Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How is the average velocity of a moving object connected to the values of its position function?
- How do we interpret the average velocity of an object geometrically with regard to the graph of its position function?
- How is the notion of instantaneous velocity connected to average velocity?

Preview Activity 0.0

Suppose that the height s of a ball (in feet) at time t (in seconds) is given by the formula $s(t) = 64 - 16(t-1)^2$.

(a) Construct an accurate graph of $y = s(t)$ on the time interval $0 \le t \le 3$. Label at least six distinct points on the graph, including the three points that correspond to when the ball was released, when the ball reaches its highest point, and when the ball lands.

(b) In everyday language, describe the behavior of the ball on the time interval $0 < t < 1$ and on time interval $1 < t < 3$. What occurs at the instant $t = 1$?

(c) Consider the expression

$$AV_{[0.5,1]} = \frac{s(1) - s(0.5)}{1 - 0.5}.$$

Compute the value of $AV_{[0.5,1]}$. What does this value measure geometrically? What does this value measure physically? In particular, what are the units on $AV_{[0.5,1]}$?

Activity 0.0–1

The following questions concern the position function given by $s(t) = 64 - 16(t-1)^2$, which is the same function considered in Preview Activity .

(a) Compute the average velocity of the ball on each of the following time intervals: $[0.4, 0.8]$, $[0.7, 0.8]$, $[0.79, 0.8]$, $[0.799, 0.8]$, $[0.8, 1.2]$, $[0.8, 0.9]$, $[0.8, 0.81]$, $[0.8, 0.801]$. Include units for each value.

(b) On the provided graph in no figure, sketch the line that passes through the points $A = (0.4, s(0.4))$ and $B = (0.8, s(0.8))$. What is the meaning of the slope of this line? In light of this meaning, what is a geometric way to interpret each of the values computed in the preceding question?

(c) Use a graphing utility to plot the graph of $s(t) = 64 - 16(t - 1)^2$ on an interval containing the value $t = 0.8$. Then, zoom in repeatedly on the point $(0.8, s(0.8))$. What do you observe about how the graph appears as you view it more and more closely?

(d) What do you conjecture is the velocity of the ball at the instant $t = 0.8$? Why?

Example 1

For a falling ball whose position function is given by $s(t) = 16 - 16t^2$ (where s is measured in feet and t in seconds), find an expression for the average velocity of the ball on a time interval of the form $[0.5, 0.5 + h]$ where $-0.5 < h < 0.5$ and $h \neq 0$. Use this expression to compute the average velocity on $[0.5, 0.75]$ and $[0.4, 0.5]$, as well as to make a conjecture about the instantaneous velocity at $t = 0.5$.

Solution. We make the assumptions that $-0.5 < h < 0.5$ and $h \neq 0$ because h cannot be zero (otherwise there is no interval on which to compute average velocity) and because the function only makes sense on the time interval $0 \leq t \leq 1$, as this is the duration of time during which the ball is falling. Observe that we want to compute and simplify

$$AV_{[0.5,0.5+h]} = \frac{s(0.5 + h) - s(0.5)}{(0.5 + h) - 0.5}.$$

The most unusual part of this computation is finding $s(0.5 + h)$. To do so, we follow the rule that defines the function s. In particular, since $s(t) = 16 - 16t^2$, we see that

$$\begin{aligned} s(0.5 + h) &= 16 - 16(0.5 + h)^2 \\ &= 16 - 16(0.25 + h + h^2) \\ &= 16 - 4 - 16h - 16h^2 \\ &= 12 - 16h - 16h^2. \end{aligned}$$

Now, returning to our computation of the average velocity, we find that

$$\begin{aligned} AV_{[0.5,0.5+h]} &= \frac{s(0.5+h)-s(0.5)}{(0.5+h)-0.5} \\ &= \frac{(12-16h-16h^2)-(16-16(0.5)^2)}{0.5+h-0.5} \\ &= \frac{12-16h-16h^2-12}{h} \\ &= \frac{-16h-16h^2}{h}. \end{aligned}$$

At this point, we note two things: first, the expression for average velocity clearly depends on h, which it must, since as h changes the average velocity will change. Further, we note that since h can never equal zero, we may further simplify the most recent expression. Removing the common factor of h from the numerator and denominator, it follows that

$$AV_{[0.5,0.5+h]} = -16 - 16h.$$

Now, for any small positive or negative value of h, we can compute the average velocity. For instance, to obtain the average velocity on $[0.5, 0.75]$, we let $h = 0.25$, and the average velocity is $-16 - 16(0.25) = -20$ ft/sec. To get the average velocity on $[0.4, 0.5]$, we let $h = -0.1$, which tells us the average velocity is $-16 - 16(-0.1) = -14.4$ ft/sec. Moreover, we can even explore what happens to $AV_{[0.5,0.5+h]}$ as h gets closer and closer to zero. As h approaches zero, $-16h$ will also approach zero, and thus it appears that the instantaneous velocity of the ball at $t = 0.5$ should be -16 ft/sec.

Summary

In this section, we encountered the following important ideas:

- The average velocity on $[a, b]$ can be viewed geometrically as the slope of the line between the points $(a, s(a))$ and $(b, s(b))$ on the graph of $y = s(t)$, as shown in Figure **??**.

- Given a moving object whose position at time t is given by a function s, the average velocity of the object on the time interval $[a, b]$ is given by $AV_{[a,b]} = \frac{s(b)-s(a)}{b-a}$. Viewing the interval $[a, b]$ as having the form $[a, a + h]$, we equivalently compute average velocity by the formula $AV_{[a,a+h]} = \frac{s(a+h)-s(a)}{h}$.

- The instantaneous velocity of a moving object at a fixed time is estimated by considering average velocities on shorter and shorter time intervals that contain the instant of interest.