.64,\ 0.736,\ \ldots\}$

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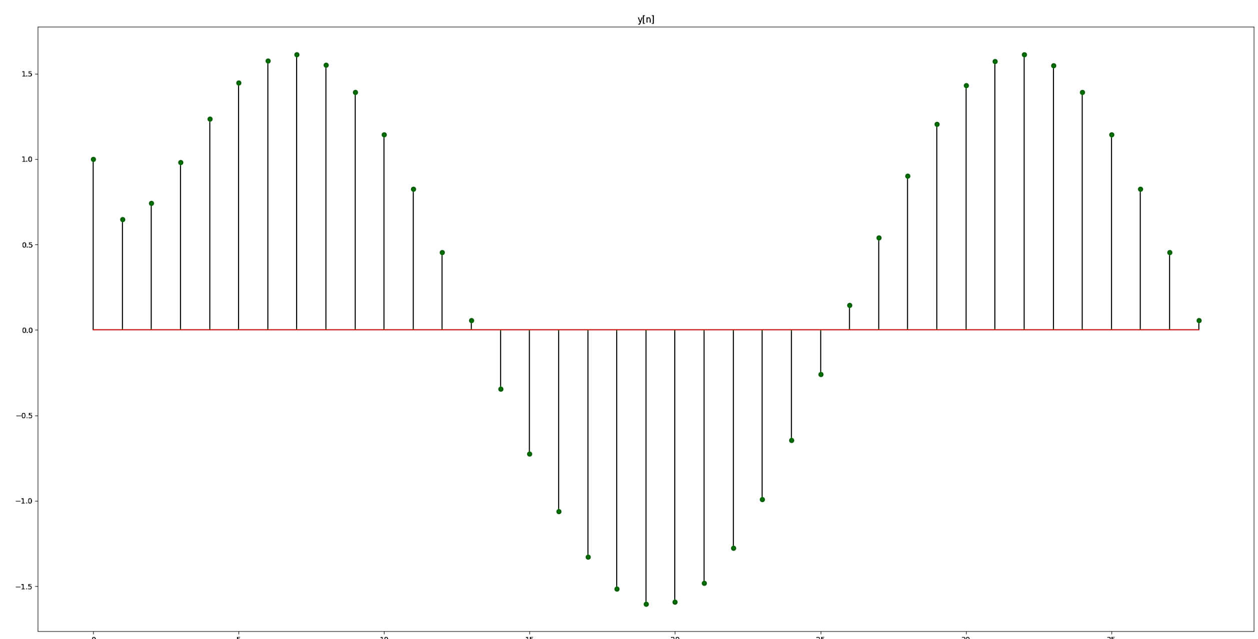
n: 1, y[n] => 0.6487   n: 2, y[n] => 0.7412   n: 3, y[n] => 0.9810
n: 4, y[n] => 1.2367   n: 5, y[n] => 1.4458   n: 6, y[n] => 1.5763
n: 7, y[n] => 1.6128   n: 8, y[n] => 1.5500   n: 9, y[n] => 1.3905
n: 10, y[n] => 1.1440  n: 11, y[n] => 0.8257  n: 12, y[n] => 0.4556
n: 13, y[n] => 0.0569  n: 14, y[n] => -0.3454 n: 15, y[n] => -0.7259
n: 16, y[n] => -1.0609 n: 17, y[n] => -1.3292 n: 18, y[n] => -1.5140
n: 19, y[n] => -1.6036 n: 20, y[n] => -1.5925 n: 21, y[n] => -1.4813
n: 22, y[n] => -1.2771 n: 23, y[n] => -0.9926 n: 24, y[n] => -0.6457
n: 25, y[n] => -0.2583 n: 26, y[n] => 0.1454  n: 27, y[n] => 0.5399
n: 28, y[n] => 0.9005  n: 29, y[n] => 1.2045  n: 30, y[n] => 1.4329
n: 31, y[n] => 1.5712  n: 32, y[n] => 1.6108  n: 33, y[n] => 1.5491
n: 34, y[n] => 1.3902  n: 35, y[n] => 1.1439  n: 36, y[n] => 0.8257
n: 37, y[n] => 0.4556  n: 38, y[n] => 0.0569
```



- c, d) If n is near zero, we expect to see behaviour like a step function, and when n is going infinity, we expect to see a periodic behaviour of the y[n].

3. The ideal lowpass filter in Example 2.18 is not implementable. A simple approximation can be achieved by truncating the impulse response. Assume a noncausal L-tap (L is odd) FIR filter is obtained by such a truncation. To make this filter causal, one can shift the impulse response to the right by (L-1)/2 samples, giving

$$h_{FIR}(n) = \frac{\sin[\omega_C(n-\frac{L-1}{2})]}{\pi(n-\frac{L-1}{2})}, \qquad 0 \le n \le L-1 .$$

a) Assuming $\omega_C=0.2\pi$, compute and plot the amplitude and phase responses of this causal FIR filter for L=19 and L=101.
b) What are the advantages and disadvantages of increasing L?
c) Plot the amplitude and phase responses of $(-1)^n h_{FIR}(n)$ and compare with the results in a).

Answer:

a)

Truncation of the impulse response equals multiplying h[n] or shifted by a rectangular window

$$h(n) \overset{\text{DTFT}}{\leftrightarrow} H(w)$$

Frequency response: $H(w) = \underbrace{H_r(w)}_{\text{Real part}} e^{j\theta(w)}$

Symmetric impulse response with M, which is an odd value and defined by the following amplitude and phase response formulas:

$$H_r(w) = h\left(\frac{M-1}{2}\right) + \sum_{n=0}^{\frac{M-3}{2}} 2h(n)\cos(w\left[n-\frac{m-1}{2}\right])$$

$$\measuredangle H_r(w) = \begin{cases} -w\left(\frac{M-1}{2}\right) & ; H_r(w) > 0 \\ -w\left(\frac{M-1}{2}\right) + \pi & ; H_r(w) < 0 \end{cases}$$

Now, we can initial constants by given values in the aforementioned previous page formulas:

$M = 19\,, w_c = 0.2\pi$

$$H_r(w) = h(9) + \sum_{n=0}^{8} 2h(n)\cos\left(\frac{\pi}{5}[n-9]\right)$$

$$\Downarrow$$

amplitude:

$$H_r(w) = \frac{\sin\left[\frac{\pi}{5}\times(9-9)\right]}{\pi\underbrace{(9-9)}_{0}} + \sum_{n=0}^{8} 2\frac{\sin\left[\frac{\pi}{5}\times(n-9)\right]}{\pi(n-9)}\cos\left(\frac{\pi}{5}[n-9]\right)$$

$$H_r(w) = \frac{1}{\pi}\sum_{n=0}^{8} 2\frac{\sin\left[\frac{\pi}{5}\times(n-9)\right]}{(n-9)}\cos\left(\frac{\pi}{5}[n-9]\right) \quad \Longrightarrow \text{(Based on } \sin 2\theta = 2\sin\theta\,\cos\theta \text{)}$$

$$= \frac{1}{\pi}\sum_{n=0}^{8}\frac{\sin 2\left[\frac{\pi}{5}\times(n-9)\right]}{(n-9)} = 0 + \frac{1}{\pi}\sum_{n=1}^{8}\frac{\sin\left[\frac{2\pi}{5}\times(n-9)\right]}{(n-9)} \quad \Longrightarrow \text{(Based on } \sum_{k=1}^{n} k = \frac{1}{2}\,n(n+1) \text{)}$$

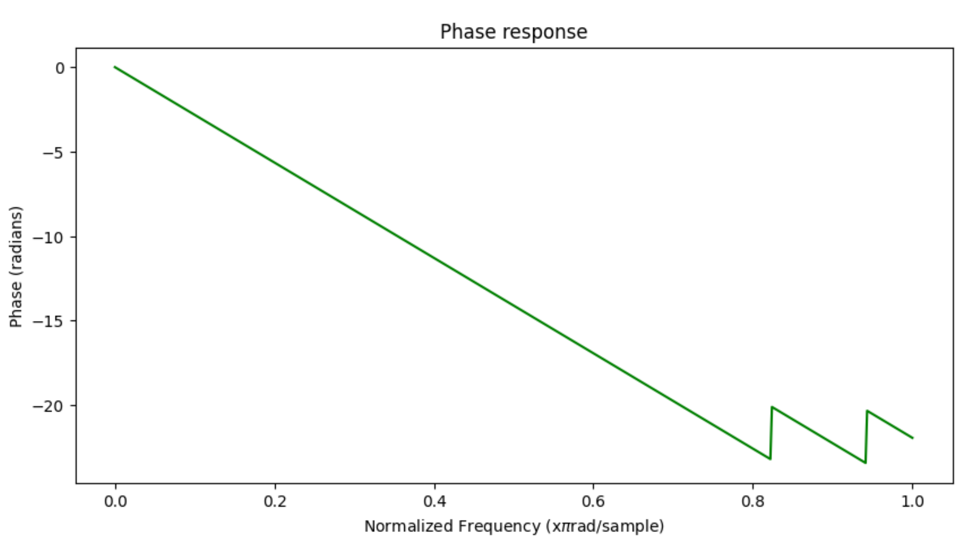
n=1 $\rightarrow \frac{1}{\pi}\frac{\sin\left[-\frac{2\pi}{5}\times(8)\right]}{-8} = 0.00676$ | n=2 $\rightarrow \frac{1}{\pi}\frac{\sin\left[-\frac{2\pi}{5}\times(7)\right]}{-7} = 0.00695$

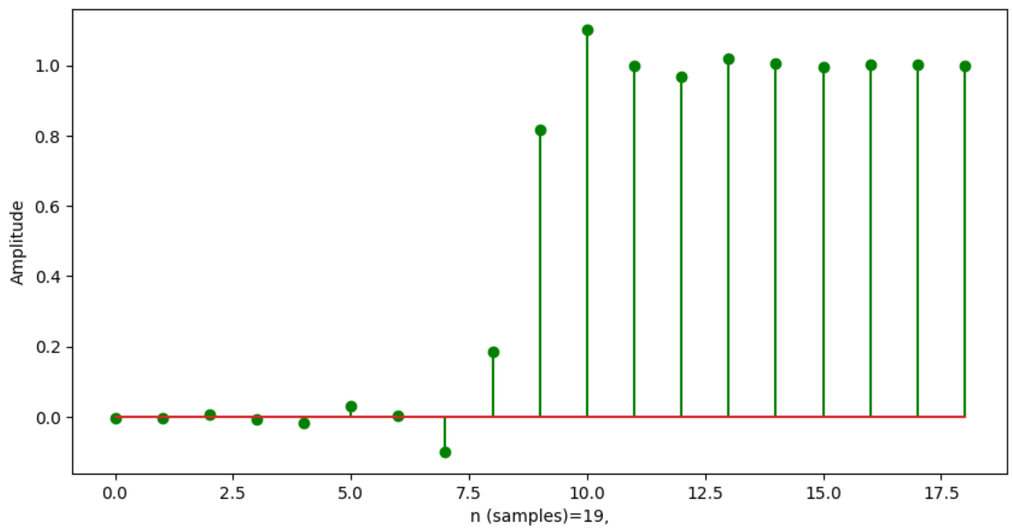
n=3 $\rightarrow \frac{1}{\pi}\frac{\sin\left[-\frac{2\pi}{5}\times(6)\right]}{-6} = 0.00696$, … n=8 $\rightarrow \frac{1}{\pi}\frac{\sin\left[-\frac{2\pi}{5}\times(1)\right]}{-1} = 0.02193$

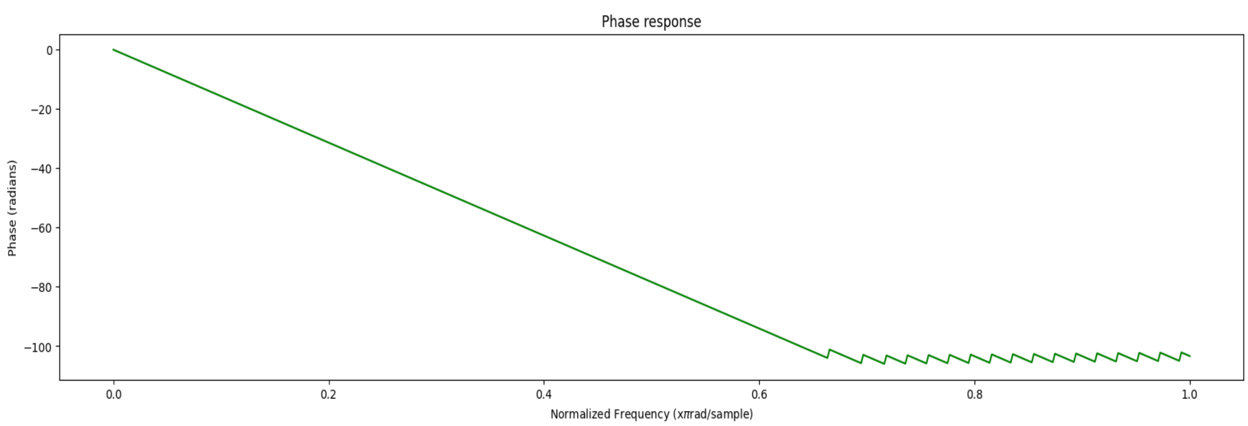
phase:

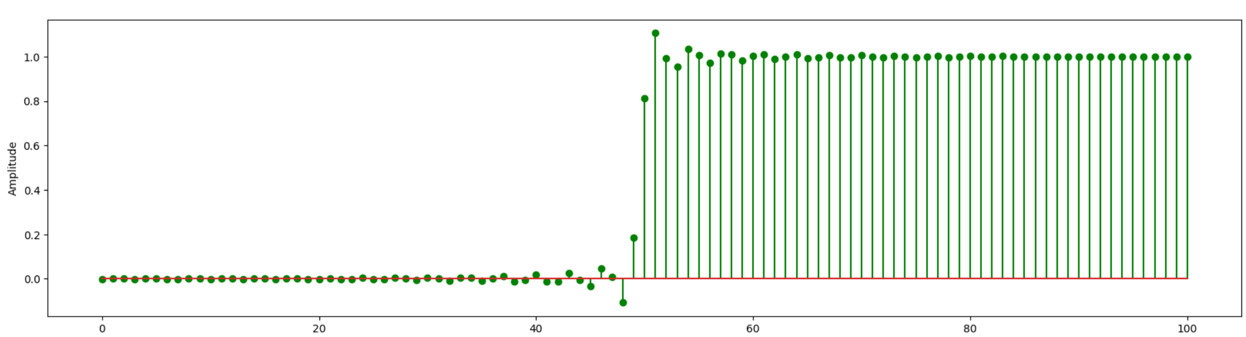
$$\measuredangle H_r(w) = \begin{cases} \dfrac{-9\pi}{5} & ;H_r(w) > 0 \\[2ex] \dfrac{-9\pi}{5} + \pi & ;H_r(w) < 0 \end{cases}$$

We should precisely repeat this process for M=101 with just different window sizes.
In the following corresponding plots are showing.




Amplitude
n (samples)=19,


Phase response
Phase (radians)
Normalized Frequency (xπrad/sample)


Amplitude
n (samples)=101,

b)

Advantages :

- This method doesn't need huge computational recourses, so it's the fastest technique in designing an FIR filter, but it's not the best.
- We receive a linear-phase frequency corresponding to delay. Therefore, we expect a smoother transition band and ripples than the ideal filter.
- Picking good coefficients improves filter performance

Disadvantages :

- The system has a delay of $\frac{L-1}{2}$ samples. For example, at n=0, the system will not react until the value $\frac{L-1}{2}$. So, in some real-time applications, it may cause a problem.
- Other drawbacks include: - inefficiency - unequal perturbation in ripples - difficulty in choosing cut-off frequency.
- Selecting a window needs a trade-off between lobe width and peak side lobe amplitude and width of the transition band.
- As L increases, the main lobe window decreases

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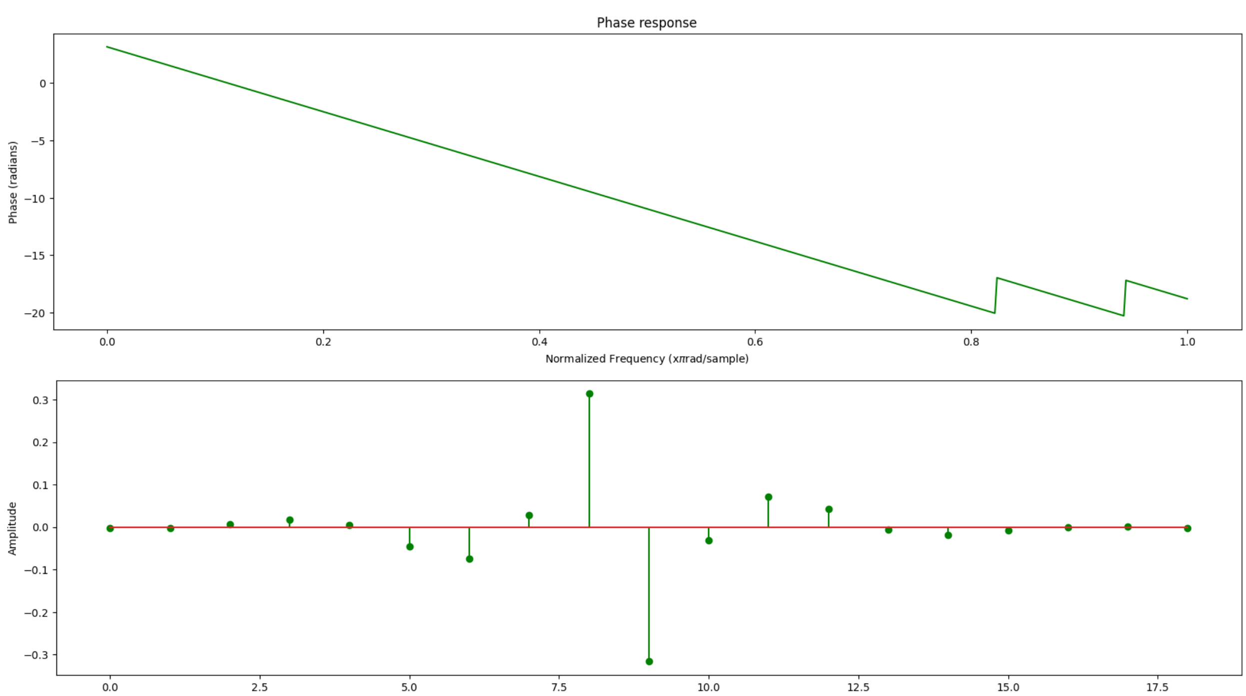
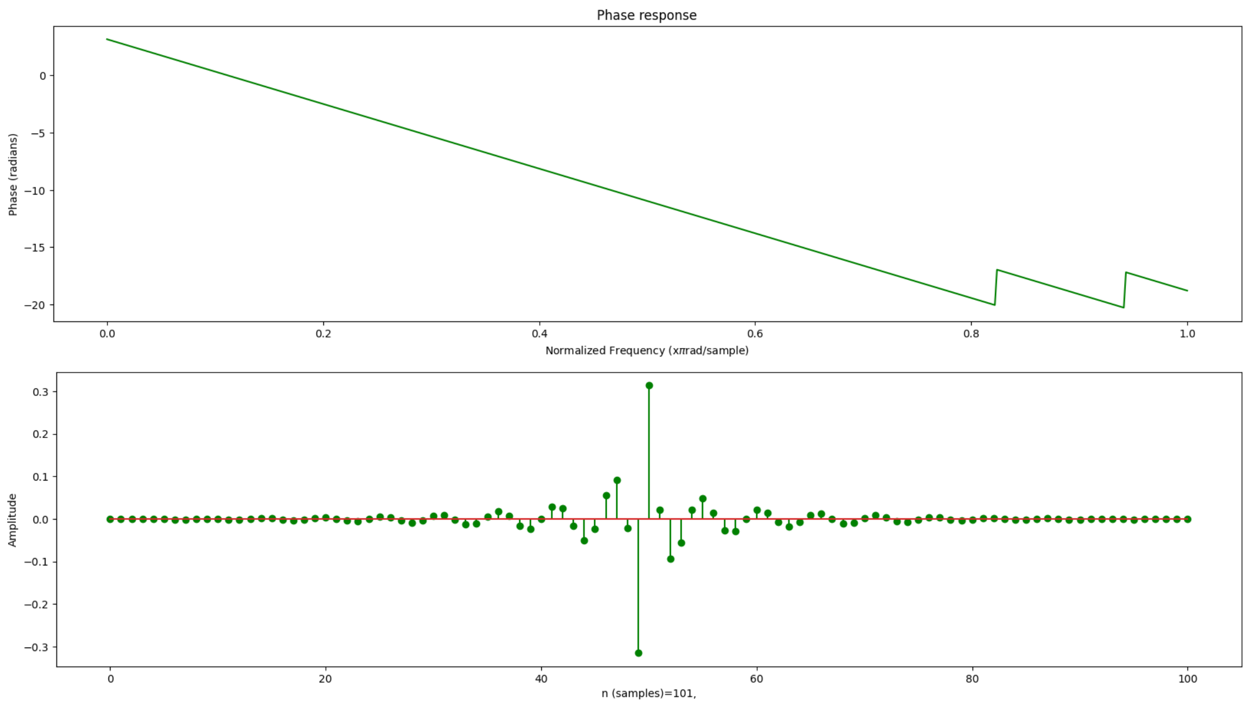
c)





- As a comparison, more delay ends up having more zero response magnitude and very narrow band-pass windowing (range), but there is no effect on phase response.