Formula derivation of Direct Sparse Visual-Inertial Odometry with Stereo Cameras

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ABSTRACT

We present Stereo VI-DSO, a novel approach for visualinertial odometry, which jointly estimates camera poses and sparse scene geometry by minimizing photometric and IMU measurement errors in a combined energy functional. The visual part of the system performs a bundle-adjustment like optimization on a sparse set of points, but unlike key-point based systems it directly minimizes a photometric error. This makes it possible for the system to track not only corners, but any pixels with large enough intensity gradients. IMU information is accumulated between several frames using measurement preintegration, and is inserted into the optimization as an additional constraint between keyframes. We explicitly include scale and gravity direction into our model and jointly optimize them together with other variables such as poses. As the scale is often not immediately observable using IMU data this allows us to initialize our visual-inertial system with an arbitrary scale instead of having to delay the initialization until everything is observable. We perform partial marginalization of old variables so that updates can be computed in a reasonable time. In order to keep the system consistent we propose a novel strategy which we call ”dynamic marginalization”. This technique allows us to use partial marginalization even in cases where the initial scale estimate is far from the optimum. We evaluate our method on the challenging EuRoC dataset, showing that VI-DSO outperforms the state of the art.

Key Words: VSLAM, IMU, Stereo vision, Mobile robot

Content

Chapter1 PHOTO RESIDUALS

1.1 INTRODUCTION

Windowed Optimization is a classic method in non-linear optimization.

1.1.1 NOTATION

Throughout the paper, we will write matrices as bold capital letters ($\mathbf{R}$) and vectors as bold lower case letters ($\boldsymbol{\xi}$), light lower-case letters to denote scalars (s). Light upper-case letters are used to represent functions (I).

Homogeneous camera calibration matrices are denoted by $\mathbf{K}$ as (2.1). Camera poses are represented by matrices of the special Euclidean group $\mathbf{T} \in SE(3)$, which transform a 3D coordinate from the camera coordinate system to the world coordinate system. In this paper, a homogeneous 2D image coordinate point $\mathbf{p}$ is represented by its image coordinate and inverse depth as (2.1) relative to its host keyframe I_i^L. The host keyframe is the frame the point got selected from. Corresponding homogeneous 3D world coordinate point $\mathbf{p}_w$ is denoted as (2.1). $\Pi_{\mathbf{K}}$ are used to denote camera projection functions. The jacobian of I_i, $\Pi_{\mathbf{K}}$ is denoted as (2.1)

$$
\begin{aligned}
&\mathbf{K} = \begin{pmatrix} f_x & 0 & c_x & 0 \\ 0 & f_y & c_y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{K}^{-1} = \begin{pmatrix} f_x^{-1} & 0 & -f_x^{-1}c_x & 0 \\ 0 & f_y^{-1} & -f_y^{-1}c_y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
&\mathbf{p} = \begin{pmatrix} u^i \\ v^i \\ 1 \\ d_{\mathbf{p}} \end{pmatrix}, \mathbf{p}_w = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}, d_{\mathbf{p}} = z^{-1}, \mathbf{p} = d_{\mathbf{p}}\mathbf{K}\mathbf{p}_w = \Pi_{\mathbf{K}}(\mathbf{p}_w) \\
&\frac{\partial I_i^L(\mathbf{p})}{\partial \mathbf{p}} = (g_x, g_y, 0, 0), \frac{\partial \mathbf{p}}{\partial \mathbf{p}_w} = \begin{pmatrix} f_x z^{-1} & 0 & -x f_x z^{-2} & 0 \\ 0 & f_y z^{-1} & -y f_y z^{-2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & z^{-2} & 0 \end{pmatrix}
\end{aligned}
\tag{2.1}
$$

1.1.2 QUESTION IMPORT

Assume we observe 5 points $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5\}$ in 4 keyframes $\boldsymbol{\mathcal{F}} = \{I_1, I_2, I_3, I_4\}$, every keyframe has stereo vision (I_i^L, I_i^R) abbreviated as (iL, iR). A point can also be observed by other frame as shown in Table(2.1). Question is how to use Windowed Optimization method to make our observation more accurate ?

Table (2.1)

Image point	Host keyframe	Observe by
$\mathbf{p}_1$	$1L$	$1R, 2L$
$\mathbf{p}_2$	$2L$	$2R, 1L, 3L$
$\mathbf{p}_3$	$2L$	$3L$
$\mathbf{p}_4$	$3L$	$1L, 2L$
$\mathbf{p}_5$	$4L$	$3L, 4L$

1.2 SOLUTION

We use direct method to construct residual, Windowed Gauss-Newton method to optimization residual。

1.2.1 CONSTRUCT RESIDUAL

Dynamic multi-view stereo residuals $E_{ij}^{\mathbf{p}}$ are defined as

$$E_{ij}^{\mathbf{p}} = w_{\mathbf{p}} ||(r_{\mathbf{p}}^d)_{ij}||_\gamma, \quad (r_{\mathbf{p}}^d)_{ij} := I_j^L(\mathbf{p}^{'}) - b_j^L - \frac{e^{a_j^L}}{e^{a_i^L}}(I_i^L(\mathbf{p}) - b_i^L) \tag{2.2}$$

γ is Huber norm. a_i^L, b_i^L is affine brightness parameters to frame iL . $w_{\mathbf{p}}$ is a gradient-dependent weighting parameters, $\mathbf{p}$ in frame I_i^L projected to I_j^L is $\mathbf{p}^{'}$ as:

$$w_{\mathbf{p}} := \frac{c^2}{c^2 + ||\nabla I_i(\mathbf{p})||_2^2}, \quad \mathbf{p}^{'} = d_{\mathbf{p}}^{jL}\mathbf{K}(\mathbf{T}_j\mathbf{T}_i^{-1}((d_{\mathbf{p}}^{iL})^{-1}\mathbf{K}^{-1}\mathbf{p})) \tag{2.3}$$

Static one-view stereo residuals $E_{is}^{\mathbf{p}}$ are modified to

$$E_{is}^{\mathbf{p}} = w_{\mathbf{p}} ||r_{\mathbf{p}}^s||_\gamma, \quad r_{\mathbf{p}}^s := I_i^R(\mathbf{p}^{'}) - b_i^R - \frac{e^{a_i^R}}{e^{a_i^L}}(I_i^L(\mathbf{p}) - b_i^L) \tag{2.2}$$

Hostframe of $\mathbf{p}$ is I_i^L. a_i^R, b_i^R is affine brightness parameters to frame iR. $\mathbf{p}$ in frame I_i^L projected to I_i^R is $\mathbf{p}^{'}$ as :

$$\mathbf{p}^{'} = d_{\mathbf{p}}^{iR}\mathbf{K}(\mathbf{T}_{RL}((d_{\mathbf{p}}^{iL})^{-1}\mathbf{K}^{-1}\mathbf{p})) \tag{2.2}$$

Total residuals

$$E(\delta) = \sum_{i \in \mathcal{F}} \sum_{\mathbf{p} \in \mathcal{P}_i} (\sum_{j \in obs^t(\mathbf{P})} E_{ij}^{\mathbf{p}} + \lambda E_{is}^{\mathbf{p}}) \tag{2.2}$$

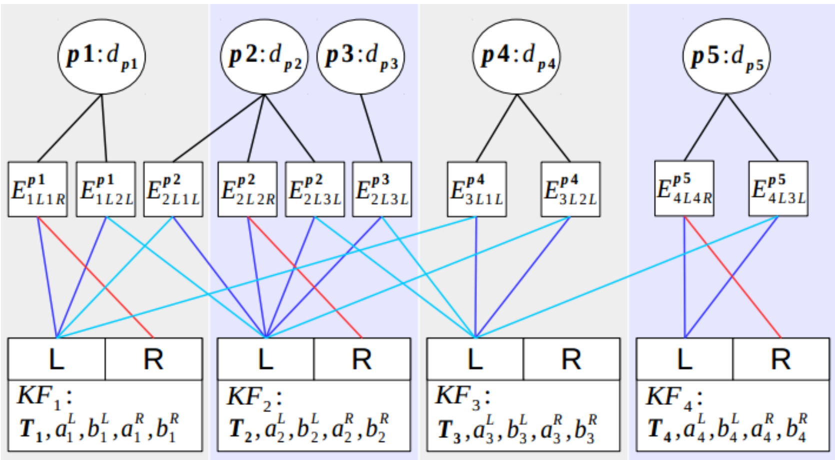
To balance the relative weights of temporal multi-view and static stereo, we introduce a coupling factor λ to weight the constraints from static stereo differently.

$$\boldsymbol{\delta} = \begin{pmatrix} (\boldsymbol{\xi}_1^T, \dots, \boldsymbol{\xi}_{N_f}^T)^T \\ (d_{\mathbf{p}_1}, \dots, d_{\mathbf{p}_{N_p}})^T \\ (a_1^L, a_1^R, b_1^L, b_1^R)^T \\ \vdots \\ (a_{N_f}^L, a_{N_f}^R, b_{N_f}^L, b_{N_f}^R)^T \\ (f_x, f_y, c_x, c_y)^T \end{pmatrix} \in \mathbb{R}^{10N_f + N_p + 4}, \boldsymbol{\xi}_i = (ln\mathbf{T}_i)^\vee \in \mathbb{R}^6 \tag{2.1}$$

$\mathcal{P}_i$ is a set of all image point host by frame iL. $obs^t(\mathbf{p})$ are the observations of $\mathbf{p}$ from temporal multi-view stereo. If there are N_p image point and N_f keyframes in $\boldsymbol{\mathcal{F}}$, optimization variable $\boldsymbol{\delta}$ is

In this example, there are 7 dynamic residuals and 3 static residuals, Factor graph of the residuals function is

Total residuals is



$$\begin{aligned} E(\delta) &= E_{1L2L}^{\mathbf{p}_1} + E_{2L1L}^{\mathbf{p}_2} + E_{2L3L}^{\mathbf{p}_2} + E_{2L3L}^{\mathbf{p}_3} + E_{3L1L}^{\mathbf{p}_4} + E_{3L2L}^{\mathbf{p}_4} + E_{4L3L}^{\mathbf{p}_5} \\ &\quad + E_{1L1R}^{\mathbf{p}_1} + E_{2L2R}^{\mathbf{p}_2} + E_{4L4R}^{\mathbf{p}_5} \\ &= E_d(\delta) + E_s(\delta) \end{aligned}$$

$$E_s(\delta) = \begin{pmatrix} r_{\mathbf{p}_1}^s \\ r_{\mathbf{p}_2}^s \\ r_{\mathbf{p}_5}^s \end{pmatrix}^T \begin{pmatrix} \lambda w_{\mathbf{p}_1} & 0 & 0 \\ 0 & \lambda w_{\mathbf{p}_2} & 0 \\ 0 & 0 & \lambda w_{\mathbf{p}_5} \end{pmatrix} \begin{pmatrix} r_{\mathbf{p}_1}^s \\ r_{\mathbf{p}_2}^s \\ r_{\mathbf{p}_5}^s \end{pmatrix} = (\mathbf{r}^s)^T \mathbf{W}^s \mathbf{r}^s$$

$$\mathbf{J}_s = \begin{pmatrix} \frac{\partial r_{\mathbf{p}_1}^s}{\partial \boldsymbol{\xi}_1} \cdots & \frac{\partial r_{\mathbf{p}_1}^s}{\partial \boldsymbol{\xi}_4} & \frac{\partial r_{\mathbf{p}_1}^s}{\partial d_{\mathbf{p}_1}} \cdots & \frac{\partial r_{\mathbf{p}_1}^s}{\partial d_{\mathbf{p}_5}} & \frac{\partial r_{\mathbf{p}_1}^s}{\partial a_1^L} \cdots & \frac{\partial r_{\mathbf{p}_1}^s}{\partial b_4^R} & \frac{\partial r_{\mathbf{p}_1}^s}{\partial f_x} \cdots & \frac{\partial r_{\mathbf{p}_1}^s}{\partial c_y} \\ \frac{\partial r_{\mathbf{p}_2}^s}{\partial \boldsymbol{\xi}_1} \cdots & \frac{\partial r_{\mathbf{p}_2}^s}{\partial \boldsymbol{\xi}_4} & \frac{\partial r_{\mathbf{p}_2}^s}{\partial d_{\mathbf{p}_1}} \cdots & \frac{\partial r_{\mathbf{p}_2}^s}{\partial d_{\mathbf{p}_5}} & \frac{\partial r_{\mathbf{p}_2}^s}{\partial a_1^L} \cdots & \frac{\partial r_{\mathbf{p}_2}^s}{\partial b_4^R} & \frac{\partial r_{\mathbf{p}_2}^s}{\partial f_x} \cdots & \frac{\partial r_{\mathbf{p}_2}^s}{\partial c_y} \\ \frac{\partial r_{\mathbf{p}_5}^s}{\partial \boldsymbol{\xi}_1} \cdots & \frac{\partial r_{\mathbf{p}_5}^s}{\partial \boldsymbol{\xi}_4} & \frac{\partial r_{\mathbf{p}_5}^s}{\partial d_{\mathbf{p}_1}} \cdots & \frac{\partial r_{\mathbf{p}_5}^s}{\partial d_{\mathbf{p}_5}} & \frac{\partial r_{\mathbf{p}_5}^s}{\partial a_1^L} \cdots & \frac{\partial r_{\mathbf{p}_5}^s}{\partial b_4^R} & \frac{\partial r_{\mathbf{p}_5}^s}{\partial f_x} \cdots & \frac{\partial r_{\mathbf{p}_5}^s}{\partial c_y} \end{pmatrix}_{3\times 49} \tag{2.2}$$

$$E_d(\delta) = \begin{pmatrix} (r_{\mathbf{p}_1}^d)_{12} \\ (r_{\mathbf{p}_1}^d)_{21} \\ \vdots \\ (r_{\mathbf{p}_5}^d)_{43} \end{pmatrix}^T \begin{pmatrix} w_{\mathbf{p}_1} & 0 & \dots & 0 \\ 0 & w_{\mathbf{p}_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & w_{\mathbf{p}_5} \end{pmatrix} \begin{pmatrix} (r_{\mathbf{p}_1}^d)_{12} \\ (r_{\mathbf{p}_1}^d)_{21} \\ \vdots \\ (r_{\mathbf{p}_5}^d)_{43} \end{pmatrix} = (\mathbf{r}^d)^T \mathbf{W}^d \mathbf{r}^d$$

$$\mathbf{J}_d = \begin{pmatrix} \frac{\partial(r^d_{\mathbf{p}_1})_{12}}{\partial\boldsymbol{\xi}_1}\cdots & \frac{\partial(r^d_{\mathbf{p}_1})_{12}}{\partial\boldsymbol{\xi}_4} & \frac{\partial(r^d_{\mathbf{p}_1})_{12}}{\partial d_{\mathbf{p}_1}}\cdots & \frac{\partial(r^d_{\mathbf{p}_1})_{12}}{\partial d_{\mathbf{p}_5}} & \frac{\partial(r^d_{\mathbf{p}_1})_{12}}{\partial a^L_1}\cdots & \frac{\partial(r^d_{\mathbf{p}_1})_{12}}{\partial b^R_4} & \frac{\partial(r^d_{\mathbf{p}_1})_{12}}{\partial f_x}\cdots & \frac{\partial(r^d_{\mathbf{p}_1})_{12}}{\partial c_y} \\ \frac{\partial(r^d_{\mathbf{p}_1})_{21}}{\partial\boldsymbol{\xi}_1}\cdots & \frac{\partial(r^d_{\mathbf{p}_1})_{21}}{\partial\boldsymbol{\xi}_4} & \frac{\partial(r^d_{\mathbf{p}_1})_{21}}{\partial d_{\mathbf{p}_1}}\cdots & \frac{\partial(r^d_{\mathbf{p}_1})_{21}}{\partial d_{\mathbf{p}_5}} & \frac{\partial(r^d_{\mathbf{p}_1})_{21}}{\partial a^L_1}\cdots & \frac{\partial(r^d_{\mathbf{p}_1})_{21}}{\partial b^R_4} & \frac{\partial(r^d_{\mathbf{p}_1})_{21}}{\partial f_x}\cdots & \frac{\partial(r^d_{\mathbf{p}_1})_{21}}{\partial c_y} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial(r^d_{\mathbf{p}_5})_{43}}{\partial\boldsymbol{\xi}_1}\cdots & \frac{\partial(r^d_{\mathbf{p}_5})_{43}}{\partial\boldsymbol{\xi}_4} & \frac{\partial(r^d_{\mathbf{p}_5})_{43}}{\partial d_{\mathbf{p}_1}}\cdots & \frac{\partial(r^d_{\mathbf{p}_5})_{43}}{\partial d_{\mathbf{p}_5}} & \frac{\partial(r^d_{\mathbf{p}_5})_{43}}{\partial a^L_1}\cdots & \frac{\partial(r^d_{\mathbf{p}_5})_{43}}{\partial b^R_4} & \frac{\partial(r^d_{\mathbf{p}_5})_{43}}{\partial f_x}\cdots & \frac{\partial(r^d_{\mathbf{p}_5})_{43}}{\partial c_y} \end{pmatrix}_{7\times 49} \quad (2.2)$$

Iteration δ^* can be calculated by

$$\begin{aligned} &(\mathbf{J}_s^T\lambda\mathbf{W}^s\mathbf{J}_s + \mathbf{J}_d^T\mathbf{W}^d\mathbf{J}_d)\delta^* = -(\mathbf{J}_s^T\lambda\mathbf{W}^s\mathbf{r}^s + \mathbf{J}_d^T\mathbf{W}^d\mathbf{r}^d) \\ &\mathbf{J}_s \in \mathbb{R}^{3\times 49}, \mathbf{W}^s \in \mathbb{R}^{3\times 3}, \mathbf{J}_s \in \mathbb{R}^{7\times 49}, \mathbf{W}^s \in \mathbb{R}^{7\times 7}, \end{aligned} \quad (2.2)$$

We construct residuals and its formulation.

1.2.2 JACOBIAN CITATION

We know for a Lie algebra $\boldsymbol{\rho} \in \mathbb{R}^3, \boldsymbol{\phi} \in \mathbb{R}^3, \boldsymbol{\xi} = \begin{pmatrix} \boldsymbol{\rho} \\ \boldsymbol{\phi} \end{pmatrix} \in \mathbb{R}^6$ and $\mathbf{p}_w$:

$$\begin{aligned} &\boldsymbol{\xi}^\wedge = \begin{pmatrix} \boldsymbol{\rho} \\ \boldsymbol{\phi} \end{pmatrix}^\wedge = \begin{pmatrix} \boldsymbol{\phi}^\wedge & \boldsymbol{\rho} \\ \mathbf{0}^T & \mathbf{0}^T \end{pmatrix} \in \mathbb{R}^{4\times 4} \\ &\boldsymbol{\epsilon} \in \mathbb{R}^3, \begin{pmatrix} \boldsymbol{\epsilon} \\ 1 \end{pmatrix}^\odot = \begin{pmatrix} \mathbf{E} & -\boldsymbol{\epsilon}^\wedge \\ \mathbf{0}^T & \mathbf{0}^T \end{pmatrix} \in \mathbb{R}^{4\times 6} \\ &\frac{\partial(exp(\xi^\wedge)\mathbf{p}_w)}{\partial\boldsymbol{\xi}} = \frac{\partial(\mathbf{T}\mathbf{p}_w)}{\partial\boldsymbol{\xi}} = (\mathbf{T}\mathbf{p}_w)^\odot \\ &\mathbf{T}\mathbf{p}_w = exp(\xi^\wedge)\mathbf{p}_w \approx (\mathbf{E}+\xi^\wedge)\mathbf{p}_w \\ &\frac{\partial(exp(\xi^\wedge)\mathbf{p}_w)}{\partial\boldsymbol{\xi}} \approx \frac{\partial(\mathbf{E}+\xi^\wedge)}{\partial\boldsymbol{\xi}} = \mathbf{0} + \frac{\partial(\xi^\wedge\mathbf{p}_w)}{\partial\boldsymbol{\xi}} \approx (\mathbf{T}\mathbf{p}_w)^\odot \\ &since, \frac{\partial(\mathbf{T}\mathbf{p}_w)}{\partial\boldsymbol{\xi}} = (\mathbf{T}^{-1}\mathbf{p}_w)^\odot = \frac{\partial(exp(-\xi^\wedge)\mathbf{p}_w)}{\partial\boldsymbol{\xi}} \\ &= \frac{\partial(\mathbf{E}-\xi^\wedge)}{\partial\boldsymbol{\xi}} = -(\mathbf{T}\mathbf{p}_w)^\odot \end{aligned} \quad (2.2)$$

1.2.3 JACOBIAN DERIVATION

1.2.3.1 Dynamic Parameter

Firstly, if $\mathbf{p}$ is neither observed by frame mL, mR nor hosted by nL, nR:

$$\frac{\partial(r^d_{\mathbf{p}})_{ij}}{\partial\boldsymbol{\xi}_{\boldsymbol{m}}} = \frac{\partial(r^d_{\mathbf{p}})_{ij}}{\partial\boldsymbol{\xi}_{\boldsymbol{n}}} = \mathbf{0}^T, so \quad \frac{\partial(r^d_{\mathbf{p}_1})_{12}}{\partial\boldsymbol{\xi}_3} = \frac{\partial r^d_{(\mathbf{p}_1)12}}{\partial\boldsymbol{\xi}_4} = \ldots = \mathbf{0}^T, \quad (2.2)$$

otherwise, we follow

$$\frac{\partial (r^d_{\mathbf{p}})_{ij}}{\partial \boldsymbol{\xi_i}} = \frac{\partial (I^L_j(\mathbf{p}^{'}))}{\partial \boldsymbol{\xi_i}} = \frac{\partial (I^L_j(\mathbf{p}^{'}))}{\partial \mathbf{p}^{'}} \frac{\partial \mathbf{p}^{'}}{\partial \mathbf{p}^{'}_w} \frac{\partial \mathbf{p}^{'}_w}{\partial \boldsymbol{\xi_i}} \qquad (2.2)$$
$$\mathbf{p}^{'}_w = \mathbf{T}_j \mathbf{T}_i^{-1} \mathbf{p}_w = \mathbf{T}_j \mathbf{T}_i^{-1} ((d^{iL}_{\mathbf{p}})^{-1} \mathbf{K}^{-1} \mathbf{p})$$

For one frame iL, we have $\mathbf{p}$ and $\mathbf{K}$, then we can get

$$\left\{ \mathbf{p}_w = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} f_x^{-1} (d^{iL}_{\mathbf{p}})^{-1} (u^i - c_x) \\ f_y^{-1} (d^{iL}_{\mathbf{p}})^{-1} (v^i - c_y) \\ (d^{iL}_{\mathbf{p}})^{-1} \\ 1 \end{pmatrix} \right. \qquad (2.2)$$

$$\frac{\partial \mathbf{p}^{'}_w}{\partial \boldsymbol{\xi_i}} = \mathbf{T}_j \frac{\partial (\mathbf{T}_i^{-1} \mathbf{p}^{'}_w)}{\partial \boldsymbol{\xi_i}} = -\mathbf{T}_j (\mathbf{T}_i \mathbf{p}_w)^{\odot}$$
$$\frac{\partial \mathbf{p}^{'}_w}{\partial \boldsymbol{\xi_j}} = \frac{\partial (\mathbf{T}_j \mathbf{T}_i^{-1} \mathbf{p}_w)}{\partial \boldsymbol{\xi_i}} = (\mathbf{T}_j \mathbf{T}_i^{-1} \mathbf{p}_w)^{\odot}$$
$$= \begin{pmatrix} x^{'} \\ y^{'} \\ z^{'} \\ 1 \end{pmatrix}^{\odot} = \begin{pmatrix} 1 & 0 & 0 & 0 & z^{'} & -y^{'} \\ 0 & 1 & 0 & -z^{'} & 0 & x^{'} \\ 0 & 1 & 0 & y^{'} & -x^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\Rightarrow \frac{\partial (r^d_{\mathbf{p}})_{ij}}{\partial \boldsymbol{\xi_j}} = \frac{\partial (I^L_j(\mathbf{p}^{'}))}{\partial \mathbf{p}^{'}} \frac{\partial \mathbf{p}^{'}}{\partial \mathbf{p}^{'}_w} \frac{\partial \mathbf{p}^{'}_w}{\partial \boldsymbol{\xi_j}} \qquad (2.2)$$
$$= (g^{'}_x, g^{'}_y, 0, 0) \begin{pmatrix} f_x (z^{'})^{-1} & 0 & -x^{'} f_x (z^{'})^{-2} & 0 \\ 0 & f_y (z^{'})^{-1} & -y^{'} f_y (z^{'})^{-2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (z^{'})^{-2} & 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 & z^{'} & -y^{'} \\ 0 & 1 & 0 & -z^{'} & 0 & x^{'} \\ 0 & 0 & 1 & y^{'} & -x^{'} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} g^{'}_x f_x (z^{'})^{-1} \\ g^{'}_y f_y (z^{'})^{-1} \\ -(g^{'}_x x^{'} f_x + g^{'}_y y^{'} f_y)(z^{'})^{-2} \\ -g^{'}_y f_y - (g^{'}_x x^{'} y^{'} f_x + g^{'}_y (y^{'})^2 f_y)(z^{'})^{-2} \\ g^{'}_x f_x + (g^{'}_x (x^{'})^2 f_x + g^{'}_y x^{'} y^{'} f_y)(z^{'})^{-2} \\ -g^{'}_x f_x y^{'} (z^{'})^{-1} + g^{'}_y f_y x^{'} (z^{'})^{-1} \end{pmatrix}^T$$

Secondly, according to

$$(r^d_{\mathbf{p}})_{ij} := I^L_j(\mathbf{p}^{'}) - b^L_j - \frac{e^{a^L_j}}{e^{a^L_i}} (I^L_i(\mathbf{p}) - b^L_i) \qquad (2.2)$$

We have:

$$\frac{\partial (r^d_{\mathbf{p}})_{ij}}{\partial a_i} = \frac{e^{a^L_j}}{e^{a^L_i}} (I^L_i(\mathbf{p}) - b^L_i), \frac{\partial (r^d_{\mathbf{p}})_{ij}}{\partial a_j} = -\frac{e^{a^L_j}}{e^{a^L_i}} (I^L_i(\mathbf{p}) - b^L_i) \qquad (2.2)$$
$$\frac{\partial (r^d_{\mathbf{p}})_{ij}}{\partial b_i} = \frac{e^{a^L_j}}{e^{a^L_i}}, \qquad \frac{\partial (r^d_{\mathbf{p}})_{ij}}{\partial b_j} = -1$$

add detail Calibration derivation……

$$
\begin{aligned}
&\mathbf{p}^{'} = d_{\mathbf{p}}^{jL}\mathbf{K}(\mathbf{T}_j\mathbf{T}_i^{-1}\mathbf{p}_w) \\
&assume : \mathbf{T}_j\mathbf{T}_i^{-1} = \begin{pmatrix} r_{11}^{ji} & r_{12}^{ji} & r_{13}^{ji} & t_1^{ji} \\ r_{21}^{ji} & r_{22}^{ji} & r_{23}^{ji} & t_2^{ji} \\ r_{31}^{ji} & r_{32}^{ji} & r_{33}^{ji} & t_3^{ji} \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&\mathbf{p}_w^{'} = \begin{pmatrix} r_{11}^{ji} f_x^{-1}(d_{\mathbf{p}}^{iL})^{-1}(u^i - c_x) + r_{12}^{ji} f_y^{-1}(d_{\mathbf{p}}^{iL})^{-1}(v^i - c_y) + r_{13}^{ji}(d_{\mathbf{p}}^{iL})^{-1} + t_1^{ji} \\ r_{21}^{ji} f_x^{-1}(d_{\mathbf{p}}^{iL})^{-1}(u^i - c_x) + r_{22}^{ji} f_y^{-1}(d_{\mathbf{p}}^{iL})^{-1}(v^i - c_y) + r_{23}^{ji}(d_{\mathbf{p}}^{iL})^{-1} + t_2^{ji} \\ r_{31}^{ji} f_x^{-1}(d_{\mathbf{p}}^{iL})^{-1}(u^i - c_x) + r_{32}^{ji} f_y^{-1}(d_{\mathbf{p}}^{iL})^{-1}(v^i - c_y) + r_{33}^{ji}(d_{\mathbf{p}}^{iL})^{-1} + t_3^{ji} \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{a}{d_{\mathbf{p}}^{iL}} + t_1^{ji} \\ \frac{b}{d_{\mathbf{p}}^{iL}} + t_2^{ji} \\ \frac{c}{d_{\mathbf{p}}^{iL}} + t_3^{ji} \\ 1 \end{pmatrix} = \begin{pmatrix} x^{'} \\ y^{'} \\ z^{'} \\ 1 \end{pmatrix} \Rightarrow d_{\mathbf{p}}^{jL} = (z^{'})^{-1}, \mathbf{p}^{'} = \begin{pmatrix} f_x x^{'} d_{\mathbf{p}}^{jL} + c_x \\ f_y y^{'} d_{\mathbf{p}}^{jL} + c_y \\ 1 \\ d_{\mathbf{p}}^{jL} \end{pmatrix} \\
&\frac{\partial(I_j^L(\mathbf{p}^{'}))}{\partial \mathbf{p}^{'}} = (g_x^{'}, g_y^{'}, 0, 0) \\
&\frac{\partial \mathbf{p}^{'}}{\partial \mathbf{p}_w^{'}} = \begin{pmatrix} f_x(z^{'})^{-1} & 0 & -x^{'} f_x (z^{'})^{-2} & 0 \\ 0 & f_y(z^{'})^{-1} & -y^{'} f_y (z^{'})^{-2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (z^{'})^{-2} & 0 \end{pmatrix}, \frac{\partial \mathbf{p}_w^{'}}{\partial d_{\mathbf{p}}^{iL}} = \begin{pmatrix} -\frac{a}{(d_{\mathbf{p}}^{iL})^2} \\ -\frac{b}{(d_{\mathbf{p}}^{iL})^2} \\ -\frac{c}{(d_{\mathbf{p}}^{iL})^2} \\ 0 \end{pmatrix} \\
&\Rightarrow \frac{\partial(r_{\mathbf{p}}^d)_{ij}}{\partial d_{\mathbf{p}}^{iL}} = \frac{\partial(I_j^L(\mathbf{p}^{'}))}{\partial d_{\mathbf{p}}^{iL}} = \frac{\partial(I_j^L(\mathbf{p}^{'}))}{\partial \mathbf{p}^{'}} \frac{\partial \mathbf{p}^{'}}{\partial \mathbf{p}_w^{'}} \frac{\partial \mathbf{p}_w^{'}}{\partial d_{\mathbf{p}}^{iL}} \\
&= -\frac{g_x^{'} f_x a}{z^{'}(d_{\mathbf{p}}^{iL})^2} - \frac{g_y^{'} f_y b}{z^{'}(d_{\mathbf{p}}^{iL})^2} + \frac{c(g_x^{'} x^{'} f_x + g_y^{'} y^{'} f_y)}{(z^{'} d_{\mathbf{p}}^{iL})^2} \\
&= \frac{c(g_x^{'} x^{'} f_x + g_y^{'} y^{'} f_y) - g_x^{'} f_x a z^{'} - g_y^{'} f_y b z^{'}}{(z^{'} d_{\mathbf{p}}^{iL})^2}
\end{aligned}
\tag{2.2}
$$

1.2.3.2 Static Parameter

Firstly, For a stereo frame i: inverse depth $d_{\mathbf{p}}^{iL} = d_{\mathbf{p}}^{iR}$, a left frame iL pixel $\mathbf{p}$ is projected to right frame iR with $\mathbf{p}^{'}$:

$$
\begin{aligned}
&\mathbf{p} = \begin{pmatrix} u^i \\ v^i \\ 1 \\ d_{\mathbf{p}}^{iL} \end{pmatrix}, \mathbf{p}_w = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}, d_{\mathbf{p}}^{iL} = z^{-1}, \mathbf{p}_w = (d_{\mathbf{p}}^{iL})^{-1}\mathbf{K}^{-1}\mathbf{p} \\
&= \begin{pmatrix} f_x^{-1}(d_{\mathbf{p}}^{iL})^{-1}(u^i - c_x) \\ f_y^{-1}(d_{\mathbf{p}}^{iL})^{-1}(v^i - c_y) \\ (d_{\mathbf{p}}^{iL})^{-1} \\ 1 \end{pmatrix}, \mathbf{T}_{RL} = \begin{pmatrix} 1 & 0 & 0 & t_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&\mathbf{p}^{'} = d_{\mathbf{p}}^{iR}\mathbf{K}(\mathbf{T}_{RL}\mathbf{p}_w) \\
&= d_{\mathbf{p}}^{iL} \begin{pmatrix} f_x & 0 & c_x & 0 \\ 0 & f_y & c_y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_x^{-1}(d_{\mathbf{p}}^{iL})^{-1}(u^i - c_x) + t_1 \\ f_y^{-1}(d_{\mathbf{p}}^{iL})^{-1}(v^i - c_y) \\ (d_{\mathbf{p}}^{iL})^{-1} \\ 1 \end{pmatrix} = \begin{pmatrix} u^i + t_1 f_x d_{\mathbf{p}}^{iL} \\ v^i \\ 1 \\ d_{\mathbf{p}}^{iL} \end{pmatrix} \\
&\frac{\partial r_{\mathbf{p}}^s}{\partial d_{\mathbf{p}}^{iL}} = \frac{\partial(I_i^R(\mathbf{p}^{'})) - \frac{e^{a_i^R}}{e^{a_i^L}}(I_i^L(\mathbf{p}))}{\partial d_{\mathbf{p}}^{iL}} = (\frac{\partial(I_i^R(\mathbf{p}^{'}))}{\partial \mathbf{p}^{'}} - \frac{e^{a_i^R}}{e^{a_i^L}}\frac{\partial(I_i^L(\mathbf{p}))}{\partial \mathbf{p}^{'}})\frac{\partial \mathbf{p}^{'}}{\partial d_{\mathbf{p}}^{iL}} \\
&= [(g_x^{iR}, g_y^{iR}, 0, 0) - \mathbf{0}^T] \begin{pmatrix} t_1 f_x \\ 0 \\ 0 \\ 1 \end{pmatrix} = g_x^{iR} t_1 f_x
\end{aligned}
\tag{1}
$$

Secondly, according to:

$$
r_{\mathbf{p}}^s := I_i^R(\mathbf{p}^{'}) - b_i^R - \frac{e^{a_i^R}}{e^{a_i^L}}(I_i^L(\mathbf{p}) - b_i^L) \tag{2.2}
$$

We have:

$$
\begin{aligned}
\frac{\partial(r_{\mathbf{p}}^s)_{ij}}{\partial a_i} &= \frac{e^{a_j^L}}{e^{a_i^L}}(I_i^L(\mathbf{p}) - b_i^L), \frac{\partial(r_{\mathbf{p}}^s)_{ij}}{\partial a_j} = -\frac{e^{a_j^L}}{e^{a_i^L}}(I_i^L(\mathbf{p}) - b_i^L) \\
\frac{\partial(r_{\mathbf{p}}^s)_{ij}}{\partial b_i} &= \frac{e^{a_j^L}}{e^{a_i^L}}, \qquad \frac{\partial(r_{\mathbf{p}}^s)_{ij}}{\partial b_j} = -1
\end{aligned}
\tag{2.2}
$$

add detail Calibration derivation……