$$
\begin{aligned}
&f_m(\boldsymbol{\delta}) = I_i(\boldsymbol{p}_m) - I_j(w(\boldsymbol{p}_m, D_i(\boldsymbol{p}_m), \boldsymbol{\delta})), \boldsymbol{p}_m \in \boldsymbol{\Omega}_{D_i}, 1 \leq m \leq n \\
&W_m(\boldsymbol{\delta}) = \left(2\sigma_I^2 + \left(\frac{\partial_{r_{p_m}}(\boldsymbol{p}_m, \boldsymbol{\delta})}{\partial D_i(\boldsymbol{p}_m)}\right)^2 V_i(\boldsymbol{p}_m)\right)^{-1} \\
&\boldsymbol{f} = \begin{pmatrix} f_1(\boldsymbol{\delta}) \\ f_2(\boldsymbol{\delta}) \\ \vdots \\ f_n(\boldsymbol{\delta}) \end{pmatrix}, \boldsymbol{W} = \begin{pmatrix} \boldsymbol{W}_1 & 0 & \cdots & 0 \\ 0 & \boldsymbol{W}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{W}_n \end{pmatrix} \\
&E(\boldsymbol{\delta}) = \boldsymbol{f}^T(\boldsymbol{\delta})\boldsymbol{W}(\boldsymbol{\delta})\boldsymbol{f}(\boldsymbol{\delta}) \\
&\boldsymbol{f}(\boldsymbol{\delta} + \triangle\boldsymbol{\delta}) \approx l(\triangle\boldsymbol{\delta}) = \boldsymbol{f}(\boldsymbol{\delta}) + \boldsymbol{J}(\boldsymbol{\delta})\triangle\boldsymbol{\delta} \\
&\frac{1}{2}E(\boldsymbol{\delta}) \quad \approx L(\triangle\boldsymbol{\delta}) = \frac{1}{2}l^T(\triangle\boldsymbol{\delta})l(\triangle\boldsymbol{\delta}) \\
&\qquad\qquad\qquad\qquad = \frac{1}{2}\boldsymbol{f}^T\boldsymbol{W}\boldsymbol{f} + \triangle\boldsymbol{\delta}^T\boldsymbol{J}^T\boldsymbol{W}\boldsymbol{f} + \frac{1}{2}\triangle\boldsymbol{\delta}^T\boldsymbol{J}^T\boldsymbol{W}\boldsymbol{J}\triangle\boldsymbol{\delta} \\
&\qquad\qquad\qquad\qquad = E + \triangle\boldsymbol{\delta}^T\boldsymbol{J}^T\boldsymbol{W}\boldsymbol{f} + \frac{1}{2}\triangle\boldsymbol{\delta}^T\boldsymbol{J}^T\boldsymbol{W}\boldsymbol{J}\triangle\boldsymbol{\delta} \\
&L^{'} = \boldsymbol{J}^T\boldsymbol{W}\boldsymbol{f} + \boldsymbol{J}^T\boldsymbol{W}\boldsymbol{J}\triangle\boldsymbol{\delta} = 0 \\
&\triangle\boldsymbol{\delta} = -(\boldsymbol{J}^T\boldsymbol{W}\boldsymbol{J})^{-1}\boldsymbol{J}^T\boldsymbol{W}\boldsymbol{f}
\end{aligned} \tag{1}
$$

$$
\begin{cases}
\boldsymbol{p} := \begin{pmatrix} \boldsymbol{p}_x \\ \boldsymbol{p}_y \\ d_p \end{pmatrix}, \boldsymbol{P} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \boldsymbol{p}_x/d_p \\ \boldsymbol{p}_y/d_p \\ 1/d_p \end{pmatrix} \\
w(\boldsymbol{p}, D_i(\boldsymbol{p}), \boldsymbol{\delta}_{ji}) := \begin{pmatrix} x^{'}/z^{'} \\ y^{'}/z^{'} \end{pmatrix}, \begin{pmatrix} x^{'} \\ y^{'} \\ z^{'} \\ 1 \end{pmatrix} = exp(\boldsymbol{\delta}) \begin{pmatrix} \boldsymbol{p}_x/d_p \\ \boldsymbol{p}_y/d_p \\ 1/d_p \\ 1 \end{pmatrix}
\end{cases} \tag{2.10}
$$

$$
\begin{aligned}
&\boldsymbol{q} = \begin{pmatrix} x^{'} \\ y^{'} \\ z^{'} \end{pmatrix}, w_m = (\boldsymbol{p}_m, D_i(\boldsymbol{p}_m), \boldsymbol{\delta}) = \begin{pmatrix} x^{'}/z^{'} \\ y^{'}/z^{'} \end{pmatrix} \\
&\frac{\partial f_m}{\partial \delta} = -\frac{\partial I_j(w_m)}{\partial \boldsymbol{\delta}} = -\frac{\partial I_j}{\partial w_m}\frac{\partial w_m}{\partial \boldsymbol{q}}\frac{\partial \boldsymbol{q}}{\partial \boldsymbol{\delta}} \\
&= -\begin{pmatrix} g_x & g_y \end{pmatrix} \begin{pmatrix} \frac{1}{z^{'}} & 0 & -\frac{x^{'}}{z^{'}} \\ 0 & \frac{1}{z^{'}} & -\frac{y^{'}}{z^{'}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & -z^{'} & y^{'} \\ 0 & 1 & 0 & z^{'} & 0 & -x^{'} \\ 0 & 0 & 1 & -y^{'} & x^{'} & 0 \end{pmatrix} \\
&= -\begin{pmatrix} \frac{g_x}{z^{'}} & \frac{g_y}{z^{'}} & -\frac{x^{'}g_x}{z^{'2}} - \frac{y^{'}g_y}{z^{'2}} & -\frac{x^{'}y^{'}g_x}{z^{'2}} - (1+\frac{y^{'2}}{z^{'2}})g_y & (1+\frac{x^{'2}}{z^{'2}})g_x + \frac{x^{'}y^{'}g_y}{z^{'2}} & -\frac{y^{'}g_x}{z^{'}} + \frac{x^{'}g_y}{z^{'}} \end{pmatrix} \\
&\boldsymbol{J} = \begin{pmatrix} \frac{\partial f_1}{\partial \delta} & \frac{\partial f_2}{\partial \delta} & \cdots & \frac{\partial f_N}{\partial \delta} \end{pmatrix}^T = \begin{pmatrix} \boldsymbol{J}_1 & \boldsymbol{J}_2 & \cdots & \boldsymbol{J}_N \end{pmatrix}^T
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
&\begin{pmatrix} x^{'} \\ y^{'} \\ z^{'} \\ 1 \end{pmatrix} = exp(\boldsymbol{\delta}) \begin{pmatrix} \boldsymbol{p}_x/d_p \\ \boldsymbol{p}_y/d_p \\ 1/d_p \\ 1 \end{pmatrix} = \frac{1}{d_p} \begin{pmatrix} \boldsymbol{R}_1 & t_1 \\ \boldsymbol{R}_2 & t_2 \\ \boldsymbol{R}_3 & t_3 \\ \boldsymbol{0} & 1 \end{pmatrix}_{4\times 4} \begin{pmatrix} \boldsymbol{P} \\ \\ d_p \end{pmatrix}_{4\times 1} = \frac{1}{d_p} \begin{pmatrix} \boldsymbol{R}_1\boldsymbol{P} + t_1 d_p \\ \boldsymbol{R}_2\boldsymbol{P} + t_2 d_p \\ \boldsymbol{R}_3\boldsymbol{P} + t_3 d_p \\ d_p \end{pmatrix} \\
&w_m = \begin{pmatrix} x^{'}/z^{'} \\ y^{'}/z^{'} \end{pmatrix} = \begin{pmatrix} \frac{\boldsymbol{R}_1\boldsymbol{P}+t_1 d_p}{\boldsymbol{R}_3\boldsymbol{P}+t_3 d_p} \\ \frac{\boldsymbol{R}_2\boldsymbol{P}+t_2 d_p}{\boldsymbol{R}_3\boldsymbol{P}+t_3 d_p} \end{pmatrix} \\
&\frac{\partial_{r_{p_m}}(\boldsymbol{p}_m, \boldsymbol{\delta})}{\partial D_i(\boldsymbol{p}_m)} := \frac{\partial f_m}{\partial d_{p_m}} = -\frac{\partial I_j}{\partial w_m}\frac{\partial w_m}{\partial d_{p_m}} \\
&= -\begin{pmatrix} g_x & g_y \end{pmatrix} \begin{pmatrix} \frac{t_1 z^{'} - x^{'} t_3}{z^{'2}} \\ \frac{t_2 z^{'} - y^{'} t_3}{z^{'2}} \end{pmatrix} = -(g_x \frac{t_1 z^{'} - x^{'} t_3}{z^{'2}} + g_y \frac{t_2 z^{'} - y^{'} t_3}{z^{'2}}) \\
&W_m := \left(2\sigma_I^2 + w_m^2 V_m)\right)^{-1}
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
\boldsymbol{J}^T\boldsymbol{W}\boldsymbol{J} &= \begin{pmatrix} \boldsymbol{J}_1^T & \boldsymbol{J}_2^T & \cdots & \boldsymbol{J}_N^T \end{pmatrix}_{6\times N} \begin{pmatrix} \boldsymbol{W}_1 & 0 & \cdots & 0 \\ 0 & \boldsymbol{W}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{W}_N \end{pmatrix}_{N\times N} \begin{pmatrix} \boldsymbol{J}_1 \\ \boldsymbol{J}_2 \\ \vdots \\ \boldsymbol{J}_N \end{pmatrix}_{N\times 6} \\
&= \sum_{i=1}^{N} (J_i^T W_i J_i)_{6\times 6} = \boldsymbol{A}_{6\times 6} \\
\boldsymbol{J}^T\boldsymbol{W} &= \sum_{i=1}^{N} (J_i^T W_i)_{6\times 1} = \boldsymbol{b}_{6\times 1} \\
& \boldsymbol{A}\boldsymbol{\delta}^* = \boldsymbol{b} \\
& \Rightarrow \boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^T\boldsymbol{\delta}^* = \boldsymbol{b} \\
& \Rightarrow (\boldsymbol{L}\boldsymbol{D}^{1/2})(\boldsymbol{L}\boldsymbol{D}^{1/2})^T\boldsymbol{\delta}^* = \boldsymbol{b} \\
& \Rightarrow \boldsymbol{G}\boldsymbol{G}^T\boldsymbol{\delta}^* = \boldsymbol{b} \\
& \Rightarrow \boldsymbol{G}\boldsymbol{\delta}^{*'} = \boldsymbol{b} \\
& \Rightarrow \boldsymbol{G}^T\boldsymbol{\delta}^* = \boldsymbol{\delta}^{*'}
\end{aligned}
\tag{4}
$$

$$
cost = \sum_{i=j=1}^{5} (A(i,j) - B(i,j))^2 \tag{4.1}
$$

$$
\begin{aligned}
\overrightarrow{s_1} &= \overrightarrow{n_1} \times \overrightarrow{n_2} = \boldsymbol{t} \times \overrightarrow{n_1} \times \overrightarrow{Op} \\
&= \begin{pmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} -at_3 + t_1 \\ -bt_3 + t_2 \\ 0 \end{pmatrix} = \begin{pmatrix} epx \\ epy \\ 0 \end{pmatrix}
\end{aligned}
\tag{4.2}
$$

$$
\begin{aligned}
& & d_2\boldsymbol{p}_{best} &= \boldsymbol{R}(d_1\boldsymbol{p}_c) + \boldsymbol{t} \\
&\Rightarrow & 0 &= [\boldsymbol{p}_{best}]_\times(d_1\boldsymbol{R}\boldsymbol{p}_c + \boldsymbol{t}) \\
&\Rightarrow & d_1[\boldsymbol{p}_{best}]_\times\boldsymbol{R}\boldsymbol{p}_c &= -[\boldsymbol{p}_{best}]_\times\boldsymbol{t} \\
&\Rightarrow & d_1\begin{pmatrix} 0 & -1 & b_1 \\ 1 & 0 & -a_1 \\ -b_1 & a_1 & 0 \end{pmatrix}\begin{pmatrix} R_{row0} \\ R_{row1} \\ R_{row2} \end{pmatrix}\boldsymbol{p}_c &= \begin{pmatrix} 0 & 1 & -b_1 \\ -1 & 0 & a_1 \\ b_1 & -a_1 & 0 \end{pmatrix}\begin{pmatrix} t_0 \\ t_1 \\ t_2 \end{pmatrix} \\
&\Rightarrow & d_1\begin{pmatrix} b_1R_{row2} - R_{row1} \\ R_{row0} - a_1R_{row2} \\ a_1R_{row1} - b_1R_{row0} \end{pmatrix}\boldsymbol{p}_c &= \begin{pmatrix} t_1 - b_1t_2 \\ a_1t_2 - t_0 \\ b_1t_0 - a_1t_1 \end{pmatrix} \\
&\Rightarrow & d_1 &= \frac{t_1 - b_1t_2}{(b_1R_{row2} - R_{row1})\boldsymbol{p}_c} \\
&\Rightarrow & &= \frac{a_1t_2 - t_0}{(R_{row0} - a_1R_{row2})\boldsymbol{p}_c} \\
&\Rightarrow & &= \frac{b_1t_0 - a_1t_1}{(a_1R_{row1} - b_1R_{row0})\boldsymbol{p}_c}
\end{aligned} \tag{4.4}
$$

$$
L := \left\{\boldsymbol{x} = \boldsymbol{p}_{close} + \lambda\begin{pmatrix} l_x \\ l_y \end{pmatrix} | \lambda = k|\mathbf{S}|, k \in \mathbf{N}^+\right\} \tag{4.4}
$$

$$
\mathcal{N}\left(\frac{\sigma_p^2 d_o + \sigma_o^2 d_p}{\sigma_p^2 + \sigma_o^2}, \frac{\sigma_p^2\sigma_o^2}{\sigma_p^2 + \sigma_o^2}\right) \tag{2.14}
$$

$$
\begin{aligned}
d_new &= (d_p^{-1} - t_z)^{-1} \\
\sigma_{new}^2 &= \left(\frac{d_{new}}{d_p}\right)^4 \sigma_p^2
\end{aligned} \tag{2.14}
$$

$$
f := \begin{cases}
X_{max} = MAX\{x_1, x_2, \dots, x_m\} \\
X_{min} = MIN\{x_1, x_2, \dots, x_m\} \\
H_{sum} = \dfrac{X_{max} - X_{min}}{H} \\
Z_{max} = MAX\{z_1, z_2, \dots, z_m\} \\
Z_{min} = MIN\{z_1, z_2, \dots, z_m\} \\
V_{sum} = \dfrac{Z_{max} - Z_{min}}{V} \\
\boldsymbol{p}_i = [x_i, z_i]^T \in \boldsymbol{P}, \boldsymbol{q}_i = [h_i, v_i]^T \in \boldsymbol{Q} \\
h_i = \left\lfloor \dfrac{x_i}{H} \right\rfloor, v_i = \left\lfloor \dfrac{z_i}{V} \right\rfloor
\end{cases} \tag{3.2}
$$

Algorithm 1 : Time optimized A* Algorithm

Require: ***Start***,***End***,***Q***

Ensure: Father node of ***End***

```
function ASTARSEARCH(Start,End,Q)
    open = binary heap containing Start node
    closed = empty set
    movecost(x, y) =  distance from node x to node y
    while End node not in open do
        i = node with min f(i) in open
        remove i from open
        add i to closed
        count = 0
        for j = neighbor node of i and not in closed and reachable in Q do
            count++
            cost = g(i) + movecost(i, j)
            if j in open and cost < g(j) then
                remove j from open
            end if
            if j not in open and not in closed then
                add j into open
                f(j) = g(j) + h(j)
                set father node of j is i
            end if
        end for
        if count == 0 then
            can't find path
            break out
        end if
    end while
end function
```

$$\ddot{X}_{t-1} \sim \mathcal{N}(0, k_1^2), \ddot{Z}_{t-1} \sim \mathcal{N}(0, k_2^2), \ddot{\theta}_{t-1} \sim \mathcal{N}(0, k_3^2),$$

$$\boldsymbol{R}_t = \begin{pmatrix} \frac{1}{4}T^4k_1^2 & \frac{1}{2}T^3k_1^2 & 0 & 0 & 0 & 0 \\ \frac{1}{2}T^3k_1^2 & T^2k_1^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4}T^4k_2^2 & \frac{1}{2}T^3k_2^2 & 0 & 0 \\ 0 & 0 & \frac{1}{2}T^3k_2^2 & T^2k_2^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4}T^4k_3^2 & \frac{1}{2}T^3k_3^2 \\ 0 & 0 & 0 & 0 & \frac{1}{2}T^3k_3^2 & T^2k_3^2 \end{pmatrix} \tag{3.4}$$

$$\begin{pmatrix} cov(\varepsilon_1, \varepsilon_1) & cov(\varepsilon_1, \varepsilon_2) & \cdots & cov(\varepsilon_1, \varepsilon_6) \\ cov(\varepsilon_2, \varepsilon_1) & cov(\varepsilon_2, \varepsilon_2) & \cdots & cov(\varepsilon_2, \varepsilon_6) \\ \vdots & \ddots & \vdots & \\ cov(\varepsilon_6, \varepsilon_1) & cov(\varepsilon_6, \varepsilon_2) & \cdots & cov(\varepsilon_6, \varepsilon_6) \end{pmatrix} \tag{3.4}$$

Algorithm 2 : Linear Kalman Filter

Require: $\boldsymbol{\mu}_{t-1}, \boldsymbol{\Sigma}_{t-1}, \boldsymbol{z}_t$

Ensure: $\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t$

function FILTER($\boldsymbol{\mu}_{t-1}, \boldsymbol{\Sigma}_{t-1}, \boldsymbol{z}_t$)

  predict $\overline{\boldsymbol{\mu}}_t = \boldsymbol{A}\boldsymbol{\mu}_{t-1}$

  $\overline{\boldsymbol{\Sigma}}_t = \boldsymbol{A}\boldsymbol{\Sigma}_{t-1}\boldsymbol{A}^T + \boldsymbol{R}_t$

  update $\boldsymbol{K}_t = \overline{\boldsymbol{\Sigma}}_t(\overline{\boldsymbol{\Sigma}}_t + \boldsymbol{Q}_t)^{-1}$

  $\boldsymbol{\mu}_t = \overline{\boldsymbol{\mu}}_t + \boldsymbol{K}_t(\boldsymbol{z}_t - \overline{\boldsymbol{\mu}}_t)$

  $\boldsymbol{\Sigma}_t = (\boldsymbol{E} - \boldsymbol{K}_t)\overline{\boldsymbol{\Sigma}}_t$

end function

$$\boldsymbol{\mathcal{F}} = \{\boldsymbol{I}_1, \boldsymbol{I}_2, \boldsymbol{I}_3, \boldsymbol{I}_4\}\xi\Pi\frac{\mathbf{p}}{\partial \mathbf{p}_w}, \begin{pmatrix} f_x z^{-1} & 0 & -x f_x z^{-2} & 0 \\ 0 & f_y z^{-1} & -y f_y z^{-2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & z^{-2} & 0 \end{pmatrix} \tag{3.4}$$

$$\Big(\sum_{j \in obs^t(\mathbf{P})} E_{ij}^{\mathbf{p}} + \lambda E_{is}^{\mathbf{p}}\Big)$$

$$\begin{aligned} E(\delta) &= E_{1L2L}^{\mathbf{p}_1} + E_{2L1L}^{\mathbf{p}_2} + E_{2L3L}^{\mathbf{p}_2} + E_{2L3L}^{\mathbf{p}_3} + E_{3L1L}^{\mathbf{p}_4} + E_{3L2L}^{\mathbf{p}_4} + E_{4L3L}^{\mathbf{p}_5} \\ &\quad + E_{1L1R}^{\mathbf{p}_1} + E_{2L2R}^{\mathbf{p}_2} + E_{4L4R}^{\mathbf{p}_5} \\ &= E_d(\delta) + E_s(\delta) \end{aligned}$$

$$E_s(\delta) = \begin{pmatrix} r^s_{\mathbf{p}_1} \\ r^s_{\mathbf{p}_2} \\ r^s_{\mathbf{p}_5} \end{pmatrix}^T \begin{pmatrix} \lambda w_{\mathbf{p}_1} & 0 & 0 \\ 0 & \lambda w_{\mathbf{p}_2} & 0 \\ 0 & 0 & \lambda w_{\mathbf{p}_5} \end{pmatrix} \begin{pmatrix} r^s_{\mathbf{p}_1} \\ r^s_{\mathbf{p}_2} \\ r^s_{\mathbf{p}_5} \end{pmatrix} = (\mathbf{r}^s)^T \mathbf{W} \mathbf{r}^s$$

$$J_s = \begin{pmatrix} \frac{\partial r^s_{\mathbf{p}_1}}{\partial \boldsymbol{\xi}_1} \cdots & \frac{\partial r^s_{\mathbf{p}_1}}{\partial \boldsymbol{\xi}_4} & \frac{\partial r^s_{\mathbf{p}_1}}{\partial d_{\mathbf{p}_1}} \cdots & \frac{\partial r^s_{\mathbf{p}_1}}{\partial d_{\mathbf{p}_5}} & \frac{\partial r^s_{\mathbf{p}_1}}{\partial a^L_1} \cdots & \frac{\partial r^s_{\mathbf{p}_1}}{\partial b^R_4} & \frac{\partial r^s_{\mathbf{p}_1}}{\partial f_x} \cdots & \frac{\partial r^s_{\mathbf{p}_1}}{\partial c_y} \\ \frac{\partial r^s_{\mathbf{p}_2}}{\partial \boldsymbol{\xi}_1} \cdots & \frac{\partial r^s_{\mathbf{p}_2}}{\partial \boldsymbol{\xi}_4} & \frac{\partial r^s_{\mathbf{p}_2}}{\partial d_{\mathbf{p}_1}} \cdots & \frac{\partial r^s_{\mathbf{p}_2}}{\partial d_{\mathbf{p}_5}} & \frac{\partial r^s_{\mathbf{p}_2}}{\partial a^L_1} \cdots & \frac{\partial r^s_{\mathbf{p}_2}}{\partial b^R_4} & \frac{\partial r^s_{\mathbf{p}_2}}{\partial f_x} \cdots & \frac{\partial r^s_{\mathbf{p}_2}}{\partial c_y} \\ \frac{\partial r^s_{\mathbf{p}_5}}{\partial \boldsymbol{\xi}_1} \cdots & \frac{\partial r^s_{\mathbf{p}_5}}{\partial \boldsymbol{\xi}_4} & \frac{\partial r^s_{\mathbf{p}_5}}{\partial d_{\mathbf{p}_1}} \cdots & \frac{\partial r^s_{\mathbf{p}_5}}{\partial d_{\mathbf{p}_5}} & \frac{\partial r^s_{\mathbf{p}_5}}{\partial a^L_1} \cdots & \frac{\partial r^s_{\mathbf{p}_5}}{\partial b^R_4} & \frac{\partial r^s_{\mathbf{p}_5}}{\partial f_x} \cdots & \frac{\partial r^s_{\mathbf{p}_5}}{\partial c_y} \end{pmatrix} \tag{2.2}$$

$$E_d(\delta) = \begin{pmatrix} (r^d_{\mathbf{p}_1})_{12} \\ (r^d_{\mathbf{p}_1})_{21} \\ \vdots \\ (r^d_{\mathbf{p}_5})_{43} \end{pmatrix}^T \begin{pmatrix} w_{\mathbf{p}_1} & 0 & \cdots & 0 \\ 0 & w_{\mathbf{p}_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_{\mathbf{p}_5} \end{pmatrix} \begin{pmatrix} (r^d_{\mathbf{p}_1})_{12} \\ (r^d_{\mathbf{p}_1})_{21} \\ \vdots \\ (r^d_{\mathbf{p}_5})_{43} \end{pmatrix} = (\mathbf{r}^d)^T \mathbf{W} \mathbf{r}^d$$

$$
J_d = \begin{pmatrix}
\frac{\partial (r^d_{\mathbf{p_1}})_{12}}{\partial \boldsymbol{\xi}_1} \cdots & \frac{\partial (r^d_{\mathbf{p_1}})_{12}}{\partial \boldsymbol{\xi}_4} & \frac{\partial (r^d_{\mathbf{p_1}})_{12}}{\partial d_{\mathbf{p_1}}} \cdots & \frac{\partial (r^d_{\mathbf{p_1}})_{12}}{\partial d_{\mathbf{p_5}}} & \frac{\partial (r^d_{\mathbf{p_1}})_{12}}{\partial a^L_1} \cdots & \frac{\partial (r^d_{\mathbf{p_1}})_{12}}{\partial b^R_4} & \frac{\partial (r^d_{\mathbf{p_1}})_{12}}{\partial f_x} \cdots & \frac{\partial (r^d_{\mathbf{p_1}})_{12}}{\partial c_y} \\
\frac{\partial (r^d_{\mathbf{p_1}})_{21}}{\partial \boldsymbol{\xi}_1} \cdots & \frac{\partial (r^d_{\mathbf{p_1}})_{21}}{\partial \boldsymbol{\xi}_4} & \frac{\partial (r^d_{\mathbf{p_1}})_{21}}{\partial d_{\mathbf{p_1}}} \cdots & \frac{\partial (r^d_{\mathbf{p_1}})_{21}}{\partial d_{\mathbf{p_5}}} & \frac{\partial (r^d_{\mathbf{p_1}})_{21}}{\partial a^L_1} \cdots & \frac{\partial (r^d_{\mathbf{p_1}})_{21}}{\partial b^R_4} & \frac{\partial (r^d_{\mathbf{p_1}})_{21}}{\partial f_x} \cdots & \frac{\partial (r^d_{\mathbf{p_1}})_{21}}{\partial c_y} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\partial (r^d_{\mathbf{p_5}})_{43}}{\partial \boldsymbol{\xi}_1} \cdots & \frac{\partial (r^d_{\mathbf{p_5}})_{43}}{\partial \boldsymbol{\xi}_4} & \frac{\partial (r^d_{\mathbf{p_5}})_{43}}{\partial d_{\mathbf{p_1}}} \cdots & \frac{\partial (r^d_{\mathbf{p_5}})_{43}}{\partial d_{\mathbf{p_5}}} & \frac{\partial (r^d_{\mathbf{p_5}})_{43}}{\partial a^L_1} \cdots & \frac{\partial (r^d_{\mathbf{p_5}})_{43}}{\partial b^R_4} & \frac{\partial (r^d_{\mathbf{p_5}})_{43}}{\partial f_x} \cdots & \frac{\partial (r^d_{\mathbf{p_5}})_{43}}{\partial c_y}
\end{pmatrix} \tag{2.2}
$$

$$
\begin{aligned}
&(\mathbf{J}_s^T \lambda \mathbf{W}^s \mathbf{J}_s + \mathbf{J}_d^T \mathbf{W}^d \mathbf{J}_d)\delta^* = -(\mathbf{J}_s^T \lambda \mathbf{W}^s \mathbf{r}_s + \mathbf{J}_d^T \mathbf{W}^d \mathbf{r}_d) \\
&\mathbf{J}_s \in \mathbb{R}^{49\times 3}, \mathbf{W}^s \in \mathbb{R}^{3\times 3}, \mathbf{J}_s \in \mathbb{R}^{49\times 7}, \mathbf{W}^s \in \mathbb{R}^{7\times 7},
\end{aligned} \tag{2.2}
$$

$$
\begin{aligned}
&\mathbf{p}^{'} = d^{iR}_{\mathbf{p}} \mathbf{K}(\mathbf{T}_{RL}((d^{iL}_{\mathbf{p}})^{-1}\mathbf{K}^{-1}\mathbf{p})) \\
&\frac{\partial \mathbf{p}^{'}}{\partial d^{iL}_{\mathbf{p}}} = 0?
\end{aligned} \tag{2.2}
$$

$$
\begin{aligned}
&\frac{\partial (r^d_{\mathbf{p}})}{\partial \boldsymbol{\xi_i}} = \frac{\partial I^L_j(\mathbf{p}^{'})}{\partial \boldsymbol{\xi_i}} = \frac{\partial I^L_j(\mathbf{p}^{'})}{\partial \mathbf{p}^{'}} \frac{\partial \mathbf{p}^{'}}{\partial \mathbf{p}^{'}_w} \frac{\partial \mathbf{p}^{'}_w}{\partial \boldsymbol{\xi_i}} \\
&\mathbf{p}^{'}_w = d^{jL}_{\mathbf{p}} \mathbf{K}(\mathbf{T}_j \mathbf{T}_i^{-1}((d^{iL}_{\mathbf{p}})^{-1}\mathbf{K}^{-1}\mathbf{p}))
\end{aligned} \tag{2.2}
$$

$$
\begin{cases}
\mathbf{p}_w = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} f_x^{-1}(d^{iL}_{\mathbf{p}})^{-1}(u^i - c_x) \\ f_y^{-1}(d^{iL}_{\mathbf{p}})^{-1}(v^i - c_y) \\ d^{iL}_{\mathbf{p}} \\ 1 \end{pmatrix} \\
\mathbf{p}^{'}_w = d^{jL}_{\mathbf{p}} \mathbf{K}(\mathbf{T}_j \mathbf{T}_i^{-1}((d^{iL}_{\mathbf{p}})^{-1}\mathbf{K}^{-1}\mathbf{p}))
\end{cases} \tag{2.2}
$$

$$
\begin{aligned}
&\xi^\wedge = \begin{pmatrix} \boldsymbol{\rho} \\ \boldsymbol{\phi} \end{pmatrix}^\wedge = \begin{pmatrix} \boldsymbol{\phi}^\wedge & \boldsymbol{\rho} \\ \mathbf{0}^T & \mathbf{0}^T \end{pmatrix} \in \mathbb{R}^{4\times 4} \\
&\boldsymbol{\epsilon} \in \mathbb{R}^3, \begin{pmatrix} \boldsymbol{\epsilon} \\ 1 \end{pmatrix}^\odot = \begin{pmatrix} \mathbf{E} & -\boldsymbol{\epsilon}^\wedge \\ \mathbf{0}^T & \mathbf{0}^T \end{pmatrix} \in \mathbb{R}^{4\times 6} \\
&\frac{\partial (exp(\xi^\wedge)\mathbf{p}_w)}{\partial \boldsymbol{\xi}} = \frac{\partial (\mathbf{T}\mathbf{p}_w)}{\partial \boldsymbol{\xi}} = (\mathbf{T}\mathbf{p}_w)^\odot \\
&\mathbf{T}\mathbf{p}_w = exp(\xi^\wedge)\mathbf{p}_w \approx (\mathbf{E} + \xi^\wedge)\mathbf{p}_w \\
&\frac{\partial (exp(\xi^\wedge)\mathbf{p}_w)}{\partial \boldsymbol{\xi}} \approx \frac{\partial (\mathbf{E} + \xi^\wedge)}{\partial \boldsymbol{\xi}} = \mathbf{0} + \frac{\partial (\xi^\wedge \mathbf{p}_w)}{\partial \boldsymbol{\xi}} \approx (\mathbf{T}\mathbf{p}_w)^\odot \\
&since, \frac{\partial (\mathbf{T}\mathbf{p}_w)}{\partial \boldsymbol{\xi}} = (\mathbf{T}^{-1}\mathbf{p}_w)^\odot = \frac{\partial (exp(-\xi^\wedge)\mathbf{p}_w)}{\partial \boldsymbol{\xi}} \\
&= \frac{\partial (\mathbf{E} - \xi^\wedge)}{\partial \boldsymbol{\xi}} = -(\mathbf{T}\mathbf{p}_w)^\odot
\end{aligned} \tag{2.2}
$$

$$
\begin{aligned}
&\mathbf{p}^{'} = d_{\mathbf{p}}^{jL}\mathbf{K}(\mathbf{T}_j\mathbf{T}_i^{-1}\mathbf{p}_w) \\
&assume : \mathbf{T}_j\mathbf{T}_i^{-1} = \begin{pmatrix} r_{11}^{ji} & r_{12}^{ji} & r_{13}^{ji} & t_1^{ji} \\ r_{21}^{ji} & r_{22}^{ji} & r_{23}^{ji} & t_2^{ji} \\ r_{31}^{ji} & r_{32}^{ji} & r_{33}^{ji} & t_3^{ji} \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&\mathbf{p}_w^{'} = \begin{pmatrix} r_{11}^{ji} f_x^{-1}(d_{\mathbf{p}}^{iL})^{-1}(u^i - c_x) + r_{12}^{ji} f_y^{-1}(d_{\mathbf{p}}^{iL})^{-1}(v^i - c_y) + r_{13}^{ji}(d_{\mathbf{p}}^{iL})^{-1} + t_1^{ji} \\ r_{21}^{ji} f_x^{-1}(d_{\mathbf{p}}^{iL})^{-1}(u^i - c_x) + r_{22}^{ji} f_y^{-1}(d_{\mathbf{p}}^{iL})^{-1}(v^i - c_y) + r_{23}^{ji}(d_{\mathbf{p}}^{iL})^{-1} + t_2^{ji} \\ r_{31}^{ji} f_x^{-1}(d_{\mathbf{p}}^{iL})^{-1}(u^i - c_x) + r_{32}^{ji} f_y^{-1}(d_{\mathbf{p}}^{iL})^{-1}(v^i - c_y) + r_{33}^{ji}(d_{\mathbf{p}}^{iL})^{-1} + t_3^{ji} \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{a}{d_{\mathbf{p}}^{iL}} + t_1^{ji} \\ \frac{b}{d_{\mathbf{p}}^{iL}} + t_2^{ji} \\ \frac{c}{d_{\mathbf{p}}^{iL}} + t_3^{ji} \\ 1 \end{pmatrix} = \begin{pmatrix} x^{'} \\ y^{'} \\ z^{'} \\ 1 \end{pmatrix} \Rightarrow d_{\mathbf{p}}^{jL} = (z^{'})^{-1}, \mathbf{p}^{'} = \begin{pmatrix} f_x x^{'} d_{\mathbf{p}}^{jL} + c_x \\ f_y y^{'} d_{\mathbf{p}}^{jL} + c_y \\ 1 \\ d_{\mathbf{p}}^{jL} \end{pmatrix} \\
&\frac{\partial(I_j^L(\mathbf{p}^{'}))}{\partial \mathbf{p}^{'}} = (g_x^{'}, g_y^{'}, 0, 0) \\
&\frac{\partial \mathbf{p}^{'}}{\partial \mathbf{p}_w^{'}} = \begin{pmatrix} f_x(z^{'})^{-1} & 0 & -x^{'} f_x (z^{'})^{-2} & 0 \\ 0 & f_y(z^{'})^{-1} & -y^{'} f_y (z^{'})^{-2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (z^{'})^{-2} & 0 \end{pmatrix}, \frac{\partial \mathbf{p}_w^{'}}{\partial d_{\mathbf{p}}^{iL}} = \begin{pmatrix} -\frac{a}{(d_{\mathbf{p}}^{iL})^2} \\ -\frac{b}{(d_{\mathbf{p}}^{iL})^2} \\ -\frac{c}{(d_{\mathbf{p}}^{iL})^2} \\ 0 \end{pmatrix} \\
&\Rightarrow \frac{\partial (r_{\mathbf{p}}^d)_{ij}}{\partial d_{\mathbf{p}}^{iL}} = \frac{\partial(I_j^L(\mathbf{p}^{'}))}{\partial d_{\mathbf{p}}^{iL}} = \frac{\partial(I_j^L(\mathbf{p}^{'}))}{\partial \mathbf{p}^{'}} \frac{\partial \mathbf{p}^{'}}{\partial \mathbf{p}_w^{'}} \frac{\partial \mathbf{p}_w^{'}}{\partial d_{\mathbf{p}}^{iL}} \\
&= -\frac{g_x^{'} f_x a}{z^{'}(d_{\mathbf{p}}^{iL})^2} - \frac{g_y^{'} f_y b}{z^{'}(d_{\mathbf{p}}^{iL})^2} + \frac{c(g_x^{'} x^{'} f_x + g_y^{'} y^{'} f_y)}{(z^{'} d_{\mathbf{p}}^{iL})^2} \\
&= \frac{c(g_x^{'} x^{'} f_x + g_y^{'} y^{'} f_y) - g_x^{'} f_x a z^{'} - g_y^{'} f_y b z^{'}}{(z^{'} d_{\mathbf{p}}^{iL})^2}
\end{aligned}
\tag{2.2}
$$

$$
\begin{aligned}
&\begin{pmatrix} x^{'} \\ y^{'} \\ z^{'} \\ 1 \end{pmatrix} = exp(\boldsymbol{\delta}) \begin{pmatrix} \boldsymbol{p}_x/d_p \\ \boldsymbol{p}_y/d_p \\ 1/d_p \\ 1 \end{pmatrix} = \frac{1}{d_p} \begin{pmatrix} \boldsymbol{R}_1 & t_1 \\ \boldsymbol{R}_2 & t_2 \\ \boldsymbol{R}_3 & t_3 \\ \mathbf{0} & 1 \end{pmatrix}_{4\times 4} \begin{pmatrix} \\ \boldsymbol{P} \\ \\ d_p \end{pmatrix}_{4\times 1} = \frac{1}{d_p} \begin{pmatrix} \boldsymbol{R}_1\boldsymbol{P} + t_1 d_p \\ \boldsymbol{R}_2\boldsymbol{P} + t_2 d_p \\ \boldsymbol{R}_3\boldsymbol{P} + t_3 d_p \\ d_p \end{pmatrix} \\
&w_m = \begin{pmatrix} x^{'}/z^{'} \\ y^{'}/z^{'} \end{pmatrix} = \begin{pmatrix} \dfrac{\boldsymbol{R}_1\boldsymbol{P} + t_1 d_p}{\boldsymbol{R}_3\boldsymbol{P} + t_3 d_p} \\ \dfrac{\boldsymbol{R}_2\boldsymbol{P} + t_2 d_p}{\boldsymbol{R}_3\boldsymbol{P} + t_3 d_p} \end{pmatrix} \\
&\frac{\partial_{r_{p_m}}(\boldsymbol{p}_m, \boldsymbol{\delta})}{\partial D_i(\boldsymbol{p}_m)} := \frac{\partial f_m}{\partial d_{p_m}} = -\frac{\partial I_j}{\partial w_m}\frac{\partial w_m}{\partial d_{p_m}} \\
&= -\begin{pmatrix} g_x & g_y \end{pmatrix} \begin{pmatrix} \frac{t_1 z^{'} - x^{'} t_3}{z^{'2}} \\ \frac{t_2 z^{'} - y^{'} t_3}{z^{'2}} \end{pmatrix} = -(g_x \frac{t_1 z^{'} - x^{'} t_3}{z^{'2}} + g_y \frac{t_2 z^{'} - y^{'} t_3}{z^{'2}})
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
&assume : \mathbf{T}_{RL} = \begin{pmatrix} 1 & 0 & 0 & t_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&\mathbf{p}^{'} = d_{\mathbf{p}}^{iR}\mathbf{K}(\mathbf{T}_{RL}\mathbf{p}_w) \\
&= d_{\mathbf{p}}^{iL} \begin{pmatrix} f_x & 0 & c_x & 0 \\ 0 & f_y & c_y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_x^{-1}(d_{\mathbf{p}}^{iL})^{-1}(u^i - c_x) + t_1 \\ f_y^{-1}(d_{\mathbf{p}}^{iL})^{-1}(v^i - c_y) \\ (d_{\mathbf{p}}^{iL})^{-1} \\ 1 \end{pmatrix} = \begin{pmatrix} u^i + t_1 f_x d_{\mathbf{p}}^{iL} \\ v^i \\ 1 \\ d_{\mathbf{p}}^{iL} \end{pmatrix} \\
&\frac{\partial r_{\mathbf{p}}^s}{\partial d_{\mathbf{p}}^{iL}} = \frac{\partial(I_j^L(\mathbf{p}^{'}))}{\partial d_{\mathbf{p}}^{iL}} = \frac{\partial(I_j^L(\mathbf{p}^{'}))}{\partial \mathbf{p}^{'}} \frac{\partial \mathbf{p}^{'}}{\partial d_{\mathbf{p}}^{iL}} = (g_x, g_y, 0, 0) \begin{pmatrix} t_1 f_x \\ 0 \\ 0 \\ 1 \end{pmatrix} = g_x t_1 f_x
\end{aligned}
\tag{2.2}
$$

$$
\frac{\partial \mathbf{p}_w^{'}}{\partial \boldsymbol{\xi_i}} = \mathbf{T}_j \frac{\partial (\mathbf{T}_i^{-1}\mathbf{p}_w^{'})}{\partial \boldsymbol{\xi_i}} = -\mathbf{T}_j(\mathbf{T}_i\mathbf{p}_w)^{\odot}
$$

$$
\frac{\partial \mathbf{p}_w^{'}}{\partial \boldsymbol{\xi_j}} = \frac{\partial (\mathbf{T}_j\mathbf{T}_i^{-1}\mathbf{p}_w)}{\partial \boldsymbol{\xi_i}} = (\mathbf{T}_j\mathbf{T}_i^{-1}\mathbf{p}_w)^{\odot}
$$

$$
= \begin{pmatrix} x^{'} \\ y^{'} \\ z^{'} \\ 1 \end{pmatrix}^{\odot} = \begin{pmatrix} 1 & 0 & 0 & 0 & z^{'} & -y^{'} \\ 0 & 1 & 0 & -z^{'} & 0 & x^{'} \\ 0 & 1 & 0 & y^{'} & -x^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
\Rightarrow \frac{\partial (r_{\mathbf{p}}^d)_{ij}}{\partial \boldsymbol{\xi_j}} = \frac{\partial (I_j^L(\mathbf{p}^{'}))}{\partial \mathbf{p}^{'}} \frac{\partial \mathbf{p}^{'}}{\partial \mathbf{p}_w^{'}} \frac{\partial \mathbf{p}_w^{'}}{\partial \boldsymbol{\xi_j}}
$$

$$
= (g_x^{'}, g_y^{'}, 0, 0) \begin{pmatrix} f_x(z^{'})^{-1} & 0 & -x^{'}f_x(z^{'})^{-2} & 0 \\ 0 & f_y(z^{'})^{-1} & -y^{'}f_y(z^{'})^{-2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (z^{'})^{-2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & z^{'} & -y^{'} \\ 0 & 1 & 0 & -z^{'} & 0 & x^{'} \\ 0 & 0 & 1 & y^{'} & -x^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} g_x^{'} f_x (z^{'})^{-1} \\ g_y^{'} f_y (z^{'})^{-1} \\ -(g_x^{'}x^{'}f_x + g_y^{'}y^{'}f_y)(z^{'})^{-2} \\ -g_y^{'}f_y - (g_x^{'}x^{'}y^{'}f_x + g_y^{'}(y^{'})^2 f_y)(z^{'})^{-2} \\ g_x^{'}f_x + (g_x^{'}(x^{'})^2 f_x + g_y^{'}x^{'}y^{'}f_y)(z^{'})^{-2} \\ -g_x^{'}f_x y^{'}(z^{'})^{-1} + g_y^{'}f_y x^{'}(z^{'})^{-1} \end{pmatrix}^T \tag{2.2}
$$

$$
\begin{aligned}
\frac{\partial \Delta \bar{R}_{0k}}{\partial \bar{b^g}} &= \sum_{m=0}^{k-1} \Delta \bar{R}_{m+1k}^T J_r^m \Delta t \\
\frac{\partial \Delta \bar{R}_{00}}{\partial \bar{b^g}} &= \Delta \bar{R}_{10}^T J_r^0 \Delta t \\
\frac{\partial \Delta \bar{R}_{01}}{\partial \bar{b^g}} &= \sum_{m=0}^{0} \Delta \bar{R}_{(m+1)1}^T J_r^m \Delta t \\
&\dots \\
\frac{\partial \Delta \bar{R}_{0(44)}}{\partial \bar{b^g}} &= \sum_{m=0}^{43} \Delta \bar{R}_{(m+1)44}^T J_r^m \Delta t
\end{aligned} \tag{2.2}
$$

$$
\Delta\bar{\mathbf{R}}_{ik} = \begin{cases} \mathbf{I}_{3\times 3}, & k = i \\ \prod_{m=i}^{k-1} \mathbf{Exp}((\tilde{\boldsymbol{\omega}}_m - \bar{\mathbf{b}}_i^g)\Delta t), & k > i \end{cases}
$$

$$
\begin{aligned}
&e.g. \quad k : 0 \to 44, i = 0 \\
&\quad \Delta\bar{\mathbf{R}}_{00} = \mathbf{I}_{3\times 3} \\
&\quad \Delta\bar{\mathbf{R}}_{01} = \mathbf{Exp}((\tilde{\boldsymbol{\omega}}_0 - \bar{\mathbf{b}}_0^g)\Delta t) \\
&\quad \Delta\bar{\mathbf{R}}_{02} = \mathbf{Exp}((\tilde{\boldsymbol{\omega}}_0 - \bar{\mathbf{b}}_0^g)\Delta t)\mathbf{Exp}((\tilde{\boldsymbol{\omega}}_1 - \bar{\mathbf{b}}_0^g)\Delta t) \\
&\quad \qquad \vdots \\
&\quad \Delta\bar{\mathbf{R}}_{0(44)} = \mathbf{Exp}((\tilde{\boldsymbol{\omega}}_0 - \bar{\mathbf{b}}_0^g)\Delta t)\mathbf{Exp}((\tilde{\boldsymbol{\omega}}_1 - \bar{\mathbf{b}}_0^g)\Delta t) \cdots \mathbf{Exp}((\tilde{\boldsymbol{\omega}}_{43} - \bar{\mathbf{b}}_0^g)\Delta t)
\end{aligned} \tag{2.2}
$$

$$
\Delta\bar{\mathbf{R}}_{kj} = \begin{cases} \prod_{m=k}^{j-1} \mathbf{Exp}((\tilde{\boldsymbol{\omega}}_m - \bar{\mathbf{b}}_i)\Delta t), & k < j \\ \mathbf{I}_{3\times 3}, & k = j \end{cases}
$$

$$
\begin{aligned}
&e.g. \quad k : 1 \to 45, j = 45, i = 0 \\
&\quad \Delta\bar{\mathbf{R}}_{1(45)} = \mathbf{Exp}((\tilde{\boldsymbol{\omega}}_1 - \bar{\mathbf{b}}_0^g)\Delta t)\mathbf{Exp}((\tilde{\boldsymbol{\omega}}_2 - \bar{\mathbf{b}}_0^g)\Delta t) \cdots \mathbf{Exp}((\tilde{\boldsymbol{\omega}}_{44} - \bar{\mathbf{b}}_0^g)\Delta t) \\
&\quad \qquad \vdots \\
&\quad \Delta\bar{\mathbf{R}}_{43(45)} = \mathbf{Exp}((\tilde{\boldsymbol{\omega}}_{43} - \bar{\mathbf{b}}_0^g)\Delta t)\mathbf{Exp}((\tilde{\boldsymbol{\omega}}_{44} - \bar{\mathbf{b}}_0^g)\Delta t) \\
&\quad \Delta\bar{\mathbf{R}}_{44(45)} = \mathbf{Exp}((\tilde{\boldsymbol{\omega}}_{44} - \bar{\mathbf{b}}_0^g)\Delta t) \\
&\quad \Delta\bar{\mathbf{R}}_{45(45)} = \mathbf{I}_{3\times 3}
\end{aligned} \tag{2.2}
$$

$$
\frac{\partial \Delta \bar{\mathbf{R}}_{ik}}{\partial \bar{\mathbf{b}}^g} = \begin{cases} \mathbf{0}_{3\times 3}, & k = i \\ \sum_{m=i}^{k-1} -\Delta \bar{\mathbf{R}}_{m+1k}^T \mathbf{J}_r^m \Delta t, & k > i \end{cases}
$$

$$
= \begin{cases} \mathbf{0}_{3\times 3}, & k = i \\ \mathbf{J}_r^0 \Delta t, & k = i+1 \\ \Delta \bar{\mathbf{R}}_{(k-1)k}^T \frac{\partial \Delta \bar{\mathbf{R}}_{i(k-1)}}{\partial \bar{\mathbf{b}}^g} + \mathbf{J}_r^{k-1} \Delta t, & k > i+1 \end{cases}
$$

$$
\begin{aligned}
&e.g. \quad i = 0, \quad k : 0 \to 45 \\
&\quad \frac{\partial \Delta \bar{\mathbf{R}}_{00}}{\partial \bar{\mathbf{b}}^g} = \mathbf{0}_{3\times 3} \\
&\quad \frac{\partial \Delta \bar{\mathbf{R}}_{01}}{\partial \bar{\mathbf{b}}^g} = \sum_{m=0}^{0} \Delta \bar{\mathbf{R}}_{(m+1)1}^T \mathbf{J}_r^m \Delta t = \Delta \bar{\mathbf{R}}_{11}^T \mathbf{J}_r^0 \Delta t = \mathbf{J}_r^0 \Delta t \\
&\quad \frac{\partial \Delta \bar{\mathbf{R}}_{02}}{\partial \bar{\mathbf{b}}^g} = \sum_{m=0}^{1} \Delta \bar{\mathbf{R}}_{(m+1)2}^T \mathbf{J}_r^m \Delta t = \Delta \bar{\mathbf{R}}_{12}^T \mathbf{J}_r^0 \Delta t + \Delta \bar{\mathbf{R}}_{22}^T \mathbf{J}_r^1 \Delta t = \Delta \bar{\mathbf{R}}_{12}^T \frac{\partial \Delta \bar{\mathbf{R}}_{01}}{\partial \bar{\mathbf{b}}^g} + \mathbf{J}_r^1 \Delta t \\
&\quad \frac{\partial \Delta \bar{\mathbf{R}}_{03}}{\partial \bar{\mathbf{b}}^g} = \sum_{m=0}^{2} \Delta \bar{\mathbf{R}}_{(m+1)2}^T \mathbf{J}_r^m \Delta t = \Delta \bar{\mathbf{R}}_{13}^T \mathbf{J}_r^0 \Delta t + \Delta \bar{\mathbf{R}}_{23}^T \mathbf{J}_r^1 \Delta t + \Delta \bar{\mathbf{R}}_{33}^T \mathbf{J}_r^2 \Delta t \\
&\qquad\qquad = \left(\Delta \bar{\mathbf{R}}_{12} \Delta \bar{\mathbf{R}}_{23}\right)^T \mathbf{J}_r^0 \Delta t + \Delta \bar{\mathbf{R}}_{23}^T \mathbf{J}_r^1 \Delta t + \mathbf{J}_r^2 \Delta t \\
&\qquad\qquad = \Delta \bar{\mathbf{R}}_{23}^T \Delta \bar{\mathbf{R}}_{12}^T \mathbf{J}_r^0 \Delta t + \Delta \bar{\mathbf{R}}_{23}^T \mathbf{J}_r^1 \Delta t + \mathbf{J}_r^2 \Delta t \\
&\qquad\qquad = \Delta \bar{\mathbf{R}}_{23}^T \frac{\partial \Delta \bar{\mathbf{R}}_{02}}{\partial \bar{\mathbf{b}}^g} + \mathbf{J}_r^2 \Delta t \\
&\qquad\qquad \vdots \\
&\quad \frac{\partial \Delta \bar{\mathbf{R}}_{0(44)}}{\partial \bar{\mathbf{b}}^g} = \sum_{m=0}^{43} \Delta \bar{\mathbf{R}}_{(m+1)44}^T \mathbf{J}_r^m \Delta t \\
&\qquad\qquad = \Delta \bar{\mathbf{R}}_{1(44)}^T \mathbf{J}_r^0 \Delta t + \Delta \bar{\mathbf{R}}_{2(44)}^T \mathbf{J}_r^1 \Delta t + \cdots + \Delta \bar{\mathbf{R}}_{43(44)}^T \mathbf{J}_r^{42} \Delta t + \Delta \bar{\mathbf{R}}_{44(44)}^T \mathbf{J}_r^{43} \Delta t \\
&\qquad\qquad = \Delta \bar{\mathbf{R}}_{43(44)}^T \frac{\partial \Delta \bar{\mathbf{R}}_{0(43)}}{\partial \bar{\mathbf{b}}^g} + \mathbf{J}_r^{43} \Delta t \\
&\quad \frac{\partial \Delta \bar{\mathbf{R}}_{0(45)}}{\partial \bar{\mathbf{b}}^g} = \sum_{m=0}^{44} \Delta \bar{\mathbf{R}}_{(m+1)45}^T \mathbf{J}_r^m \Delta t \\
&\qquad\qquad = \Delta \bar{\mathbf{R}}_{44(45)}^T \frac{\partial \Delta \bar{\mathbf{R}}_{0(44)}}{\partial \bar{\mathbf{b}}^g} + \mathbf{J}_r^{44} \Delta t
\end{aligned} \tag{2.2}
$$

$$
\begin{aligned}
\frac{\partial \Delta \bar{\mathbf{V}}_{ik}}{\partial \bar{\mathbf{b}}^{g}} &= \begin{cases} \mathbf{0}_{3\times 3}, & k=i \\ -\sum_{m=i}^{k-1} \Delta \bar{\mathbf{R}}_{im}^{T} (\tilde{\boldsymbol{a}}_m - \bar{\mathbf{b}}_i^{g})^{\wedge} \frac{\partial \Delta \bar{\mathbf{R}}_{im}}{\partial \bar{\mathbf{b}}^{g}} \Delta t, & k>i \end{cases} \\
&= \begin{cases} \mathbf{0}_{3\times 3}, & k=i \\ \frac{\partial \Delta \bar{\mathbf{V}}_{i(k-1)}}{\partial \bar{\mathbf{b}}^{g}} + \Delta \bar{\mathbf{R}}_{i(k-1)}^{T} (\tilde{\boldsymbol{a}}_{k-1} - \bar{\mathbf{b}}_i^{g})^{\wedge} \frac{\partial \Delta \bar{\mathbf{R}}_{i(k-1)}}{\partial \bar{\mathbf{b}}^{g}} \Delta t, & k>i \end{cases} \\
& e.g. \quad i=0, \quad k: 0 \to 44 \\
& \quad \frac{\partial \Delta \bar{\mathbf{V}}_{00}}{\partial \bar{\mathbf{b}}^{g}} = \mathbf{0}_{3\times 3} \\
& \quad \frac{\partial \Delta \bar{\mathbf{V}}_{01}}{\partial \bar{\mathbf{b}}^{g}} = -\Delta \bar{\mathbf{R}}_{00}^{T} (\tilde{\boldsymbol{a}}_0 - \bar{\mathbf{b}}_0^{g})^{\wedge} \frac{\partial \Delta \bar{\mathbf{R}}_{00}}{\partial \bar{\mathbf{b}}^{g}} \Delta t \\
& \quad \frac{\partial \Delta \bar{\mathbf{V}}_{02}}{\partial \bar{\mathbf{b}}^{g}} = -\Delta \bar{\mathbf{R}}_{00}^{T} (\tilde{\boldsymbol{a}}_0 - \bar{\mathbf{b}}_0^{g})^{\wedge} \frac{\partial \Delta \bar{\mathbf{R}}_{00}}{\partial \bar{\mathbf{b}}^{g}} \Delta t + \Delta \bar{\mathbf{R}}_{01}^{T} (\tilde{\boldsymbol{a}}_1 - \bar{\mathbf{b}}_0^{g})^{\wedge} \frac{\partial \Delta \bar{\mathbf{R}}_{01}}{\partial \bar{\mathbf{b}}^{g}} \Delta t \\
& \qquad\qquad = \frac{\partial \Delta \bar{\mathbf{V}}_{01}}{\partial \bar{\mathbf{b}}^{g}} + \Delta \bar{\mathbf{R}}_{01}^{T} (\tilde{\boldsymbol{a}}_1 - \bar{\mathbf{b}}_0^{g})^{\wedge} \frac{\partial \Delta \bar{\mathbf{R}}_{01}}{\partial \bar{\mathbf{b}}^{g}} \Delta t \\
& \quad \frac{\partial \Delta \bar{\mathbf{V}}_{03}}{\partial \bar{\mathbf{b}}^{g}} = \frac{\partial \Delta \bar{\mathbf{V}}_{02}}{\partial \bar{\mathbf{b}}^{g}} + \Delta \bar{\mathbf{R}}_{02}^{T} (\tilde{\boldsymbol{a}}_2 - \bar{\mathbf{b}}_0^{g})^{\wedge} \frac{\partial \Delta \bar{\mathbf{R}}_{02}}{\partial \bar{\mathbf{b}}^{g}} \Delta t \\
& \qquad\quad \vdots \\
& \quad \frac{\partial \Delta \bar{\mathbf{V}}_{0(44)}}{\partial \bar{\mathbf{b}}^{g}} == \frac{\partial \Delta \bar{\mathbf{V}}_{0(43)}}{\partial \bar{\mathbf{b}}^{g}} + \Delta \bar{\mathbf{R}}_{0(43)}^{T} (\tilde{\boldsymbol{a}}_{43} - \bar{\mathbf{b}}_0^{g})^{\wedge} \frac{\partial \Delta \bar{\mathbf{R}}_{0(43)}}{\partial \bar{\mathbf{b}}^{g}} \Delta t
\end{aligned} \tag{2.2}
$$

$$
\begin{aligned}
\mathbf{r}_{\Delta \mathbf{R}_{ij}} &= \mathtt{Log}\left(\left(\Delta \widetilde{\mathbf{R}}_{ij} (\bar{\mathbf{b}}_i^{g}) \mathtt{Exp}\left(\tfrac{\partial \Delta \bar{\mathbf{R}}_{ij}}{\partial \mathbf{b}^{g}} \delta \mathbf{b}^{g} \right) \right)^{T} \mathbf{R}_i^{T} \mathbf{R}_j \right) \\
\mathbf{r}_{\Delta \mathbf{v}_{ij}} &= \mathbf{R}_i^{T} (\mathbf{v}_j - \mathbf{v}_i - \mathbf{g} \Delta t_{ij}) \\
&\quad - [\Delta \widetilde{\mathbf{v}}_{ij} (\bar{\mathbf{b}}_i^{g}, \bar{\mathbf{b}}_i^{a}) + \tfrac{\partial \Delta \bar{\mathbf{v}}_{ij}}{\partial \mathbf{b}^{g}} \delta \mathbf{b}^{g}) + \tfrac{\partial \Delta \bar{\mathbf{v}}_{ij}}{\partial \mathbf{b}^{a}} \delta \mathbf{b}^{a})] \\
\mathbf{r}_{\Delta \mathbf{p}_{ij}} &= \mathbf{R}_i^{T} (\mathbf{p}_j - \mathbf{p}_i - \mathbf{v}_i \Delta t_{ij} - \tfrac{1}{2} \mathbf{g} \Delta t_{ij}^{2}) \\
&\quad - [\Delta \widetilde{\mathbf{p}}_{ij} (\bar{\mathbf{b}}_i^{g}, \bar{\mathbf{b}}_i^{a}) + \tfrac{\partial \Delta \bar{\mathbf{p}}_{ij}}{\partial \mathbf{b}^{g}} \delta \mathbf{b}^{g}) + \tfrac{\partial \Delta \bar{\mathbf{p}}_{ij}}{\partial \mathbf{b}^{a}} \delta \mathbf{b}^{a})]
\end{aligned} \tag{2.2}
$$

$$
\begin{aligned}
&\mathtt{Exp}(\boldsymbol{\xi}^\wedge) = \mathbf{T} = \begin{pmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 0 \end{pmatrix}_{4\times 4}, \mathtt{Exp}(\delta\boldsymbol{\xi}^\wedge) = \begin{pmatrix} \delta\boldsymbol{\phi}^\wedge & \delta\boldsymbol{\rho} \\ \mathbf{0}^T & 0 \end{pmatrix}_{4\times 4}, \mathbf{p} \in \mathbb{R}^3 \\
&\frac{\partial(\mathbf{T}\mathbf{p})}{\partial\delta\boldsymbol{\xi}} = \lim_{\delta\boldsymbol{\xi}\to\mathbf{0}} \frac{\mathtt{Exp}(\boldsymbol{\xi}^\wedge)\mathtt{Exp}(\delta\boldsymbol{\xi}^\wedge)\mathbf{p} - \mathtt{Exp}(\boldsymbol{\xi}^\wedge)\mathbf{p}}{\delta\boldsymbol{\xi}} \\
&\approx \lim_{\delta\boldsymbol{\xi}\to\mathbf{0}} \frac{\mathtt{Exp}(\boldsymbol{\xi}^\wedge)(\mathbf{I} - \delta\boldsymbol{\xi}^\wedge)\mathbf{p} - \mathtt{Exp}(\boldsymbol{\xi}^\wedge)\mathbf{p}}{\delta\boldsymbol{\xi}} = \lim_{\delta\boldsymbol{\xi}\to\mathbf{0}} - \frac{\mathtt{Exp}(\boldsymbol{\xi}^\wedge)\delta\boldsymbol{\xi}^\wedge\mathbf{p}}{\delta\boldsymbol{\xi}} \\
&= \lim_{\delta\boldsymbol{\xi}\to\mathbf{0}} - \frac{\begin{pmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 0 \end{pmatrix}\begin{pmatrix} \delta\boldsymbol{\phi}^\wedge\mathbf{p} + \delta\boldsymbol{\rho} \\ 1 \end{pmatrix}}{\delta\boldsymbol{\xi}} = \lim_{\delta\boldsymbol{\xi}\to\mathbf{0}} - \frac{\begin{pmatrix} \mathbf{R}\delta\boldsymbol{\phi}^\wedge\mathbf{p} + \mathbf{R}\delta\boldsymbol{\rho} + \mathbf{t} \\ \mathbf{0}^T \end{pmatrix}}{\delta\boldsymbol{\xi}} \\
&= \lim_{\delta\boldsymbol{\xi}\to\mathbf{0}} - \frac{\begin{pmatrix} -\mathbf{R}\mathbf{p}^\wedge\delta\boldsymbol{\phi} + \mathbf{R}\delta\boldsymbol{\rho} + \mathbf{t} \\ \mathbf{0}^T \end{pmatrix}}{\begin{pmatrix} \delta\boldsymbol{\rho} \\ \delta\boldsymbol{\phi} = \end{pmatrix}} = \begin{pmatrix} -\mathbf{R} & \mathbf{R}\mathbf{p}^\wedge \\ \mathbf{0}^T & \mathbf{0}^T \end{pmatrix}_{4\times 6}
\end{aligned}
\tag{1.1}
$$

Algorithm 1 Time-closest measurements selection

Require: $gyro\_list$,$acc\_list[s]$(an element in acc_list)
Ensure: $gyro\_measure$(time closest element in gyro_list)

function TIME_CLOSEST_SELECT($gyro\_list, i$)
  $t \leftarrow acc\_list[s].timestamp, i \leftarrow s$
  while $true$ **do**
    if $i >= gyro\_list.size$ **then**
      return $gyro\_list.back$
    else
      $t_{now} \leftarrow gyro\_list[i].timestamp$
      $t_{next} \leftarrow gyro\_list[i+1].timestamp$
      if $t_{now} < t$ **then**
        if $t_{next} > t$ **then**
          $t_{front} \leftarrow abs(t_{now} - t), t_{back} \leftarrow abs(t_{next} - t)$
          return $t_{front} > t_{back} ? gyro\_list[i] : gyro\_list[i+1]$
        else
          $i = i + 1$
        end if
      else if $t_{now} > t$ **then**
        $i = i - 1$
      else
        return $gyro\_list[i]$
      end if
    end if
  end while
end function

Algorithm 1 On-Manifold Preintegeration for IMU

Require: $gyro\_list, acc\_list, m, n, rotate\_list$

Ensure: $(\Delta\bar{\mathbf{R}}_{ij}, \frac{\partial\Delta\bar{\mathbf{R}}_{ij}}{\partial\mathbf{b}^g}, \frac{\partial\Delta\bar{\mathbf{v}}_{ij}}{\partial\mathbf{b}^a}, \frac{\partial\Delta\bar{\mathbf{p}}_{ij}}{\partial\mathbf{b}^a}), (\Delta\bar{\mathbf{v}}_{ij}, \frac{\partial\Delta\bar{\mathbf{v}}_{ij}}{\partial\mathbf{b}^g}, \Delta\bar{\mathbf{p}}_{ij}, \frac{\partial\Delta\bar{\mathbf{p}}_{ij}}{\partial\mathbf{b}^g}), \Sigma_{ij}$

function IMU_PREINTEGERATION($gyro\_list, acc\_list, m, n, rotate\_list$)

  for all $gyro\_list[i], i : 0 \to m$ **do**

    $last\_r \leftarrow rotate\_list[i-1]$

    $rot.timestamp \leftarrow gyro\_list[i].timestamp$

    $rot.\omega \leftarrow gyro\_list[i].\omega - \bar{\mathbf{b}}_i^g$

    $rot.\Delta\bar{\mathbf{R}}_{ik} \leftarrow last\_r.\Delta\bar{\mathbf{R}}_{ik} * \mathtt{Exp}(rot.\omega * \Delta t)$

    $rot.\Delta\bar{\mathbf{R}}_{(k-1)k} \leftarrow \mathtt{Exp}(rot.\omega * \Delta t)$

    $rot.\frac{\partial\Delta\bar{\mathbf{R}}_{ik}}{\partial\mathbf{b}^g} \leftarrow \Delta\bar{\mathbf{R}}_{(k-1)k}^T * last\_r.\frac{\partial\Delta\bar{\mathbf{R}}_{ik}}{\partial\mathbf{b}^g} - \mathtt{J}_r(rot.\omega * \Delta t) * \Delta t$

    $rot.\frac{\partial\Delta\bar{\mathbf{v}}_{ik}}{\partial\mathbf{b}^a} \leftarrow last\_r.\frac{\partial\Delta\bar{\mathbf{v}}_{ik}}{\partial\mathbf{b}^a} - last\_r.\Delta\bar{\mathbf{R}}_{ik} * \Delta t$

    $rot.\frac{\partial\Delta\bar{\mathbf{p}}_{ik}}{\partial\mathbf{b}^a} \leftarrow last\_r.\frac{\partial\Delta\bar{\mathbf{p}}_{ik}}{\partial\mathbf{b}^a} + last\_r.\frac{\partial\Delta\bar{\mathbf{v}}_{ik}}{\partial\mathbf{b}^a} * \Delta t - \frac{1}{2} last\_r.\Delta\bar{\mathbf{R}}_{ik} * \Delta t^2$

    $rotate\_list.push(rot)$

  end for

  $\Delta\bar{\mathbf{R}}_{ij} = rotate\_list.end.\Delta\bar{\mathbf{R}}_{ik}$

  $\frac{\partial\Delta\bar{\mathbf{R}}_{ij}}{\partial\mathbf{b}^g} = rotate\_list.end.\frac{\partial\Delta\bar{\mathbf{R}}_{ik}}{\partial\mathbf{b}^g}$

  $\frac{\partial\Delta\bar{\mathbf{v}}_{ij}}{\partial\mathbf{b}^a} = rotate\_list.end.\frac{\partial\Delta\bar{\mathbf{v}}_{ik}}{\partial\mathbf{b}^a}$

  $\frac{\partial\Delta\bar{\mathbf{p}}_{ij}}{\partial\mathbf{b}^a} = rotate\_list.end.\frac{\partial\Delta\bar{\mathbf{p}}_{ik}}{\partial\mathbf{b}^a}$

  for all $acc\_list[i], i : 0 \to n$ **do**

    $cls\_r \leftarrow time\_closest\_select(rotate\_list, acc\_list[i])$

    $acc \leftarrow acc\_list[i] - \bar{\mathbf{b}}_i^a$

    $\Delta\bar{\mathbf{v}}_{ij} += cls\_r.\Delta\bar{\mathbf{R}}_{ik} * acc * \Delta t$

    $\frac{\partial\Delta\bar{\mathbf{v}}_{ij}}{\partial\mathbf{b}^g} -= cls\_r.\Delta\bar{\mathbf{R}}_{ik} * acc^\wedge * cls\_r.\frac{\partial\Delta\bar{\mathbf{R}}_{ik}}{\partial\mathbf{b}^g} * \Delta t$

    $\Delta\bar{\mathbf{p}}_{ij} += \Delta\bar{\mathbf{v}}_{ij} * \Delta t + \frac{1}{2} cls\_r.\Delta\bar{\mathbf{R}}_{ik} * acc * \Delta t^2$

    $\frac{\partial\Delta\bar{\mathbf{p}}_{ij}}{\partial\mathbf{b}^g} += cls\_r.\frac{\partial\Delta\bar{\mathbf{v}}_{ik}}{\partial\mathbf{b}^g}\Delta t - \frac{1}{2} cls\_r.\Delta\bar{\mathbf{R}}_{ik} * acc^\wedge * cls\_r.\frac{\partial\Delta\bar{\mathbf{R}}_{ik}}{\partial\mathbf{b}^g} * \Delta t^2$

    $$A = \begin{pmatrix} cls\_r.\Delta\bar{\mathbf{R}}_{(k-1)k}^T & \mathbf{0} & \mathbf{0} \\ -cls\_r.\Delta\bar{\mathbf{R}}_{ik} * acc^\wedge * \Delta t & \mathbf{I} & \mathbf{0} \\ -\frac{1}{2} cls\_r.\Delta\bar{\mathbf{R}}_{ik} * acc^\wedge * \Delta t^2 & \Delta t\mathbf{I} & \mathbf{I} \end{pmatrix}$$

    $$B = \begin{pmatrix} \mathtt{J}_r(rot.\omega * \Delta t) * \Delta t & \mathbf{0} \\ \mathbf{0} & cls\_r.\Delta\bar{\mathbf{R}}_{ik} * \Delta t \\ \mathbf{0} & \frac{1}{2} cls\_r.\Delta\bar{\mathbf{R}}_{ik} * \Delta t^2 \end{pmatrix}$$

    $\Sigma_{ij} = A * \Sigma_{ij} * A^T + B * \Sigma_\eta * B^T$

  end for

end function

$$
\begin{aligned}
\frac{\partial \mathbf{r}_{\Delta \mathbf{p}_{ij}}}{\partial \delta \mathbf{p}_i} &= -\mathbf{I} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{p}_{ij}}}{\partial \delta \boldsymbol{\phi}_i} &= (\mathbf{R}_i^T(\mathbf{p}_j - \mathbf{p}_i - \mathbf{v}_i \Delta t_{ij} - \frac{1}{2}\mathbf{g}\Delta t_{ij}^2))^\wedge \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{p}_{ij}}}{\partial \delta \mathbf{v}_i} &= -\mathbf{R}_i^T \Delta t_{ij} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{p}_{ij}}}{\partial \delta \mathbf{p}_j} &= \mathbf{R}_i^T \mathbf{R}_j \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{p}_{ij}}}{\partial \delta \boldsymbol{\phi}_j} &= \mathbf{0} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{p}_{ij}}}{\partial \delta \mathbf{v}_j} &= \mathbf{0} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{p}_{ij}}}{\partial \delta \mathbf{b}_i^a} &= -\frac{\partial \Delta \bar{\mathbf{p}}_{ij}}{\partial \mathbf{b}^a} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{p}_{ij}}}{\partial \delta \mathbf{b}_i^g} &= -\frac{\partial \Delta \bar{\mathbf{p}}_{ij}}{\partial \mathbf{b}^g}
\end{aligned} \tag{2.2}
$$

$$
\begin{aligned}
\frac{\partial \mathbf{r}_{\Delta \mathbf{v}_{ij}}}{\partial \delta \mathbf{p}_i} &= \mathbf{0} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{v}_{ij}}}{\partial \delta \boldsymbol{\phi}_i} &= (\mathbf{R}_i^T(\mathbf{v}_j - \mathbf{v}_i - \mathbf{g}\Delta t_{ij}))^\wedge \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{v}_{ij}}}{\partial \delta \mathbf{v}_i} &= -\mathbf{R}_i^T \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{v}_{ij}}}{\partial \delta \mathbf{p}_j} &= \mathbf{0} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{v}_{ij}}}{\partial \delta \boldsymbol{\phi}_j} &= \mathbf{0} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{v}_{ij}}}{\partial \delta \mathbf{v}_j} &= \mathbf{R}_i^T \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{v}_{ij}}}{\partial \delta \mathbf{b}_i^a} &= -\frac{\partial \Delta \bar{\mathbf{v}}_{ij}}{\partial \mathbf{b}^a} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{v}_{ij}}}{\partial \delta \mathbf{b}_i^g} &= -\frac{\partial \Delta \bar{\mathbf{v}}_{ij}}{\partial \mathbf{b}^g}
\end{aligned} \tag{2.2}
$$

$$
\begin{aligned}
\frac{\partial \mathbf{r}_{\Delta \mathbf{R}_{ij}}}{\partial \delta \mathbf{p}_i} &= \mathbf{0} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{R}_{ij}}}{\partial \delta \boldsymbol{\phi}_i} &= -\mathtt{J}_r^{-1}(\mathbf{r}_{\Delta \mathbf{R}_{ij}})\mathbf{R}_j^T \mathbf{R}_i \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{R}_{ij}}}{\partial \delta \mathbf{v}_i} &= \mathbf{0} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{R}_{ij}}}{\partial \delta \mathbf{p}_j} &= \mathbf{0} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{R}_{ij}}}{\partial \delta \boldsymbol{\phi}_j} &= \mathtt{J}_r^{-1}(\mathbf{r}_{\Delta \mathbf{R}_{ij}}) \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{R}_{ij}}}{\partial \delta \mathbf{v}_j} &= \mathbf{0} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{R}_{ij}}}{\partial \delta \mathbf{b}_i^a} &= \mathbf{0} \\
\frac{\partial \mathbf{r}_{\Delta \mathbf{R}_{ij}}}{\partial \delta \mathbf{b}_i^g} &= -\mathtt{J}_r^{-1}(\mathbf{r}_{\Delta \mathbf{R}_{ij}})\mathtt{Exp}(\mathbf{r}_{\Delta \mathbf{R}_{ij}})^T \mathtt{J}_r(\frac{\partial \Delta \bar{\mathbf{R}}_{ij}}{\partial \mathbf{b}^g}\delta \mathbf{b}_i^g)\frac{\partial \Delta \bar{\mathbf{R}}_{ij}}{\partial \mathbf{b}^g}
\end{aligned}
\tag{2.2}
$$

$$
\begin{aligned}
&\mathtt{Exp}(\boldsymbol{\xi}^\wedge) = \mathbf{T} = \begin{pmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{pmatrix}_{4\times 4}, \mathtt{Exp}(\delta\boldsymbol{\xi}^\wedge) = \begin{pmatrix} \delta\boldsymbol{\phi}^\wedge & \delta\boldsymbol{\rho} \\ \mathbf{0}^T & 1 \end{pmatrix}_{4\times 4}, \mathbf{p} \in \mathbb{R}^3 \\
&\frac{\partial(\mathbf{T}\mathbf{p})}{\partial \delta\boldsymbol{\xi}} = \lim_{\delta\boldsymbol{\xi}\to\mathbf{0}} \frac{\mathtt{Exp}(\boldsymbol{\xi}^\wedge)\mathtt{Exp}(\delta\boldsymbol{\xi}^\wedge)\mathbf{p} - \mathtt{Exp}(\boldsymbol{\xi}^\wedge)\mathbf{p}}{\delta\boldsymbol{\xi}} \\
&\approx \lim_{\delta\boldsymbol{\xi}\to\mathbf{0}} \frac{\mathtt{Exp}(\boldsymbol{\xi}^\wedge)(\mathbf{I}+\delta\boldsymbol{\xi}^\wedge)\mathbf{p} - \mathtt{Exp}(\boldsymbol{\xi}^\wedge)\mathbf{p}}{\delta\boldsymbol{\xi}} = \lim_{\delta\boldsymbol{\xi}\to\mathbf{0}} \frac{\mathtt{Exp}(\boldsymbol{\xi}^\wedge)\delta\boldsymbol{\xi}^\wedge\mathbf{p}}{\delta\boldsymbol{\xi}} \\
&= \lim_{\delta\boldsymbol{\xi}\to\mathbf{0}} \frac{\begin{pmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{pmatrix}\begin{pmatrix} \delta\boldsymbol{\phi}^\wedge\mathbf{p} + \delta\boldsymbol{\rho} \\ 1 \end{pmatrix}}{\delta\boldsymbol{\xi}} = \lim_{\delta\boldsymbol{\xi}\to\mathbf{0}} \frac{\begin{pmatrix} \mathbf{R}\delta\boldsymbol{\phi}^\wedge\mathbf{p} + \mathbf{R}\delta\boldsymbol{\rho} + \mathbf{t} \\ 1 \end{pmatrix}}{\delta\boldsymbol{\xi}} \\
&= \lim_{\delta\boldsymbol{\xi}\to\mathbf{0}} \frac{\begin{pmatrix} -\mathbf{R}\mathbf{p}^\wedge\delta\boldsymbol{\phi} + \mathbf{R}\delta\boldsymbol{\rho} + \mathbf{t} \\ 1 \end{pmatrix}}{\begin{pmatrix} \delta\boldsymbol{\rho} \\ \delta\boldsymbol{\phi} \end{pmatrix}} = \begin{pmatrix} \mathbf{R} & -\mathbf{R}\mathbf{p}^\wedge \\ \mathbf{0}^T & \mathbf{0}^T \end{pmatrix}_{4\times 6}
\end{aligned}
\tag{1.1}
$$

$$
\begin{aligned}
\frac{\partial(\mathbf{T}^{-1}\mathbf{p})}{\partial\delta\boldsymbol{\xi}} &= \lim_{\delta\boldsymbol{\xi}\to\mathbf{0}} \frac{(\texttt{Exp}(\boldsymbol{\xi}^\wedge)\texttt{Exp}(\delta\boldsymbol{\xi}^\wedge))^{-1}\mathbf{p} - \texttt{Exp}(-\boldsymbol{\xi}^\wedge)\mathbf{p}}{\delta\boldsymbol{\xi}} \\
&= \lim_{\delta\boldsymbol{\xi}\to\mathbf{0}} \frac{(\texttt{Exp}(\delta\boldsymbol{\xi}^\wedge))^{-1}(\texttt{Exp}(\boldsymbol{\xi}^\wedge))^{-1}\mathbf{p} - \texttt{Exp}(-\boldsymbol{\xi}^\wedge)\mathbf{p}}{\delta\boldsymbol{\xi}} \\
&= \lim_{\delta\boldsymbol{\xi}\to\mathbf{0}} \frac{\texttt{Exp}(-\delta\boldsymbol{\xi}^\wedge)\texttt{Exp}(-\boldsymbol{\xi}^\wedge)\mathbf{p} - \texttt{Exp}(-\boldsymbol{\xi}^\wedge)\mathbf{p}}{\delta\boldsymbol{\xi}} \\
&\approx \lim_{\delta\boldsymbol{\xi}\to\mathbf{0}} \frac{(\mathbf{I} - \delta\boldsymbol{\xi}^\wedge)\texttt{Exp}(-\boldsymbol{\xi}^\wedge)\mathbf{p} - \texttt{Exp}(-\boldsymbol{\xi}^\wedge)\mathbf{p}}{\delta\boldsymbol{\xi}} = \lim_{\delta\boldsymbol{\xi}\to\mathbf{0}} - \frac{\delta\boldsymbol{\xi}^\wedge\texttt{Exp}(-\boldsymbol{\xi}^\wedge)\mathbf{p}}{\delta\boldsymbol{\xi}} \\
&= \lim_{\delta\boldsymbol{\xi}\to\mathbf{0}} - \frac{\begin{pmatrix} \delta\boldsymbol{\phi}^\wedge & \delta\boldsymbol{\rho} \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} \mathbf{R}^{-1} & -\mathbf{R}^{-1}\mathbf{t} \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ 1 \end{pmatrix}}{\delta\boldsymbol{\xi}} \\
&= \lim_{\delta\boldsymbol{\xi}\to\mathbf{0}} - \frac{\begin{pmatrix} \delta\boldsymbol{\phi}^\wedge & \delta\boldsymbol{\rho} \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} \mathbf{R}^{-1}\mathbf{p} - \mathbf{R}^{-1}\mathbf{t} \\ 1 \end{pmatrix}}{\delta\boldsymbol{\xi}} \\
&= \lim_{\delta\boldsymbol{\xi}\to\mathbf{0}} - \frac{\begin{pmatrix} \delta\boldsymbol{\phi}^\wedge(\mathbf{R}^{-1}\mathbf{p} - \mathbf{R}^{-1}\mathbf{t}) + \delta\boldsymbol{\rho} \\ 1 \end{pmatrix}}{\delta\boldsymbol{\xi}} \\
&= \lim_{\delta\boldsymbol{\xi}\to\mathbf{0}} - \frac{\begin{pmatrix} -(\mathbf{R}^{-1}\mathbf{p} - \mathbf{R}^{-1}\mathbf{t})^\wedge\delta\boldsymbol{\phi} + \delta\boldsymbol{\rho} \\ 1 \end{pmatrix}}{\begin{pmatrix} \delta\boldsymbol{\rho} \\ \delta\boldsymbol{\phi} \end{pmatrix}} = \begin{pmatrix} -\mathbf{I}_3 & (\mathbf{R}^{-1}\mathbf{p} - \mathbf{R}^{-1}\mathbf{t})^\wedge \\ \mathbf{0}^T & \mathbf{0}^T \end{pmatrix}_{4\times 6}
\end{aligned}
\tag{1.2}
$$

Algorithm 3 Levenberg-Marquardt 迭代法

Require: $\mathcal{K}_i$,$\mathcal{I}_j$,$\boldsymbol{\delta}_{i(i-1)}$,k_{max}(迭代法)
Ensure: $\boldsymbol{\delta}_{ji}$

function TRACKFRAME($Array, left, middle, right$)
  $v \leftarrow 2$
  $\boldsymbol{\delta} \leftarrow \boldsymbol{\delta}_{i(i-1)}$
  $\boldsymbol{f}_i$
  for $i = 0 \rightarrow k_{max} - 1$ **do**
    $\boldsymbol{J}^T\boldsymbol{W}\boldsymbol{J}$
    while $true$ **do**
      slove $\boldsymbol{\delta}^*$
      update
      $\boldsymbol{f}_i$
      $error$=caclweight
      if $error < lastError$ **then**
        $lastResidual$=$lastError$=$error$
        if $\lambda <= 0.2$ **then**
          λ=0
        else
          λ=$\frac{1}{2}\lambda$
        end if
        break
      else
        if 迭代法< ε_1 **then**
          i=$k_{max} - 1$
          break
        end if
        if λ==0 **then**
          λ=0.2
        else
          λ=2*inctry
        end if
      end if
    end while
  end for
  return $\boldsymbol{\delta}$
end function