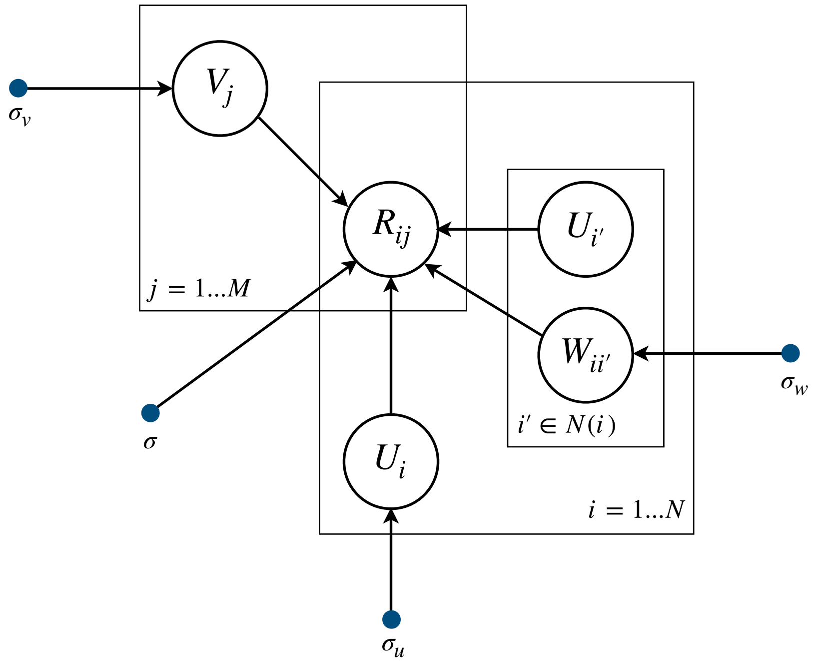
Defining the model by



where

$$\begin{cases} R_{ij} \sim Normal\left(U_i^T V_j + \sum_{i' \in N(i)} W_{ii'} U_{i'}^T V_j, \sigma\right) \\ U_i \sim Normal\left(0_D, \sigma_u I_D\right) \\ V_j \sim Normal\left(0_D, \sigma_v I_D\right) \\ W_{ii'} \sim Normal\left(0, \sigma_w\right) \end{cases},$$

predictive distribution will defined by

$$P(R_{ij}^+ \,|\, R, N) = \int P(R_{ij}^+ \,|\, U, V, W, N) P(U, V, W \,|\, R, N)\, d\{U, V, W\}$$

where N is social network. We approximate the above integral using the expected values of variational parameters in the model[1] by

$$P(R_{ij}^+ \,|\, R, N) \approx P(R_{ij}^+ \,|\, E_Q[U], E_Q[V], E_Q[W]).$$

Also R_{ij}^+ is estimated by

$$\hat{R_{ij}^+} = E_Q[U_i]^T E_Q[V_j] + \sum_{i' \in N(i)} E_Q[W_{ii'}] E_Q[U_{i'}]^T E_Q[V_j],$$

[1] Edward Snelson, Zoubin Ghahramani, Variational Bayes for predictive distributions, Gatsby Computational Neuroscience Unit, 2005.

which comes from the fact that the mean of a normal distribution has highest probability. Variational distributions will defined by below equations. See appendix for derivations.

$$Q(U_i) \sim N(\mu_i^*, (\Lambda_i^*)^{-1}),$$

$$\begin{cases} \Lambda_i^* = \frac{1}{\sigma_u} I + \frac{1}{\sigma} \sum_j I_{ij} E\left[V_j V_j^T\right] + \frac{1}{\sigma} \sum_{i'' \in N(i)} \sum_j I_{i''j} E\left[W_{i''i}^2\right] E\left[V_j V_j^T\right] \\ \mu_i = (\Lambda_i^*)^{-1} \left[\frac{1}{\sigma} \sum_j I_{ij} \big(R_{ij} E\left[V_j\right] - E\left[V_j V_j^T\right] E\left[S_i\right]^T \big) \right. \\ \qquad \left. + \frac{1}{\sigma} \sum_{i'' \in N(i)} \sum_j I_{i''j} E\left[W_{i''i}\right] \big(R_{i''j} E\left[V_j\right] - E\left[V_j V_j^T\right] \big(E\left[U_{i''}\right] + E\left[S_{i''}^{-i}\right]^T \big) \big) \right] \end{cases}$$

$$Q(V_j) \sim N(\mu_j^*, (\Lambda_j^*)^{-1}), \begin{cases} \Lambda_j^* = \frac{1}{\sigma_v} I + \frac{1}{\sigma} \sum_i I_{ij} \big(E\left[U_i U_i^T\right] + 2E\left[U_i\right] E\left[S_i\right] + E\left[S_i^T S_i\right] \big) \\ \mu_j = (\Lambda_j^*)^{-1} \frac{1}{\sigma} \sum_i I_{ij} R_{ij} \big(E\left[U_i\right] + E\left[S_i\right]^T \big) \end{cases}$$

$$Q(W_{ii'}) \sim N(\mu_{ii'}^*, \frac{1}{\rho_{ii'}^*}),$$

$$\begin{cases} \rho_{ii'}^* = \frac{1}{\sigma_w} + \frac{1}{\sigma} \sum_j I_{ij} E\left[U_{i'}^T V_j V_j^T U_{i'}\right] \\ \mu_{ii'} = \frac{1}{\rho_{ii'}^*} \frac{1}{\sigma} \sum_j I_{ij} \big(E\left[U_{i'}\right]^T E\left[V_j\right] R_{ij} - E\left[U_i\right]^T E\left[V_j V_j^T\right] E\left[U_{i'}\right] - E\left[U_{i'}\right]^T E\left[V_j V_j^T\right] E\left[S_i^{-i'}\right]^T \big) \Big] \end{cases}$$

where $S_i^{-k} = \sum_{i' \in N(i)/k} W_{ii'} U_{i'}$ and $S_i = \sum_{i' \in N(i)} W_{ii'} U_{i'}$.

We initialize model parameters of $U_i, V_j, W_{ii'}$ uniformly in [0, 1], and update them according to above equations in 20 steps.¶

Appendix

Defining $Q(U,V,W)=\prod_i Q(U_i)\prod_j Q(V)\prod_{i,i'} Q(W_{ii'})$ by naive mean field assumption, each $Q(Z_i)$ obtained by $Q(Z_i)\propto exp\left[E_{-i}[lnP(Z_i|Z_{-i},X)]\right]$ using coordinate ascent. For simplicity we define

$$\begin{cases} m_{ij} \triangleq \sum_{i'\in N(i)} w_{ii'}U_{i'}^T V_j \\ m_{ij}^{-k} \triangleq m_{ij} - W_{ik}U_k^T V_j \\ S_i \triangleq \sum_{i'\in N(i)} w_{ii'}U_{i'}^T \\ S_i^{-k} \triangleq S_i - W_{ik}U_k^T \end{cases},$$

- For $P(U_i|U_{-i},V,R,W;\Theta_0)$,

$$\propto P(U,V,R,W;\Theta_0)$$

$$\propto P(U_i;\sigma_u)\prod_i\prod_j N(U_i^TV_j+m_{ij},\sigma)^{I_{ij}}$$

$$\propto P(U_i;\sigma_u)\Big[\prod_j N(U_i^TV_j+m_{ij},\sigma)^{I_{ij}}\Big]\Big[\prod_{i'':i\in N(i'')}\prod_j N(U_{i''}^TV_j+m_{i''j},\sigma)^{I_{i''j}}\Big]$$

$$\Rightarrow logP(U_i|U_{-i},V,R,W;\Theta_0)$$

$$\propto logP(U_i;\sigma_u)+\Big[\sum_j I_{ij}log\big(N(U_i^TV_j+m_{ij},\sigma)\big)\Big]+\Big[\sum_{i'':i\in N(i'')}\sum_j I_{i''j}log\big(N(U_{i''}^TV_j+m_{i''j},\sigma)\big)\Big]$$

$$\propto -\frac{1}{2\sigma_u}U_i^TU_i+\sum_j I_{ij}\Big[-\frac{1}{2\sigma}\big(-2R_{ij}U_i^TV_j+U_i^TV_jV_j^TU_i+2m_{ij}U_i^TV_j\big)\Big]$$

$$+\sum_{i'':i\in N(i'')}\sum_j I_{i''j}\Big[-\frac{1}{2\sigma}\big(-2R_{i''j}W_{i''i}U_i^TV_j+2W_{i''i}U_i^TV_jU_{i''}^TV_j+W_{i''i}^2U_i^TV_jV_j^TU_i+2W_{i''i}U_i^TV_jm_{i''j}^{-i}\Big]$$

$$\propto -\frac{1}{2}U_i^T\Big[\frac{1}{\sigma_u}I+\frac{1}{\sigma}\sum_j I_{ij}V_jV_j^T+\frac{1}{\sigma}\sum_{i'':i\in N(i'')}\sum_j I_{i''j}W_{i''i}^2V_jV_j^T\Big]U_i$$

$$+\Big[\frac{1}{\sigma}\sum_j I_{ij}\big(R_{ij}V_j^T-S_iV_jV_j^T\big)+\frac{1}{\sigma}\sum_{i'':i\in N(i'')}\sum_j I_{i''j}W_{i''i}\big(R_{i''j}V_j^T-U_{i''}^TV_jV_j^T-S_{i''}^{-i}V_jV_j^T\big)\Big]U_i$$

$$\Rightarrow P(U_i|U_{-i},V,R,W;\Theta_0)\sim N(m_i,J_i^{-1}),$$

$$\begin{cases} J_i=\frac{1}{\sigma_u}I+\frac{1}{\sigma}\sum_j I_{ij}V_jV_j^T+\frac{1}{\sigma}\sum_{i'':i\in N(i'')}\sum_j I_{i''j}W_{i''i}^2V_jV_j^T \\ m_i=J_i^{-1}\Big[\frac{1}{\sigma}\sum_j I_{ij}\big(R_{ij}V_j-V_jV_j^TS_i^T\big)+\frac{1}{\sigma}\sum_{i'':i\in N(i'')}\sum_j I_{i''j}W_{i''i}\big(R_{i''j}V_j-V_jV_j^T\big(U_{i''}+S_{i''}^{-i^T}\big)\big)\Big] \end{cases}$$

So $Q(U_i)\sim N(\mu_i^*,(\Lambda_i^*)^{-1})$, where

$$\begin{cases} \Lambda_i^* = \frac{1}{\sigma_u} I + \frac{1}{\sigma} \sum_j I_{ij} E\left[V_j V_j^T\right] + \frac{1}{\sigma} \sum_{i'':i \in N(i'')} \sum_j I_{i''j} E\left[W_{i''i}^2\right] E\left[V_j V_j^T\right] \\ \mu_i = (\Lambda_i^*)^{-1} \left[\frac{1}{\sigma} \sum_j I_{ij} \left(R_{ij} E\left[V_j\right] - E\left[V_j V_j^T\right] E\left[S_i\right]^T \right) \right. \\ \qquad \left. + \frac{1}{\sigma} \sum_{i'':i \in N(i'')} \sum_j I_{i''j} E\left[W_{i''i}\right] \left(R_{i''j} E\left[V_j\right] - E\left[V_j V_j^T\right] \left(E\left[U_{i''}\right] + E\left[S_{i''}^{-i}\right]^T \right) \right) \right] \end{cases}$$

Also note that

$$E\left[S_i\right] = \sum_{i' \in N(i)} E\left[w_{ii'}\right] E\left[U_{i'}\right]^T,$$

$$E\left[S_i^{-k}\right] = \sum_{i' \in N(i),\, i' \neq k} E\left[w_{ii'}\right] E\left[U_{i'}\right]^T.$$

For $P(V_j | U, V_{-j}, R, W; \Theta_0)$,

$\propto P(U, V, R, W; \Theta_0)$

$\propto P(V_j; \sigma_v) \prod_i N(U_i^T V_j + m_{ij}, \sigma)^{I_{ij}}$

$\Rightarrow logP(V_j | U, V_{-j}, R, W; \Theta_0) \propto logP(V_j; \sigma_v) + \sum_i I_{ij}\, logN(U_i^T V_j + m_{ij}, \sigma)$

$\propto -\frac{1}{2\sigma_v} V_j^T V_j + \sum_i -\frac{I_{ij}}{2\sigma} \left[-2R_{ij} \left(U_i^T V_j + S_i V_j \right) + V_j^T U_i U_i^T V_j + 2 V_j^T U_i S_i V_j + V_j^T S_i^T S_i V_j \right]$

$\propto -\frac{1}{2} V_j^T \left[\frac{1}{\sigma_v} I + \frac{1}{\sigma} \sum_i I_{ij} \left(U_i U_i^T + 2 U_i S_i + S_i^T S_i \right) \right] V_j + \left[\frac{1}{\sigma} \sum_i I_{ij} R_{ij} \left(U_i^T + S_i \right) \right] V_j$

$\Rightarrow P(V_j | U, V_{-j}, R, W; \Theta_0) \sim N(m_j, J_j), \begin{cases} J_j = \frac{1}{\sigma_v} I + \frac{1}{\sigma} \sum_i I_{ij} \left(U_i U_i^T + 2 U_i S_i + S_i^T S_i \right) \\ m_j = J_j^{-1} \frac{1}{\sigma} \sum_i I_{ij} R_{ij} \left(U_i + S_i^T \right) \end{cases}$

So

$$Q(V_j) \sim N(\mu_j^*, (\Lambda_j^*)^{-1}), \begin{cases} \Lambda_j^* = \frac{1}{\sigma_v} I + \frac{1}{\sigma} \sum_i I_{ij} \left(E\left[U_i U_i^T\right] + 2E\left[U_i\right] E\left[S_i\right] + E\left[S_i^T S_i\right] \right) \\ \mu_j = (\Lambda_j^*)^{-1} \frac{1}{\sigma} \sum_i I_{ij} R_{ij} \left(E\left[U_i\right] + E\left[S_i\right]^T \right) \end{cases}.$$

Also for $E\left[S_i^T S_i\right]$, we have

$$E\left[S_i^T S_i\right] = \sum_{i', i''} E[W_{ii'} W_{ii''}] E[U_{i'} U_{i''}^T] = \sum_{i'} E[W_{ii'}^2] E[U_{i'} U_{i'}^T] + \sum_{i' \neq i''} E[W_{ii'}] E[W_{ii''}] E[U_{i'}] E[U_{i''}]^T$$

For $P(W_{ii'} | U, V, R, W_{-ii'}; \Theta_0)$,

$\propto P(U, V, R, W; \Theta_0)$

$\propto P(W_{ii'}; \sigma_w) \prod_j N(U_i^T V_j + m_{ij}, \sigma)^{I_{ij}}$

$$\Rightarrow logP(W_{ii'}|U,V,R,W_{-ii'};\Theta_0) \propto logp(W_{ii'};\sigma_w) + \sum_j I_{ij}\, logN(U_i^TV_j + m_{ij},\sigma)$$

$$\propto -\frac{1}{2\sigma_w}W_{ii'}^2 + \sum_j -\frac{I_{ij}}{2\sigma}\Big[-2W_{ii'}U_{i'}^TV_jR_{ij} + 2U_i^TV_jW_{ii'}U_{i'}V_j + W_{ii'}^2U_{i'}^TV_jV_j^TU_{i'} + 2W_{ii'}U_{i'}^TV_jm_{ij}^{-i'}\Big]$$

$$\propto -\frac{1}{2}\Big[\frac{1}{\sigma_w} + \frac{1}{\sigma}\sum_j I_{ij}U_{i'}^TV_jV_j^TU_{i'}\Big]W_{ii'}^2 + \Big[\frac{1}{\sigma}\sum_j I_{ij}\big(U_{i'}^TV_jR_{ij} - U_i^TV_jV_j^TU_{i'} - U_{i'}^TV_jV_j^TS_i^{-i'}\big)\Big]W_{ii'}$$

$$\Rightarrow P(W_{ii'}|U,V,R,W_{-ii'};\Theta_0) \sim N(m_{ii'},J_{ii'}^{-1}),\ \begin{cases} J_{ii'} = \frac{1}{\sigma_w} + \frac{1}{\sigma}\sum_j I_{ij}U_{i'}^TV_jV_j^TU_{i'} \\ m_{ii'} = J_{ii'}^{-1}\frac{1}{\sigma}\sum_j I_{ij}\big(U_{i'}^TV_jR_{ij} - U_i^TV_jV_j^TU_{i'} - U_{i'}^TV_jV_j^TS_i^{-i'T}\big)\Big] \end{cases}$$

So,

$$Q(W_{ii'}) \sim N(\mu_{ii'}^*, \frac{1}{\rho_{ii'}^*}),$$

$$\begin{cases} \rho_{ii'}^* = \frac{1}{\sigma_w} + \frac{1}{\sigma}\sum_j I_{ij}E\big[U_{i'}^TV_jV_j^TU_{i'}\big] \\ \mu_{ii'} = \frac{1}{\rho_{ii'}^*}\frac{1}{\sigma}\sum_j I_{ij}\big(E\big[U_{i'}\big]^TE\big[V_j\big]R_{ij} - E\big[U_i\big]^TEE\big[V_jV_j^T\big]E\big[U_{i'}\big] - E\big[U_{i'}\big]^TE\big[V_jV_j^T\big]E\big[S_i^{-i'}\big]^T\big)\Big] \end{cases}$$

Also note that

$$E\big[U_i^TV_jV_j^TU_i\big] = \sum_{a,b,c,d}E\big[U_{ia}U_{ib}\big]E\big[V_{jc}V_{jd}\big] = \sum_{a,b}E\big[U_{ia}U_{ib}\big]\sum_{c,d}E\big[V_{jc}V_{jd}\big]$$

$$= \big(e^T\Big[cov(U_i) + E[U_i]E[U_i]^T\Big]e\big)\big(e^T\Big[Cov(V_j) + E[V_j]E[V_j]^T\Big]e\big),$$

where e is column of one(s).