1. Demonstration of Aliasing

A continuous-time signal is given by $x(t)=\cos(0.5\pi t)u(t)$,. The sequence $x(n)$ is obtained by sampling $x(t)$ with period T.

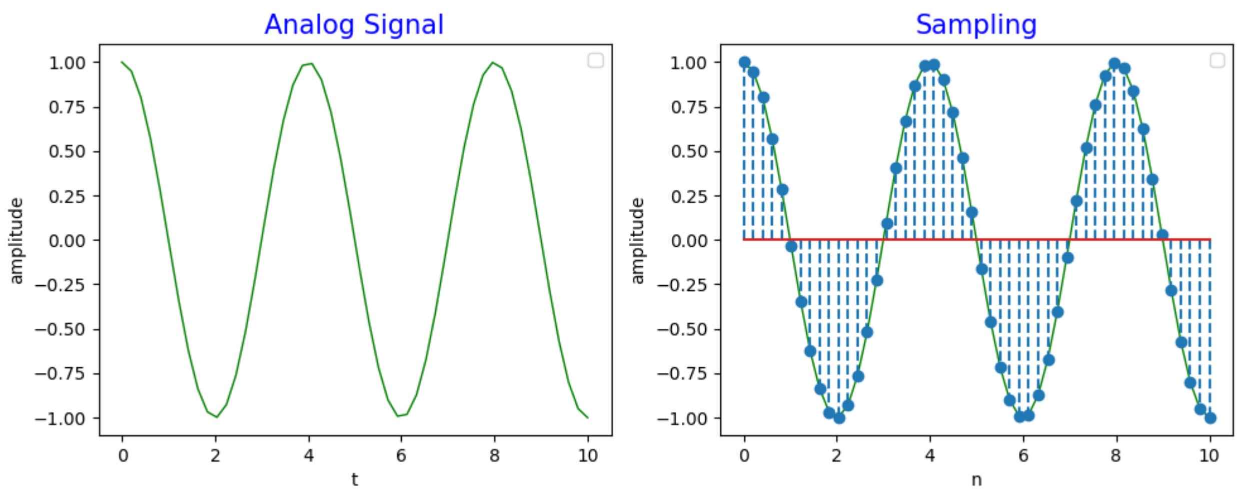
a) Plot the spectrums of $x(t)$;

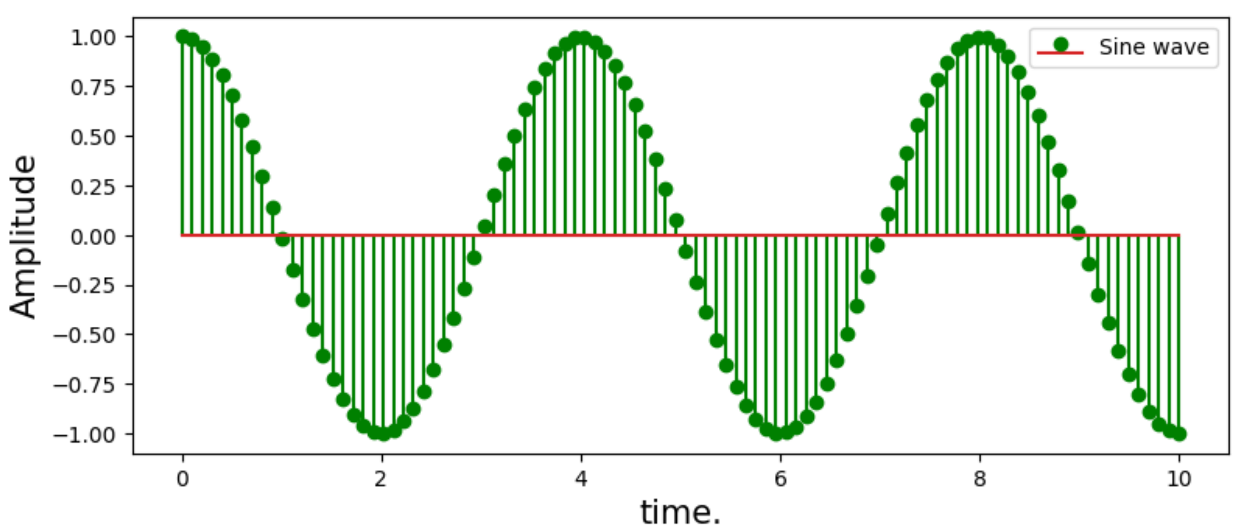
b) Plot the spectrums of $x(n)$ for $T_1=0.2$, $T_2=0.05$, and $T_3=0.01$, and discuss the results;

c) Which sampling period leads to higher aliasing? Is it possible to completely eliminate the aliasing?

Answer)

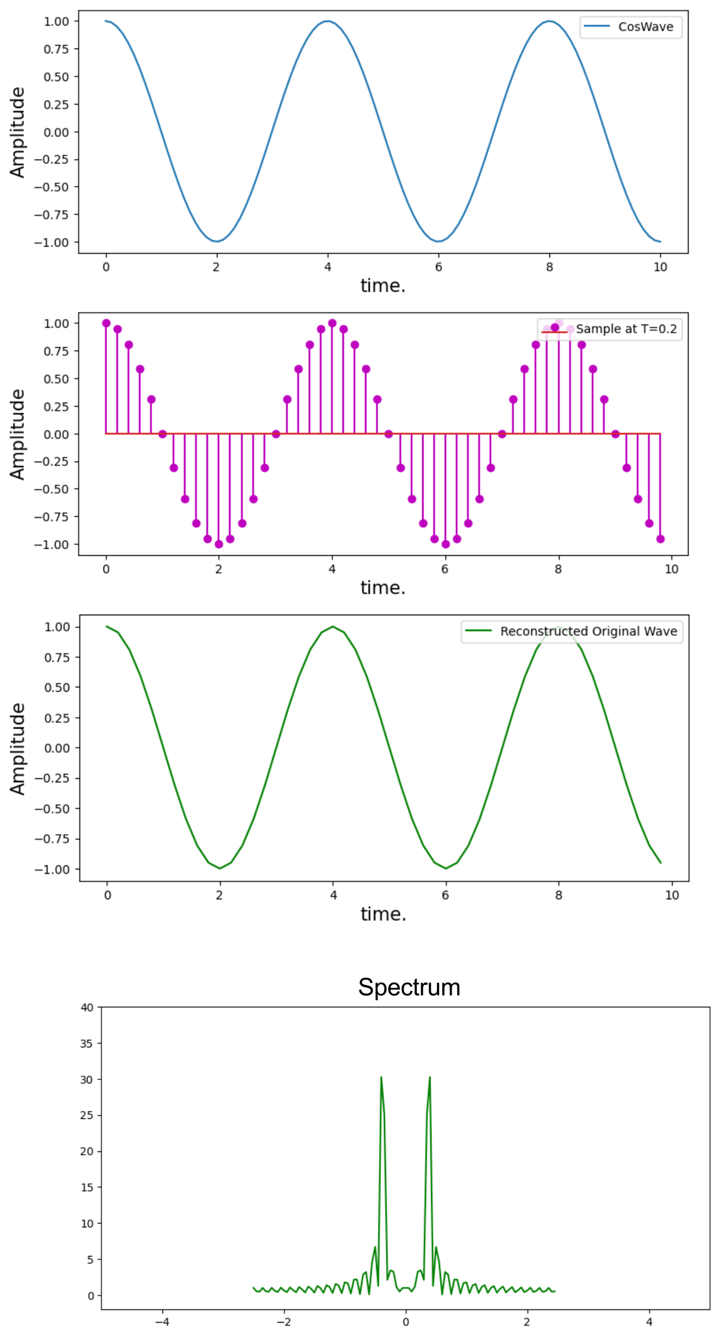
a)



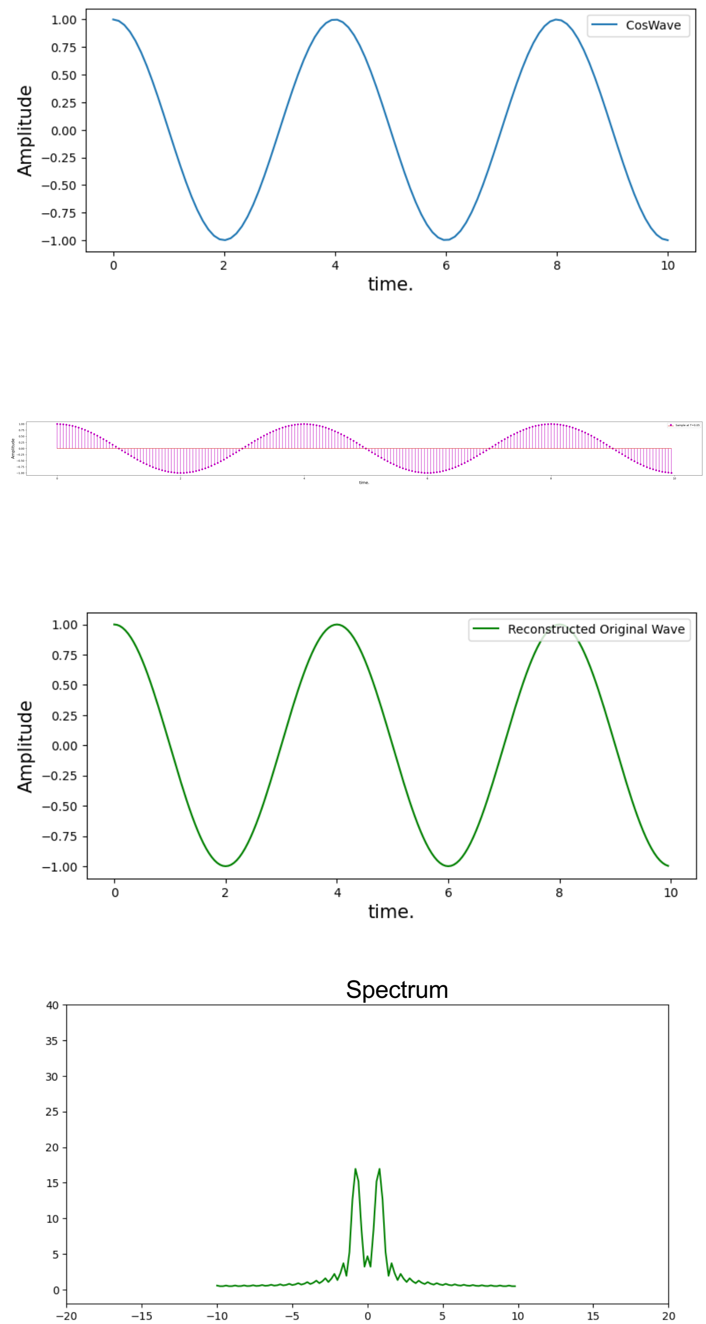


b)

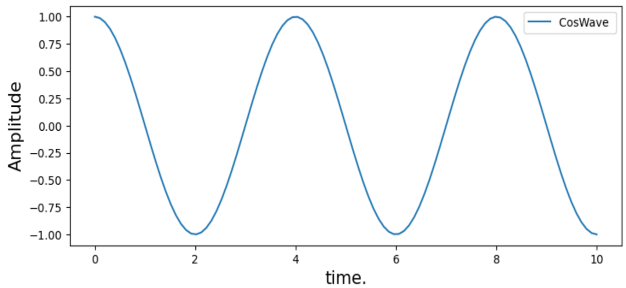
$T_1 = 0.2$

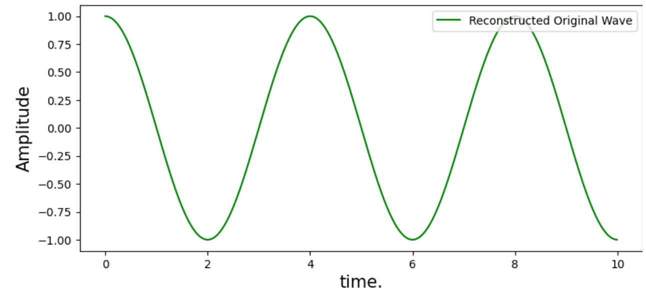

CosWave
Amplitude
time.
Sample at T=0.2
Amplitude
time.
Reconstructed Original Wave
Amplitude
time.
Spectrum

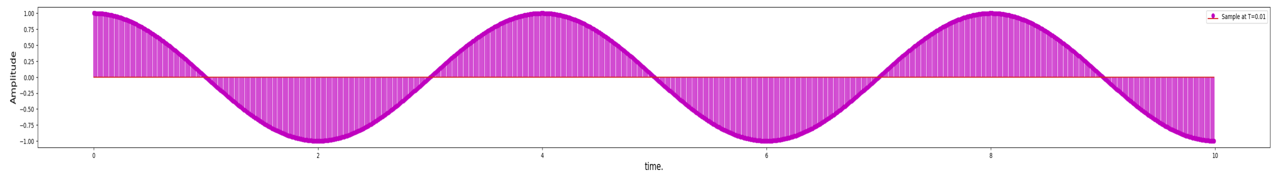
$T_1 = 0.05$

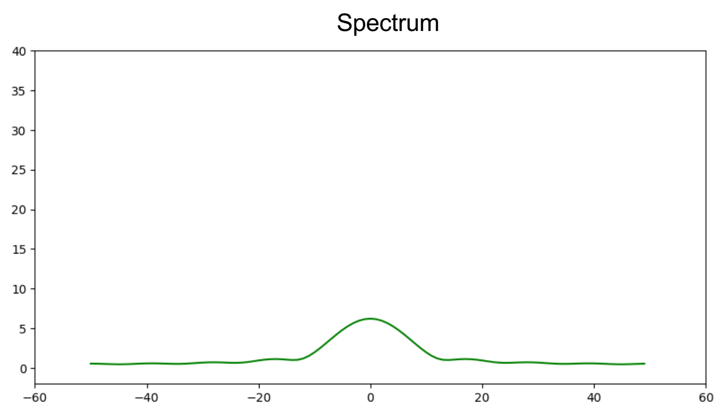

CosWave
Amplitude
time.
Reconstructed Original Wave
Amplitude
time.
Spectrum

$T_1 = 0.01$









Impulse sampling is named ideal sampling. In this technique, the sampling function trains impulses, known as the multiplication principle.
As T is decreasing reconstructed signal is more like the original, and for T equals 0.2, we can see break lines on positive/negative peakness due to insufficient sampling time.
In this problem, T is calculated by :

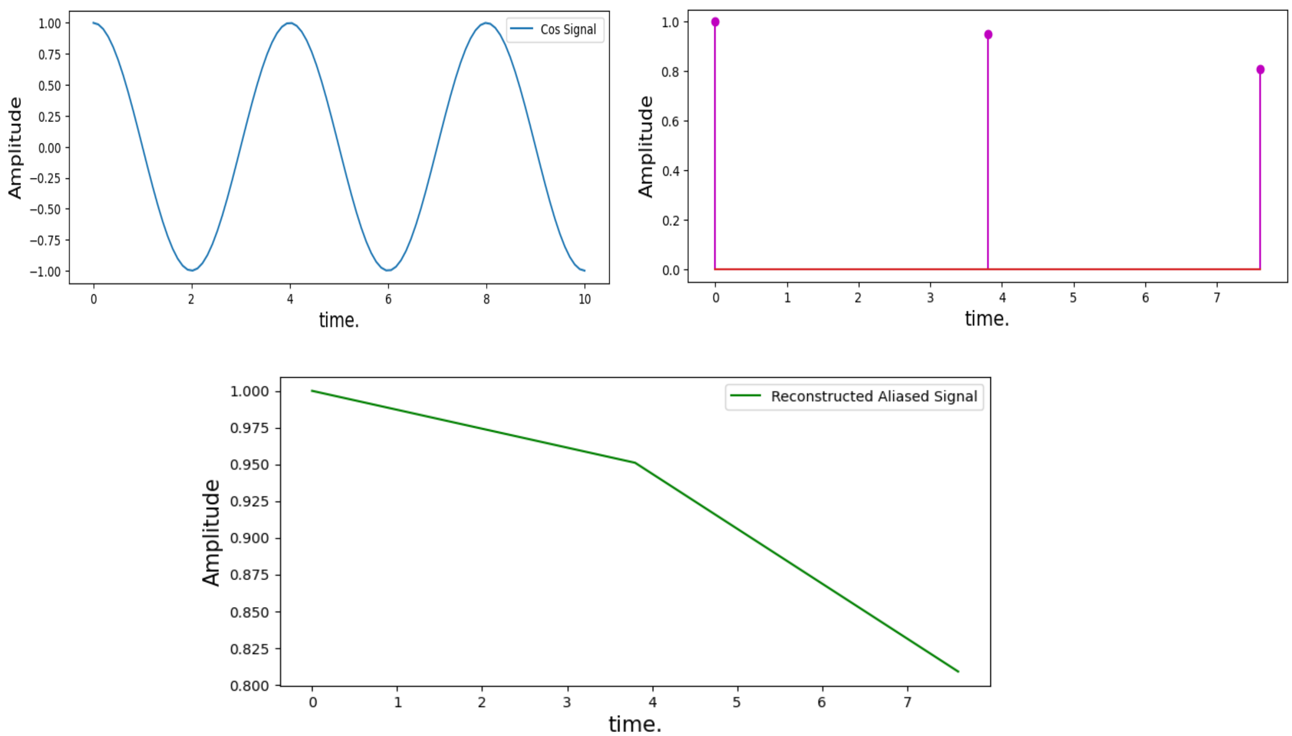
$$T = \frac{2\pi}{w} \rightarrow \frac{2\pi}{\frac{\pi}{2}} = 4 \rightarrow \varphi = 0.25$$
$$So\ \varphi_s \geq 2\varphi_p$$

For this section, all of them satisfy this condition, and we don’t have aliasing. The difference is just a more precise reconstructed signal as an output.
But the drawback is in size memory for saving computation which more samples need higher computation and space, so it’s expensive.

c)

the minimum sampling rate is Nyquist frequency. This limit is double the maximum frequency of the original signal. So, in choosing $\varphi_s \leq 0.5$, aliasing occurs. aliasing occurs when higher frequency components are placed into another signal pass-band limit and create interposition. To eliminate the effect of aliasing, we should be sure that there is no φ_p bigger than half of φ_s(sampling frequency).

For example, by choosing $\varphi_s = 3.8$, we see the effect of wrong reconstruction due to aliasing for which the higher frequency is placed in the wrong phase



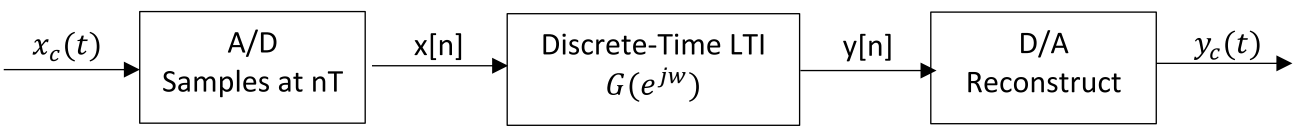
2. <u>Recovery of Continuous-Time Signal from Sequence</u>

To demonstrate the recovery of x(t) from x(nT), we use x(t)=cos(ω_1t)+ cos(ω_2t). Choose proper values for ω_1 , ω_2 and T such that x(t) is sampled without aliasing. Theoretically, we need an ideal analog LPF to recover x(t) from x(n). Practically, you can use a high-order FIR filter to do this job but the sampling rate of the FIR filter should be high enough so that you can see an approximate "analog" signal x(nT$_1$) (T$_1$<<T, you may consider T=8T$_1$). This procedure also simulates the interpolation.

Determine the FIR filter and plot the output signal. Comment on your results.
(The real analog signal can be obtained by following the FIR filter with a D/A converter).
(You may also try this method for Q.1).

Answer)

We should implement an equivalent Analog filter by this schema:

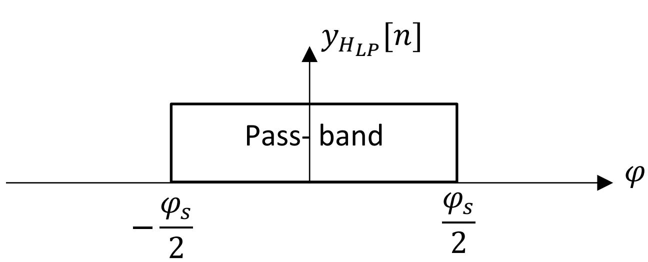


$$x_c(t) = \cos(w_1 t) + \cos(w_2 t)$$

We should transfer continuous filter specification to direct time filter.
First, we remove high-frequency noise, for example, $|\varphi| > 100Hz$. So we want to generate a low-pass filter to pass $|\varphi| < 100Hz$. We assume $X_c(\varphi) = 0$ for $|\varphi| > 8000Hz$ and $\varphi_s = 20000 \frac{rad}{sec}$.
So, we satisfy the sampling theorem of $\varphi_s \geq 2\varphi_f$ for sampling without aliasing. We should choose FIR filter components properly as well.
Some facts:

$$x_c(t) \leftrightarrow X_c(\varphi) \quad : \quad x[n] \leftrightarrow X(e^{jw})$$
$$y_c(t) \leftrightarrow Y_c(\varphi) \quad : \quad y[n] \leftrightarrow y(e^{jw})$$

$w = \varphi T$ for an ideal reconstruction by analog low-pass filter:



$$Y_c(\varphi) = H_{LP}(\varphi)\, Y\left(e^{jw}\right) \ |_{w=\varphi T} = H_{LP}(\varphi) Y(e^{j\varphi T})$$

$$Y\left(e^{jw}\right) = G\left(e^{jw}\right)\ X(e^{jw})$$

$$Y_c(\varphi) = H_{LP}(\varphi)\, G\left(e^{j\varphi T}\right) X\left(e^{j\varphi T}\right)$$

By sampling operation

Recall: $X_{cx}(\varphi) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(\varphi - k\varphi_s)$

We know: $X\left(e^{jw}\right) = X_{cs}(\varphi)|_{\varphi=\frac{w}{T}} = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(\frac{w}{T} - k\varphi_s)$

So: $X\left(e^{j\varphi T}\right) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(\varphi - k\varphi_s)$

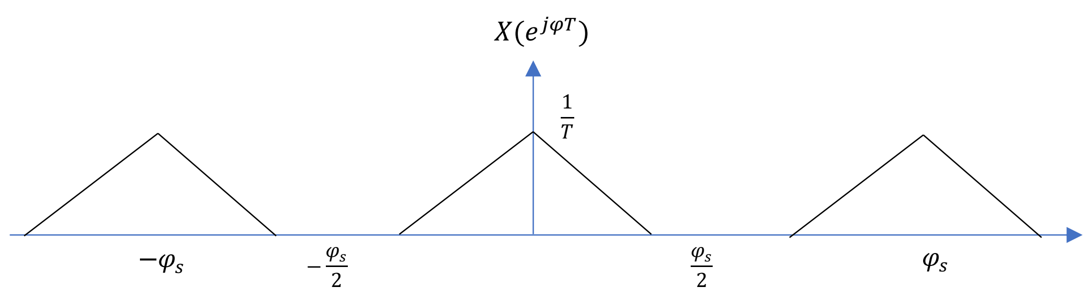
Replacing output furrier transform:

$$Y_c(\varphi) = H_{LP}(\varphi)\, G\left(e^{j\varphi T}\right) \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(\frac{w}{T} - k\varphi_s)$$

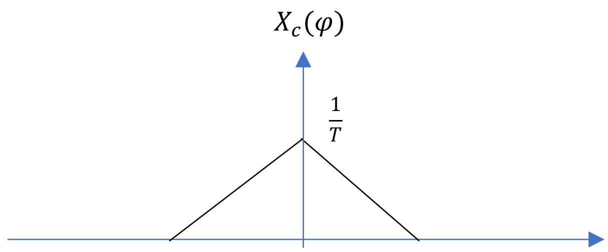
This expression tells us how the output spectrum is related to the input spectrum.
To analyze further, we assume $|X_c(\varphi)| = for\ |\varphi| > \frac{\varphi_s}{2}$ (sampling theorem) in multiplication low-pass filter response $(H_{LP}(\varphi))$ with Fourier transfer of input.

$$H_{LP}(\varphi)\, \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(\varphi - k\varphi_s) = X_c(\varphi)$$

This expression tells us, we can reconstruct original spectrum from sampling.



By applying LPF, we have:

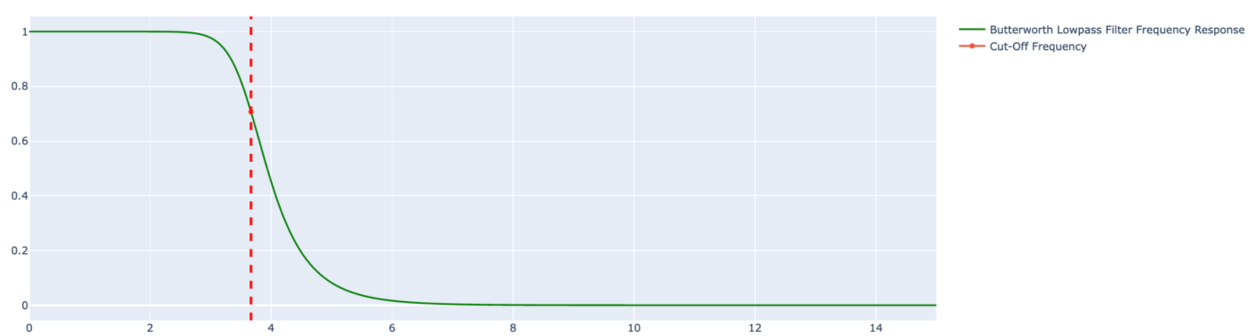


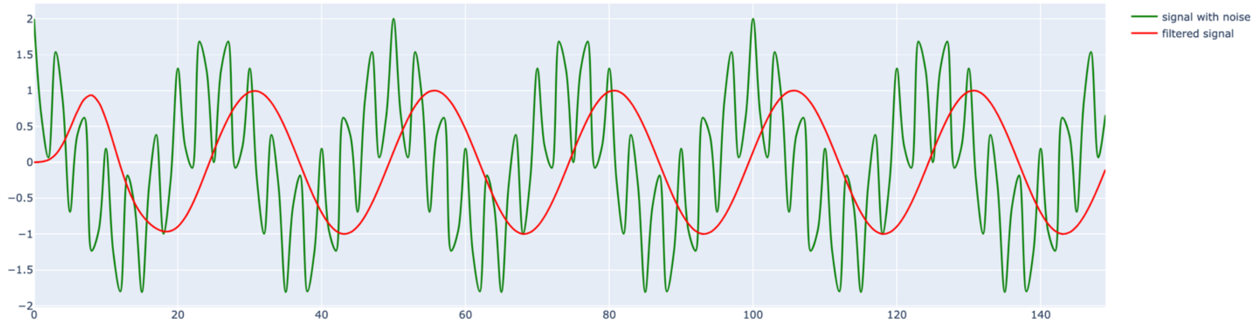
Finally, the output $Y_c(\varphi) = G(e^{j\varphi T})\ X_c(\varphi)$. We use the specification of continues time filter response $H(\varphi)$ by transferring directly to $G(e^{j\varphi T})$ (discrete time filter) for $|\varphi| < \frac{\varphi_s}{2}$.
Note: $G(e^{jw})$ is always 2π periodic and $G(e^{j\varphi T})$ has period $\frac{2\pi}{T} = \varphi_s$.
So, we showed that we can implement a discrete time filter as a continuous time filter.

In this problem, we assume noisy data for our function by $x_c(t)$. We set T=5, W_1= 2.4π, W_2= 18π. The sample rate must be greater or equal to twice the signal's bandwidth. Therefore, for $\varphi_s < 10$, there is an aliasing error, as below:

```
ValueError: Digital filter critical frequencies must be 0 < Wn < fs/2 (fs=6.0 -> fs/2=3.0)
```

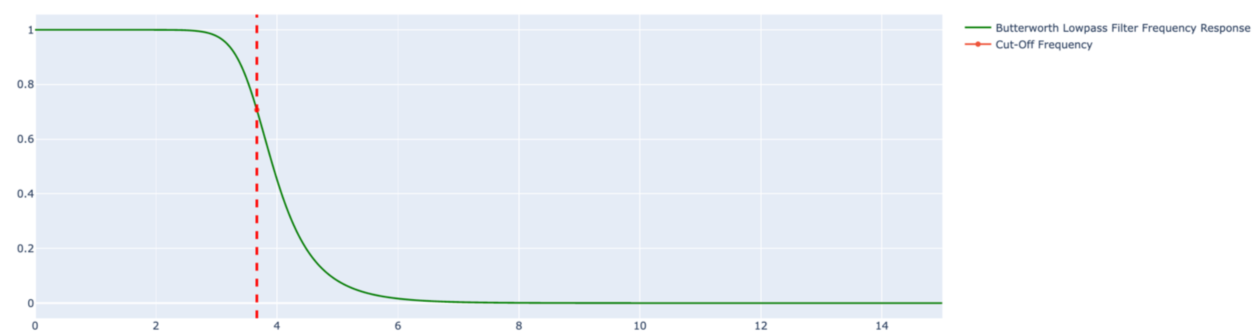
This tells us the band-limited should be between half the sampling frequency and zero. The cut-off frequency should be less or equal to half of the sampling rate (Nyquist rate). We choose cut-off$\approx 3.667\ Hz$ for the sampling rate of 30 Hz as below. The implemented filter order is 7. Filter order can be approximated by inversely proportional to the ratio of transition-band width to the sample rates. A desired approach is to choose the lowest cut-off and highest order which can be tolerated.

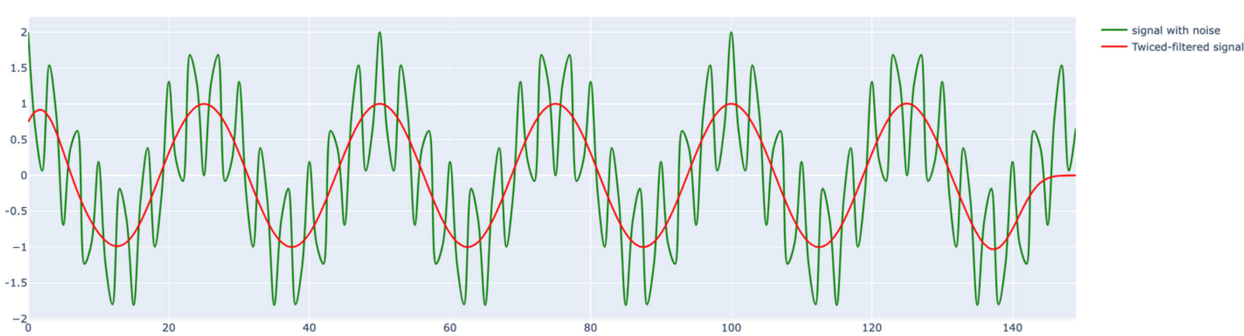




As we increase the cut-off frequency, we see an output the same as the input signal. But generally, the lower cut-off frequency is better.

I utilized the Butterworth filter to remove ripples in the passband and make the transition Band fall toward zero smooth. Also, two consecutive Low-pass filters, “former” and “reserve,” are utilized to compensate for frequency offset, which creates a much better filter’s output and increases the amplifier within the matrix multiplication operation. The corresponding desired filter results are shown by:





3. <u>Filtering and downsampling of discrete sinusoidal signals</u>

A continuous time signal x(t) consists of the sum of 2 sinusoidal signals each with zero initial phase and magnitude 1. The frequencies of these sinusoids are 500 Hz, 3250 Hz.

a) x(t) is sampled at sampling rate 2000 Hz. Determine the resulting discrete time signal x[n].

b) x[n] is passed through an LTI system whose frequency response is

$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| < \pi/2 \\ 0 & \pi/2 < |\omega| < \pi \end{cases}$$

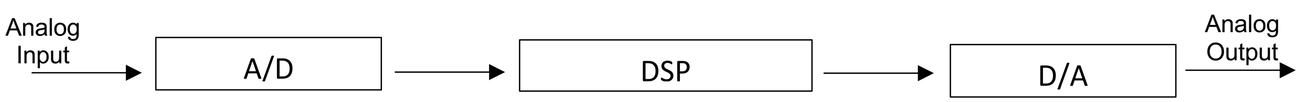
Determine the output of that LTI system. Call it y[n].

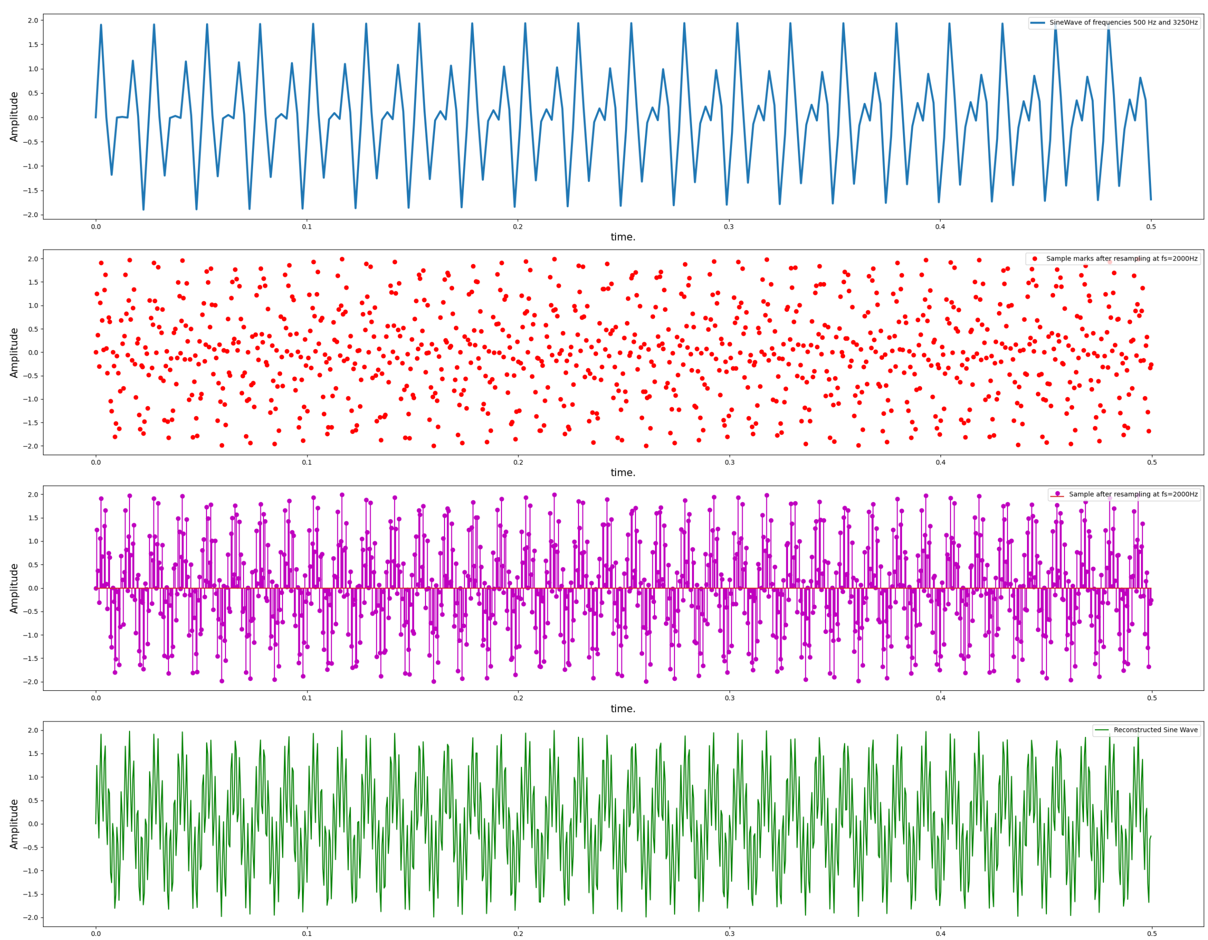
c) y[n] is downsampled by a factor of 2. The output of the downsampler is called w[n]. Write an expression for w[n].

d) Sketch the magnitude spectrum of w[n].

Answer)

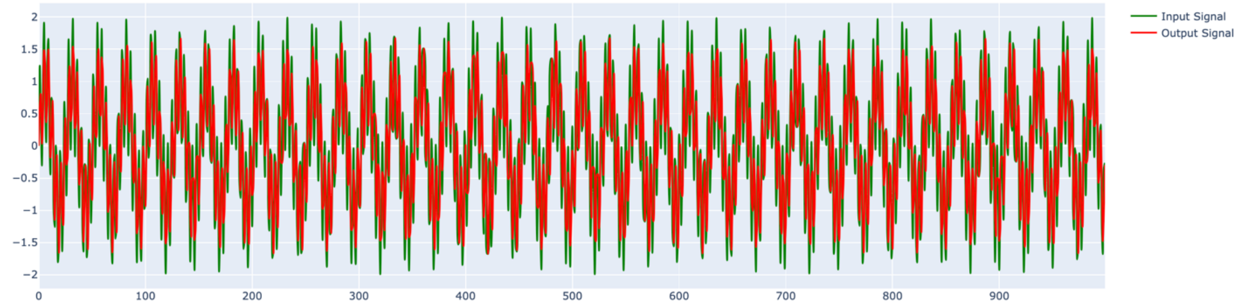
a) we need to pass the signal through a structure like this below schema:





b)

The Low-pass filter order is 1. For choosing cut-off frequency, we shall obtain the rule of $f_m < f_c < f_s - f_m$ in selecting proper components. Also, we set: $f_s = 2000\ Hz$, $f_m = 4Hz$ and based on $0 < f_c < 1000$, I chose 500Hz for cut-off frequency.



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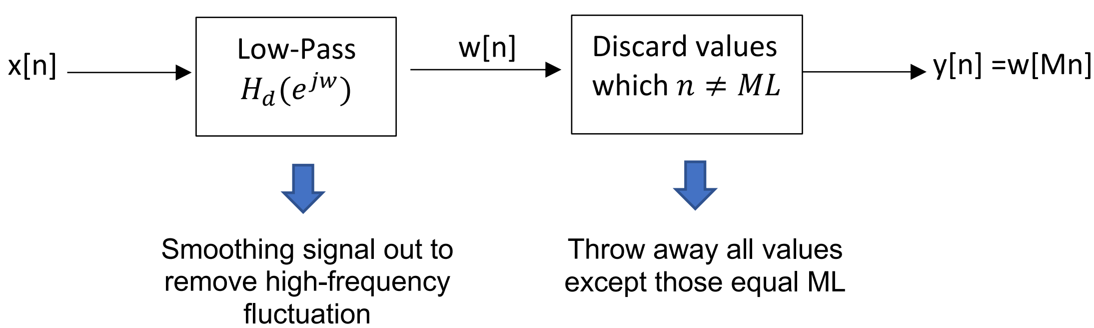
c)

M=downsampling rate

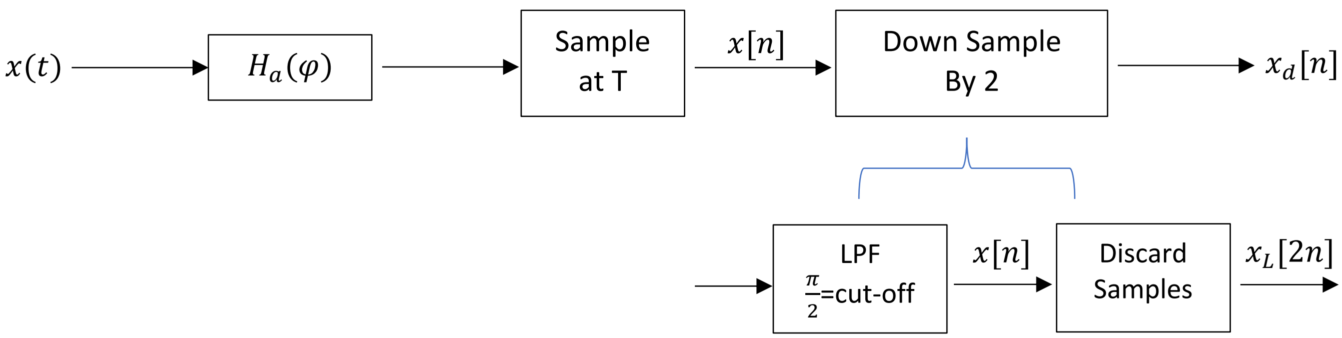


Recall: if $y[n] = W[mn]$, then $\underbrace{Y\left(e^{jw}\right)}_{\text{Discrete-time Fourier transform}} = \frac{1}{M}\sum_{m=0}^{M-1} W(\underbrace{e^{j(w-m2t\pi)/M}}_{\text{Stretching f frequency axis}})$

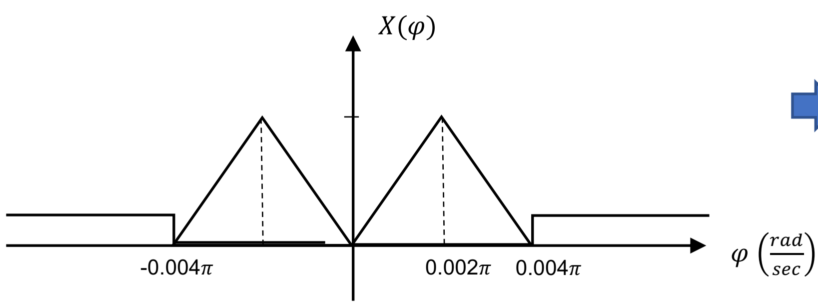
In this problem: we assume $x(t) = \sin(w_1 2\pi t) + \sin(w_2 2\pi t)$, $w_1 = 500Hz, w_2 = 3250Hz$,

$L = 2\ and\ \varphi_s = 2000Hz.$

So, we have: T= $\frac{1}{200} sec$, $\varphi_s = 0.004\pi$, $\varphi_s = 0.0006\pi$. We follow this below structure:

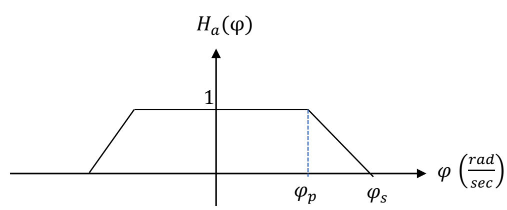


Input Spectrum



We find DTFT and FT Representations for $x[n]$, $x_d[n]$

We pass the input signal Through Low-pass filter

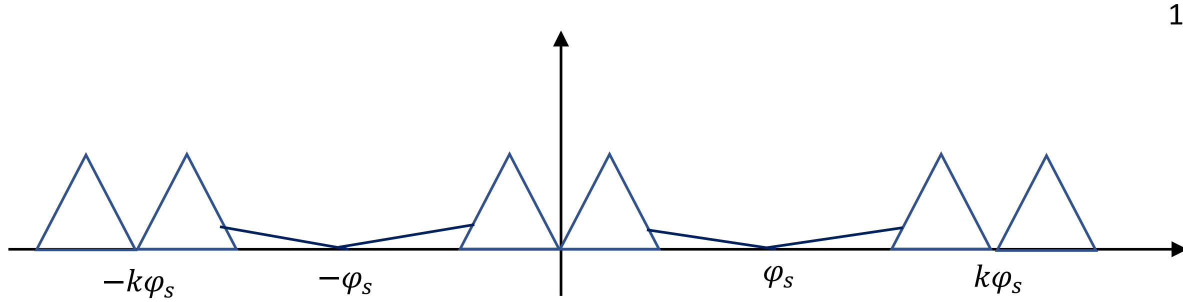


By $X_a(\varphi) = X(\varphi)\ H_a(\varphi)$

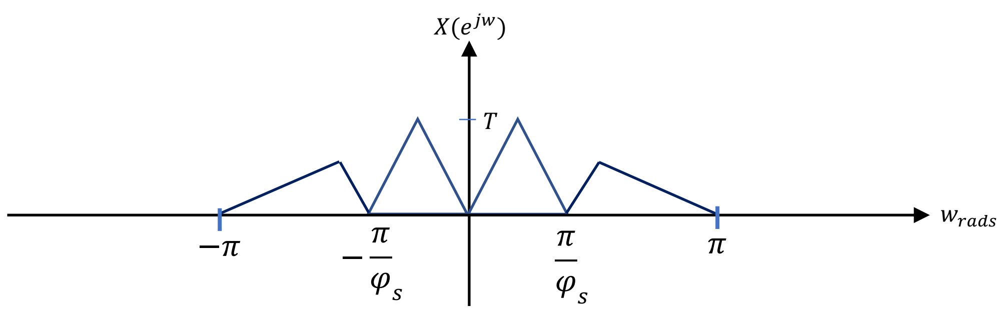
Then:

$$x[n] \overset{\text{DTFT}}{\rightarrow} X_s(\varphi) = \frac{1}{T}\sum_{h=-\infty}^{\infty} X_a(\varphi - k\varphi_s)$$

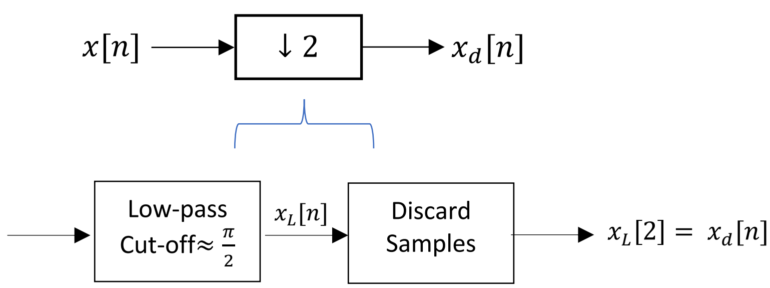
Fourier transformation
For sampled data

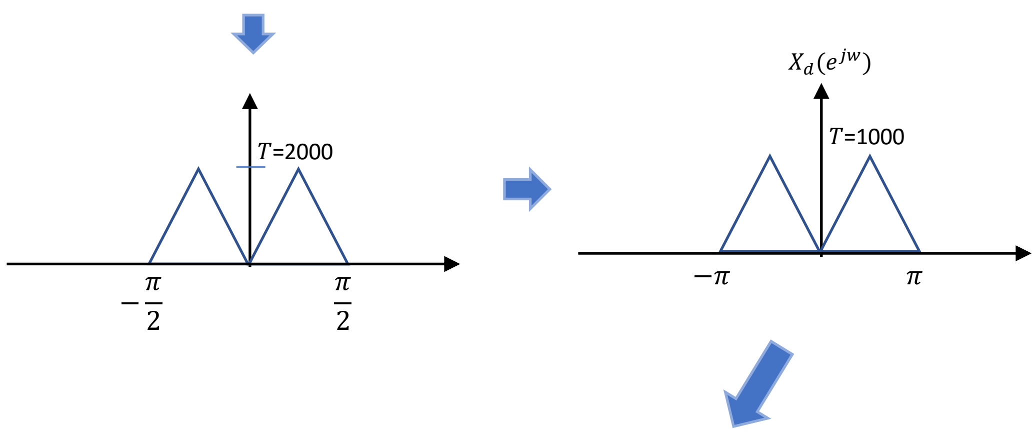


Now: $x[n] \overset{\text{DTFT}}{\rightarrow} X(e^{jw}) = X_s(\varphi)|_{\varphi = w/T}$



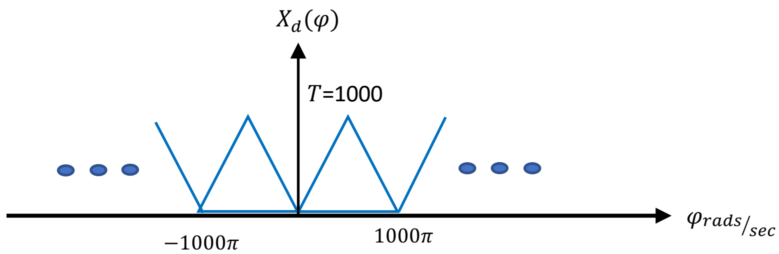
Next Step is :





$$X_d(e^{jw}) = \frac{1}{2}\sum_{m=0}^{1} X_L(e^{j(w-m2\pi)/2}$$

Finally, by converting to Fourier transform, we replace $X_d(\varphi) = X_d\left(e^{jw}\right)|_{w=\varphi T'\,;\,T'=\frac{1}{1000}}$



d)