Deterministic Independent Component Analysis (ICA)

Ruitong Huang   András György   Csaba Szepesvári

University of Alberta

July 17, 2015

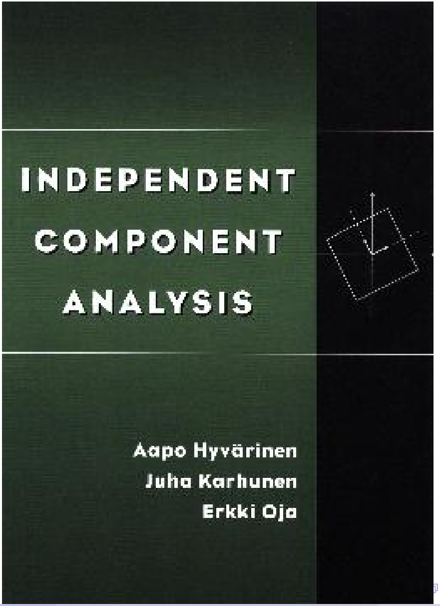
UNIVERSITY OF ALBERTA
DEPARTMENT OF COMPUTING SCIENCE

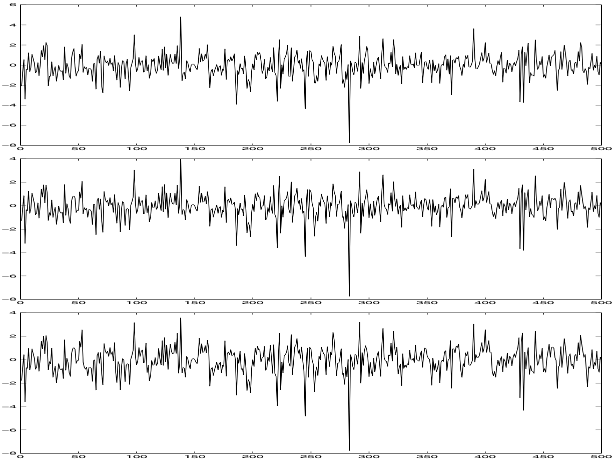
ALBERTA INNOVATES CENTRE FOR MACHINE LEARNING

Outline

1. Introduction
   - What is ICA, really?
   - Deterministic ICA
   - Has not this been done previously?
2. Results
   - Our algorithm: Deterministic ICA
3. Empirical Illustration
4. Conclusions

What is Independent Component Analysis (ICA)?





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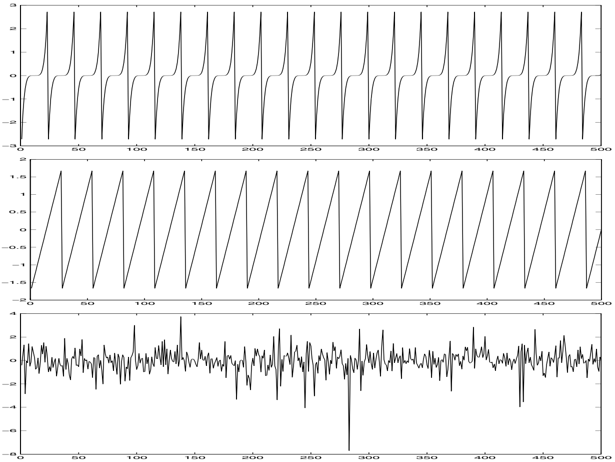

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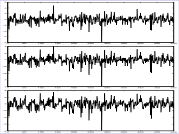

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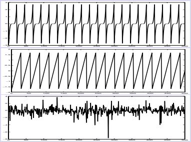

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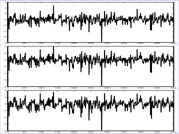

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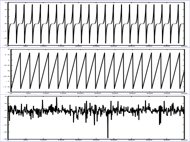

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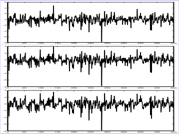

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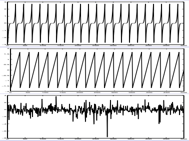

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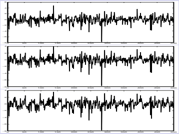

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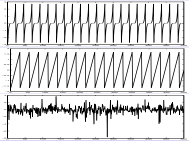

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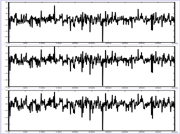

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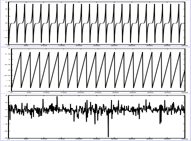

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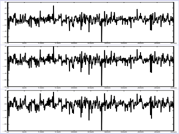

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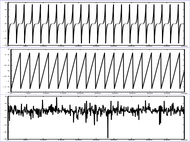

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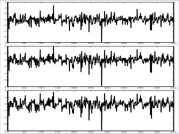

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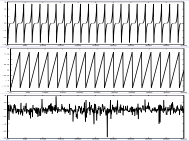

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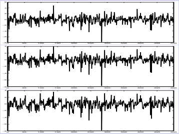

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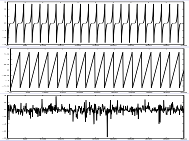

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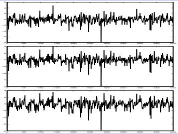

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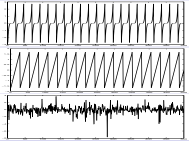

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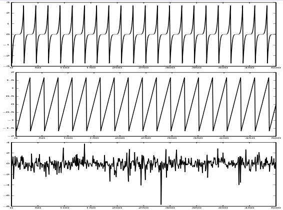
where A is some unknown matrix. Independent component analysis now consists of estimating both the matrix A and the $s_i(t)$, when we only observe the $x_i(t)$.

Good?

Independence?



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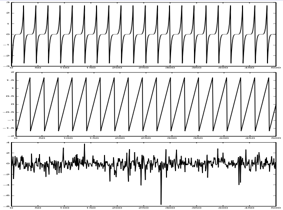

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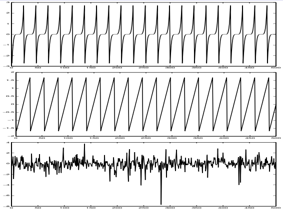

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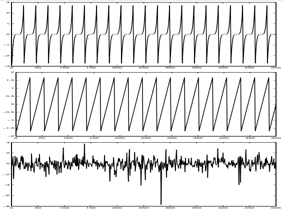

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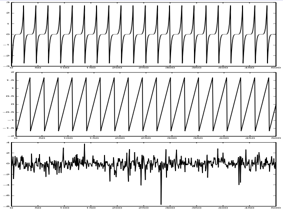

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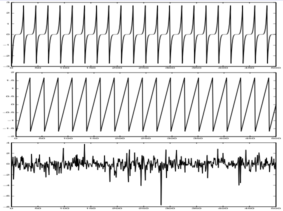

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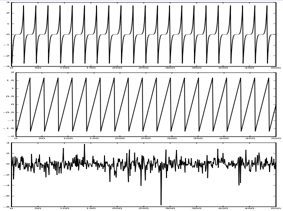

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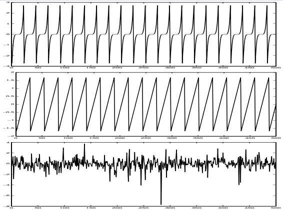

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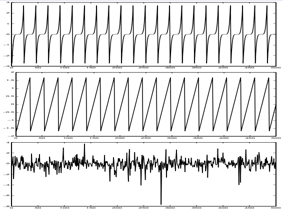

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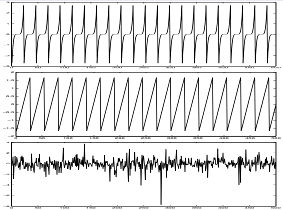

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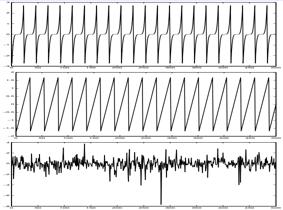

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- Here: Let's go beyond statistics!

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$$D_4 = \inf_{\mu} \sup_{f \in \mathcal{F}} \left| \int f(s) d\nu_T^{(s)} - \int f(s) d\mu(s) \right|,$$

$\mathcal{F}$: the set of all monomials up to degree 4;
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Note: When $s(t)$ has independent components and $s(1), \ldots, s(T)$ are iid, $D_4 = O(1/\sqrt{T})$.

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Accuracy measure:

$$d(\hat{A}, A) = \inf_{\substack{\pi \in \mathrm{Perm}([d]) \\ c \in \mathbb{R}^d}} \max_k ||c_k A_{:\pi(k)} - A_{:k}||_2 \,.$$

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Why care about a free parameters? ... unsupervised learning

Outline

Result

There exist a randomized method to estimate A from $(x(t))_{t=1}^{T}$, with $x(t) = As(t) + \epsilon(t),\ t = 1, \ldots, T$ s.t.:

- The computational complexity is $O(d^3 T)$;

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- Here, θ_1, θ_2 problem dependent, polynomial in the parameters.

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For $x \sim \mu = \mu_1 \otimes \cdots \otimes \mu_d$, $\eta \in \mathbb{R}^d$, let

$$f(\eta) = \mathbb{E}\left[(\eta^\top x)^4\right] - 3\mathbb{E}\left[(\eta^\top x)^2\right]^2 .$$

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Given finite sample $(x(t))_{t=1}^T$, replace $\mathbb{E}[\cdot]$ with $\mathbb{E}_n[\cdot]$.

Our issue with the HK method

Problem: Minimal gap of the eigenvalues.

- Theoretical analysis shows that the performance depends on γ_A^{-1}, where
$$\gamma_A = \min_{i \neq j} |\lambda_i - \lambda_j| \,.$$

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Goal: Avoid dependency on γ_A^{-1}!

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- Recursive version is also developed in the paper, based on the idea of Vempala and Xiao [2014].

Outline

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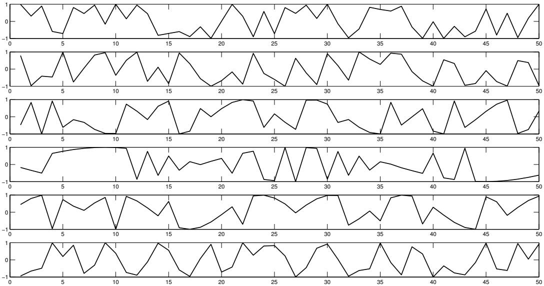
- DICA/DICA.R;
- MDICA/MDICA.R (heuristic version of DICA)

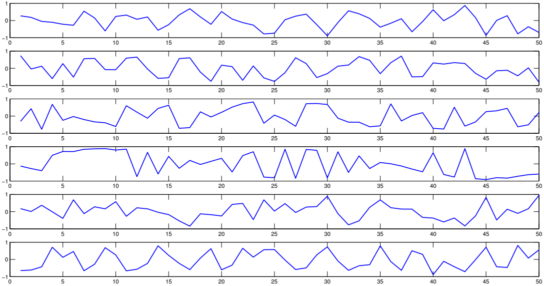
Sources

BPSK sources, $T = 20,000$:

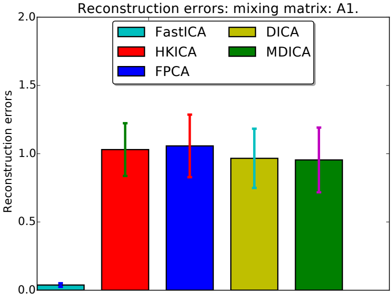


Example reconstructions: Noise-free case

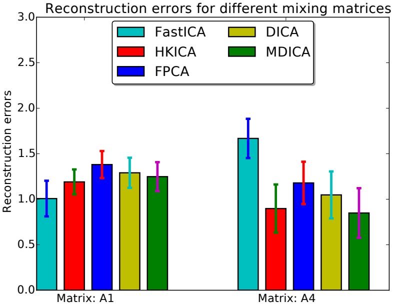
Reconstruction:



Turf: Noise and coherence



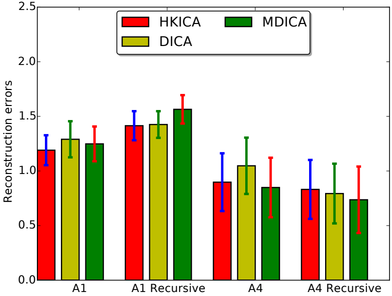
Noise-free



Noisy

Reconstruction error of “random” matrix: 2.2.
Error bars are based on 150 runs.

Recursive versions



Outline

Conclusions and Open Questions

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Questions?

Code: `https://github.com/Armstring/Deterministic-ICA`

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