Spans, bases and linear independence

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Basis of a vector space

Let V be a vector space over a field F.

Definition: A set of vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n} \in V$ forms a basis if **both** the following two conditions are satisfied:

- $\text{Span}(\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}) = V$
  - That is, every vector in V can be represented as a linear combination of the given vectors
  - If $\text{Span}(\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}) = V$ then the set of vectors $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ is called as a spanning set
- The set of vectors $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ is linearly independent
  - That is, $r_1\vec{v_1} \oplus r_2\vec{v_2} \oplus \ldots \oplus r_n\vec{v_n} = \vec{0}$ implies $r_1 = r_2 = \ldots = r_n = 0$

I Otherwise, the set of vectors is said to be linearly dependent

Consider our standard examples of the vector spaces:

- $\mathbb{Q}^2$ over $\mathbb{Q}$, i.e., 2-tuples of rational numbers over the field of rational numbers
- $\mathbb{Q}^3$ over $\mathbb{Q}$, i.e., 3-tuples of rational numbers over the field of rational numbers

Two questions about these standard examples of vector spaces:

- Do these vector spaces have a basis?
- If yes, can you find one?

Q^2 over Q

$\boxed{\{(0,1),(1,0)\} \quad \text{Yes}}$

Linearly independent

$a(0,1) + b(1,0) =$ anything in Q^2 span

Q^3 over Q

$\boxed{\{(0,0,1),(0,1,0),(1,0,0)\} \quad \text{Yes}}$

$a(0,0,1) + b(0,1,0) + c(1,0,0)$

$=$ any number in Q^2 span

Linearly independent

Basis of a vector space

Let V be a vector space over a field F.

Definition: A set of vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n} \in V$ forms a basis if the following two conditions are satisfied:

- $\text{Span}(\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}) = V$
  - That is, every vector in V can be represented as a linear combination of the given vectors
- This set of vectors is linearly independent

Number of vectors in basis = dimension

Consider our standard example of the vector space $\mathbb{Q}^2$ over $\mathbb{Q}$

- Can you think of a basis for this vector space?

$\boxed{\{(0,1),(1,0)\}}$

However, we can define (inner) product of two vectors for our two go-to examples of vector spaces:

- Vector space $\mathbb{Q}^2$ of two-tuples of rational numbers over the field of rational numbers $\mathbb{Q}$
  - If $\vec{u} = \binom{u_1}{u_2}$ and $\vec{v} = \binom{v_1}{v_2}$, then the inner product of $\vec{u}$ and $\vec{v}$ is defined as $\vec{u} \cdot \vec{v} = (u_1 \times v_1) + (u_2 \times v_2)$ pick $\vec{x} = (x_1, x_2, x_3) = (x, y, z)$
  - Example: $\vec{x} \cdot \vec{x} = x \cdot x + y \cdot y + z \cdot z = x^2 + y^2 + z^2 = \|\vec{x}\|^2$ ← norm
- Vector space $\mathbb{Q}^3$ of two-tuples of rational numbers over the field of rational numbers $\mathbb{Q}$

$\binom{0}{1} \cdot \binom{1}{0} = 0 \cdot 1 + 1 \cdot 0 = 0$

  - If $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, then the inner product of $\vec{u}$ and $\vec{v}$ is defined as $\vec{u} \cdot \vec{v} = (u_1 \times v_1) + (u_2 \times v_2) + (u_3 \times v_3)$

$\binom{1}{1} \cdot \binom{-1}{1} = 1 \cdot (-1) + 1 \cdot 1 = 0$

  - Example:

Orthogonal basis (if pairwise independent) $\vec{e}_i$
Orthonormal if in addition all $\|\vec{e}_i\| = 1$

$\vec{u} \begin{pmatrix} 8 \\ 3 \\ 24 \end{pmatrix}$ $\vec{v} \begin{pmatrix} 109 \\ 21 \\ 50 \end{pmatrix}$

$$\vec{u}\,\vec{v} = (8 \times 109) + (3 \times 21) + (24 \times 50) = \boxed{2135}$$

Motivation: Linear Spaces (Vector Spaces)

Consider liner combinations

$$a \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$$

Can we reach every point in the plane ? Yes
in $\mathbb{R}^3$? No

We could also use

$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \ldots$

to span $\mathbb{R}^3$ we need to add e.g. $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

What is the advantage of an orthogonal basis?

Consider the vector space $\mathbb{Q}^2$ of two-tuples of rational numbers over the field of rational numbers $\mathbb{Q}$

- Consider the orthogonal basis given by $\vec{u} = \binom{1}{1}$ and $\vec{v} = \binom{1}{-1}$ is orthogonal
- I want to find how $\vec{w} = \binom{11}{-9}$ can be represented as linear combination of
  - Since $\{\vec{u}, \vec{v}\}$ form a basis we know $\vec{w} \in \text{Span}(\vec{u}, \vec{v})$
- Suppose $\vec{w} = r\vec{u} \oplus s\vec{v}$. How can we find r and s easily?

$$r \begin{pmatrix} 1 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 11 \\ -9 \end{pmatrix}$$

$r + s = 11$

$r - s = -9$

$s - r = 9$

$r = s - 9$

$s - 9 + s = 11$

$2s = 20$

$\boxed{s = 10 \quad r = 1}$

Consider the vector space $\mathbb{Q}^3$ over $\mathbb{Q}$ of 3-tuples of rational numbers.

- Consider the vectors $\vec{v_1} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} 5 \\ 6 \\ 10 \end{pmatrix}, \vec{v_3} = \begin{pmatrix} 11 \\ 300 \\ 14 \end{pmatrix}$ from $\mathbb{Q}^3$
- Want to check if these three vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$ are linearly independent
- How can we do that?
-

$$a\vec{V_1} + b\vec{V_2} + c\vec{V_3} = \vec{0}$$

$$ax_1 + bx_2 + cx_3 = 0$$

$$ay_1 + by_2 + cy_3 = 0$$

$$az_1 + bz_2 + cz_3 = 0$$

$$\underbrace{\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}}_{M} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If $g(x) \neq 0$ it is linearly independent

If $\det(M) \neq 0$

then M^{-1} exists

If $g(x) = 0$, it is linearly dependent

$$x_1 \begin{vmatrix} y_2 & y_3 \\ z_2 & z_3 \end{vmatrix} - x_2 \begin{vmatrix} y_1 & y_3 \\ z_1 & z_3 \end{vmatrix} + x_3 \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} = g(x)$$

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Two theorems about basis of vector spaces

Two theorems about basis of vector spaces

- Theorem 1: Every vector space has a basis
- Theorem 2: Every basis of a vector space has the same number of vectors
  - The number of vectors in a basis of a vector space is called as its dimension

Proofs of Theorem 1 and Theorem 2 are beyond the scope of this module!

Questions:

- What is the dimension of the vector space $\mathbb{Q}^2$ of two-tuples of rational numbers over the field of rational numbers?
  - Answer: 2
- What is the dimension of the vector space $\mathbb{Q}^3$ of three-tuples of rational numbers over the field of rational numbers?
  - Answer: 3

- Answer: 2
- What is the dimension of the vector space $\mathbb{Q}^3$ of three-tuples of rational numbers over the field of rational numbers?
  - Answer: 3

Definition: Orthogonal basis

We can define what it means for a basis to be orthogonal if there is some notion of inner product between vectors

Hence, we can define orthogonal basis for our two go-to examples of vector spaces:

- Vector space $\mathbb{Q}^2$ of two-tuples of rational numbers over the field of rational numbers $\mathbb{Q}$
  - A basis $\{\vec{u}, \vec{v}\}$ for $\mathbb{Q}^2$ over $\mathbb{Q}$ is orthogonal if $\vec{u} \cdot \vec{v} = 0$
  - Question: Does this vector space have an orthogonal basis?

$\{(1,0),(0,-1)\}$ Linearly independent
Spans Q^2 Linear comb of vectors $= 0$

- Vector space $\mathbb{Q}^3$ of two-tuples of rational numbers over the field of rational numbers $\mathbb{Q}$
  - A basis $\{\vec{u}, \vec{v}, \vec{w}\}$ for $\mathbb{Q}^3$ over $\mathbb{Q}$ is orthogonal if $\vec{u} \cdot \vec{v} = 0$, $\vec{u} \cdot \vec{w} = 0$ and $\vec{v} \cdot \vec{w} = 0$
  - Question: Does this vector space have an orthogonal basis?

$\{\underbrace{(0,0,1)}_{u}, \underbrace{(0,1,0)}_{v}, \underbrace{(1,0,0)}_{w}\}$

Span:

$u \cdot v \cdot w = 0$

Linearly independent