Direct exponential integration for linear SWE on sphere

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There was a certain interest in using rational approximations for resembling exponential integrators with the SWE on the rotating sphere. Here, it was assumed that rational approximations must be used for solving the linearized SWE. This work investigates a way with the spherical harmonics transformations to directly exponentially integrate the linearized SWE without the Coriolis effect.

1 Linear operator

Using the vorticity-divergence formulation, the linear operator is then given by

$$\begin{bmatrix} \partial_t \Phi' \\ \partial_t \delta \end{bmatrix} = \begin{bmatrix} & -c\overline{\Phi} \\ -c\nabla^2 & \end{bmatrix} \begin{bmatrix} \Phi' \\ \delta \end{bmatrix}.$$

Using spherical harmonics, the Laplace operator ∇^2 is also diagonal. Therefore, we can rearrange this system in 2×2 blocked matrices along the diagonal with each block relating to one particular associated Legendre polynomial P_n^m (ALP).

2 Exp. integrators

With exponential time integrators, we can write

$$U_t = LU$$

with

$$L = \begin{bmatrix} & -\overline{\Phi} \\ -\nabla^2 & \end{bmatrix}$$

and a direct solution

$$U(t) = \exp(tL)U(0).$$

For sake of convenience, we use the letters $D = -\nabla^2$ and $G = -\overline{\Phi}$, giving

$$L = \begin{bmatrix} & G \\ D & \end{bmatrix}$$

2.1 Eigen diagonalization

Again, we can rewrite this to a 2×2 matrix with all blocks independent to the other ones. From hereon, we assume L, G and D related to one particular ALP. Finally, we can work with an Eigen diagonalization of the 2×2 matrices, giving

$$U(t) = Q \exp(t\Lambda) Q^{-1} U(0) = \exp(tL) U(0).$$

where $LQ = Q\Lambda$ an Eigenvalue decomposition with

$$Q = \begin{bmatrix} -\frac{\sqrt{DG}}{D} & \frac{\sqrt{DG}}{D} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{D}\sqrt{G}}{\sqrt{D}\sqrt{D}} & \frac{\sqrt{D}\sqrt{G}}{\sqrt{D}\sqrt{D}} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -\sqrt{\frac{G}{D}} & +\sqrt{\frac{G}{D}} \\ 1 & 1 \end{bmatrix}$$

and

$$\Lambda = \begin{bmatrix} -\sqrt{DG} & \\ & \sqrt{DG} \end{bmatrix}$$

and finallyf

$$Q^{-1} = \begin{bmatrix} -\frac{1}{2}\frac{D}{\sqrt{DG}} & \frac{1}{2} \\ \frac{1}{2}\frac{D}{\sqrt{DG}} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\frac{\sqrt{D}\sqrt{D}}{\sqrt{DG}} & \frac{1}{2} \\ \frac{1}{2}\frac{\sqrt{D}\sqrt{D}}{\sqrt{DG}} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{\frac{D}{G}} & \frac{1}{2} \\ -\frac{1}{2}\sqrt{\frac{D}{G}} & \frac{1}{2} \end{bmatrix}$$

where we had a change of signs by “moving” an imaginary number from the denominator to the nominator.

Using this, we can directly exponentially integrate the linearized SWE on the non-rotating sphere.