

$$\begin{aligned}
f_m(\boldsymbol{\delta}) &= I_i(\mathbf{p}_m) - I_j(w(\mathbf{p}_m, D_i(\mathbf{p}_m), \boldsymbol{\delta})), \mathbf{p}_m \in \boldsymbol{\Omega}_{D_i}, 1 \leq m \leq n \\
W_m(\boldsymbol{\delta}) &= \left(2\sigma_I^2 + \left(\frac{\partial_{r_{p_m}}(\mathbf{p}_m, \boldsymbol{\delta})}{\partial D_i(\mathbf{p}_m)} \right)^2 V_i(\mathbf{p}_m) \right)^{-1} \\
\mathbf{f} &= \begin{pmatrix} f_1(\boldsymbol{\delta}) \\ f_2(\boldsymbol{\delta}) \\ \vdots \\ f_n(\boldsymbol{\delta}) \end{pmatrix}, \mathbf{W} = \begin{pmatrix} \mathbf{W}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{W}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{W}_n \end{pmatrix} \\
E(\boldsymbol{\delta}) &= \mathbf{f}^T(\boldsymbol{\delta}) \mathbf{W}(\boldsymbol{\delta}) \mathbf{f}(\boldsymbol{\delta}) \\
\mathbf{f}(\boldsymbol{\delta} + \Delta\boldsymbol{\delta}) &\approx l(\Delta\boldsymbol{\delta}) = \mathbf{f}(\boldsymbol{\delta}) + \mathbf{J}(\boldsymbol{\delta}) \Delta\boldsymbol{\delta} \\
\frac{1}{2}E(\boldsymbol{\delta}) &\approx L(\Delta\boldsymbol{\delta}) = \frac{1}{2}l^T(\Delta\boldsymbol{\delta})l(\Delta\boldsymbol{\delta}) \\
&= \frac{1}{2}\mathbf{f}^T \mathbf{W} \mathbf{f} + \Delta\boldsymbol{\delta}^T \mathbf{J}^T \mathbf{W} \mathbf{f} + \frac{1}{2}\Delta\boldsymbol{\delta}^T \mathbf{J}^T \mathbf{W} \mathbf{J} \Delta\boldsymbol{\delta} \\
&= E + \Delta\boldsymbol{\delta}^T \mathbf{J}^T \mathbf{W} \mathbf{f} + \frac{1}{2}\Delta\boldsymbol{\delta}^T \mathbf{J}^T \mathbf{W} \mathbf{J} \Delta\boldsymbol{\delta} \\
L' &= \mathbf{J}^T \mathbf{W} \mathbf{f} + \mathbf{J}^T \mathbf{W} \mathbf{J} \Delta\boldsymbol{\delta} = 0 \\
\Delta\boldsymbol{\delta} &= -(\mathbf{J}^T \mathbf{W} \mathbf{J})^{-1} \mathbf{J}^T \mathbf{W} \mathbf{f}
\end{aligned} \tag{1}$$

$$\left\{ \begin{aligned} \mathbf{p} &:= \begin{pmatrix} p_x \\ p_y \\ d_p \end{pmatrix}, \mathbf{P} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} p_x/d_p \\ p_y/d_p \\ 1/d_p \end{pmatrix} \\ w(\mathbf{p}, D_i(\mathbf{p}), \boldsymbol{\delta}_{ji}) &:= \begin{pmatrix} x'/z' \\ y'/z' \end{pmatrix}, \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \exp(\boldsymbol{\delta}) \begin{pmatrix} p_x/d_p \\ p_y/d_p \\ 1/d_p \\ 1 \end{pmatrix} \end{aligned} \right. \tag{2.10}$$

$$\begin{aligned}
\mathbf{q} &= \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}, w_m = (\mathbf{p}_m, D_i(\mathbf{p}_m), \boldsymbol{\delta}) = \begin{pmatrix} x'/z' \\ y'/z' \end{pmatrix} \\
\frac{\partial f_m}{\partial \boldsymbol{\delta}} &= -\frac{\partial I_j(w_m)}{\partial \boldsymbol{\delta}} = -\frac{\partial I_j}{\partial w_m} \frac{\partial w_m}{\partial \mathbf{q}} \frac{\partial \mathbf{q}}{\partial \boldsymbol{\delta}} \\
&= -\begin{pmatrix} g_x & g_y \end{pmatrix} \begin{pmatrix} \frac{1}{z'} & 0 & -\frac{x'}{z'^2} \\ 0 & \frac{1}{z'} & -\frac{y'}{z'^2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & -z' & y' \\ 0 & 1 & 0 & z' & 0 & -x' \\ 0 & 0 & 1 & -y' & x' & 0 \end{pmatrix} \\
&= -\begin{pmatrix} \frac{g_x}{z'} & \frac{g_y}{z'} & -\frac{x'g_x}{z'^2} - \frac{y'g_y}{z'^2} & -\frac{x'y'g_x}{z'^2} - (1 + \frac{y'^2}{z'^2})g_y & (1 + \frac{x'^2}{z'^2})g_x + \frac{x'y'g_y}{z'^2} & -\frac{y'g_x}{z'} + \frac{x'g_y}{z'} \end{pmatrix} \\
\mathbf{J} &= \begin{pmatrix} \frac{\partial f_1}{\partial \boldsymbol{\delta}} & \frac{\partial f_2}{\partial \boldsymbol{\delta}} & \dots & \frac{\partial f_N}{\partial \boldsymbol{\delta}} \end{pmatrix}^T = \begin{pmatrix} \mathbf{J}_1 & \mathbf{J}_2 & \dots & \mathbf{J}_N \end{pmatrix}^T
\end{aligned} \tag{2}$$

$$\begin{aligned}
\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} &= \exp(\boldsymbol{\delta}) \begin{pmatrix} \mathbf{p}_x/d_p \\ \mathbf{p}_y/d_p \\ 1/d_p \\ 1 \end{pmatrix} = \frac{1}{d_p} \begin{pmatrix} \mathbf{R}_1 & t_1 \\ \mathbf{R}_2 & t_2 \\ \mathbf{R}_3 & t_3 \\ \mathbf{0} & 1 \end{pmatrix}_{4 \times 4} \begin{pmatrix} \mathbf{P} \\ d_p \end{pmatrix}_{4 \times 1} = \frac{1}{d_p} \begin{pmatrix} \mathbf{R}_1 \mathbf{P} + t_1 d_p \\ \mathbf{R}_2 \mathbf{P} + t_2 d_p \\ \mathbf{R}_3 \mathbf{P} + t_3 d_p \\ d_p \end{pmatrix} \\
w_m &= \begin{pmatrix} x'/z' \\ y'/z' \end{pmatrix} = \begin{pmatrix} \frac{\mathbf{R}_1 \mathbf{P} + t_1 d_p}{\mathbf{R}_3 \mathbf{P} + t_3 d_p} \\ \frac{\mathbf{R}_2 \mathbf{P} + t_2 d_p}{\mathbf{R}_3 \mathbf{P} + t_3 d_p} \end{pmatrix} \\
\frac{\partial_{r_{p_m}}(\mathbf{p}_m, \boldsymbol{\delta})}{\partial D_i(\mathbf{p}_m)} &:= \frac{\partial f_m}{\partial d_{p_m}} = -\frac{\partial I_j}{\partial w_m} \frac{\partial w_m}{\partial d_{p_m}} \\
&= -\begin{pmatrix} g_x & g_y \end{pmatrix} \begin{pmatrix} \frac{t_1 z' - x' t_3}{z'^2} \\ \frac{t_2 z' - y' t_3}{z'^2} \end{pmatrix} = -(g_x \frac{t_1 z' - x' t_3}{z'^2} + g_y \frac{t_2 z' - y' t_3}{z'^2}) \\
W_m &:= (2\sigma_I^2 + w_m^2 V_m)^{-1}
\end{aligned} \tag{3}$$

$$\begin{aligned}
\mathbf{J}^T \mathbf{W} \mathbf{J} &= \begin{pmatrix} \mathbf{J}_1^T & \mathbf{J}_2^T & \cdots & \mathbf{J}_N^T \end{pmatrix}_{6 \times N} \begin{pmatrix} \mathbf{W}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{W}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{W}_N \end{pmatrix}_{N \times N} \begin{pmatrix} \mathbf{J}_1 \\ \mathbf{J}_2 \\ \vdots \\ \mathbf{J}_N \end{pmatrix}_{N \times 6} \\
&= \sum_{i=1}^N (\mathbf{J}_i^T \mathbf{W}_i \mathbf{J}_i)_{6 \times 6} = \mathbf{A}_{6 \times 6} \\
\mathbf{J}^T \mathbf{W} &= \sum_{i=1}^N (\mathbf{J}_i^T \mathbf{W}_i)_{6 \times 1} = \mathbf{b}_{6 \times 1} \\
\mathbf{A} \boldsymbol{\delta}^* &= \mathbf{b} \\
\Rightarrow \mathbf{L} \mathbf{D} \mathbf{L}^T \boldsymbol{\delta}^* &= \mathbf{b} \\
\Rightarrow (\mathbf{L} \mathbf{D}^{1/2}) (\mathbf{L} \mathbf{D}^{1/2})^T \boldsymbol{\delta}^* &= \mathbf{b} \\
\Rightarrow \mathbf{G} \mathbf{G}^T \boldsymbol{\delta}^* &= \mathbf{b} \\
\Rightarrow \mathbf{G} \boldsymbol{\delta}^{*'} &= \mathbf{b} \\
\Rightarrow \mathbf{G}^T \boldsymbol{\delta}^* &= \boldsymbol{\delta}^{*'}
\end{aligned} \tag{4}$$

$$cost = \sum_{i=j=1}^5 (A(i, j) - B(i, j))^2 \tag{4.1}$$

$$\begin{aligned}
\vec{s}_1 &= \vec{n}_1 \times \vec{n}_2 = \mathbf{t} \times \vec{n}_1 \times \vec{Op} \\
&= \begin{pmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} -at_3 + t_1 \\ -bt_3 + t_2 \\ 0 \end{pmatrix} = \begin{pmatrix} ep_x \\ ep_y \\ 0 \end{pmatrix}
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
& d_2 \mathbf{p}_{best} = \mathbf{R}(d_1 \mathbf{p}_c) + \mathbf{t} \\
\Rightarrow & 0 = [\mathbf{p}_{best}]_{\times} (d_1 \mathbf{R} \mathbf{p}_c + \mathbf{t}) \\
\Rightarrow & d_1 [\mathbf{p}_{best}]_{\times} \mathbf{R} \mathbf{p}_c = -[\mathbf{p}_{best}]_{\times} \mathbf{t} \\
\Rightarrow & d_1 \begin{pmatrix} 0 & -1 & b_1 \\ 1 & 0 & -a_1 \\ -b_1 & a_1 & 0 \end{pmatrix} \begin{pmatrix} R_{row0} \\ R_{row1} \\ R_{row2} \end{pmatrix} \mathbf{p}_c = \begin{pmatrix} 0 & 1 & -b_1 \\ -1 & 0 & a_1 \\ b_1 & -a_1 & 0 \end{pmatrix} \begin{pmatrix} t_0 \\ t_1 \\ t_2 \end{pmatrix} \\
\Rightarrow & d_1 \begin{pmatrix} b_1 R_{row2} - R_{row1} \\ R_{row0} - a_1 R_{row2} \\ a_1 R_{row1} - b_1 R_{row0} \end{pmatrix} \mathbf{p}_c = \begin{pmatrix} t_1 - b_1 t_2 \\ a_1 t_2 - t_0 \\ b_1 t_0 - a_1 t_1 \end{pmatrix} \\
\Rightarrow & d_1 = \frac{t_1 - b_1 t_2}{(b_1 R_{row2} - R_{row1}) \mathbf{p}_c} \\
\Rightarrow & = \frac{a_1 t_2 - t_0}{(R_{row0} - a_1 R_{row2}) \mathbf{p}_c} \\
\Rightarrow & = \frac{b_1 t_0 - a_1 t_1}{(a_1 R_{row1} - b_1 R_{row0}) \mathbf{p}_c}
\end{aligned} \tag{4.4}$$

$$L := \left\{ \mathbf{x} = \mathbf{p}_{close} + \lambda \begin{pmatrix} l_x \\ l_y \end{pmatrix} \mid \lambda = k|\mathbf{S}|, k \in \mathbf{N}^+ \right\} \tag{4.4}$$

$$\mathcal{N}\left(\frac{\sigma_p^2 d_o + \sigma_o^2 d_p}{\sigma_p^2 + \sigma_o^2}, \frac{\sigma_p^2 \sigma_o^2}{\sigma_p^2 + \sigma_o^2}\right) \tag{2.14}$$

$$\begin{aligned}
d_{new} &= (d_p^{-1} - t_z)^{-1} \\
\sigma_{new}^2 &= \left(\frac{d_{new}}{d_p} \right)^4 \sigma_p^2
\end{aligned} \tag{2.14}$$

$$f := \begin{cases} X_{max} = MAX\{x_1, x_2, \dots, x_m\} \\ X_{min} = MIN\{x_1, x_2, \dots, x_m\} \\ H_{sum} = \frac{X_{max} - X_{min}}{H} \\ Z_{max} = MAX\{z_1, z_2, \dots, z_m\} \\ Z_{min} = MIN\{z_1, z_2, \dots, z_m\} \\ V_{sum} = \frac{Z_{max} - Z_{min}}{V} \\ \mathbf{p}_i = [x_i, z_i]^T \in \mathbf{P}, \mathbf{q}_i = [h_i, v_i]^T \in \mathbf{Q} \\ h_i = \left\lfloor \frac{x_i}{H} \right\rfloor, v_i = \left\lfloor \frac{z_i}{V} \right\rfloor \end{cases} \tag{3.2}$$

Algorithm 1 : Time optimized A* Algorithm

Require: *Start*, *End*, *Q*

Ensure: Father node of *End*

```

1: function ASTARSEARCH(Start, End, Q)
2:   open = binary heap containing Start node
3:   closed = empty set
4:   movecost(x, y) = distance from node x to node y
5:   while End node not in open do
6:     i = node with min f(i) in open
7:     remove i from open
8:     add i to closed
9:     count = 0
10:    for j = neighbor node of i and not in closed and reachable in Q do
11:      count++
12:      cost = g(i) + movecost(i, j)
13:      if j in open and cost < g(j) then
14:        remove j from open
15:      end if
16:      if j not in open and not in closed then
17:        add j into open
18:        f(j) = g(j) + h(j)
19:        set father node of j is i
20:      end if
21:    end for
22:    if count == 0 then
23:      can't find path
24:      break out
25:    end if
26:  end while
27: end function

```

$$\ddot{X}_{t-1} \sim \mathcal{N}(0, k_1^2), \ddot{Z}_{t-1} \sim \mathcal{N}(0, k_2^2), \ddot{\theta}_{t-1} \sim \mathcal{N}(0, k_3^2),$$

$$\mathbf{R}_t = \begin{pmatrix} \frac{1}{4}T^4k_1^2 & \frac{1}{2}T^3k_1^2 & 0 & 0 & 0 & 0 \\ \frac{1}{2}T^3k_1^2 & T^2k_1^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4}T^4k_2^2 & \frac{1}{2}T^3k_2^2 & 0 & 0 \\ 0 & 0 & \frac{1}{2}T^3k_2^2 & T^2k_2^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4}T^4k_3^2 & \frac{1}{2}T^3k_3^2 \\ 0 & 0 & 0 & 0 & \frac{1}{2}T^3k_3^2 & T^2k_3^2 \end{pmatrix} \quad (3.4)$$

$$\begin{pmatrix} cov(\varepsilon_1, \varepsilon_1) & cov(\varepsilon_1, \varepsilon_2) & \cdots & cov(\varepsilon_1, \varepsilon_6) \\ cov(\varepsilon_2, \varepsilon_1) & cov(\varepsilon_2, \varepsilon_2) & \cdots & cov(\varepsilon_2, \varepsilon_6) \\ \vdots & \ddots & \vdots & \\ cov(\varepsilon_6, \varepsilon_1) & cov(\varepsilon_6, \varepsilon_2) & \cdots & cov(\varepsilon_6, \varepsilon_6) \end{pmatrix} \quad (3.4)$$

Algorithm 2 : Linear Kalman Filter

Require: $\mu_{t-1}, \Sigma_{t-1}, z_t$
Ensure: μ_t, Σ_t

```

1: function FILTER( $\mu_{t-1}, \Sigma_{t-1}, z_t$ )
2:   predict  $\bar{\mu}_t = A\mu_{t-1}$ 
3:    $\bar{\Sigma}_t = A\Sigma_{t-1}A^T + R_t$ 
4:   update  $K_t = \bar{\Sigma}_t(\bar{\Sigma}_t + Q_t)^{-1}$ 
5:    $\mu_t = \bar{\mu}_t + K_t(z_t - \bar{\mu}_t)$ 
6:    $\Sigma_t = (E - K_t)\bar{\Sigma}_t$ 
7: end function

```

$$\mathcal{F} = \{I_1, I_2, I_3, I_4\} \xi \Pi \frac{\mathbf{P}}{\partial \mathbf{p}_w}, \begin{pmatrix} f_x z^{-1} & 0 & -x f_x z^{-2} & 0 \\ 0 & f_y z^{-1} & -y f_y z^{-2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & z^{-2} & 0 \end{pmatrix} \quad (3.4)$$

$$\left(\sum_{j \in \text{obs}^t(\mathbf{P})} E_{ij}^{\mathbf{P}} + \lambda E_{is}^{\mathbf{P}} \right)$$

$$\begin{aligned} E(\delta) &= E_{1L2L}^{\mathbf{P}_1} + E_{2L1L}^{\mathbf{P}_2} + E_{2L3L}^{\mathbf{P}_2} + E_{2L3L}^{\mathbf{P}_3} + E_{3L1L}^{\mathbf{P}_4} + E_{3L2L}^{\mathbf{P}_4} + E_{4L3L}^{\mathbf{P}_5} \\ &\quad + E_{1L1R}^{\mathbf{P}_1} + E_{2L2R}^{\mathbf{P}_2} + E_{4L4R}^{\mathbf{P}_5} \\ &= E_d(\delta) + E_s(\delta) \end{aligned}$$

$$E_s(\delta) = \begin{pmatrix} r_{\mathbf{p}_1}^s \\ r_{\mathbf{p}_2}^s \\ r_{\mathbf{p}_5}^s \end{pmatrix}^T \begin{pmatrix} \lambda w_{\mathbf{p}_1} & 0 & 0 \\ 0 & \lambda w_{\mathbf{p}_2} & 0 \\ 0 & 0 & \lambda w_{\mathbf{p}_5} \end{pmatrix} \begin{pmatrix} r_{\mathbf{p}_1}^s \\ r_{\mathbf{p}_2}^s \\ r_{\mathbf{p}_5}^s \end{pmatrix} = (\mathbf{r}^s)^T \mathbf{W} \mathbf{r}^s$$

$$J_s = \begin{pmatrix} \frac{\partial r_{\mathbf{p}_1}^s}{\partial \xi_1} \dots \frac{\partial r_{\mathbf{p}_1}^s}{\partial \xi_4} & \frac{\partial r_{\mathbf{p}_1}^s}{\partial d_{\mathbf{p}_1}} \dots \frac{\partial r_{\mathbf{p}_1}^s}{\partial d_{\mathbf{p}_5}} & \frac{\partial r_{\mathbf{p}_1}^s}{\partial a_1^L} \dots \frac{\partial r_{\mathbf{p}_1}^s}{\partial b_4^R} & \frac{\partial r_{\mathbf{p}_1}^s}{\partial f_x} \dots \frac{\partial r_{\mathbf{p}_1}^s}{\partial c_y} \\ \frac{\partial r_{\mathbf{p}_2}^s}{\partial \xi_1} \dots \frac{\partial r_{\mathbf{p}_2}^s}{\partial \xi_4} & \frac{\partial r_{\mathbf{p}_2}^s}{\partial d_{\mathbf{p}_1}} \dots \frac{\partial r_{\mathbf{p}_2}^s}{\partial d_{\mathbf{p}_5}} & \frac{\partial r_{\mathbf{p}_2}^s}{\partial a_1^L} \dots \frac{\partial r_{\mathbf{p}_2}^s}{\partial b_4^R} & \frac{\partial r_{\mathbf{p}_2}^s}{\partial f_x} \dots \frac{\partial r_{\mathbf{p}_2}^s}{\partial c_y} \\ \frac{\partial r_{\mathbf{p}_5}^s}{\partial \xi_1} \dots \frac{\partial r_{\mathbf{p}_5}^s}{\partial \xi_4} & \frac{\partial r_{\mathbf{p}_5}^s}{\partial d_{\mathbf{p}_1}} \dots \frac{\partial r_{\mathbf{p}_5}^s}{\partial d_{\mathbf{p}_5}} & \frac{\partial r_{\mathbf{p}_5}^s}{\partial a_1^L} \dots \frac{\partial r_{\mathbf{p}_5}^s}{\partial b_4^R} & \frac{\partial r_{\mathbf{p}_5}^s}{\partial f_x} \dots \frac{\partial r_{\mathbf{p}_5}^s}{\partial c_y} \end{pmatrix} \quad (2.2)$$

$$E_d(\delta) = \begin{pmatrix} (r_{\mathbf{p}_1}^d)_{12} \\ (r_{\mathbf{p}_1}^d)_{21} \\ \vdots \\ (r_{\mathbf{p}_5}^d)_{43} \end{pmatrix}^T \begin{pmatrix} w_{\mathbf{p}_1} & 0 & \dots & 0 \\ 0 & w_{\mathbf{p}_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & w_{\mathbf{p}_5} \end{pmatrix} \begin{pmatrix} (r_{\mathbf{p}_1}^d)_{12} \\ (r_{\mathbf{p}_1}^d)_{21} \\ \vdots \\ (r_{\mathbf{p}_5}^d)_{43} \end{pmatrix} = (\mathbf{r}^d)^T \mathbf{W} \mathbf{r}^d$$

$$J_d = \begin{pmatrix} \frac{\partial(r_{\mathbf{p}_1}^d)_{12}}{\partial \xi_1} \dots & \frac{\partial(r_{\mathbf{p}_1}^d)_{12}}{\partial \xi_4} & \frac{\partial(r_{\mathbf{p}_1}^d)_{12}}{\partial d_{\mathbf{p}_1}} \dots & \frac{\partial(r_{\mathbf{p}_1}^d)_{12}}{\partial d_{\mathbf{p}_5}} & \frac{\partial(r_{\mathbf{p}_1}^d)_{12}}{\partial a_1^L} \dots & \frac{\partial(r_{\mathbf{p}_1}^d)_{12}}{\partial b_4^R} & \frac{\partial(r_{\mathbf{p}_1}^d)_{12}}{\partial f_x} \dots & \frac{\partial(r_{\mathbf{p}_1}^d)_{12}}{\partial c_y} \\ \frac{\partial(r_{\mathbf{p}_1}^d)_{21}}{\partial \xi_1} \dots & \frac{\partial(r_{\mathbf{p}_1}^d)_{21}}{\partial \xi_4} & \frac{\partial(r_{\mathbf{p}_1}^d)_{21}}{\partial d_{\mathbf{p}_1}} \dots & \frac{\partial(r_{\mathbf{p}_1}^d)_{21}}{\partial d_{\mathbf{p}_5}} & \frac{\partial(r_{\mathbf{p}_1}^d)_{21}}{\partial a_1^L} \dots & \frac{\partial(r_{\mathbf{p}_1}^d)_{21}}{\partial b_4^R} & \frac{\partial(r_{\mathbf{p}_1}^d)_{21}}{\partial f_x} \dots & \frac{\partial(r_{\mathbf{p}_1}^d)_{21}}{\partial c_y} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial(r_{\mathbf{p}_5}^d)_{43}}{\partial \xi_1} \dots & \frac{\partial(r_{\mathbf{p}_5}^d)_{43}}{\partial \xi_4} & \frac{\partial(r_{\mathbf{p}_5}^d)_{43}}{\partial d_{\mathbf{p}_1}} \dots & \frac{\partial(r_{\mathbf{p}_5}^d)_{43}}{\partial d_{\mathbf{p}_5}} & \frac{\partial(r_{\mathbf{p}_5}^d)_{43}}{\partial a_1^L} \dots & \frac{\partial(r_{\mathbf{p}_5}^d)_{43}}{\partial b_4^R} & \frac{\partial(r_{\mathbf{p}_5}^d)_{43}}{\partial f_x} \dots & \frac{\partial(r_{\mathbf{p}_5}^d)_{43}}{\partial c_y} \end{pmatrix} \quad (2.2)$$

$$(\mathbf{J}_s^T \lambda \mathbf{W}^s \mathbf{J}_s + \mathbf{J}_d^T \mathbf{W}^d \mathbf{J}_d) \delta^* = -(\mathbf{J}_s^T \lambda \mathbf{W}^s \mathbf{r}_s + \mathbf{J}_d^T \mathbf{W}^d \mathbf{r}_d) \quad (2.2)$$

$$\mathbf{J}_s \in \mathbb{R}^{49 \times 3}, \mathbf{W}^s \in \mathbb{R}^{3 \times 3}, \mathbf{J}_s \in \mathbb{R}^{49 \times 7}, \mathbf{W}^s \in \mathbb{R}^{7 \times 7},$$

$$\mathbf{p}' = d_{\mathbf{p}}^{iR} \mathbf{K}(\mathbf{T}_{RL}((d_{\mathbf{p}}^{iL})^{-1} \mathbf{K}^{-1} \mathbf{p}))$$

$$\frac{\partial \mathbf{p}'}{\partial d_{\mathbf{p}}^{iL}} = 0? \quad (2.2)$$

$$\frac{\partial(r_{\mathbf{p}}^d)}{\partial \xi_i} = \frac{\partial I_j^L(\mathbf{p}')}{\partial \xi_i} = \frac{\partial I_j^L(\mathbf{p}')}{\partial \mathbf{p}'} \frac{\partial \mathbf{p}'}{\partial \mathbf{p}_w'} \frac{\partial \mathbf{p}_w'}{\partial \xi_i} \quad (2.2)$$

$$\mathbf{p}_w' = d_{\mathbf{p}}^{jL} \mathbf{K}(\mathbf{T}_j \mathbf{T}_i^{-1}((d_{\mathbf{p}}^{iL})^{-1} \mathbf{K}^{-1} \mathbf{p}))$$

$$\begin{cases} \mathbf{p}_w = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} f_x^{-1}(d_{\mathbf{p}}^{iL})^{-1}(u^i - c_x) \\ f_y^{-1}(d_{\mathbf{p}}^{iL})^{-1}(v^i - c_y) \\ d_{\mathbf{p}}^{iL} \\ 1 \end{pmatrix} \\ \mathbf{p}_w' = d_{\mathbf{p}}^{jL} \mathbf{K}(\mathbf{T}_j \mathbf{T}_i^{-1}((d_{\mathbf{p}}^{iL})^{-1} \mathbf{K}^{-1} \mathbf{p})) \end{cases} \quad (2.2)$$

$$\xi^\wedge = \begin{pmatrix} \rho \\ \phi \end{pmatrix}^\wedge = \begin{pmatrix} \phi^\wedge & \rho \\ \mathbf{0}^T & \mathbf{0}^T \end{pmatrix} \in \mathbb{R}^{4 \times 4}$$

$$\epsilon \in \mathbb{R}^3, \begin{pmatrix} \epsilon \\ 1 \end{pmatrix}^\odot = \begin{pmatrix} \mathbf{E} & -\epsilon^\wedge \\ \mathbf{0}^T & \mathbf{0}^T \end{pmatrix} \in \mathbb{R}^{4 \times 6}$$

$$\frac{\partial(\exp(\xi^\wedge) \mathbf{p}_w)}{\partial \xi} = \frac{\partial(\mathbf{T} \mathbf{p}_w)}{\partial \xi} = (\mathbf{T} \mathbf{p}_w)^\odot$$

$$\mathbf{T} \mathbf{p}_w = \exp(\xi^\wedge) \mathbf{p}_w \approx (\mathbf{E} + \xi^\wedge) \mathbf{p}_w \quad (2.2)$$

$$\frac{\partial(\exp(\xi^\wedge) \mathbf{p}_w)}{\partial \xi} \approx \frac{\partial(\mathbf{E} + \xi^\wedge)}{\partial \xi} = \mathbf{0} + \frac{\partial(\xi^\wedge \mathbf{p}_w)}{\partial \xi} \approx (\mathbf{T} \mathbf{p}_w)^\odot$$

$$\text{since, } \frac{\partial(\mathbf{T} \mathbf{p}_w)}{\partial \xi} = (\mathbf{T}^{-1} \mathbf{p}_w)^\odot = \frac{\partial(\exp(-\xi^\wedge) \mathbf{p}_w)}{\partial \xi}$$

$$= \frac{\partial(\mathbf{E} - \xi^\wedge)}{\partial \xi} = -(\mathbf{T} \mathbf{p}_w)^\odot$$

$$\begin{aligned}
\mathbf{p}' &= d_{\mathbf{p}}^{jL} \mathbf{K}(\mathbf{T}_j \mathbf{T}_i^{-1} \mathbf{p}_w) \\
\text{assume : } \mathbf{T}_j \mathbf{T}_i^{-1} &= \begin{pmatrix} r_{11}^{ji} & r_{12}^{ji} & r_{13}^{ji} & t_1^{ji} \\ r_{21}^{ji} & r_{22}^{ji} & r_{23}^{ji} & t_2^{ji} \\ r_{31}^{ji} & r_{32}^{ji} & r_{33}^{ji} & t_3^{ji} \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
\mathbf{p}'_w &= \begin{pmatrix} r_{11}^{ji} f_x^{-1} (d_{\mathbf{p}}^{iL})^{-1} (u^i - c_x) + r_{12}^{ji} f_y^{-1} (d_{\mathbf{p}}^{iL})^{-1} (v^i - c_y) + r_{13}^{ji} (d_{\mathbf{p}}^{iL})^{-1} + t_1^{ji} \\ r_{21}^{ji} f_x^{-1} (d_{\mathbf{p}}^{iL})^{-1} (u^i - c_x) + r_{22}^{ji} f_y^{-1} (d_{\mathbf{p}}^{iL})^{-1} (v^i - c_y) + r_{23}^{ji} (d_{\mathbf{p}}^{iL})^{-1} + t_2^{ji} \\ r_{31}^{ji} f_x^{-1} (d_{\mathbf{p}}^{iL})^{-1} (u^i - c_x) + r_{32}^{ji} f_y^{-1} (d_{\mathbf{p}}^{iL})^{-1} (v^i - c_y) + r_{33}^{ji} (d_{\mathbf{p}}^{iL})^{-1} + t_3^{ji} \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{a}{d_{\mathbf{p}}^{iL}} + t_1^{ji} \\ \frac{b}{d_{\mathbf{p}}^{iL}} + t_2^{ji} \\ \frac{c}{d_{\mathbf{p}}^{iL}} + t_3^{ji} \\ 1 \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} \Rightarrow d_{\mathbf{p}}^{jL} = (z')^{-1}, \mathbf{p}' = \begin{pmatrix} f_x x' d_{\mathbf{p}}^{jL} + c_x \\ f_y y' d_{\mathbf{p}}^{jL} + c_y \\ 1 \\ d_{\mathbf{p}}^{jL} \end{pmatrix} \\
\frac{\partial(I_j^L(\mathbf{p}'))}{\partial \mathbf{p}'} &= (g'_x, g'_y, 0, 0) \\
\frac{\partial \mathbf{p}'}{\partial \mathbf{p}'_w} &= \begin{pmatrix} f_x (z')^{-1} & 0 & -x' f_x (z')^{-2} & 0 \\ 0 & f_y (z')^{-1} & -y' f_y (z')^{-2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (z')^{-2} & 0 \end{pmatrix}, \frac{\partial \mathbf{p}'_w}{\partial d_{\mathbf{p}}^{iL}} = \begin{pmatrix} -\frac{a}{(d_{\mathbf{p}}^{iL})^2} \\ -\frac{b}{(d_{\mathbf{p}}^{iL})^2} \\ -\frac{c}{(d_{\mathbf{p}}^{iL})^2} \\ 0 \end{pmatrix} \\
\Rightarrow \frac{\partial(r_{\mathbf{p}}^d)_{ij}}{\partial d_{\mathbf{p}}^{iL}} &= \frac{\partial(I_j^L(\mathbf{p}'))}{\partial d_{\mathbf{p}}^{iL}} = \frac{\partial(I_j^L(\mathbf{p}'))}{\partial \mathbf{p}'} \frac{\partial \mathbf{p}'}{\partial \mathbf{p}'_w} \frac{\partial \mathbf{p}'_w}{\partial d_{\mathbf{p}}^{iL}} \\
&= -\frac{g'_x f_x a}{z' (d_{\mathbf{p}}^{iL})^2} - \frac{g'_y f_y b}{z' (d_{\mathbf{p}}^{iL})^2} + \frac{c(g'_x x' f_x + g'_y y' f_y)}{(z' d_{\mathbf{p}}^{iL})^2} \\
&= \frac{c(g'_x x' f_x + g'_y y' f_y) - g'_x f_x a z' - g'_y f_y b z'}{(z' d_{\mathbf{p}}^{iL})^2}
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} &= exp(\boldsymbol{\delta}) \begin{pmatrix} \mathbf{p}_x/d_p \\ \mathbf{p}_y/d_p \\ 1/d_p \\ 1 \end{pmatrix} = \frac{1}{d_p} \begin{pmatrix} \mathbf{R}_1 & t_1 \\ \mathbf{R}_2 & t_2 \\ \mathbf{R}_3 & t_3 \\ \mathbf{0} & 1 \end{pmatrix}_{4 \times 4} \begin{pmatrix} \mathbf{P} \\ d_p \end{pmatrix}_{4 \times 1} = \frac{1}{d_p} \begin{pmatrix} \mathbf{R}_1 \mathbf{P} + t_1 d_p \\ \mathbf{R}_2 \mathbf{P} + t_2 d_p \\ \mathbf{R}_3 \mathbf{P} + t_3 d_p \\ d_p \end{pmatrix} \\
w_m &= \begin{pmatrix} x'/z' \\ y'/z' \end{pmatrix} = \begin{pmatrix} \frac{\mathbf{R}_1 \mathbf{P} + t_1 d_p}{\mathbf{R}_3 \mathbf{P} + t_3 d_p} \\ \frac{\mathbf{R}_2 \mathbf{P} + t_2 d_p}{\mathbf{R}_3 \mathbf{P} + t_3 d_p} \end{pmatrix} \\
\frac{\partial_{r_{p_m}}(\mathbf{p}_m, \boldsymbol{\delta})}{\partial D_i(\mathbf{p}_m)} &:= \frac{\partial f_m}{\partial d_{p_m}} = - \frac{\partial I_j}{\partial w_m} \frac{\partial w_m}{\partial d_{p_m}} \\
&= - \begin{pmatrix} g_x & g_y \end{pmatrix} \begin{pmatrix} \frac{t_1 z' - x' t_3}{z'^2} \\ \frac{t_2 z' - y' t_3}{z'^2} \end{pmatrix} = - (g_x \frac{t_1 z' - x' t_3}{z'^2} + g_y \frac{t_2 z' - y' t_3}{z'^2})
\end{aligned} \tag{3}$$

$$assume : \mathbf{T}_{RL} = \begin{pmatrix} 1 & 0 & 0 & t_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
\mathbf{p}' &= d_{\mathbf{p}}^{iR} \mathbf{K}(\mathbf{T}_{RL} \mathbf{p}_w) \\
&= d_{\mathbf{p}}^{iL} \begin{pmatrix} f_x & 0 & c_x & 0 \\ 0 & f_y & c_y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_x^{-1}(d_{\mathbf{p}}^{iL})^{-1}(u^i - c_x) + t_1 \\ f_y^{-1}(d_{\mathbf{p}}^{iL})^{-1}(v^i - c_y) \\ (d_{\mathbf{p}}^{iL})^{-1} \\ 1 \end{pmatrix} = \begin{pmatrix} u^i + t_1 f_x d_{\mathbf{p}}^{iL} \\ v^i \\ 1 \\ d_{\mathbf{p}}^{iL} \end{pmatrix}
\end{aligned}$$

$$\frac{\partial r_{\mathbf{p}}^s}{\partial d_{\mathbf{p}}^{iL}} = \frac{\partial(I_j^L(\mathbf{p}'))}{\partial d_{\mathbf{p}}^{iL}} = \frac{\partial(I_j^L(\mathbf{p}'))}{\partial \mathbf{p}'} \frac{\partial \mathbf{p}'}{\partial d_{\mathbf{p}}^{iL}} = (g_x, g_y, 0, 0) \begin{pmatrix} t_1 f_x \\ 0 \\ 0 \\ 1 \end{pmatrix} = g_x t_1 f_x$$

(2.2)

$$\begin{aligned}
\frac{\partial \mathbf{p}'_w}{\partial \xi_i} &= \mathbf{T}_j \frac{\partial (\mathbf{T}_i^{-1} \mathbf{p}'_w)}{\partial \xi_i} = -\mathbf{T}_j (\mathbf{T}_i \mathbf{p}_w)^\odot \\
\frac{\partial \mathbf{p}'_w}{\partial \xi_j} &= \frac{\partial (\mathbf{T}_j \mathbf{T}_i^{-1} \mathbf{p}_w)}{\partial \xi_i} = (\mathbf{T}_j \mathbf{T}_i^{-1} \mathbf{p}_w)^\odot \\
&= \begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix}^\odot = \begin{pmatrix} 1 & 0 & 0 & 0 & z' & -y' \\ 0 & 1 & 0 & -z' & 0 & x' \\ 0 & 1 & 0 & y' & -x' & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
&\Rightarrow \frac{\partial (r_{\mathbf{p}}^d)_{ij}}{\partial \xi_j} = \frac{\partial (I_j^L(\mathbf{p}'))}{\partial \mathbf{p}'} \frac{\partial \mathbf{p}'}{\partial \mathbf{p}'_w} \frac{\partial \mathbf{p}'_w}{\partial \xi_j} \\
&= (g'_x, g'_y, 0, 0) \begin{pmatrix} f_x(z')^{-1} & 0 & -x' f_x(z')^{-2} & 0 \\ 0 & f_y(z')^{-1} & -y' f_y(z')^{-2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (z')^{-2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & z' & -y' \\ 0 & 1 & 0 & -z' & 0 & x' \\ 0 & 0 & 1 & y' & -x' & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} g'_x f_x(z')^{-1} \\ g'_y f_y(z')^{-1} \\ -(g'_x x' f_x + g'_y y' f_y)(z')^{-2} \\ -g'_y f_y - (g'_x x' y' f_x + g'_y (y')^2 f_y)(z')^{-2} \\ g'_x f_x + (g'_x (x')^2 f_x + g'_y x' y' f_y)(z')^{-2} \\ -g'_x f_x y' (z')^{-1} + g'_y f_y x' (z')^{-1} \end{pmatrix}^T
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
\frac{\partial \Delta \bar{R}_{0k}}{\partial b^g} &= \sum_{m=0}^{k-1} \Delta \bar{R}_{m+1k}^T J_r^m \Delta t \\
\frac{\partial \Delta \bar{R}_{00}}{\partial b^g} &= \Delta \bar{R}_{10}^T J_r^0 \Delta t \\
\frac{\partial \Delta \bar{R}_{01}}{\partial b^g} &= \sum_{m=0}^0 \Delta \bar{R}_{(m+1)1}^T J_r^m \Delta t \\
&\dots \\
\frac{\partial \Delta \bar{R}_{0(44)}}{\partial b^g} &= \sum_{m=0}^{43} \Delta \bar{R}_{(m+1)44}^T J_r^m \Delta t
\end{aligned} \tag{2.2}$$

$$\Delta \bar{\mathbf{R}}_{ik} = \begin{cases} \mathbf{I}_{3 \times 3}, & k = i \\ \prod_{m=i}^{k-1} \mathbf{Exp}((\tilde{\omega}_m - \bar{\mathbf{b}}_i^g) \Delta t), & k > i \end{cases}$$

e.g. $k : 0 \rightarrow 44, i = 0$

$$\Delta \bar{\mathbf{R}}_{00} = \mathbf{I}_{3 \times 3}$$

$$\Delta \bar{\mathbf{R}}_{01} = \mathbf{Exp}((\tilde{\omega}_0 - \bar{\mathbf{b}}_0^g) \Delta t)$$

$$\Delta \bar{\mathbf{R}}_{02} = \mathbf{Exp}((\tilde{\omega}_0 - \bar{\mathbf{b}}_0^g) \Delta t) \mathbf{Exp}((\tilde{\omega}_1 - \bar{\mathbf{b}}_0^g) \Delta t)$$

\vdots

$$\Delta \bar{\mathbf{R}}_{0(44)} = \mathbf{Exp}((\tilde{\omega}_0 - \bar{\mathbf{b}}_0^g) \Delta t) \mathbf{Exp}((\tilde{\omega}_1 - \bar{\mathbf{b}}_0^g) \Delta t) \cdots \mathbf{Exp}((\tilde{\omega}_{43} - \bar{\mathbf{b}}_0^g) \Delta t) \quad (2.2)$$

$$\Delta \bar{\mathbf{R}}_{kj} = \begin{cases} \prod_{m=k}^{j-1} \mathbf{Exp}((\tilde{\omega}_m - \bar{\mathbf{b}}_i) \Delta t), & k < j \\ \mathbf{I}_{3 \times 3}, & k = j \end{cases}$$

e.g. $k : 1 \rightarrow 45, j = 45, i = 0$

$$\Delta \bar{\mathbf{R}}_{1(45)} = \mathbf{Exp}((\tilde{\omega}_1 - \bar{\mathbf{b}}_0^g) \Delta t) \mathbf{Exp}((\tilde{\omega}_2 - \bar{\mathbf{b}}_0^g) \Delta t) \cdots \mathbf{Exp}((\tilde{\omega}_{44} - \bar{\mathbf{b}}_0^g) \Delta t)$$

\vdots

$$\Delta \bar{\mathbf{R}}_{43(45)} = \mathbf{Exp}((\tilde{\omega}_{43} - \bar{\mathbf{b}}_0^g) \Delta t) \mathbf{Exp}((\tilde{\omega}_{44} - \bar{\mathbf{b}}_0^g) \Delta t)$$

$$\Delta \bar{\mathbf{R}}_{44(45)} = \mathbf{Exp}((\tilde{\omega}_{44} - \bar{\mathbf{b}}_0^g) \Delta t)$$

$$\Delta \bar{\mathbf{R}}_{45(45)} = \mathbf{I}_{3 \times 3}$$

(2.2)

$$\begin{aligned}
\frac{\partial \Delta \bar{\mathbf{R}}_{ik}}{\partial \bar{\mathbf{b}}^g} &= \begin{cases} \mathbf{0}_{3 \times 3}, & k = i \\ \sum_{m=i}^{k-1} -\Delta \bar{\mathbf{R}}_{m+1k}^T \mathbf{J}_r^m \Delta t, & k > i \end{cases} \\
&= \begin{cases} \mathbf{0}_{3 \times 3}, & k = i \\ \mathbf{J}_r^0 \Delta t, & k = i + 1 \\ \Delta \bar{\mathbf{R}}_{(k-1)k}^T \frac{\partial \Delta \bar{\mathbf{R}}_{i(k-1)}}{\partial \bar{\mathbf{b}}^g} + \mathbf{J}_r^{k-1} \Delta t, & k > i + 1 \end{cases} \\
e.g. \quad i = 0, \quad k : 0 \rightarrow 45 \\
\frac{\partial \Delta \bar{\mathbf{R}}_{00}}{\partial \bar{\mathbf{b}}^g} &= \mathbf{0}_{3 \times 3} \\
\frac{\partial \Delta \bar{\mathbf{R}}_{01}}{\partial \bar{\mathbf{b}}^g} &= \sum_{m=0}^0 \Delta \bar{\mathbf{R}}_{(m+1)1}^T \mathbf{J}_r^m \Delta t = \Delta \bar{\mathbf{R}}_{11}^T \mathbf{J}_r^0 \Delta t = \mathbf{J}_r^0 \Delta t \\
\frac{\partial \Delta \bar{\mathbf{R}}_{02}}{\partial \bar{\mathbf{b}}^g} &= \sum_{m=0}^1 \Delta \bar{\mathbf{R}}_{(m+1)2}^T \mathbf{J}_r^m \Delta t = \Delta \bar{\mathbf{R}}_{12}^T \mathbf{J}_r^0 \Delta t + \Delta \bar{\mathbf{R}}_{22}^T \mathbf{J}_r^1 \Delta t = \Delta \bar{\mathbf{R}}_{12}^T \frac{\partial \Delta \bar{\mathbf{R}}_{01}}{\partial \bar{\mathbf{b}}^g} + \mathbf{J}_r^1 \Delta t \\
\frac{\partial \Delta \bar{\mathbf{R}}_{03}}{\partial \bar{\mathbf{b}}^g} &= \sum_{m=0}^2 \Delta \bar{\mathbf{R}}_{(m+1)3}^T \mathbf{J}_r^m \Delta t = \Delta \bar{\mathbf{R}}_{13}^T \mathbf{J}_r^0 \Delta t + \Delta \bar{\mathbf{R}}_{23}^T \mathbf{J}_r^1 \Delta t + \Delta \bar{\mathbf{R}}_{33}^T \mathbf{J}_r^2 \Delta t \\
&= (\Delta \bar{\mathbf{R}}_{12} \Delta \bar{\mathbf{R}}_{23})^T \mathbf{J}_r^0 \Delta t + \Delta \bar{\mathbf{R}}_{23}^T \mathbf{J}_r^1 \Delta t + \mathbf{J}_r^2 \Delta t \\
&= \Delta \bar{\mathbf{R}}_{23}^T \Delta \bar{\mathbf{R}}_{12}^T \mathbf{J}_r^0 \Delta t + \Delta \bar{\mathbf{R}}_{23}^T \mathbf{J}_r^1 \Delta t + \mathbf{J}_r^2 \Delta t \\
&= \Delta \bar{\mathbf{R}}_{23}^T \frac{\partial \Delta \bar{\mathbf{R}}_{02}}{\partial \bar{\mathbf{b}}^g} + \mathbf{J}_r^2 \Delta t \\
&\vdots \\
\frac{\partial \Delta \bar{\mathbf{R}}_{0(44)}}{\partial \bar{\mathbf{b}}^g} &= \sum_{m=0}^{43} \Delta \bar{\mathbf{R}}_{(m+1)44}^T \mathbf{J}_r^m \Delta t \\
&= \Delta \bar{\mathbf{R}}_{1(44)}^T \mathbf{J}_r^0 \Delta t + \Delta \bar{\mathbf{R}}_{2(44)}^T \mathbf{J}_r^1 \Delta t + \cdots + \Delta \bar{\mathbf{R}}_{43(44)}^T \mathbf{J}_r^{42} \Delta t + \Delta \bar{\mathbf{R}}_{44(44)}^T \mathbf{J}_r^{43} \Delta t \\
&= \Delta \bar{\mathbf{R}}_{43(44)}^T \frac{\partial \Delta \bar{\mathbf{R}}_{0(43)}}{\partial \bar{\mathbf{b}}^g} + \mathbf{J}_r^{43} \Delta t \\
\frac{\partial \Delta \bar{\mathbf{R}}_{0(45)}}{\partial \bar{\mathbf{b}}^g} &= \sum_{m=0}^{44} \Delta \bar{\mathbf{R}}_{(m+1)45}^T \mathbf{J}_r^m \Delta t \\
&= \Delta \bar{\mathbf{R}}_{44(45)}^T \frac{\partial \Delta \bar{\mathbf{R}}_{0(44)}}{\partial \bar{\mathbf{b}}^g} + \mathbf{J}_r^{44} \Delta t
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
\frac{\partial \Delta \bar{\mathbf{V}}_{ik}}{\partial \bar{\mathbf{b}}^g} &= \begin{cases} \mathbf{0}_{3 \times 3}, & k = i \\ -\sum_{m=i}^{k-1} \Delta \bar{\mathbf{R}}_{im}^T (\tilde{\mathbf{a}}_m - \bar{\mathbf{b}}_i^g)^\wedge \frac{\partial \Delta \bar{\mathbf{R}}_{im}}{\partial \bar{\mathbf{b}}^g} \Delta t, & k > i \end{cases} \\
&= \begin{cases} \mathbf{0}_{3 \times 3}, & k = i \\ \frac{\partial \Delta \bar{\mathbf{V}}_{i(k-1)}}{\partial \bar{\mathbf{b}}^g} + \Delta \bar{\mathbf{R}}_{i(k-1)}^T (\tilde{\mathbf{a}}_{k-1} - \bar{\mathbf{b}}_i^g)^\wedge \frac{\partial \Delta \bar{\mathbf{R}}_{i(k-1)}}{\partial \bar{\mathbf{b}}^g} \Delta t, & k > i \end{cases} \\
e.g. \quad i = 0, \quad k : 0 \rightarrow 44 \\
\frac{\partial \Delta \bar{\mathbf{V}}_{00}}{\partial \bar{\mathbf{b}}^g} &= \mathbf{0}_{3 \times 3} \\
\frac{\partial \Delta \bar{\mathbf{V}}_{01}}{\partial \bar{\mathbf{b}}^g} &= -\Delta \bar{\mathbf{R}}_{00}^T (\tilde{\mathbf{a}}_0 - \bar{\mathbf{b}}_0^g)^\wedge \frac{\partial \Delta \bar{\mathbf{R}}_{00}}{\partial \bar{\mathbf{b}}^g} \Delta t \\
\frac{\partial \Delta \bar{\mathbf{V}}_{02}}{\partial \bar{\mathbf{b}}^g} &= -\Delta \bar{\mathbf{R}}_{00}^T (\tilde{\mathbf{a}}_0 - \bar{\mathbf{b}}_0^g)^\wedge \frac{\partial \Delta \bar{\mathbf{R}}_{00}}{\partial \bar{\mathbf{b}}^g} \Delta t + \Delta \bar{\mathbf{R}}_{01}^T (\tilde{\mathbf{a}}_1 - \bar{\mathbf{b}}_0^g)^\wedge \frac{\partial \Delta \bar{\mathbf{R}}_{01}}{\partial \bar{\mathbf{b}}^g} \Delta t \\
&= \frac{\partial \Delta \bar{\mathbf{V}}_{01}}{\partial \bar{\mathbf{b}}^g} + \Delta \bar{\mathbf{R}}_{01}^T (\tilde{\mathbf{a}}_1 - \bar{\mathbf{b}}_0^g)^\wedge \frac{\partial \Delta \bar{\mathbf{R}}_{01}}{\partial \bar{\mathbf{b}}^g} \Delta t \\
\frac{\partial \Delta \bar{\mathbf{V}}_{03}}{\partial \bar{\mathbf{b}}^g} &= \frac{\partial \Delta \bar{\mathbf{V}}_{02}}{\partial \bar{\mathbf{b}}^g} + \Delta \bar{\mathbf{R}}_{02}^T (\tilde{\mathbf{a}}_2 - \bar{\mathbf{b}}_0^g)^\wedge \frac{\partial \Delta \bar{\mathbf{R}}_{02}}{\partial \bar{\mathbf{b}}^g} \Delta t \\
&\vdots \\
\frac{\partial \Delta \bar{\mathbf{V}}_{0(44)}}{\partial \bar{\mathbf{b}}^g} &= \frac{\partial \Delta \bar{\mathbf{V}}_{0(43)}}{\partial \bar{\mathbf{b}}^g} + \Delta \bar{\mathbf{R}}_{0(43)}^T (\tilde{\mathbf{a}}_{43} - \bar{\mathbf{b}}_0^g)^\wedge \frac{\partial \Delta \bar{\mathbf{R}}_{0(43)}}{\partial \bar{\mathbf{b}}^g} \Delta t
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
\mathbf{r}_{\Delta \mathbf{R}_{ij}} &= \text{Log}((\Delta \tilde{\mathbf{R}}_{ij}(\bar{\mathbf{b}}_i^g) \text{Exp}(\frac{\partial \Delta \tilde{\mathbf{R}}_{ij}}{\partial \bar{\mathbf{b}}^g} \delta \mathbf{b}^g))^T \mathbf{R}_i^T \mathbf{R}_j) \\
\mathbf{r}_{\Delta \mathbf{v}_{ij}} &= \mathbf{R}_i^T (\mathbf{v}_j - \mathbf{v}_i - \mathbf{g} \Delta t_{ij}) \\
&\quad - [\Delta \tilde{\mathbf{v}}_{ij}(\bar{\mathbf{b}}_i^g, \bar{\mathbf{b}}_i^a) + \frac{\partial \Delta \tilde{\mathbf{v}}_{ij}}{\partial \bar{\mathbf{b}}^g} \delta \mathbf{b}^g + \frac{\partial \Delta \tilde{\mathbf{v}}_{ij}}{\partial \bar{\mathbf{b}}^a} \delta \mathbf{b}^a] \\
\mathbf{r}_{\Delta \mathbf{p}_{ij}} &= \mathbf{R}_i^T (\mathbf{p}_j - \mathbf{p}_i - \mathbf{v}_i \Delta t_{ij} - \frac{1}{2} \mathbf{g} \Delta t_{ij}^2) \\
&\quad - [\Delta \tilde{\mathbf{p}}_{ij}(\bar{\mathbf{b}}_i^g, \bar{\mathbf{b}}_i^a) + \frac{\partial \Delta \tilde{\mathbf{p}}_{ij}}{\partial \bar{\mathbf{b}}^g} \delta \mathbf{b}^g + \frac{\partial \Delta \tilde{\mathbf{p}}_{ij}}{\partial \bar{\mathbf{b}}^a} \delta \mathbf{b}^a]
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
\text{Exp}(\xi^\wedge) &= \mathbf{T} = \begin{pmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 0 \end{pmatrix}_{4 \times 4}, \text{Exp}(\delta \xi^\wedge) = \begin{pmatrix} \delta \phi^\wedge & \delta \rho \\ \mathbf{0}^T & 0 \end{pmatrix}_{4 \times 4}, \mathbf{p} \in \mathbb{R}^3 \\
\frac{\partial(\mathbf{T}\mathbf{p})}{\partial \delta \xi} &= \lim_{\delta \xi \rightarrow 0} \frac{\text{Exp}(\xi^\wedge) \text{Exp}(\delta \xi^\wedge) \mathbf{p} - \text{Exp}(\xi^\wedge) \mathbf{p}}{\delta \xi} \\
&\approx \lim_{\delta \xi \rightarrow 0} \frac{\text{Exp}(\xi^\wedge)(\mathbf{I} - \delta \xi^\wedge) \mathbf{p} - \text{Exp}(\xi^\wedge) \mathbf{p}}{\delta \xi} = \lim_{\delta \xi \rightarrow 0} - \frac{\text{Exp}(\xi^\wedge) \delta \xi^\wedge \mathbf{p}}{\delta \xi} \\
&= \lim_{\delta \xi \rightarrow 0} - \frac{\begin{pmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 0 \end{pmatrix} \begin{pmatrix} \delta \phi^\wedge \mathbf{p} + \delta \rho \\ 1 \end{pmatrix}}{\delta \xi} = \lim_{\delta \xi \rightarrow 0} - \frac{\begin{pmatrix} \mathbf{R} \delta \phi^\wedge \mathbf{p} + \mathbf{R} \delta \rho + \mathbf{t} \\ \mathbf{0}^T \end{pmatrix}}{\delta \xi} \quad (1.1) \\
&= \lim_{\delta \xi \rightarrow 0} - \frac{\begin{pmatrix} -\mathbf{R} \mathbf{p}^\wedge \delta \phi + \mathbf{R} \delta \rho + \mathbf{t} \\ \mathbf{0}^T \end{pmatrix}}{\begin{pmatrix} \delta \rho \\ \delta \phi \end{pmatrix}} = \begin{pmatrix} -\mathbf{R} & \mathbf{R} \mathbf{p}^\wedge \\ \mathbf{0}^T & \mathbf{0}^T \end{pmatrix}_{4 \times 6}
\end{aligned}$$

Algorithm 1 Time-closest measurements selection

Require: *gyro_list*, *acc_list*[*s*] (an element in *acc_list*)

Ensure: *gyro_measure* (time closest element in *gyro_list*)

```

1: function TIME_CLOSEST_SELECT(gyro_list, i)
2:   t ← acc_list[s].timestamp, i ← s
3:   while true do
4:     if i ≥ gyro_list.size then
5:       return gyro_list.back
6:     else
7:       tnow ← gyro_list[i].timestamp
8:       tnext ← gyro_list[i + 1].timestamp
9:       if tnow < t then
10:        if tnext > t then
11:          tfront ← abs(tnow − t), tback ← abs(tnext − t)
12:          return tfront > tback ? gyro_list[i] : gyro_list[i + 1]
13:        else
14:          i = i + 1
15:        end if
16:      else if tnow > t then
17:        i = i − 1
18:      else
19:        return gyro_list[i]
20:      end if
21:    end if
22:  end while
23: end function

```

Algorithm 1 On-Manifold Preintegration for IMU

Require: $gyro_list, acc_list, m, n, rotate_list$

Ensure: $(\Delta \bar{\mathbf{R}}_{ij}, \frac{\partial \Delta \bar{\mathbf{R}}_{ij}}{\partial \mathbf{b}^g}, \frac{\partial \Delta \bar{\mathbf{v}}_{ij}}{\partial \mathbf{b}^a}, \frac{\partial \Delta \bar{\mathbf{p}}_{ij}}{\partial \mathbf{b}^a}), (\Delta \bar{\mathbf{v}}_{ij}, \frac{\partial \Delta \bar{\mathbf{v}}_{ij}}{\partial \mathbf{b}^g}, \Delta \bar{\mathbf{p}}_{ij}, \frac{\partial \Delta \bar{\mathbf{p}}_{ij}}{\partial \mathbf{b}^g}), \Sigma_{ij}$

- 1: **function** IMU_PREINTEGRATION($gyro_list, acc_list, m, n, rotate_list$)
- 2: **for all** $gyro_list[i], i : 0 \rightarrow m$ **do**
- 3: $last_r \leftarrow rotate_list[i - 1]$
- 4: $rot.timestamp \leftarrow gyro_list[i].timestamp$
- 5: $rot.\omega \leftarrow gyro_list[i].\omega - \mathbf{b}_i^g$
- 6: $rot.\Delta \bar{\mathbf{R}}_{ik} \leftarrow last_r.\Delta \bar{\mathbf{R}}_{ik} * \text{Exp}(rot.\omega * \Delta t)$
- 7: $rot.\Delta \bar{\mathbf{R}}_{(k-1)k} \leftarrow \text{Exp}(rot.\omega * \Delta t)$
- 8: $rot.\frac{\partial \Delta \bar{\mathbf{R}}_{ik}}{\partial \mathbf{b}^g} \leftarrow \Delta \bar{\mathbf{R}}_{(k-1)k}^T * last_r.\frac{\partial \Delta \bar{\mathbf{R}}_{ik}}{\partial \mathbf{b}^g} - \mathbf{J}_r(rot.\omega * \Delta t) * \Delta t$
- 9: $rot.\frac{\partial \Delta \bar{\mathbf{v}}_{ik}}{\partial \mathbf{b}^a} \leftarrow last_r.\frac{\partial \Delta \bar{\mathbf{v}}_{ik}}{\partial \mathbf{b}^a} - last_r.\Delta \bar{\mathbf{R}}_{ik} * \Delta t$
- 10: $rot.\frac{\partial \Delta \bar{\mathbf{p}}_{ik}}{\partial \mathbf{b}^a} \leftarrow last_r.\frac{\partial \Delta \bar{\mathbf{p}}_{ik}}{\partial \mathbf{b}^a} + last_r.\frac{\partial \Delta \bar{\mathbf{v}}_{ik}}{\partial \mathbf{b}^a} * \Delta t - \frac{1}{2}last_r.\Delta \bar{\mathbf{R}}_{ik} * \Delta t^2$
- 11: $rotate_list.push(rot)$
- 12: **end for**
- 13: $\Delta \bar{\mathbf{R}}_{ij} = rotate_list.end.\Delta \bar{\mathbf{R}}_{ik}$
- 14: $\frac{\partial \Delta \bar{\mathbf{R}}_{ij}}{\partial \mathbf{b}^g} = rotate_list.end.\frac{\partial \Delta \bar{\mathbf{R}}_{ik}}{\partial \mathbf{b}^g}$
- 15: $\frac{\partial \Delta \bar{\mathbf{v}}_{ij}}{\partial \mathbf{b}^a} = rotate_list.end.\frac{\partial \Delta \bar{\mathbf{v}}_{ik}}{\partial \mathbf{b}^a}$
- 16: $\frac{\partial \Delta \bar{\mathbf{p}}_{ij}}{\partial \mathbf{b}^a} = rotate_list.end.\frac{\partial \Delta \bar{\mathbf{p}}_{ik}}{\partial \mathbf{b}^a}$
- 17: **for all** $acc_list[i], i : 0 \rightarrow n$ **do**
- 18: $cls_r \leftarrow time_closest_select(rotate_list, acc_list[i])$
- 19: $acc \leftarrow acc_list[i] - \mathbf{b}_i^a$
- 20: $\Delta \bar{\mathbf{v}}_{ij} += cls_r.\Delta \bar{\mathbf{R}}_{ik} * acc * \Delta t$
- 21: $\frac{\partial \Delta \bar{\mathbf{v}}_{ij}}{\partial \mathbf{b}^g} += cls_r.\Delta \bar{\mathbf{R}}_{ik} * acc^\wedge * cls_r.\frac{\partial \Delta \bar{\mathbf{R}}_{ik}}{\partial \mathbf{b}^g} * \Delta t$
- 22: $\Delta \bar{\mathbf{p}}_{ij} += \Delta \bar{\mathbf{v}}_{ij} * \Delta t + \frac{1}{2}cls_r.\Delta \bar{\mathbf{R}}_{ik} * acc * \Delta t^2$
- 23: $\frac{\partial \Delta \bar{\mathbf{p}}_{ij}}{\partial \mathbf{b}^g} += cls_r.\frac{\partial \Delta \bar{\mathbf{v}}_{ik}}{\partial \mathbf{b}^a} \Delta t - \frac{1}{2}cls_r.\Delta \bar{\mathbf{R}}_{ik} * acc^\wedge * cls_r.\frac{\partial \Delta \bar{\mathbf{R}}_{ik}}{\partial \mathbf{b}^g} * \Delta t^2$
- 24:
$$A = \begin{pmatrix} cls_r.\Delta \bar{\mathbf{R}}_{(k-1)k}^T & \mathbf{0} & \mathbf{0} \\ -cls_r.\Delta \bar{\mathbf{R}}_{ik} * acc^\wedge * \Delta t & \mathbf{I} & \mathbf{0} \\ -\frac{1}{2}cls_r.\Delta \bar{\mathbf{R}}_{ik} * acc^\wedge * \Delta t^2 & \Delta t \mathbf{I} & \mathbf{I} \end{pmatrix}$$
- 25:
$$B = \begin{pmatrix} \mathbf{J}_r(rot.\omega * \Delta t) * \Delta t & \mathbf{0} \\ \mathbf{0} & cls_r.\Delta \bar{\mathbf{R}}_{ik} * \Delta t \\ \mathbf{0} & \frac{1}{2}cls_r.\Delta \bar{\mathbf{R}}_{ik} * \Delta t^2 \end{pmatrix}$$
- 26: $\Sigma_{ij} = A * \Sigma_{ij} * A^T + B * \Sigma_\eta * B^T$
- 27: **end for**
- 28: **end function**

$$\begin{aligned}
\frac{\partial \mathbf{r}_{\Delta \mathbf{p}_{ij}}}{\partial \delta \mathbf{p}_i} &= -\mathbf{I} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{p}_{ij}}}{\partial \delta \phi_i} &= (\mathbf{R}_i^T (\mathbf{p}_j - \mathbf{p}_i - \mathbf{v}_i \Delta t_{ij} - \frac{1}{2} \mathbf{g} \Delta t_{ij}^2))^\wedge \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{p}_{ij}}}{\partial \delta \mathbf{v}_i} &= -\mathbf{R}_i^T \Delta t_{ij} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{p}_{ij}}}{\partial \delta \mathbf{p}_j} &= \mathbf{R}_i^T \mathbf{R}_j \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{p}_{ij}}}{\partial \delta \phi_j} &= \mathbf{0} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{p}_{ij}}}{\partial \delta \mathbf{v}_j} &= \mathbf{0} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{p}_{ij}}}{\partial \delta \mathbf{b}_i^a} &= -\frac{\partial \Delta \bar{\mathbf{p}}_{ij}}{\partial \mathbf{b}^a} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{p}_{ij}}}{\partial \delta \mathbf{b}_i^g} &= -\frac{\partial \Delta \bar{\mathbf{p}}_{ij}}{\partial \mathbf{b}^g}
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
\frac{\partial \mathbf{r}_{\Delta \mathbf{v}_{ij}}}{\partial \delta \mathbf{p}_i} &= \mathbf{0} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{v}_{ij}}}{\partial \delta \phi_i} &= (\mathbf{R}_i^T (\mathbf{v}_j - \mathbf{v}_i - \mathbf{g} \Delta t_{ij}))^\wedge \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{v}_{ij}}}{\partial \delta \mathbf{v}_i} &= -\mathbf{R}_i^T \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{v}_{ij}}}{\partial \delta \mathbf{p}_j} &= \mathbf{0} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{v}_{ij}}}{\partial \delta \phi_j} &= \mathbf{0} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{v}_{ij}}}{\partial \delta \mathbf{v}_j} &= \mathbf{R}_i^T \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{v}_{ij}}}{\partial \delta \mathbf{b}_i^a} &= -\frac{\partial \Delta \bar{\mathbf{v}}_{ij}}{\partial \mathbf{b}^a} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{v}_{ij}}}{\partial \delta \mathbf{b}_i^g} &= -\frac{\partial \Delta \bar{\mathbf{v}}_{ij}}{\partial \mathbf{b}^g}
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
\frac{\partial \mathbf{r}_{\Delta \mathbf{R}_{ij}}}{\partial \delta \mathbf{p}_i} &= \mathbf{0} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{R}_{ij}}}{\partial \delta \phi_i} &= -\mathbf{J}_r^{-1}(\mathbf{r}_{\Delta \mathbf{R}_{ij}}) \mathbf{R}_j^T \mathbf{R}_i \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{R}_{ij}}}{\partial \delta \mathbf{v}_i} &= \mathbf{0} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{R}_{ij}}}{\partial \delta \mathbf{p}_j} &= \mathbf{0} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{R}_{ij}}}{\partial \delta \phi_j} &= \mathbf{J}_r^{-1}(\mathbf{r}_{\Delta \mathbf{R}_{ij}}) \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{R}_{ij}}}{\partial \delta \mathbf{v}_j} &= \mathbf{0} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{R}_{ij}}}{\partial \delta \mathbf{b}_i^a} &= \mathbf{0} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{R}_{ij}}}{\partial \delta \mathbf{b}_i^g} &= -\mathbf{J}_r^{-1}(\mathbf{r}_{\Delta \mathbf{R}_{ij}}) \text{Exp}(\mathbf{r}_{\Delta \mathbf{R}_{ij}})^T \mathbf{J}_r \left(\frac{\partial \Delta \bar{\mathbf{R}}_{ij}}{\partial \mathbf{b}^g} \delta \mathbf{b}_i^g \right) \frac{\partial \Delta \bar{\mathbf{R}}_{ij}}{\partial \mathbf{b}^g}
\end{aligned} \tag{2.2}$$

Algorithm 3 Levenberg-Marquardt 迭代法

Require: $\mathcal{K}_i, \mathcal{I}_j, \delta_{i(i-1)}, k_{max}$ (迭代法)**Ensure:** δ_{ji}

```
1: function TRACKFRAME(Array, left, middle, right)
2:    $v \leftarrow 2$ 
3:    $\delta \leftarrow \delta_{i(i-1)}$ 
4:    $\mathbf{f}_i$ 
5:   for  $i = 0 \rightarrow k_{max} - 1$  do
6:      $\mathbf{J}^T \mathbf{W} \mathbf{J}$ 
7:     while true do
8:       solve  $\delta^*$ 
9:       update
10:       $\mathbf{f}_i$ 
11:       $error = \text{cacldweight}$ 
12:      if  $error < lastError$  then
13:         $lastResidual = lastError = error$ 
14:        if  $\lambda \leq 0.2$  then
15:           $\lambda = 0$ 
16:        else
17:           $\lambda = \frac{1}{2} \lambda$ 
18:        end if
19:        break
20:      else
21:        if 迭代法  $< \varepsilon_1$  then
22:           $i = k_{max} - 1$ 
23:          break
24:        end if
25:        if  $\lambda == 0$  then
26:           $\lambda = 0.2$ 
27:        else
28:           $\lambda = 2 * \text{incry}$ 
29:        end if
30:      end if
31:    end while
32:  end for
33:  return  $\delta$ 
34: end function
```
