LORENZ ATTRACTOR (IDC402)

Akhil Ratan Mishra Ms17202

The Lorenz model is a system of three coupled first-order non-linear differential equations used to model fluid convection. Here x, y, z represent the rate of convection, variation in horizontal temperature and variation in vertical temperature respectively. Given by:

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = rx - y - xz \qquad [1]$$

$$\dot{z} = xy - bz$$

Here, σ is the Prandtl number,r is the Rayleigh number and $b > 0$. Figure 1 displays a plot of Lorenz's equation.

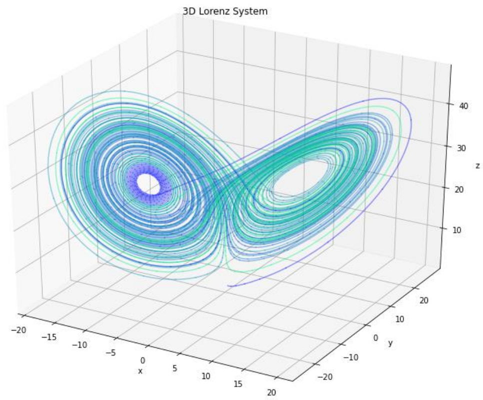

Fig:1 Example of 3D Lorenz System σ, b , r = [10, 8/3 28], with starting condition x, y, z = [0,1,1.05]

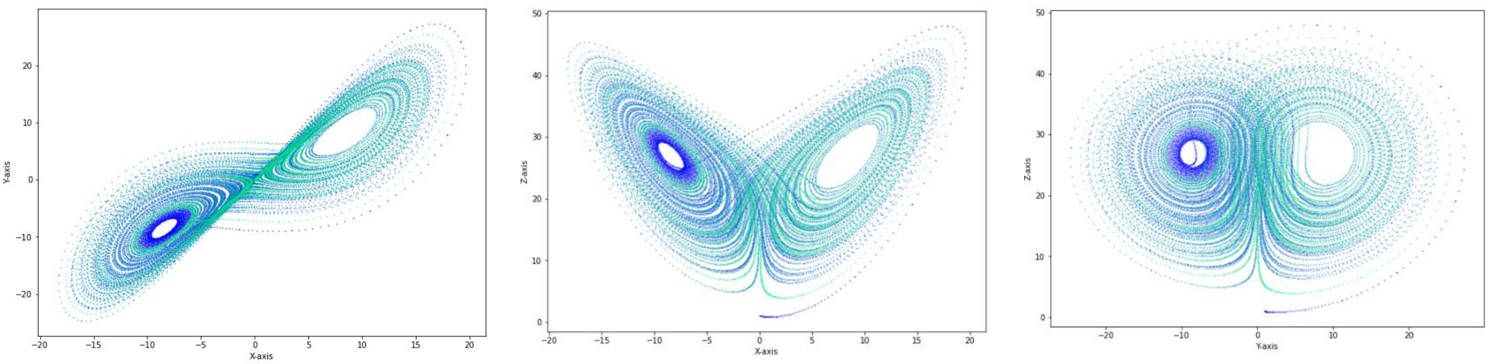

Fig:2 2D scatter plots with the above parameters in the order Left to Right, Y vs X , Z vs X and Z vs Y.

PROPERTIES OF LORENZ SYSTEM:

Even though the system is linear there is no way to write down a solution because of the presence of two non-linear terms xz and xy. So numerical approximation methods are used to solve this system.

- Lorenz equations are symmetric so, if $(x, y, z) \rightarrow (-x, -y, -z)$.
- The z-axis is invariant.
- The flow is volume contracting since $\nabla \cdot X = -(\sigma + b + 1) < 0$, where X is the vector field.

LORENZ SYSTEM FIXED POINTS AND STABILITY ANALYSIS:

Equilibrium points or fixed points, occur when all the below given equations becomes zero simultaneously.

$$\dot{x} = \sigma(y - x) = 0$$

$$\dot{y} = rx - y - xz = 0$$

$$\dot{z} = xy - bz = 0$$

The Lorenz system ([1]) has the following fixed points:

$$x_1^* = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } x_{2,3}^* = \begin{pmatrix} \pm\sqrt{b(r-1)} \\ \pm\sqrt{b(r-1)} \\ r-1 \end{pmatrix} \text{ if } r \geq 1 \tag{2}$$

Here, the positive value in the $x_{2,3}^*$ The stability of these fixed points can be determined by the Jacobian matrix $J(x)$, given below $J_L(x)$:

$$J(x) = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} & \frac{\partial \dot{x}}{\partial z} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} & \frac{\partial \dot{y}}{\partial z} \\ \frac{\partial \dot{z}}{\partial x} & \frac{\partial \dot{z}}{\partial y} & \frac{\partial z}{\partial z} \end{pmatrix} \qquad J_L(x) = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix} \tag{3}$$

EIGENVALUES:

- For, fixed point $x_1^* = (0,0,0)$. We can obtain the corresponding eigenvalues and eigenvectors as:

$$\nabla_{x_1^*} f(x^*) = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} \to (-b-\lambda)(\lambda^2 + (r+1)\lambda + (1-\sigma)r) = 0 \tag{4}$$

$$\lambda_1 = -b \text{ and } \lambda_{2,3} = \frac{-(r+1) \pm \sqrt{(r+1)^2 - 4(1-\sigma)r}}{2} \tag{5}$$

At, σ, b , r = [10, 8/3 28] , $\lambda_1 = -\frac{8}{3}, \lambda_2 = 7, \lambda_3 = -36$. Here two of the three eigenvalues are real-negative numbers. The vector filed proceed towards this point along two of the directions and are pushed away across one direction.

- For fixed points, x_2^*, x_3^*, we can similarly find the matrix $\nabla_{x_2^*} f(x^*)$. Computing eigenvalues for $\nabla_{x_2^*} f(x^*)$ we get:

$$\nabla_{x_2^*} f(x^*) = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-27 & -1 & -6\sqrt{2} \\ 6\sqrt{2} & 6\sqrt{2} & -b \end{pmatrix} \to det \begin{pmatrix} -10-\lambda & 10 & 0 \\ 1 & -1-\lambda & -6\sqrt{2} \\ 6\sqrt{2} & 6\sqrt{2} & -\frac{8}{3} - \lambda \end{pmatrix} = 0$$

The eigen values are $\lambda_1 \approx -13.855, \lambda_2 \approx 0.094 - 10.194i$, $\lambda_3 \approx 0.094 + 10.194i$. Note that we have 2 complex eigen values and one real negative eigenvalue. So, along one dimension the critical point is attracting and along the other two dimension it is a stable spiral.

BIFURCATIONS:

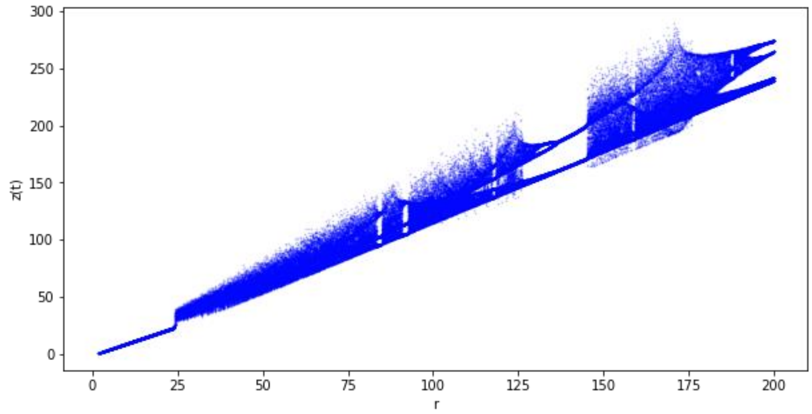

Fig: 3 Bifurcation diagram for the Lorenz system by using r as the order parameter.

- **Pitchfork Bifurcations**

At, $r = 1$, the system undergoes a pitchfork bifurcation where two additional fixed points appear (Eq. (2)). Graphically they look as:

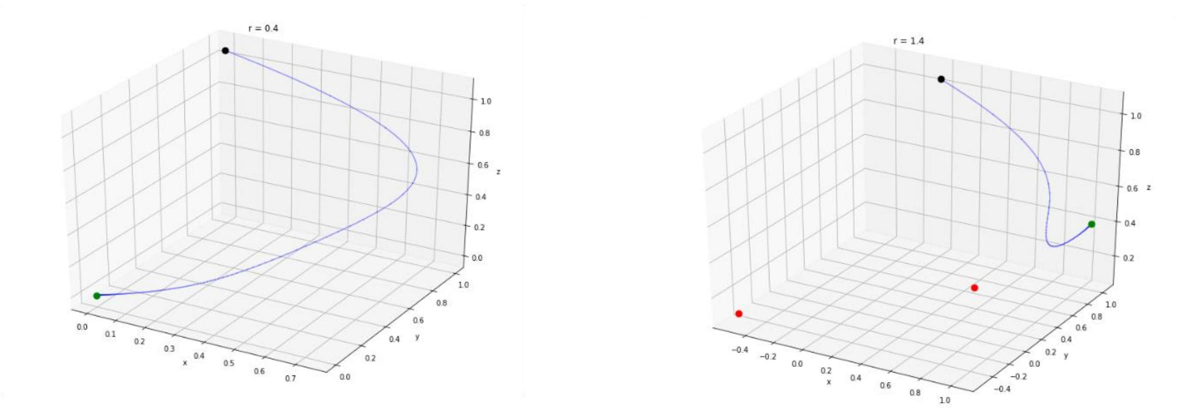

Fig 4: We can see Pitchfork bifurcations at r = 1. By the emergence of 2 critical points(red). Right image r = 0.4, Left image r = 1.4.

- **Hopf Bifurcations:**

Hopf bifurcation characterizes the change in stability as r varies. Since the origin is stable for $0 < r < 1$ and unstable for $r > 1$ where the two additional fixed points x_2^* and x_3^* exist and are stable for $1 < r < r_H$. (Hopf Bifurcation).

The instability points are characterized by a vanishing real part of the largest eigenvalue, $\Re\{\lambda\} = 0$. Then, substituting λ in (6) by $i\omega$ and after equation real and imaginary parts with zero, we get

$$det\left|J\left(x_{2,3}^*\right) - \lambda I\right| = 0 \quad \rightarrow \quad \lambda^3 + \lambda^2(\sigma + b + 1) + \lambda b(\sigma + r) + 2\sigma b(r - 1) = 0 \qquad (6)$$

$$\frac{2\sigma b(r-1)}{\sigma+b+1} = b(\sigma + r) \quad \text{putting the initial conditions, } \sigma = 10 \;\; b = \frac{8}{3} \;\; \rightarrow r_H \approx 24.74 \qquad (7)$$

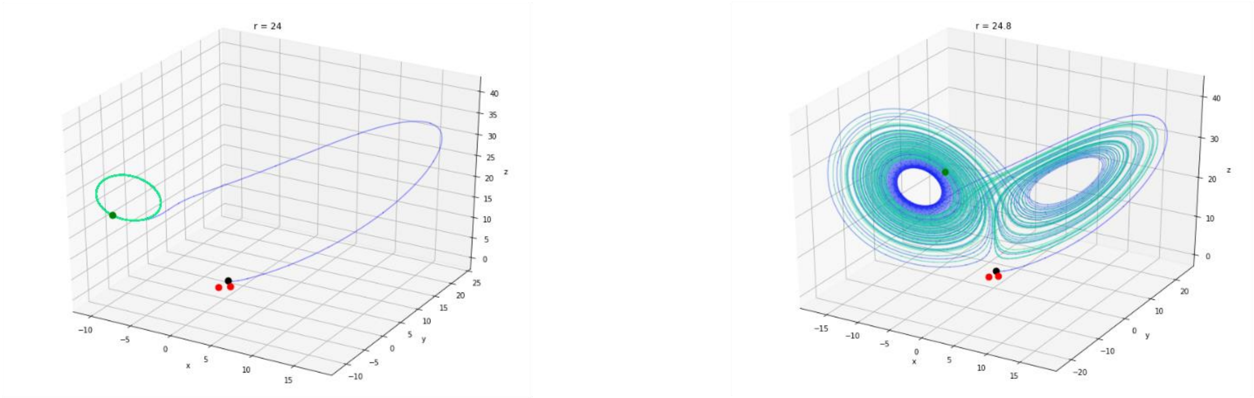

Fig: 5 The red points denote the critical points, (Left: Sub critical) for r = 24 (Right Critical) r = 24.8. We can see Hopf bifurcation at $r > r_H$

The system behaves differently for different ranges of r values. It is called sub-critical $(r < r_H)$ and super-critical $(r > r_H)$ conditions.

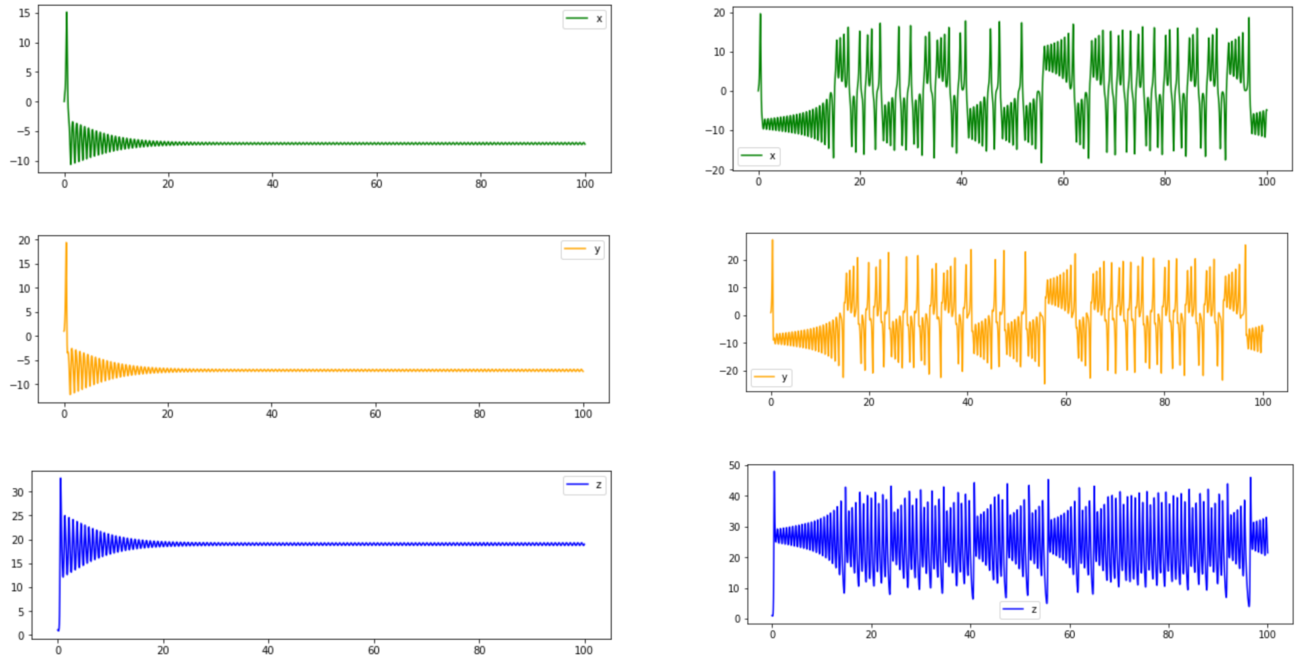

Fig 6: Variation in the x, y, z values in the: Subcritical Conditions (Left), Critical Conditions (Right)

POINCARE SECTION

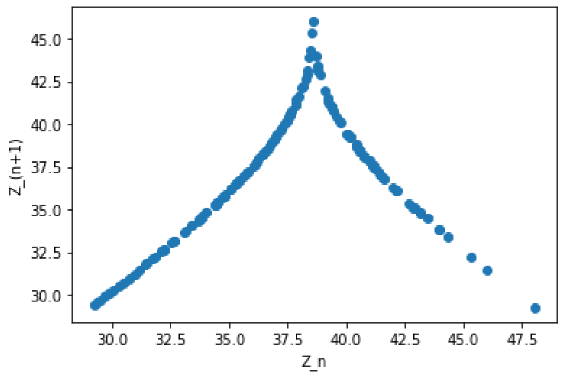

Fig:7 Plot of relation between successive maxima ZN for the Lorenz system

Poincare map is used to characterize the complex behaviour of the unstable trajectories Fig:6 (Poincare map of Lorenz system). This figure has a form of a one-dimensional map (TENT MAP).

LYAPUNOV EXPONENT:

The maximal Lyapunov exponent of the system is the number λ, if it exists, such that:

$$|\delta(t)| = |\delta(t)|e^{\lambda t}$$

Here, the word maximal has been used dynamical systems don't just have a single Lyapunov exponent. Rather, every dynamical system has a spectrum of Lyapunov exponents, one for each dimension of its phase space. For example, Negative Lyapunov exponents are associated with dissipative systems; Lyapunov exponents equal to zero are associated with conservative systems; and positive Lyapunov exponents are associated with chaotic systems (provided the system has an attractor).

Now, we will estimate the maximal Lorentz exponent for nearby trajectories with initial separation 10^{-9} on a large time interval $t \in [0,30]$

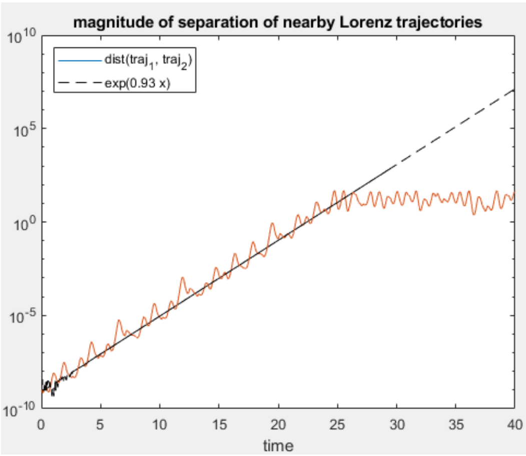

Fig 8: Magnitude of separation of nearby Lorenz trajectories.

The log of the distance between trajectories can be approximated by a straight line. So the Lorenz system has a positive Lyapunov exponent. Notice, however, that the positive slope only holds up for the first 25-time units or so. After that, the curve levels off. That is because all trajectories of the Lorenz system wind up in its strange attractor.

REFERENCES:

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2. https://www2.physics.ox.ac.uk/sites/default/files/profiles/read/lect6-43147.pdf
3. Numerical Computation of Lyapunov Exponents and Dimension in Nonlinear Dynamics and Chaos, Dr. Rob Sturman.