Supplementary Material to:
Direct Sparse Visual-Inertial Odometry with Stereo Cameras

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Content

Chapter1 PHOTO RESIDUALS

1.1 INTRODUCTION

Windowed Optimization is a classic method in non-linear optimization.

1.1.1 NOTATION

Throughout the paper, we will write matrices as bold capital letters ($\mathbf{R}$) and vectors as bold lower case letters ($\boldsymbol{\xi}$), light lower-case letters to denote scalars (s). Light upper-case letters are used to represent functions (I).

Homogeneous camera calibration matrices are denoted by $\mathbf{K}$ as (2.1). Camera poses are represented by matrices of the special Euclidean group $\mathbf{T} \in SE(3)$, which transform a 3D coordinate from the camera coordinate system to the world coordinate system. In this paper, a homogeneous 2D image coordinate point $\mathbf{p}$ is represented by its image coordinate and inverse depth as (2.1) relative to its host keyframe I_i^L. The host keyframe is the frame the point got selected from. Corresponding homogeneous 3D world coordinate point $\mathbf{p}_w$ is denoted as (2.1). $\Pi_{\mathbf{K}}$ are used to denote camera projection functions. The jacobian of I_i, $\Pi_{\mathbf{K}}$ is denoted as (2.1)

$$
\begin{gathered}
\mathbf{K} = \begin{pmatrix} f_x & 0 & c_x & 0 \\ 0 & f_y & c_y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{K}^{-1} = \begin{pmatrix} f_x^{-1} & 0 & -f_x^{-1}c_x & 0 \\ 0 & f_y^{-1} & -f_y^{-1}c_y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
\mathbf{p} = \begin{pmatrix} u^i \\ v^i \\ 1 \\ d_{\mathbf{p}} \end{pmatrix}, \mathbf{p}_w = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}, d_{\mathbf{p}} = z^{-1}, \mathbf{p} = d_{\mathbf{p}}\mathbf{K}\mathbf{p}_w = \Pi_{\mathbf{K}}(\mathbf{p}_w) \\
\frac{\partial I_i^L(\mathbf{p})}{\partial \mathbf{p}} = (g_x, g_y, 0, 0), \frac{\partial \mathbf{p}}{\partial \mathbf{p}_w} = \begin{pmatrix} f_x z^{-1} & 0 & -x f_x z^{-2} & 0 \\ 0 & f_y z^{-1} & -y f_y z^{-2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & z^{-2} & 0 \end{pmatrix}
\end{gathered}
\tag{2.1}
$$

1.1.2 QUESTION IMPORT

Assume we observe 5 points $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5\}$ in 4 keyframes $\boldsymbol{\mathcal{F}} = \{I_1, I_2, I_3, I_4\}$, every keyframe has stereo vision (I_i^L, I_i^R) abbreviated as (iL, iR). A point can also be observed by other frame as shown in Table(2.1). Question is how to use Windowed Optimization method to make our observation more accurate ?

Table (2.1)

Image point	Host keyframe	Observe by
$\mathbf{p}_1$	$1L$	$1R, 2L$
$\mathbf{p}_2$	$2L$	$2R, 1L, 3L$
$\mathbf{p}_3$	$2L$	$3L$
$\mathbf{p}_4$	$3L$	$1L, 2L$
$\mathbf{p}_5$	$4L$	$3L, 4L$

1.2 SOLUTION

We use direct method to construct residual, Windowed Gauss-Newton method to optimization residual。

1.2.1 CONSTRUCT RESIDUAL

Dynamic multi-view stereo residuals $E_{ij}^{\mathbf{p}}$ are defined as

$$E_{ij}^{\mathbf{p}} = w_{\mathbf{p}} ||(r_{\mathbf{p}}^d)_{ij}||_\gamma, \quad (r_{\mathbf{p}}^d)_{ij} := I_j^L(\mathbf{p}') - b_j^L - \frac{e^{a_j^L}}{e^{a_i^L}}(I_i^L(\mathbf{p}) - b_i^L) \tag{2.2}$$

γ is Huber norm. a_i^L, b_i^L is affine brightness parameters to frame iL . $w_{\mathbf{p}}$ is a gradient-dependent weighting parameters, $\mathbf{p}$ in frame I_i^L projected to I_j^L is $\mathbf{p}'$ as:

$$w_{\mathbf{p}} := \frac{c^2}{c^2 + ||\nabla I_i(\mathbf{p})||_2^2}, \quad \mathbf{p}' = d_{\mathbf{p}}^{jL}\mathbf{K}(\mathbf{T}_j\mathbf{T}_i^{-1}((d_{\mathbf{p}}^{iL})^{-1}\mathbf{K}^{-1}\mathbf{p})) \tag{2.3}$$

Static one-view stereo residuals $E_{is}^{\mathbf{p}}$ are modified to

$$E_{is}^{\mathbf{p}} = w_{\mathbf{p}} ||r_{\mathbf{p}}^s||_\gamma, \quad r_{\mathbf{p}}^s := I_i^R(\mathbf{p}') - b_i^R - \frac{e^{a_i^R}}{e^{a_i^L}}(I_i^L(\mathbf{p}) - b_i^L) \tag{2.2}$$

Hostframe of $\mathbf{p}$ is I_i^L. a_i^R, b_i^R is affine brightness parameters to frame iR. $\mathbf{p}$ in frame I_i^L projected to I_i^R is $\mathbf{p}'$ as :

$$\mathbf{p}' = d_{\mathbf{p}}^{iR}\mathbf{K}(\mathbf{T}_{RL}((d_{\mathbf{p}}^{iL})^{-1}\mathbf{K}^{-1}\mathbf{p})) \tag{2.2}$$

Total residuals

$$E(\delta) = \sum_{i \in \mathcal{F}} \sum_{\mathbf{p} \in \mathcal{P}_i} (\sum_{j \in obs^t(\mathbf{P})} E_{ij}^{\mathbf{p}} + \lambda E_{is}^{\mathbf{p}}) \tag{2.2}$$

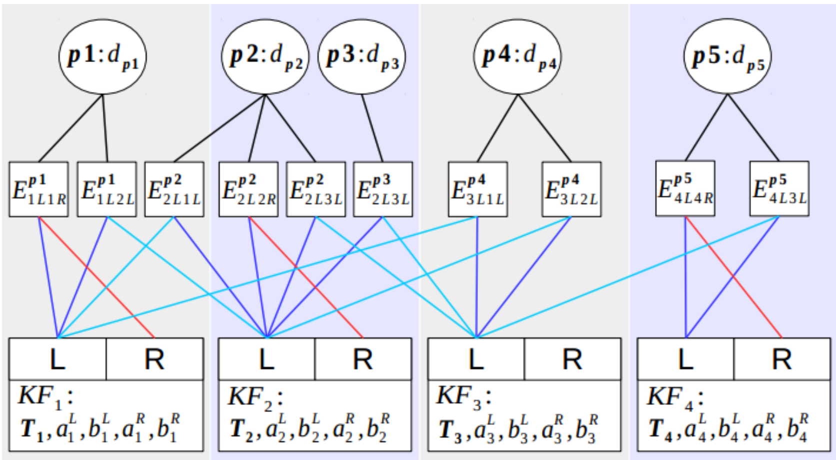
To balance the relative weights of temporal multi-view and static stereo, we introduce a coupling factor λ to weight the constraints from static stereo differently.

$$\boldsymbol{\delta} = \begin{pmatrix} (\boldsymbol{\xi}_1^T, \ldots, \boldsymbol{\xi}_{N_f}^T)^T \\ (d_{\mathbf{p}_1}, \ldots, d_{\mathbf{p}_{N_p}})^T \\ (a_1^L, a_1^R, b_1^L, b_1^R)^T \\ \vdots \\ (a_{N_f}^L, a_{N_f}^R, b_{N_f}^L, b_{N_f}^R)^T \\ (f_x, f_y, c_x, c_y)^T \end{pmatrix} \in \mathbb{R}^{10N_f+N_p+4}, \boldsymbol{\xi}_i = (ln\mathbf{T}_i)^{\vee} \in \mathbb{R}^6 \tag{2.1}$$

$\mathcal{P}_i$ is a set of all image point host by frame iL. $obs^t(\mathbf{p})$ are the observations of $\mathbf{p}$ from temporal multi-view stereo. If there are N_p image point and N_f keyframes in $\boldsymbol{\mathcal{F}}$, optimization variable $\boldsymbol{\delta}$ is

In this example, there are 7 dynamic residuals and 3 static residuals, Factor graph of the residuals function is

Total residuals is



$$\begin{aligned} E(\delta) &= E^{\mathbf{p}_1}_{1L2L} + E^{\mathbf{p}_2}_{2L1L} + E^{\mathbf{p}_2}_{2L3L} + E^{\mathbf{p}_3}_{2L3L} + E^{\mathbf{p}_4}_{3L1L} + E^{\mathbf{p}_4}_{3L2L} + E^{\mathbf{p}_5}_{4L3L} \\ &\quad + E^{\mathbf{p}_1}_{1L1R} + E^{\mathbf{p}_2}_{2L2R} + E^{\mathbf{p}_5}_{4L4R} \\ &= E_d(\delta) + E_s(\delta) \end{aligned}$$

$$E_s(\delta) = \begin{pmatrix} r^s_{\mathbf{p}_1} \\ r^s_{\mathbf{p}_2} \\ r^s_{\mathbf{p}_5} \end{pmatrix}^T \begin{pmatrix} \lambda w_{\mathbf{p}_1} & 0 & 0 \\ 0 & \lambda w_{\mathbf{p}_2} & 0 \\ 0 & 0 & \lambda w_{\mathbf{p}_5} \end{pmatrix} \begin{pmatrix} r^s_{\mathbf{p}_1} \\ r^s_{\mathbf{p}_2} \\ r^s_{\mathbf{p}_5} \end{pmatrix} = (\mathbf{r}^s)^T \mathbf{W}^s \mathbf{r}^s$$

$$\mathbf{J}_s = \begin{pmatrix} \frac{\partial r^s_{\mathbf{p}_1}}{\partial \boldsymbol{\xi}_1} \cdots & \frac{\partial r^s_{\mathbf{p}_1}}{\partial \boldsymbol{\xi}_4} & \frac{\partial r^s_{\mathbf{p}_1}}{\partial d_{\mathbf{p}_1}} \cdots & \frac{\partial r^s_{\mathbf{p}_1}}{\partial d_{\mathbf{p}_5}} & \frac{\partial r^s_{\mathbf{p}_1}}{\partial a^L_1} \cdots & \frac{\partial r^s_{\mathbf{p}_1}}{\partial b^R_4} & \frac{\partial r^s_{\mathbf{p}_1}}{\partial f_x} \cdots & \frac{\partial r^s_{\mathbf{p}_1}}{\partial c_y} \\ \frac{\partial r^s_{\mathbf{p}_2}}{\partial \boldsymbol{\xi}_1} \cdots & \frac{\partial r^s_{\mathbf{p}_2}}{\partial \boldsymbol{\xi}_4} & \frac{\partial r^s_{\mathbf{p}_2}}{\partial d_{\mathbf{p}_1}} \cdots & \frac{\partial r^s_{\mathbf{p}_2}}{\partial d_{\mathbf{p}_5}} & \frac{\partial r^s_{\mathbf{p}_2}}{\partial a^L_1} \cdots & \frac{\partial r^s_{\mathbf{p}_2}}{\partial b^R_4} & \frac{\partial r^s_{\mathbf{p}_2}}{\partial f_x} \cdots & \frac{\partial r^s_{\mathbf{p}_2}}{\partial c_y} \\ \frac{\partial r^s_{\mathbf{p}_5}}{\partial \boldsymbol{\xi}_1} \cdots & \frac{\partial r^s_{\mathbf{p}_5}}{\partial \boldsymbol{\xi}_4} & \frac{\partial r^s_{\mathbf{p}_5}}{\partial d_{\mathbf{p}_1}} \cdots & \frac{\partial r^s_{\mathbf{p}_5}}{\partial d_{\mathbf{p}_5}} & \frac{\partial r^s_{\mathbf{p}_5}}{\partial a^L_1} \cdots & \frac{\partial r^s_{\mathbf{p}_5}}{\partial b^R_4} & \frac{\partial r^s_{\mathbf{p}_5}}{\partial f_x} \cdots & \frac{\partial r^s_{\mathbf{p}_5}}{\partial c_y} \end{pmatrix}_{3\times 49} \tag{2.2}$$

$$E_d(\delta) = \begin{pmatrix} (r^d_{\mathbf{p}_1})_{12} \\ (r^d_{\mathbf{p}_1})_{21} \\ \vdots \\ (r^d_{\mathbf{p}_5})_{43} \end{pmatrix}^T \begin{pmatrix} w_{\mathbf{p}_1} & 0 & \ldots & 0 \\ 0 & w_{\mathbf{p}_1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & w_{\mathbf{p}_5} \end{pmatrix} \begin{pmatrix} (r^d_{\mathbf{p}_1})_{12} \\ (r^d_{\mathbf{p}_1})_{21} \\ \vdots \\ (r^d_{\mathbf{p}_5})_{43} \end{pmatrix} = (\mathbf{r}^d)^T \mathbf{W}^d \mathbf{r}^d$$

$$
\mathbf{J}_d = \begin{pmatrix}
\frac{\partial (r^d_{\mathbf{p}_1})_{12}}{\partial \boldsymbol{\xi}_1} \dots & \frac{\partial (r^d_{\mathbf{p}_1})_{12}}{\partial \boldsymbol{\xi}_4} & \frac{\partial (r^d_{\mathbf{p}_1})_{12}}{\partial d_{\mathbf{p}_1}} \dots & \frac{\partial (r^d_{\mathbf{p}_1})_{12}}{\partial d_{\mathbf{p}_5}} & \frac{\partial (r^d_{\mathbf{p}_1})_{12}}{\partial a^L_1} \dots & \frac{\partial (r^d_{\mathbf{p}_1})_{12}}{\partial b^R_4} & \frac{\partial (r^d_{\mathbf{p}_1})_{12}}{\partial f_x} \dots & \frac{\partial (r^d_{\mathbf{p}_1})_{12}}{\partial c_y} \\
\frac{\partial (r^d_{\mathbf{p}_1})_{21}}{\partial \boldsymbol{\xi}_1} \dots & \frac{\partial (r^d_{\mathbf{p}_1})_{21}}{\partial \boldsymbol{\xi}_4} & \frac{\partial (r^d_{\mathbf{p}_1})_{21}}{\partial d_{\mathbf{p}_1}} \dots & \frac{\partial (r^d_{\mathbf{p}_1})_{21}}{\partial d_{\mathbf{p}_5}} & \frac{\partial (r^d_{\mathbf{p}_1})_{21}}{\partial a^L_1} \dots & \frac{\partial (r^d_{\mathbf{p}_1})_{21}}{\partial b^R_4} & \frac{\partial (r^d_{\mathbf{p}_1})_{21}}{\partial f_x} \dots & \frac{\partial (r^d_{\mathbf{p}_1})_{21}}{\partial c_y} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\partial (r^d_{\mathbf{p}_5})_{43}}{\partial \boldsymbol{\xi}_1} \dots & \frac{\partial (r^d_{\mathbf{p}_5})_{43}}{\partial \boldsymbol{\xi}_4} & \frac{\partial (r^d_{\mathbf{p}_5})_{43}}{\partial d_{\mathbf{p}_1}} \dots & \frac{\partial (r^d_{\mathbf{p}_5})_{43}}{\partial d_{\mathbf{p}_5}} & \frac{\partial (r^d_{\mathbf{p}_5})_{43}}{\partial a^L_1} \dots & \frac{\partial (r^d_{\mathbf{p}_5})_{43}}{\partial b^R_4} & \frac{\partial (r^d_{\mathbf{p}_5})_{43}}{\partial f_x} \dots & \frac{\partial (r^d_{\mathbf{p}_5})_{43}}{\partial c_y}
\end{pmatrix}_{7\times 49} \tag{2.2}
$$

Iteration δ^* can be calculated by

$$
\begin{aligned}
&(\mathbf{J}_s^T \lambda \mathbf{W}^s \mathbf{J}_s + \mathbf{J}_d^T \mathbf{W}^d \mathbf{J}_d)\delta^* = -(\mathbf{J}_s^T \lambda \mathbf{W}^s \mathbf{r}^s + \mathbf{J}_d^T \mathbf{W}^d \mathbf{r}^d) \\
&\mathbf{J}_s \in \mathbb{R}^{3\times 49}, \mathbf{W}^s \in \mathbb{R}^{3\times 3}, \mathbf{J}_s \in \mathbb{R}^{7\times 49}, \mathbf{W}^s \in \mathbb{R}^{7\times 7},
\end{aligned} \tag{2.2}
$$

We construct residuals and its formulation.

1.2.2 JACOBIAN CITATION

We know for a Lie algebra $\boldsymbol{\rho} \in \mathbb{R}^3, \boldsymbol{\phi} \in \mathbb{R}^3, \boldsymbol{\xi} = \begin{pmatrix} \boldsymbol{\rho} \\ \boldsymbol{\phi} \end{pmatrix} \in \mathbb{R}^6$ and $\mathbf{p}_w$:

$$
\begin{aligned}
&\boldsymbol{\xi}^\wedge = \begin{pmatrix} \boldsymbol{\rho} \\ \boldsymbol{\phi} \end{pmatrix}^\wedge = \begin{pmatrix} \boldsymbol{\phi}^\wedge & \boldsymbol{\rho} \\ \mathbf{0}^T & \mathbf{0}^T \end{pmatrix} \in \mathbb{R}^{4\times 4} \\
&\boldsymbol{\epsilon} \in \mathbb{R}^3, \begin{pmatrix} \boldsymbol{\epsilon} \\ 1 \end{pmatrix}^\odot = \begin{pmatrix} \mathbf{E} & -\boldsymbol{\epsilon}^\wedge \\ \mathbf{0}^T & \mathbf{0}^T \end{pmatrix} \in \mathbb{R}^{4\times 6} \\
&\frac{\partial (exp(\xi^\wedge)\mathbf{p}_w)}{\partial \boldsymbol{\xi}} = \frac{\partial (\mathbf{T}\mathbf{p}_w)}{\partial \boldsymbol{\xi}} = (\mathbf{T}\mathbf{p}_w)^\odot \\
&\mathbf{T}\mathbf{p}_w = exp(\xi^\wedge)\mathbf{p}_w \approx (\mathbf{E} + \xi^\wedge)\mathbf{p}_w \\
&\frac{\partial (exp(\xi^\wedge)\mathbf{p}_w)}{\partial \boldsymbol{\xi}} \approx \frac{\partial (\mathbf{E} + \xi^\wedge)}{\partial \boldsymbol{\xi}} = \mathbf{0} + \frac{\partial (\xi^\wedge \mathbf{p}_w)}{\partial \boldsymbol{\xi}} \approx (\mathbf{T}\mathbf{p}_w)^\odot \\
&since, \frac{\partial (\mathbf{T}\mathbf{p}_w)}{\partial \boldsymbol{\xi}} = (\mathbf{T}^{-1}\mathbf{p}_w)^\odot = \frac{\partial (exp(-\xi^\wedge)\mathbf{p}_w)}{\partial \boldsymbol{\xi}} \\
&= \frac{\partial (\mathbf{E} - \xi^\wedge)}{\partial \boldsymbol{\xi}} = -(\mathbf{T}\mathbf{p}_w)^\odot
\end{aligned} \tag{2.2}
$$

1.2.3 JACOBIAN DERIVATION

1.2.3.1 Dynamic Parameter

Firstly, if $\mathbf{p}$ is neither observed by frame mL, mR nor hosted by nL, nR:

$$
\frac{\partial (r^d_{\mathbf{p}})_{ij}}{\partial \boldsymbol{\xi_m}} = \frac{\partial (r^d_{\mathbf{p}})_{ij}}{\partial \boldsymbol{\xi_n}} = \mathbf{0}^T, so \quad \frac{\partial (r^d_{\mathbf{p}_1})_{12}}{\partial \boldsymbol{\xi}_3} = \frac{\partial r^d_{(\mathbf{p}_1)_{12}}}{\partial \boldsymbol{\xi}_4} = \ldots = \mathbf{0}^T, \tag{2.2}
$$

otherwise, we follow

$$\frac{\partial (r^d_{\mathbf{p}})_{ij}}{\partial \boldsymbol{\xi_i}} = \frac{\partial (I^L_j(\mathbf{p}^{'}))}{\partial \boldsymbol{\xi_i}} = \frac{\partial (I^L_j(\mathbf{p}^{'}))}{\partial \mathbf{p}^{'}} \frac{\partial \mathbf{p}^{'}}{\partial \mathbf{p}^{'}_w} \frac{\partial \mathbf{p}^{'}_w}{\partial \boldsymbol{\xi_i}}$$
$$\mathbf{p}^{'}_w = \mathbf{T}_j \mathbf{T}_i^{-1} \mathbf{p}_w = \mathbf{T}_j \mathbf{T}_i^{-1} ((d^{iL}_{\mathbf{p}})^{-1} \mathbf{K}^{-1} \mathbf{p}) \tag{2.2}$$

For one frame iL, we have $\mathbf{p}$ and $\mathbf{K}$, then we can get

$$\left\{ \mathbf{p}_w = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} f_x^{-1} (d^{iL}_{\mathbf{p}})^{-1} (u^i - c_x) \\ f_y^{-1} (d^{iL}_{\mathbf{p}})^{-1} (v^i - c_y) \\ (d^{iL}_{\mathbf{p}})^{-1} \\ 1 \end{pmatrix} \right. \tag{2.2}$$

$$\frac{\partial \mathbf{p}^{'}_w}{\partial \boldsymbol{\xi_i}} = \mathbf{T}_j \frac{\partial (\mathbf{T}_i^{-1} \mathbf{p}^{'}_w)}{\partial \boldsymbol{\xi_i}} = -\mathbf{T}_j (\mathbf{T}_i \mathbf{p}_w)^{\odot}$$

$$\frac{\partial \mathbf{p}^{'}_w}{\partial \boldsymbol{\xi_j}} = \frac{\partial (\mathbf{T}_j \mathbf{T}_i^{-1} \mathbf{p}_w)}{\partial \boldsymbol{\xi_i}} = (\mathbf{T}_j \mathbf{T}_i^{-1} \mathbf{p}_w)^{\odot}$$

$$= \begin{pmatrix} x^{'} \\ y^{'} \\ z^{'} \\ 1 \end{pmatrix}^{\odot} = \begin{pmatrix} 1 & 0 & 0 & 0 & z^{'} & -y^{'} \\ 0 & 1 & 0 & -z^{'} & 0 & x^{'} \\ 0 & 1 & 0 & y^{'} & -x^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \frac{\partial (r^d_{\mathbf{p}})_{ij}}{\partial \boldsymbol{\xi_j}} = \frac{\partial (I^L_j(\mathbf{p}^{'}))}{\partial \mathbf{p}^{'}} \frac{\partial \mathbf{p}^{'}}{\partial \mathbf{p}^{'}_w} \frac{\partial \mathbf{p}^{'}_w}{\partial \boldsymbol{\xi_j}}$$
$$= (g^{'}_x, g^{'}_y, 0, 0) \begin{pmatrix} f_x (z^{'})^{-1} & 0 & -x^{'} f_x (z^{'})^{-2} & 0 \\ 0 & f_y (z^{'})^{-1} & -y^{'} f_y (z^{'})^{-2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (z^{'})^{-2} & 0 \end{pmatrix} \tag{2.2}$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 & z^{'} & -y^{'} \\ 0 & 1 & 0 & -z^{'} & 0 & x^{'} \\ 0 & 0 & 1 & y^{'} & -x^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} g^{'}_x f_x (z^{'})^{-1} \\ g^{'}_y f_y (z^{'})^{-1} \\ -(g^{'}_x x^{'} f_x + g^{'}_y y^{'} f_y)(z^{'})^{-2} \\ -g^{'}_y f_y - (g^{'}_x x^{'} y^{'} f_x + g^{'}_y (y^{'})^2 f_y)(z^{'})^{-2} \\ g^{'}_x f_x + (g^{'}_x (x^{'})^2 f_x + g^{'}_y x^{'} y^{'} f_y)(z^{'})^{-2} \\ -g^{'}_x f_x y^{'} (z^{'})^{-1} + g^{'}_y f_y x^{'} (z^{'})^{-1} \end{pmatrix}^T$$

Secondly, according to

$$(r^d_{\mathbf{p}})_{ij} := I^L_j(\mathbf{p}^{'}) - b^L_j - \frac{e^{a^L_j}}{e^{a^L_i}} (I^L_i(\mathbf{p}) - b^L_i) \tag{2.2}$$

We have:

$$\frac{\partial (r^d_{\mathbf{p}})_{ij}}{\partial a_i} = \frac{e^{a^L_j}}{e^{a^L_i}} (I^L_i(\mathbf{p}) - b^L_i), \frac{\partial (r^d_{\mathbf{p}})_{ij}}{\partial a_j} = -\frac{e^{a^L_j}}{e^{a^L_i}} (I^L_i(\mathbf{p}) - b^L_i)$$
$$\frac{\partial (r^d_{\mathbf{p}})_{ij}}{\partial b_i} = \frac{e^{a^L_j}}{e^{a^L_i}}, \qquad \frac{\partial (r^d_{\mathbf{p}})_{ij}}{\partial b_j} = -1 \tag{2.2}$$

add detail Calibration derivation……

$$
\begin{aligned}
&\mathbf{p}^{'} = d_{\mathbf{p}}^{jL}\mathbf{K}(\mathbf{T}_j\mathbf{T}_i^{-1}\mathbf{p}_w) \\
&assume: \mathbf{T}_j\mathbf{T}_i^{-1} = \begin{pmatrix} r_{11}^{ji} & r_{12}^{ji} & r_{13}^{ji} & t_1^{ji} \\ r_{21}^{ji} & r_{22}^{ji} & r_{23}^{ji} & t_2^{ji} \\ r_{31}^{ji} & r_{32}^{ji} & r_{33}^{ji} & t_3^{ji} \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&\mathbf{p}_w^{'} = \begin{pmatrix} r_{11}^{ji}f_x^{-1}(d_{\mathbf{p}}^{iL})^{-1}(u^i - c_x) + r_{12}^{ji}f_y^{-1}(d_{\mathbf{p}}^{iL})^{-1}(v^i - c_y) + r_{13}^{ji}(d_{\mathbf{p}}^{iL})^{-1} + t_1^{ji} \\ r_{21}^{ji}f_x^{-1}(d_{\mathbf{p}}^{iL})^{-1}(u^i - c_x) + r_{22}^{ji}f_y^{-1}(d_{\mathbf{p}}^{iL})^{-1}(v^i - c_y) + r_{23}^{ji}(d_{\mathbf{p}}^{iL})^{-1} + t_2^{ji} \\ r_{31}^{ji}f_x^{-1}(d_{\mathbf{p}}^{iL})^{-1}(u^i - c_x) + r_{32}^{ji}f_y^{-1}(d_{\mathbf{p}}^{iL})^{-1}(v^i - c_y) + r_{33}^{ji}(d_{\mathbf{p}}^{iL})^{-1} + t_3^{ji} \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{a}{d_{\mathbf{p}}^{iL}} + t_1^{ji} \\ \frac{b}{d_{\mathbf{p}}^{iL}} + t_2^{ji} \\ \frac{c}{d_{\mathbf{p}}^{iL}} + t_3^{ji} \\ 1 \end{pmatrix} = \begin{pmatrix} x^{'} \\ y^{'} \\ z^{'} \\ 1 \end{pmatrix} \Rightarrow d_{\mathbf{p}}^{jL} = (z^{'})^{-1}, \mathbf{p}^{'} = \begin{pmatrix} f_x x^{'} d_{\mathbf{p}}^{jL} + c_x \\ f_y y^{'} d_{\mathbf{p}}^{jL} + c_y \\ 1 \\ d_{\mathbf{p}}^{jL} \end{pmatrix} \\
&\frac{\partial(I_j^L(\mathbf{p}^{'}))}{\partial\mathbf{p}^{'}} = (g_x^{'}, g_y^{'}, 0, 0) \\
&\frac{\partial\mathbf{p}^{'}}{\partial\mathbf{p}_w^{'}} = \begin{pmatrix} f_x(z^{'})^{-1} & 0 & -x^{'}f_x(z^{'})^{-2} & 0 \\ 0 & f_y(z^{'})^{-1} & -y^{'}f_y(z^{'})^{-2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (z^{'})^{-2} & 0 \end{pmatrix}, \frac{\partial\mathbf{p}_w^{'}}{\partial d_{\mathbf{p}}^{iL}} = \begin{pmatrix} -\frac{a}{(d_{\mathbf{p}}^{iL})^2} \\ -\frac{b}{(d_{\mathbf{p}}^{iL})^2} \\ -\frac{c}{(d_{\mathbf{p}}^{iL})^2} \\ 0 \end{pmatrix} \\
&\Rightarrow \frac{\partial(r_{\mathbf{p}}^d)_{ij}}{\partial d_{\mathbf{p}}^{iL}} = \frac{\partial(I_j^L(\mathbf{p}^{'}))}{\partial d_{\mathbf{p}}^{iL}} = \frac{\partial(I_j^L(\mathbf{p}^{'}))}{\partial\mathbf{p}^{'}}\frac{\partial\mathbf{p}^{'}}{\partial\mathbf{p}_w^{'}}\frac{\partial\mathbf{p}_w^{'}}{\partial d_{\mathbf{p}}^{iL}} \\
&= -\frac{g_x^{'}f_x a}{z^{'}(d_{\mathbf{p}}^{iL})^2} - \frac{g_y^{'}f_y b}{z^{'}(d_{\mathbf{p}}^{iL})^2} + \frac{c(g_x^{'}x^{'}f_x + g_y^{'}y^{'}f_y)}{(z^{'}d_{\mathbf{p}}^{iL})^2} \\
&= \frac{c(g_x^{'}x^{'}f_x + g_y^{'}y^{'}f_y) - g_x^{'}f_x a z^{'} - g_y^{'}f_y b z^{'}}{(z^{'}d_{\mathbf{p}}^{iL})^2}
\end{aligned}
\tag{2.2}
$$

1.2.3.2 Static Parameter

Firstly, For a stereo frame i: inverse depth $d_{\mathbf{p}}^{iL} = d_{\mathbf{p}}^{iR}$, a left frame iL pixel $\mathbf{p}$ is projected to right frame iR with $\mathbf{p}^{'}$:

$$
\begin{aligned}
\mathbf{p} &= \begin{pmatrix} u^i \\ v^i \\ 1 \\ d_{\mathbf{p}}^{iL} \end{pmatrix}, \mathbf{p}_w = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}, d_{\mathbf{p}}^{iL} = z^{-1}, \mathbf{p}_w = (d_{\mathbf{p}}^{iL})^{-1}\mathbf{K}^{-1}\mathbf{p} \\
&= \begin{pmatrix} f_x^{-1}(d_{\mathbf{p}}^{iL})^{-1}(u^i - c_x) \\ f_y^{-1}(d_{\mathbf{p}}^{iL})^{-1}(v^i - c_y) \\ (d_{\mathbf{p}}^{iL})^{-1} \\ 1 \end{pmatrix}, \mathbf{T}_{RL} = \begin{pmatrix} 1 & 0 & 0 & t_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
\mathbf{p}^{'} &= d_{\mathbf{p}}^{iR}\mathbf{K}(\mathbf{T}_{RL}\mathbf{p}_w) \\
&= d_{\mathbf{p}}^{iL} \begin{pmatrix} f_x & 0 & c_x & 0 \\ 0 & f_y & c_y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_x^{-1}(d_{\mathbf{p}}^{iL})^{-1}(u^i - c_x) + t_1 \\ f_y^{-1}(d_{\mathbf{p}}^{iL})^{-1}(v^i - c_y) \\ (d_{\mathbf{p}}^{iL})^{-1} \\ 1 \end{pmatrix} = \begin{pmatrix} u^i + t_1 f_x d_{\mathbf{p}}^{iL} \\ v^i \\ 1 \\ d_{\mathbf{p}}^{iL} \end{pmatrix} \\
\frac{\partial r_{\mathbf{p}}^s}{\partial d_{\mathbf{p}}^{iL}} &= \frac{\partial(I_i^R(\mathbf{p}^{'})) - \frac{e^{a_i^R}}{e^{a_i^L}}(I_i^L(\mathbf{p}))}{\partial d_{\mathbf{p}}^{iL}} = (\frac{\partial(I_i^R(\mathbf{p}^{'}))}{\partial \mathbf{p}^{'}} - \frac{e^{a_i^R}}{e^{a_i^L}}\frac{\partial(I_i^L(\mathbf{p}))}{\partial \mathbf{p}^{'}})\frac{\partial \mathbf{p}^{'}}{\partial d_{\mathbf{p}}^{iL}} \\
&= [(g_x^{iR}, g_y^{iR}, 0, 0) - \mathbf{0}^T] \begin{pmatrix} t_1 f_x \\ 0 \\ 0 \\ 1 \end{pmatrix} = g_x^{iR} t_1 f_x
\end{aligned} \tag{1}
$$

Secondly, according to:

$$
r_{\mathbf{p}}^s := I_i^R(\mathbf{p}^{'}) - b_i^R - \frac{e^{a_i^R}}{e^{a_i^L}}(I_i^L(\mathbf{p}) - b_i^L) \tag{2.2}
$$

We have:

$$
\begin{aligned}
\frac{\partial(r_{\mathbf{p}}^s)_{ij}}{\partial a_i} &= \frac{e^{a_j^L}}{e^{a_i^L}}(I_i^L(\mathbf{p}) - b_i^L), \frac{\partial(r_{\mathbf{p}}^s)_{ij}}{\partial a_j} = -\frac{e^{a_j^L}}{e^{a_i^L}}(I_i^L(\mathbf{p}) - b_i^L) \\
\frac{\partial(r_{\mathbf{p}}^s)_{ij}}{\partial b_i} &= \frac{e^{a_j^L}}{e^{a_i^L}}, \qquad \frac{\partial(r_{\mathbf{p}}^s)_{ij}}{\partial b_j} = -1
\end{aligned} \tag{2.2}
$$

add detail Calibration derivation……