

Assignment - 2

Assigned: 2nd April 2024Due: 12th April 2024

Max. marks: 100

General Instructions

- Submit a hard-copy of the solutions either in-class or in TAs office NAC-1, Room no. 322.
- Upload a scanned copy of the solutions and the codes you have developed as a single zipped folder on Moodle.
- Do not email your solutions/codes. Emailed submissions will not be graded.

Programming Questions

1. (40 points) Consider the following one-dimensional first-order traveling wave equation that convects to the right,

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0, \quad (1)$$

with $c = 1.0$ in a domain of length $L = 2.0$. The initial condition for the wave is given as follows,

$$u(x, t = 0) = u_0(x) = \begin{cases} \sin(4\pi x), & \text{for } 0 \leq x \leq 0.5 \\ 0, & \text{for } 0.5 \leq x \leq 2.0 \end{cases}.$$

The exact solution at any time instant, t , is given by $u(x, t) = u_0(x - ct)$. Consider the numerical solution of this equation using Euler explicit time integration method and two different spatial discretization schemes (i) first-order upwind scheme and (ii) third-order accurate QUICK scheme. The Euler explicit time discretization is given as follows:

$$\frac{\partial u}{\partial t} \approx \frac{u_i^{n+1} - u_i^n}{\Delta t}, \quad (2)$$

and the first-order upwind scheme is given by,

$$\frac{\partial u}{\partial x} \approx \frac{u_i^n - u_{i-1}^n}{\Delta x}, \quad (3)$$

and the QUICK scheme is given by,

$$\frac{\partial u}{\partial x} \approx \frac{1}{\Delta x} \left(\frac{3}{8} u_i^n - \frac{7}{8} u_{i-1}^n + \frac{1}{8} u_{i-2}^n + \frac{3}{8} u_{i+1}^n \right) \quad (4)$$

where Δx is the grid spacing and Δt is the time-step and they can be taken as $\Delta x = 0.002$ and $\Delta t = 0.001$. The boundary conditions on the left and right-ends of the domain are given as $u(x = 0, t) = 0$ and $u(x = L, t) = 0$.

- (a) Develop a serial program to solve this problem using upwind and QUICK schemes. Plot the analytical solution and the numerical solution obtained at several times $t = 0, 0.5, 1.0$ in the same graph.
- (b) Develop an MPI program to solve this problem using both the spatial discretization schemes as above. Plot the analytical solution and the numerical solutions at times $t = 0, 0.5, 1.0$ in the same graph for number of threads $p = 2$ and 4. Use appropriate number of halo/virtual points on both sides of the parallel domains.
- (c) Comment on the differences observed between the solutions for upwind and QUICK schemes.

2. (60 points) Consider the solution of the following Poisson equation,

$$\nabla^2 \phi = -q; \quad q = (x^2 + y^2); \quad (5)$$

in the domain shown in Figure 1 and the boundary conditions are as specified in the figure. The discretized equation using Jacobi or Gauss-Seidel method can be written as follows,

$$\phi_{i,j}^{(k+1)} = \frac{1}{4} \left[\phi_{i+1,j}^{(k)} + \phi_{i-1,j}^{(k),(k+1)} + \phi_{i,j+1}^{(k)} + \phi_{i,j-1}^{(k),(k+1)} \right] + \frac{\Delta^2}{4} q_{i,j} \quad (6)$$

where $\Delta = \Delta x = \Delta y$. Consider an initial guess of $\phi(x, y) = 0$ everywhere. Use double precision arithmetic. On the $x = 1$ boundary, approximate the first derivative $\frac{\partial \phi}{\partial x}$ using the second-order accurate one-sided formula as given below,

$$\phi_i = \frac{4\phi_{i-1} - \phi_{i-2}}{3} \quad (7)$$

- Develop a serial program using Jacobi iterative method. Test your program using $\Delta = \Delta x = \Delta y = 0.1$ (that is 21 points along each of the directions). Report the number of iterations for the solution between successive iterations to converge to 10^{-4} , using any norm. Plot the numerical solution of ϕ vs x for $y = 0.0$ and ϕ vs y for $x = 0.0$.
- Develop an MPI program using Jacobi iterative method. Test your program using $\Delta = \Delta x = \Delta y = 0.01$ (that is 201 points along each of the directions). Report the number of iterations for the solution between successive iterations to converge to 10^{-4} , using any norm. Plot the numerical solution obtained using the Jacobi iterative method run on parallel system using $p = 2, 4, 8$ for ϕ vs x for $y = 0.0$ and ϕ vs y for $x = 0.0$. Use a one-dimensional domain decomposition (either row-wise block decomposition or column-wise block decomposition on the two-dimensional mesh) only. In the same plots compare the parallel solution obtained with the serial solution for the same grid sizes of 0.01.
- Repeat the above step using Gauss-Seidel red-black coloring approach. Comment on the number of iterations required by Gauss-Seidel when compared the Jacobi method.
- Run parallel versions of the Jacobi and Gauss-Seidel methods for $\Delta = 0.005$ using $p = 2, 4, 8$ and 16 threads. Plot the speed-up, $\psi(n, p)$, obtained by each of the methods as a function of the number of processors. Do you see any improvement in performance? Which method is better? Comment on the speedup obtained as the problem size is increased.

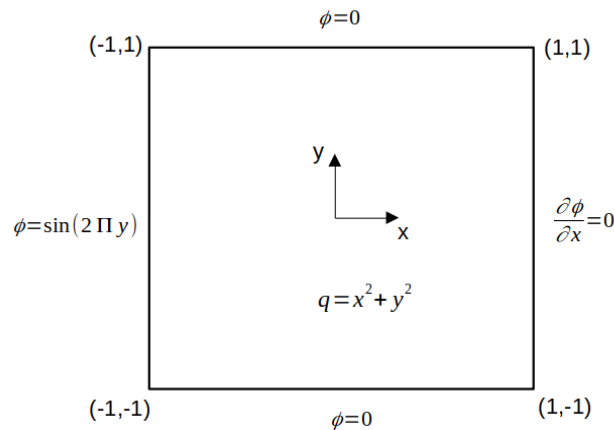


Figure 1: Domain and boundary conditions for Poisson equation