All-Pass System

1. A second-order all-pass function has two poles at $z=0.8\exp(\pm j\pi/8)$.
   a) Find the system function and plot the amplitude and phase response
   b) Plot the output sequence y(n) for

$$x_1(n) = [\cos(6\pi n/40) + \cos(18\pi n/40)]u(n)$$
$$x_2(n) = [\cos(6\pi n/80) + \cos(18\pi n/80)]u(n)$$

   c) Compare the input and output signals. Has any distortion been caused by the system?
   d) Now, plot the output sequence for each sinusoidal input sequence in $x_1(n)$ separately. Compare the input and output signals. Has any distortion been caused by the system?
   e) Repeat d) for $x_2(n)$

Answer)

a)

poles and zeros are in conjugate reciprocal pairs :

$$H_{ap}(z) = \prod_{i=1}^{P} \frac{z^{-1} - c_i^*}{1 - c_i^{z^{-1}}}$$

Second order all-pass filter has transfer function:

$$H_d(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

There is the symmetry of coefficients $(b_0, b_1, b_2) = (a_2, a_1, 1)$ so, transfer function becomes :

$$H_d(z) = \frac{a_2 + a_1 z^{-1} + z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

Pole at $z = z_1$ means a pole at $z = z_1^*$ and zeros at $z = {}^1\!/_{z^*}$, $z = {}^1\!/_{z_1}$ so, a pole at $0.8e^{\pm j\pi/8}$, means zeros at $1.25e^{\pm j\pi/8}$.

we can represent every all-pass filter as a cascade of first and second-order all-pass filters.

A Z-transform of all-pass filter is $H(z) = \frac{z^{-1} - \overline{z_0}}{1 - z_0 z^{-1}}$. A zero at $^1/_{z_0^*}$, and a pole is z_0. Poles and Zeros are at conjugate reverse locations.

To implement second-order, we can cascade two first-order filters as:

$$H(z) = \frac{z^{-1} - \overline{z_0}}{1 - \ z_0 z^1} \times \frac{z^{-1} - z_0}{1 - \overline{z_0} z^{-1}} = \frac{z^{-2} - 2R(z_0)z^{-1} + |z_0|^2}{1 - 2R(z_0)z^{-1} + |z_0|^2 z^{-2}}$$

Which is equivalent by difference equation as:

$$y[k] - 2R(z_0)y[k-1] + |z_0|^2 y[k-2] = \ x[k-2] - 2R(z_0)x[k-1] + |z_0|^2 x[k]$$

$y[k]$ is an output and $x[k]$ is an input.

In this problem z_0 and $\overline{z_0}$ = $0.8e^{\pm j\pi/_8}$ $\approx 1.1847781.$

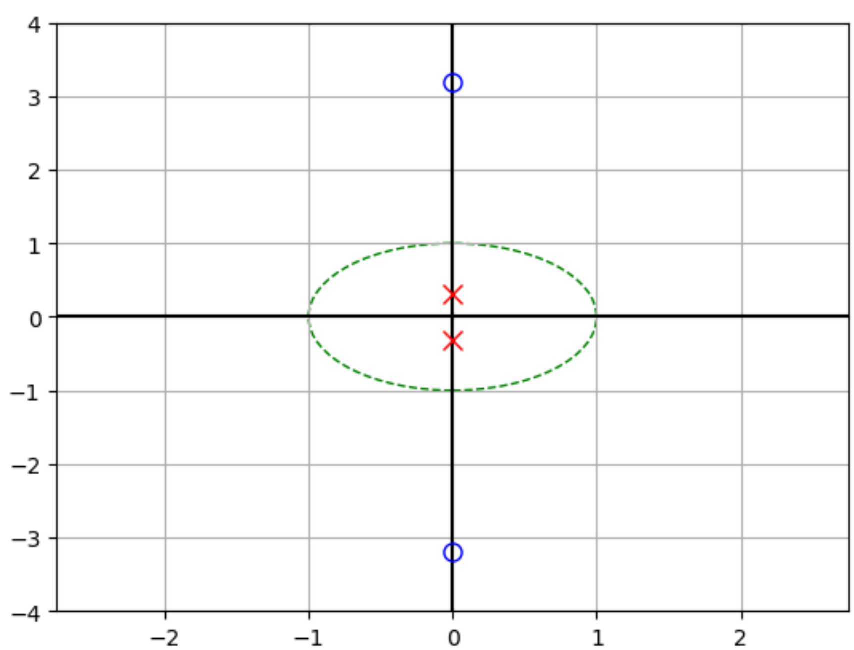
In this IIR all-pass filter, different slopes or derivatives of the phase function with respect to frequency show a variety of group delays for different coefficients.

The phase formula for the second-order all-pass filter based on the book Oppenheim/Schafer “Discrete-Time Signal Processing”, is:

$$\emptyset(\omega) = \ -2\omega - 2\arctan\left(\frac{rsin(\omega - \theta)}{1 - rcos(\omega - \theta)}\right) - 2\arctan[\frac{rsin(\omega + \theta)}{1 - rcos(\omega + \theta)}]$$

r is magnitude and θ is the phase of coefficient a ($a = r.e^{j\theta}$).

Visualizing Poles and Zeros



System Function and Phase Plot:

```
\Omega
Hap=
```

$$\frac{\left(-\overline{a}+e^{-i\Omega}\right)^{2}}{\left(-ae^{i\Omega}+1\right)^{2}}$$

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Magnitude=
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$$\sqrt{\left(a-e^{i\Omega}\right)^{2}\left(\overline{a}-e^{-i\Omega}\right)^{2}}\left|\frac{1}{\left(ae^{i\Omega}-1\right)^{2}}\right|$$

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=
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$$1.0$$

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Phase:
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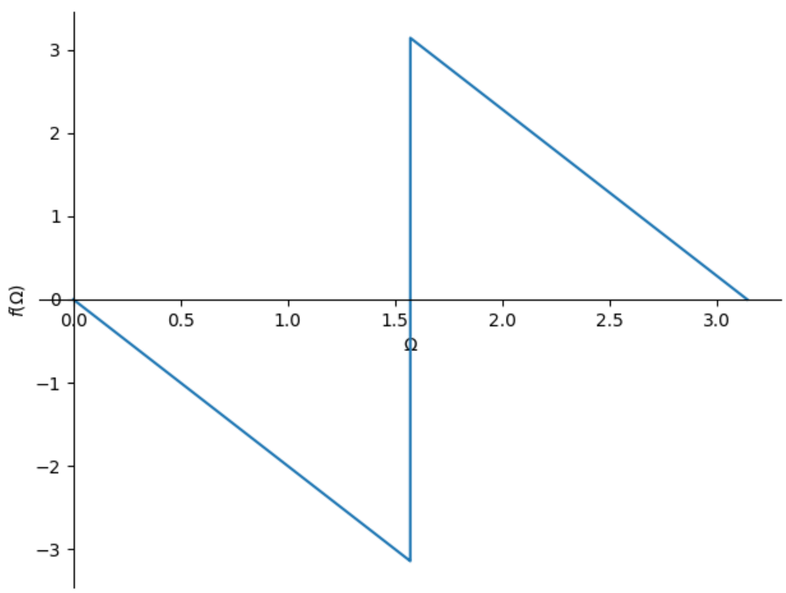
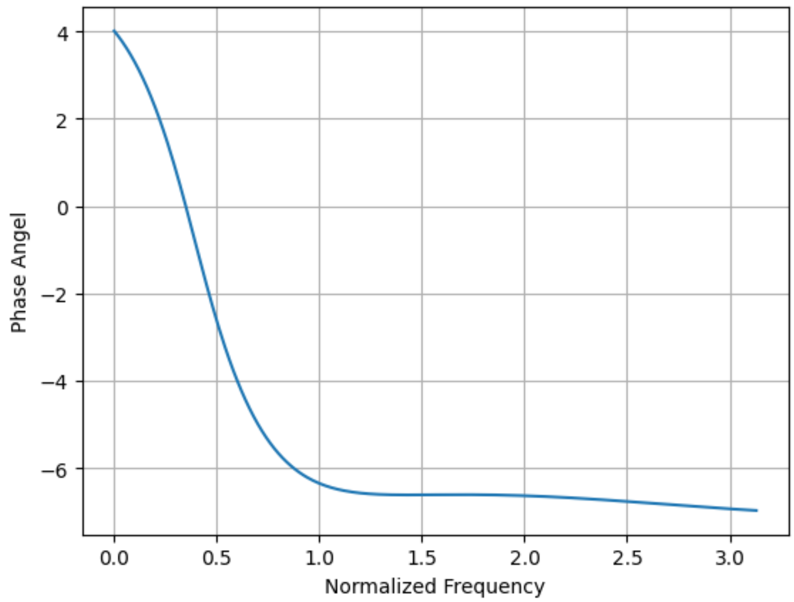
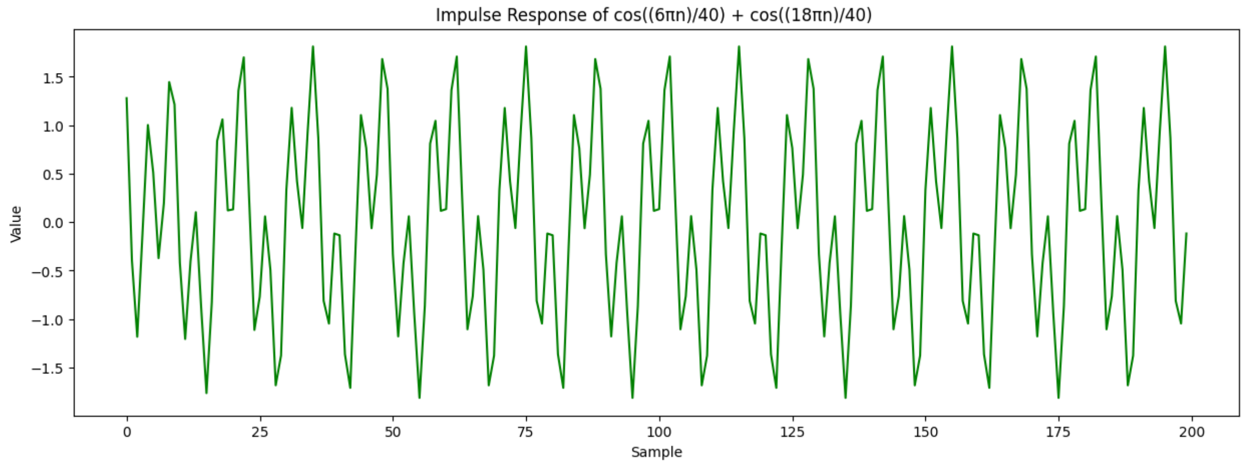
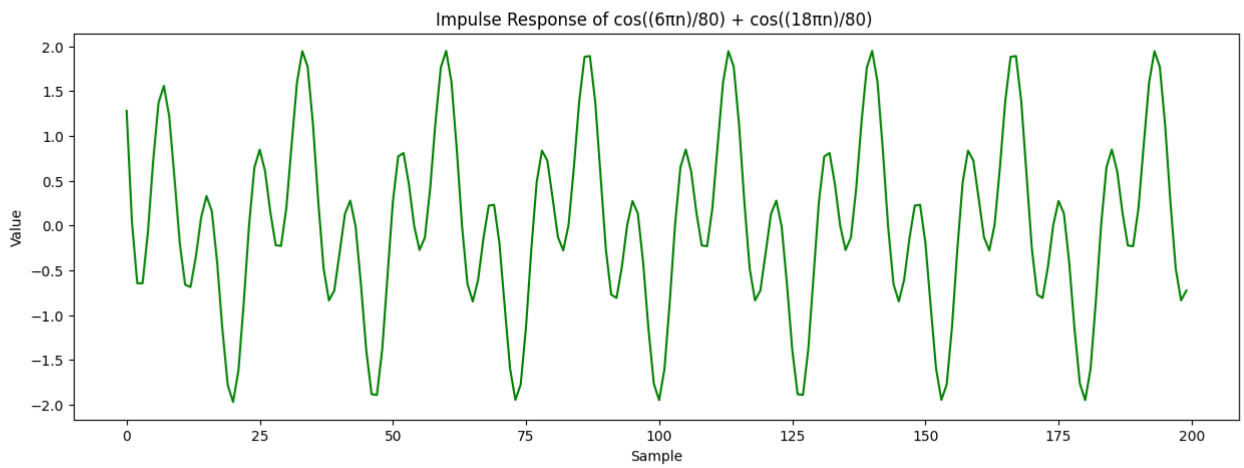


We stretch low-frequency by implementing a warping function for normalized frequency, as below:

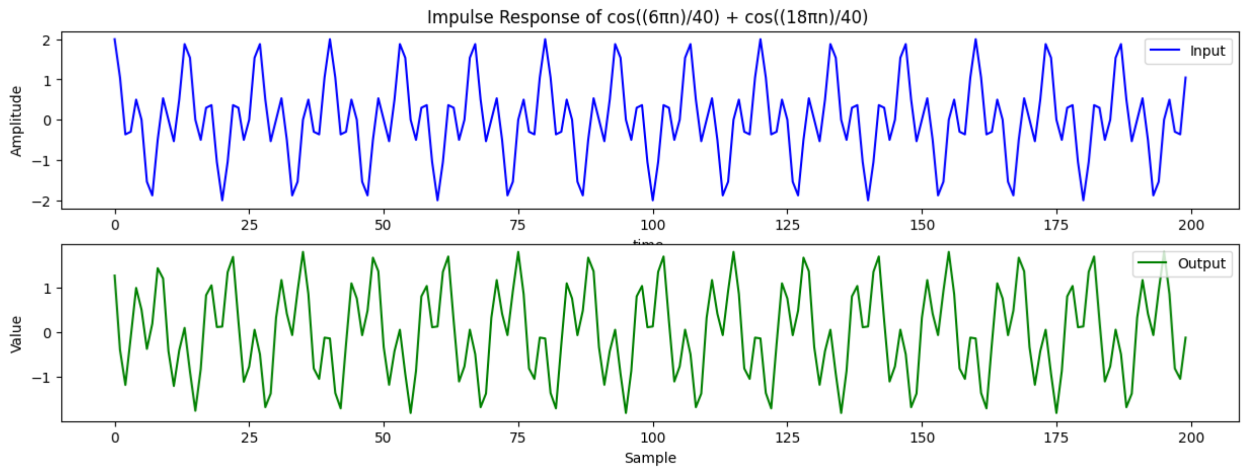


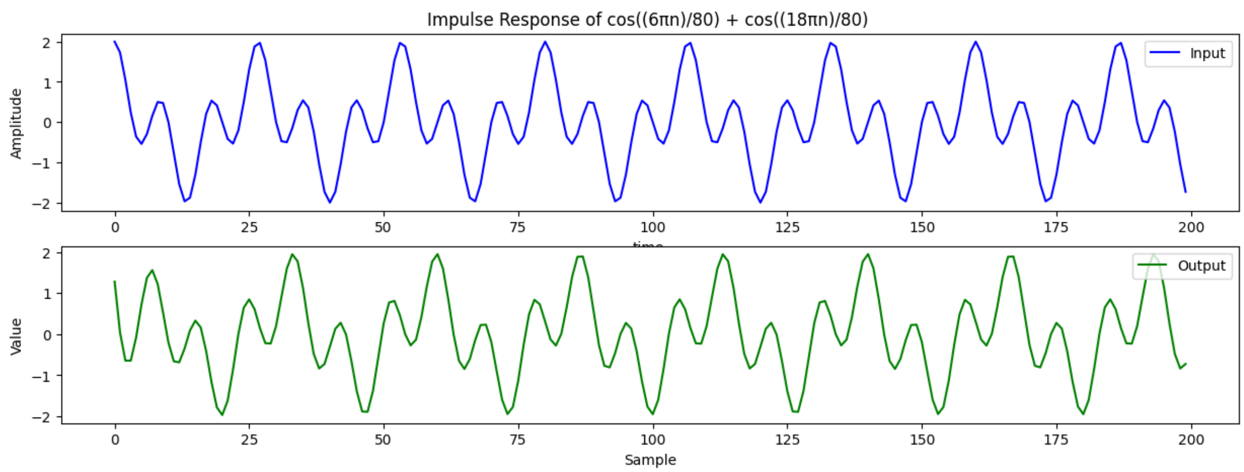
b)


Impulse Response of cos((6πn)/40) + cos((18πn)/40)
Value
Sample
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0
25
50
75
100
125
150
175
200


Impulse Response of cos((6πn)/80) + cos((18πn)/80)
Value
Sample
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
0
25
50
75
100
125
150
175
200

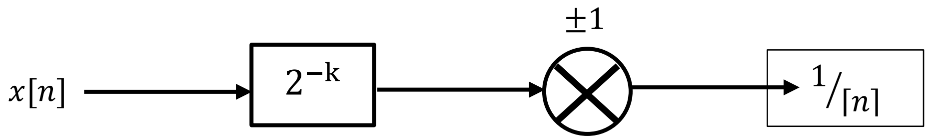
c)





Yes, there is distortion caused by the system. We see a phase shift. They can compensate for phase distortion (or group delay). Actually, the thing all-pass systems do is introduce a frequency-dependent delay (some frequencies have more delays). Since, in a causal all-pass system, there is a positivity of group delay.

FIR All-pass filter :



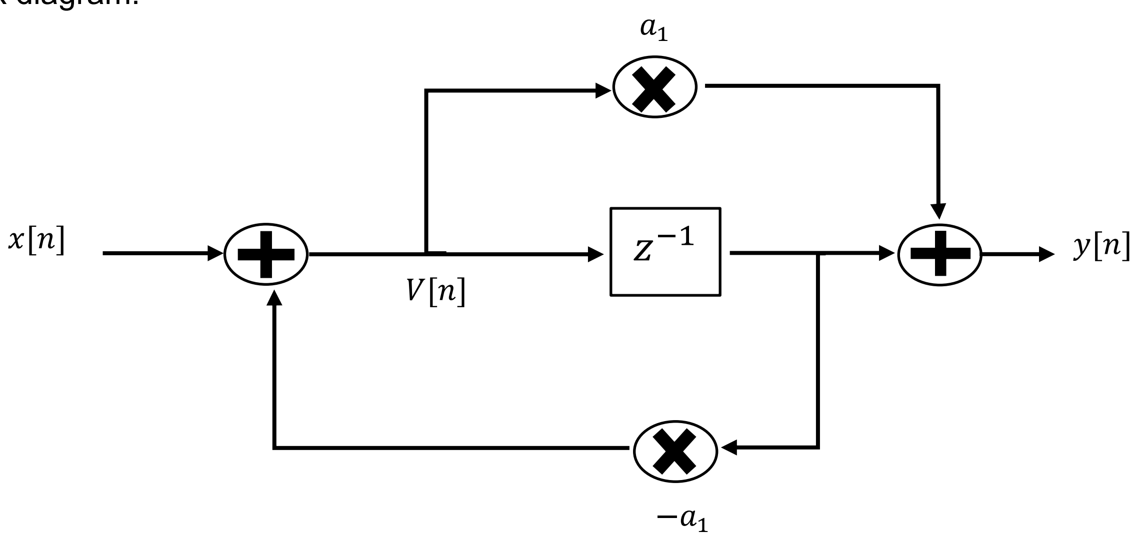
The transfer function is $H(z) = \pm z^{-k}$ with delay $k \geq 0$

But, when we are talking about all-pass filters, we are referring to IIR all-pass filters by:

Transfer function: $H_{AP_1}(z) = \frac{a_1 + z^{-1}}{1 + a_1 z^{-1}}$

Difference equation $y[n] = a_1 x[n] + x[n-1] - a_1 y[n-1]$

Block diagram:

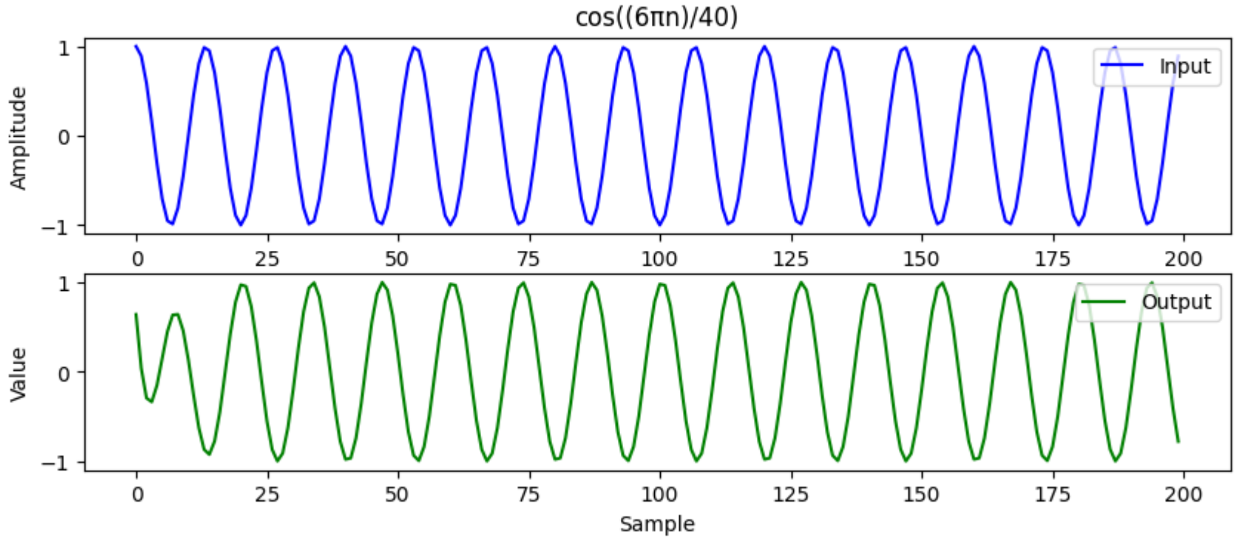


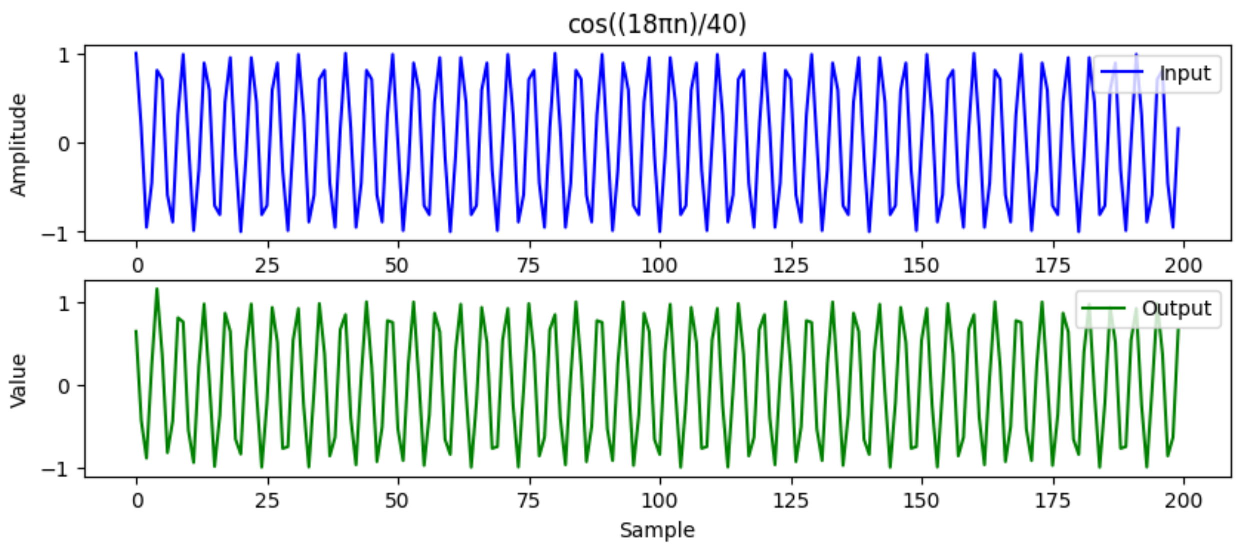
Feedforward part: $a_1.V[n] + V[n-1]$

Feedback part: $x[n] - a_1.V[n-1] = V[n]$

The power of all-pass filters lies in their phase response by filtering the phase delay of each frequency component that passes through all-pass filter

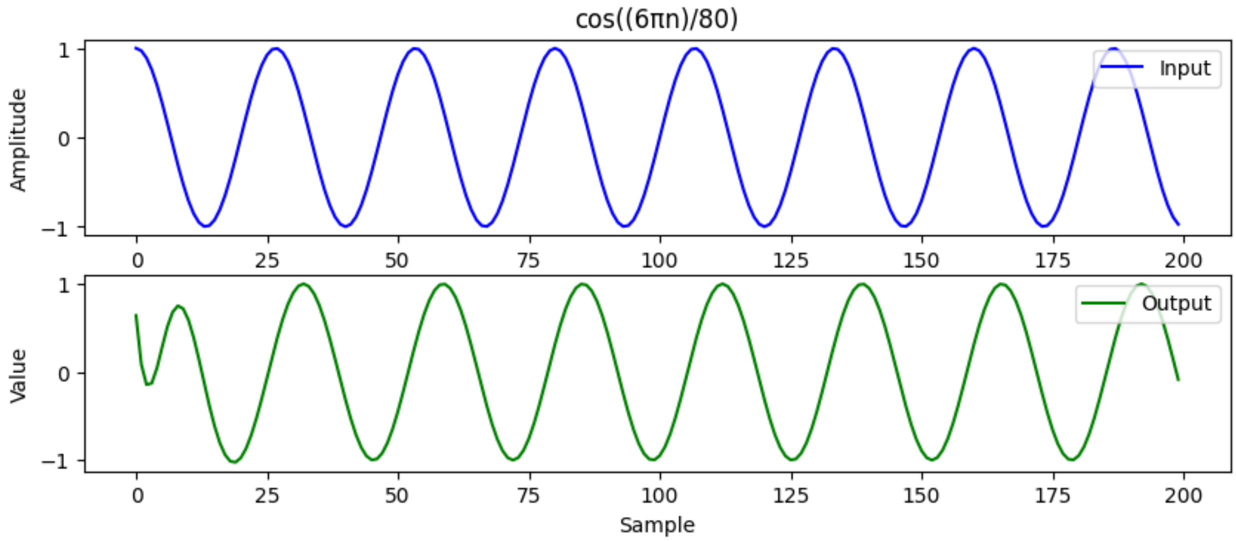
d)

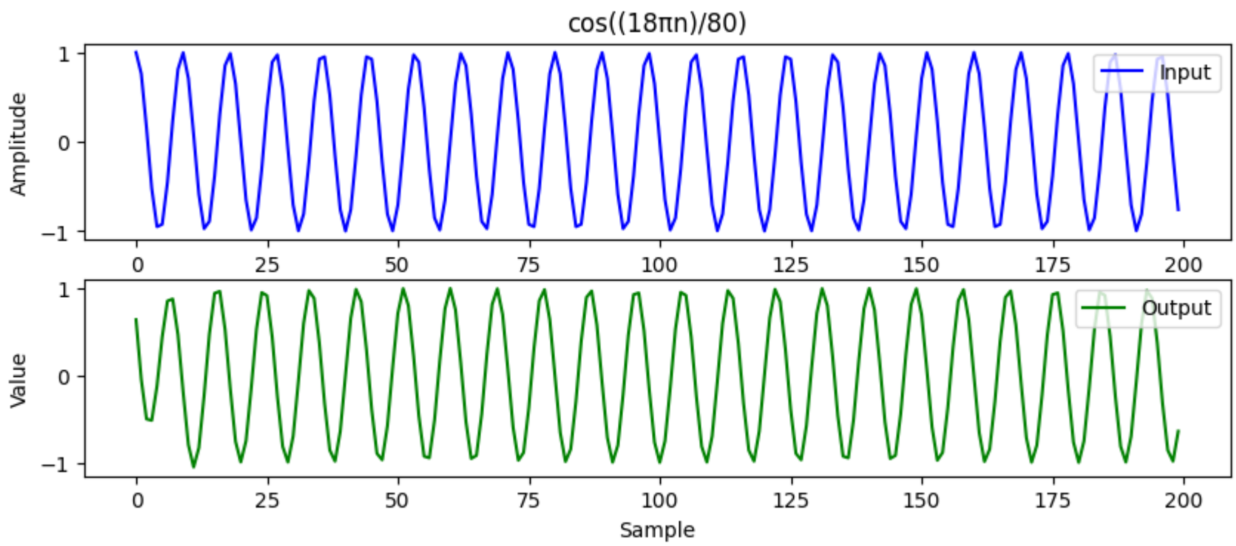




As we can see, there is only a small distortion at first.

e)





As with the same previous problem, we can see only a small distortion at first.

Design of a Hilbert transformer

2. Using Hanning window and Kaiser window, design a digital Hilbert transformer. The ideal frequency response of a linear-phase Hilbert transformer is given by,

$$H_d(e^{-j\omega}) = \begin{cases} -j & 0 < \omega < \pi \\ j & -\pi < \omega < 0 \end{cases}$$

Choose M=24 and M=25.

a) Plot the resulting amplitude responses and the impulse responses.
b) Which M results in a better approximation to the ideal Hilbert transformer? Why?
c) Which window function results in a better approximation to the ideal Hilbert transformer? Why?

Hint: You can find the impulse response of the filter in Eq. (12.64) of the textbook.

Answer)

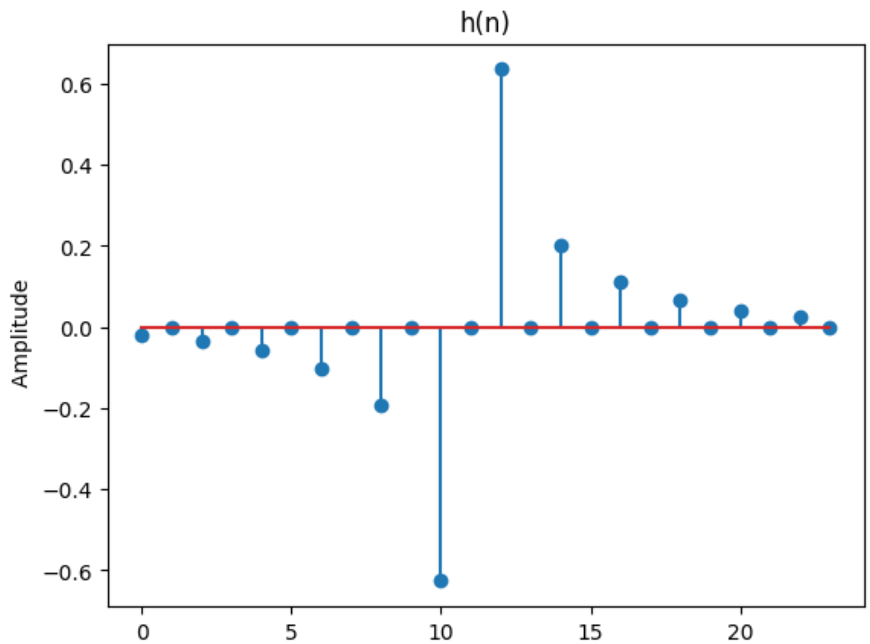
a)

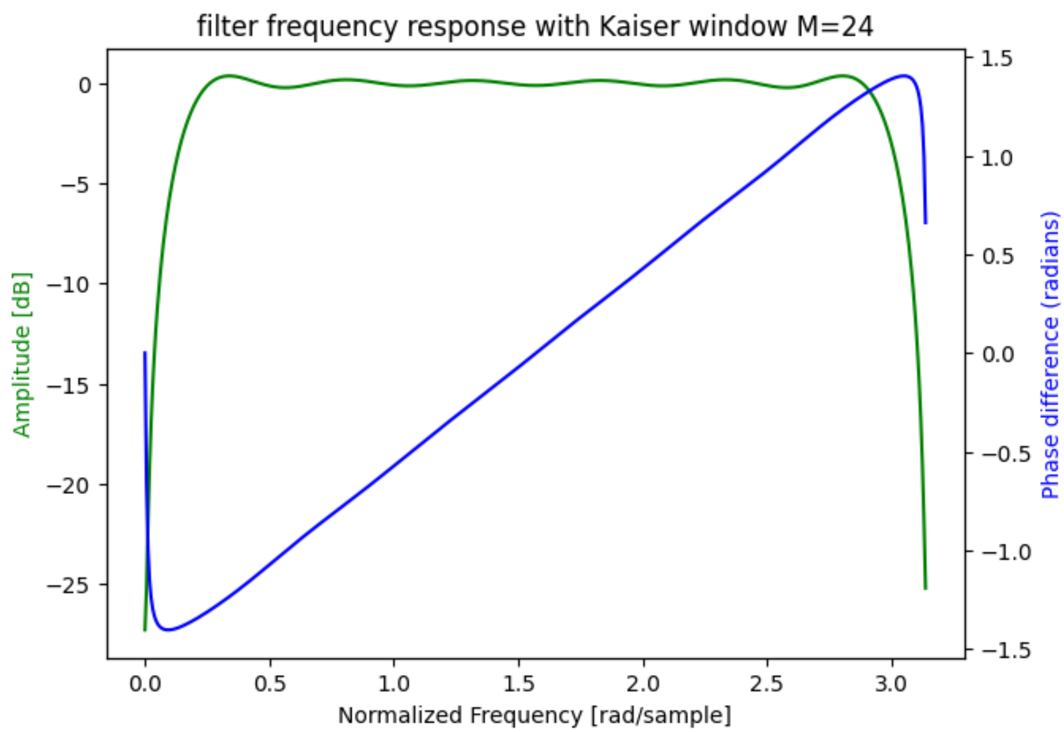
$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{0} je^{jwn}dw - \frac{1}{2\pi}\int_{0}^{\pi} je^{jwn}\,dw$$

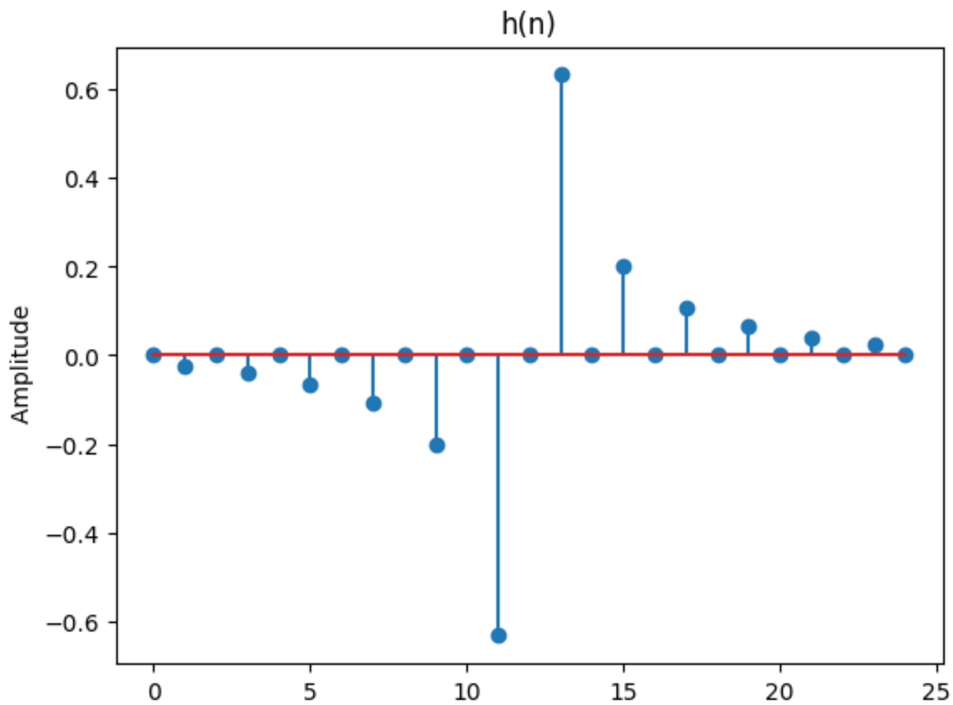
$$h[n] = \begin{cases} \dfrac{2}{\pi}\,\dfrac{\sin^2(\pi n/2)}{n} & n \neq 0 \\ 0 & n = 0 \end{cases}$$

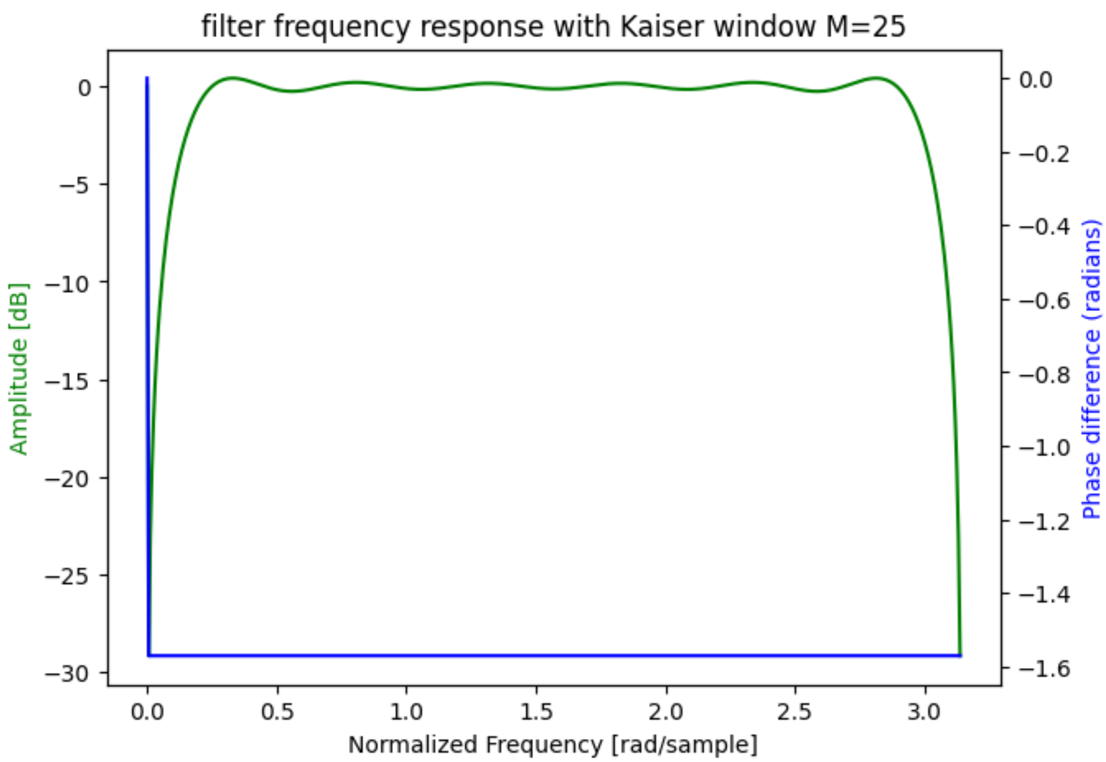
Since the ideal Hilbert transform is not causal, so, a constant phase shifted in time by half of the filter length to make a causal filter. This operation in the frequency domain changes to multiplication with a complex transform. Transfer function multiplied by $e^{i\omega N}$ (N samples) for a negative delay.

Kaiser Windowing

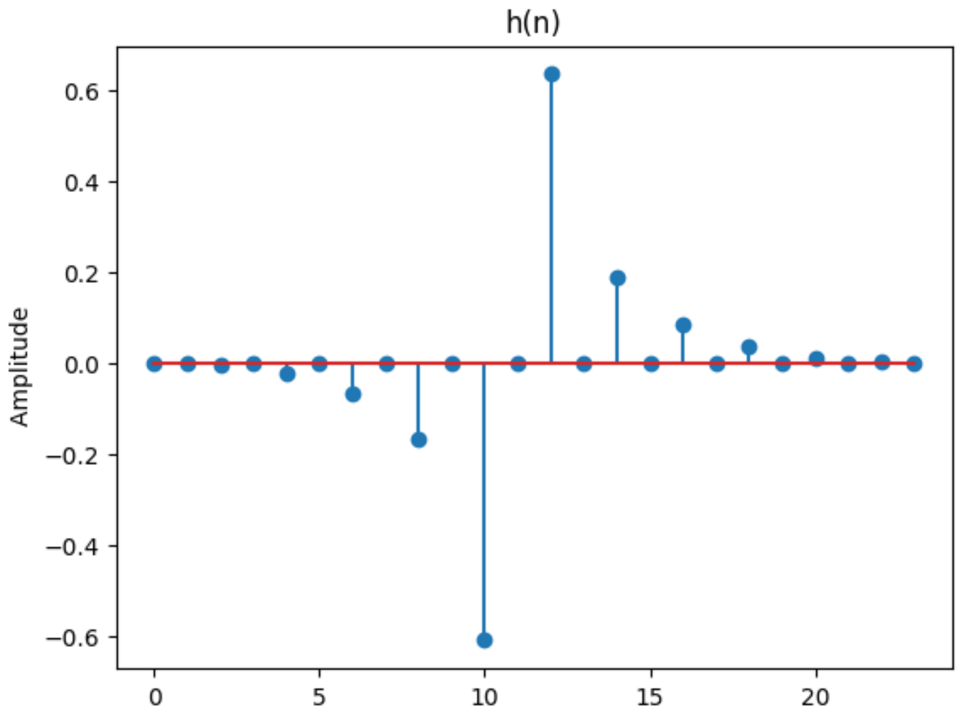


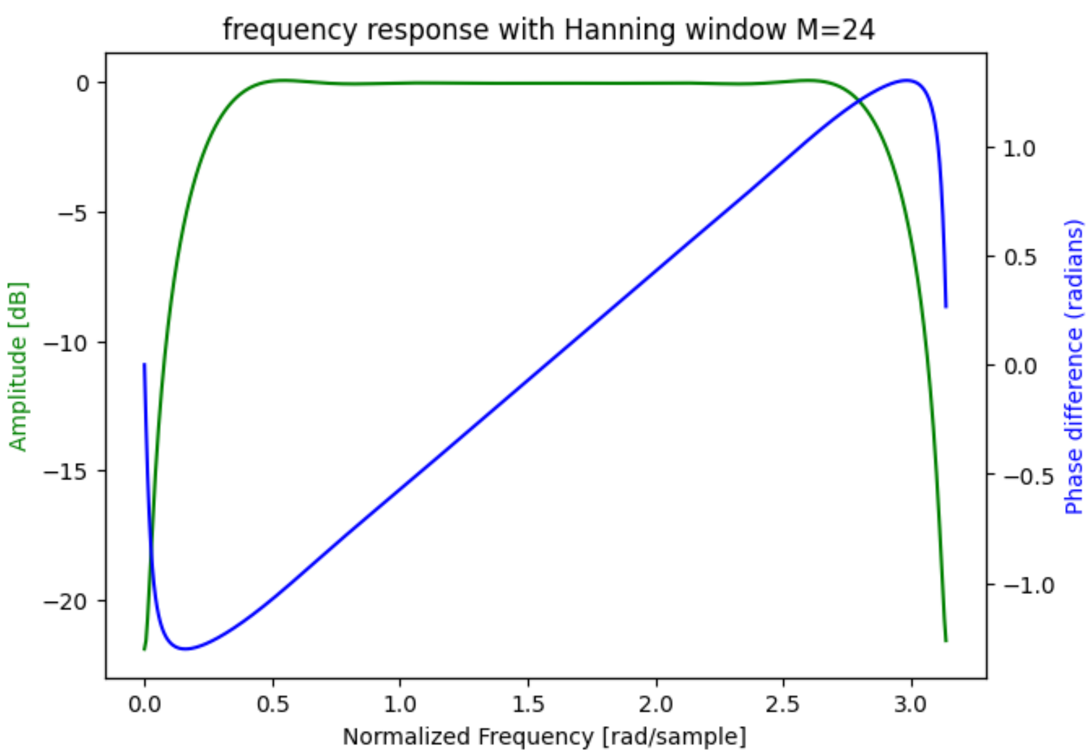


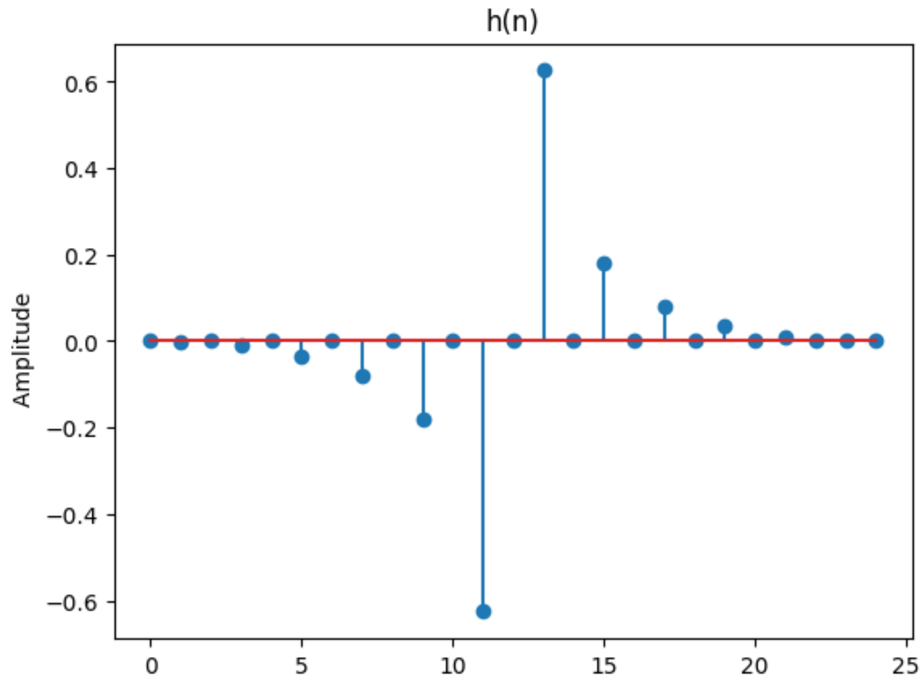

h(n)
Amplitude
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
0
5
10
15
20
25

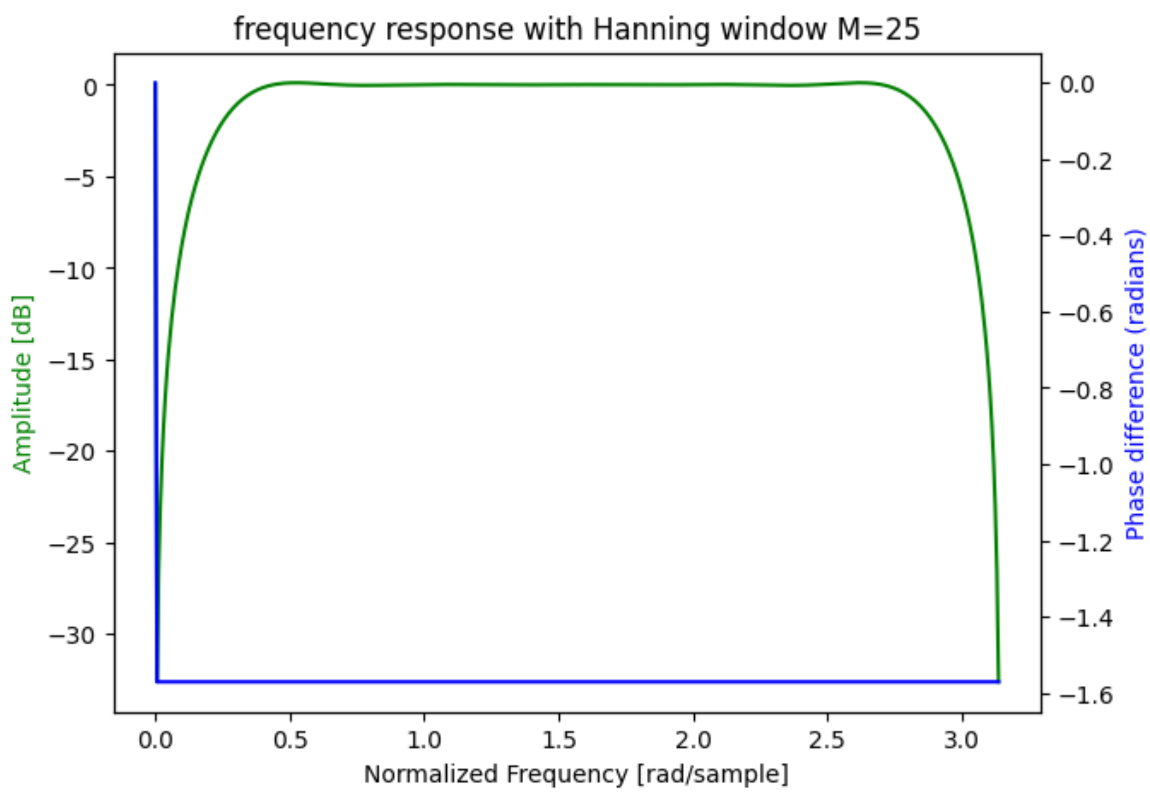

filter frequency response with Kaiser window M=25
Amplitude [dB]
Phase difference (radians)
Normalized Frequency [rad/sample]
0
−5
−10
−15
−20
−25
−30
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
−1.2
−1.4
−1.6
0.0
0.5
1.0
1.5
2.0
2.5
3.0

Hanning Windowing






h(n)
Amplitude
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
0
5
10
15
20
25


frequency response with Hanning window M=25
Amplitude [dB]
Phase difference (radians)
Normalized Frequency [rad/sample]
0
−5
−10
−15
−20
−25
−30
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
−1.2
−1.4
−1.6
0.0
0.5
1.0
1.5
2.0
2.5
3.0

b)

Window functions are used to reduce the Gibbs phenomenon. As M increases, the main lobe decreases, and amplitude increases which provide a more confident frequency domain and the side-lobe level remains unchanged. In frequency resolution problems, a narrower main lobe is required.

Selecting an appropriate window function decreases spectral leakage of the Fourier transform. Most window functions like triangular, Hamming, Blackmand and Kaiser try to have an effect on the center of data but do not edge.

If M is even, the system is a type three FIR. If M is an odd integer, we obtain a type four system. So, a better approximation is made around $\omega = \pi$. The phase response is exactly 90^o at all frequencies. For $M = 25$, we see $\frac{25}{2} = 12.8$ samples delay.

We now type three FIR has a more computational advantage over type 4 systems because even sample indexed of impulse responses are exactly zero.

I think M (window length) should be computed by beta coefficient with a customizable algorithm to implement a filter with correct performance.

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c)

Hanning window is good for controlling the leakage and for compact space signals. It's always recommended.
the difference between window types is in the method of tapering near edges. Kaiser parameters have maximum energy concentration in the main lobe. Kaiser's advantage is maximized in the ratio of the main lob energy to the sidelobe energy.
By keeping window length fixed, we can adjust a factor for the passband and stopband ripples, although in other windows is a tradeoff between main-lobe width and ripple ratios. Kaiser-Bessel window exploits better amplitude accuracy by a balance among side lobe distance and side lobe height.
They have the same transition bandwidth and fewer ripples in PB & SB compared to the Hanning window.
Hanning window, sometimes called Hann window or random window, has a sinusoidal shape and touches zero at both ends. The signal will still have little discontinuity, resulting in nice low side lobes and a wide peak, but this window shouldn't be used for strictly periodic signals. Hanning advantage is that it can control the leakage, which may destroy close signal components and small magnitudes. Hanning window is almost always recommended, and it's not useful for frequencies in which the component's space is less than bandwidth.