What is a matrix?

For us, matrix is a two-dimensional array whose entries come from a field (say $\mathbb{Q}$ or $\mathbb{R}$)

- We can read it row-wise
- or column-wise

Size of a matrix
- Number of rows

x
- Number of columns

We can add two matrices if they have the same size

- Number of rows must be same
- Number of columns must also be the same

This is done by adding the corresponding entries from each matrix!

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+1 & b \\ c & d+1 \end{pmatrix}$$

Matrix multiplication by a scalar:

Let A be any $(m \times n)$ matrix with entries from $\mathbb{Q}$

For any $r \in \mathbb{Q}$, we define the product of the scalar $r \in Q$ and the matrix A as the $(m \times n)$ matrix $B := rA$ obtained as follows:

- Each entry of B is obtained by multiplying the corresponding entry of A by r
- That is, for each $1 \leq i \leq m$; $1 \leq j \leq n$ we define $b_{i,j} = r \times a_{i,j}$

$$\underset{\in \mathbb{R}}{\frac{1}{2}} \cdot \underset{\mathbb{R}^{2 \times 3}}{\begin{pmatrix} 2 & 3 & 2 \\ 0 & 0 & 1 \end{pmatrix}} = \begin{pmatrix} 2\frac{1}{2} & 3\frac{1}{2} & 2\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

We now define matrix multiplication:

- Let A be an $m \times n$ matrix whose entry in row i and column j is given by a_{ij}
- Let B be an $n \times p$ matrix whose entry in row j and column k is given by b_{jk}
- Then the result of multiplying A and B is a $(n \times p)$ matrix C whose entry c_{ik} in row i and column k is given by

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$$c_{ik} \quad = \quad \sum_{j=1}^{n} a_{ij} \times b_{jk} \quad = \quad (a_{i1} \times b_{1k}) + (a_{i2} \times b_{2k}) + (a_{i3} \times b_{3k}) + \ldots + (a_{in} \times b_{nk})$$

- That is, the entry in row i and column k of the matrix AB is defined to be the inner product of row i of A with column k of B

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & c \\ a & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ c \\ a \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 2 & 6 \end{pmatrix} \quad AB \neq BA$$

But matrix multiplication is associative!

- Let A be a $(m \times n)$ matrix
- Let B be a $(n \times p)$ matrix
- Let C be a $(p \times s)$ matrix
- Then $A(BC) = (AB)C$

$$\underbrace{(A \cdot B)}_{\substack{m \times p \\ \text{matrix}}} \cdot C \quad = A \cdot (B \cdot C)$$

A sanity check in linear algebra typically involves verifying that operations are mathematically valid and consistent with the rules of matrix dimensions and operations.

Sanity checks:

- BC is well-defined and a $(n \times s)$ matrix
- So $A(BC)$ is well-defined and a $(m \times s)$ matrix
- AB is well-defined and a $(m \times p)$ matrix
- So $(AB)C$ is well-defined and a $(m \times s)$ matrix

Proof is not very hard, but we will not cover it in this module!

Vector space of matrices over the field $\mathbb{Q}$

Another example of a vector space:

- Fix any $m, n \geq 1$
- Then the set of all $(m \times n)$ matrices whose entries are from $\mathbb{Q}$
  - Each vector in this vector space is a $(m \times n)$ matrix
  - Each scalar in this vector space is a rational number

To show this is a vector space, we need to verify the following 8 conditions

For any vectors $\vec{u}, \vec{v}, \vec{w} \in V$ and any scalars $r, s \in F$

(1) Commutativity of vector addition: $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$
(2) Associativity of vector addition: $(\vec{u} \oplus \vec{v}) \oplus \vec{w} = \vec{u} \oplus (\vec{v} \oplus \vec{w})$
(3) Existence of Additive identity: $\vec{0} \oplus \vec{v} = \vec{v}$
(4) Existence of additive inverse: for each $\vec{x}$, there exists $\overrightarrow{-x}$ such that $\vec{x} \oplus \overrightarrow{-x} = \vec{0}$
(5) Associativity of multiplication of scalar & vector: $r(s\vec{v}) = (rs)\vec{v}$
(6) Distributivity of scalar sums: $(r+s)\vec{v} = r\vec{v} \oplus s\vec{v}$
(7) Distributivity of vector sums: $r(\vec{u} \oplus \vec{v}) = r\vec{u} \oplus r\vec{v}$
(8) Existence of identity of multiplication of scalar & vector: $1\vec{v} = \vec{v}$

First, we need to define two operations for above 8 conditions to make sense:

- Vector addition: for each $\vec{u}, \vec{v}$ a vector from V is assigned to $\vec{u} \oplus \vec{v}$
- Multiplication of a scalar by a vector: for each $s \in F$ and $\vec{v} \in V$, a vector from V is assigned to $s\vec{v}$

Identity matrix for (2×2) matrices

Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Let A be any (2×2) matrix over the field of rational numbers

- Show that $AI = A = IA$

$$A = \begin{matrix} 30 & 15 \\ 4 & 47 \end{matrix}$$

$$AI = \begin{matrix} 30 & 15 \\ 4 & 47 \end{matrix} \times \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} = \begin{matrix} (30x1)+(15x0) & (30x0)+(15x1) \\ (4x1)+(47x0) & (4x0)+(47x1) \end{matrix} = \begin{matrix} 30 & 15 \\ 4 & 47 \end{matrix} = A$$

$$IA = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \times \begin{matrix} 30 & 15 \\ 4 & 47 \end{matrix} = \begin{matrix} (1x30)+(0x4) & (1x15)+(0x47) \\ (0x30)+(1x4) & (0x15)+(1x47) \end{matrix} = \begin{matrix} 30 & 15 \\ 4 & 47 \end{matrix} = AI$$

Therefore

AI = A = IA

Identity matrix for (3×3) matrices

Let $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & x \end{pmatrix} \cdot \begin{pmatrix} 1 & & 0 \\ 0 & 1 & \\ & & 1 \end{pmatrix}$$

Let A be any (3×3) matrix over the field of rational numbers

- Show that $AI = A = IA$

A =

30 15 26

4	47	3
38	35	21

	30	15	26		1	0	0		(30x1)+(15x0)+(26x0)	(30x0)+(15x1)+(26x0)	(30x0)+(15x0)+(26x1)		30	15	26	
AI =	4	47	3	x	0	1	0	=	(4x1)+(47x0)+(3x0)	(4x0)+(47x1)+(3x0)	(4x0)+(47x0)+(3x1)	=	4	47	3	= A
	38	35	21		0	0	1		(38x1)+(35x0)+(21x0)	(38x0)+(35x1)+(21x0)	(38x0)+(35x0)+(21x1)		38	35	21	

	1	0	0		30	15	26		(1x30)+(0x4)+(0x38)	(1x15)+(0x47)+(0x35)	(1x26)+(0x3)+(1x21)		30	15	26	
IA =	0	1	0	x	4	47	3	=	(0x30)+(1x47)+(0x35)	(0x15)+(1x47)+(0x35)	(0x26)+(1x3)+(0x21)	=	4	47	3	= AI
	0	0	1		38	35	21		(0x30)+(0x4)+(1x38)	(0x15)+(0x47)+(1x35)	(0x26)+(0x3)+(1x21)		38	35	21	

Therefore

AI = A = IA

Inverse of a (2×2) matrix

Let A be a (2×2) matrix.

Then a matrix B is said to be inverse of A if $BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$

IF

$A = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}$?

What would it's inverse be? And does it even have one?

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \overset{!}{=} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix} = \begin{pmatrix} a+3b & 2a+8b \\ c+3d & 2c+8d \end{pmatrix}$$

$$\begin{aligned} a+3b &\overset{!}{=} 1 \\ c+3d &= 0 \\ 2a+8b &= 0 \\ 2c+8d &= 1 \end{aligned}$$

Easier way:

det(A) = ad - bc

$$A^{-1} = \frac{1}{\det(A)} * \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Inverse of a matrix need not always exist!

Show that the (2×2) matrix $A = \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$ does not have an inverse.

Need to show that there is no (2×2) matrix B with entries which are rational numbers such that $BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

If $B = \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$

<u>In the easier method:</u>

It's simple.
Since 1/det(B) is involved in the calculation
det(B) cannot be equal to 0 to output a
defined result
det(A) = 0:
Meaning that the matrix B does NOT
have an inverse

<u>In their method:</u>

Assumption:
The matrix B has an inverse

Equations from that:

$6a + 12b = 1$
$a + 2b = 0$
$6c + 12d = 0$
$c + 2d = 1$

$a = -2b$
$-12b + 12b = 1$ {This is not possible
because -12b and 12b cancel out, equalling 0
in every possible scenario not 1}

c = -2d
-12c + 12d = 0

Since one of the equations yielded an impossible answer, there is a contradiction
Meaning the original assumption is false.
Meaning that the matrix B does NOT have an inverse

Inverse of a 3x3 matrix:

If A = matrix

a	b	c
d	e	f
g	h	i

Find det(A):

a (det (b, c, e , f)) - b (det (d , f , g , i)) + c (det (d , e , g , h))

Example:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{bmatrix}$$

$$\det(A) = 1\begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 \\ -5 & 4 \end{vmatrix}$$

$$= 1(-4) - 2(11) + (-1)(12)$$

Adjust signs:
Based on pattern below, if the sign is a +, multiply by +1, which in effect does nothing. If the sign is a -, multiply by -1, which reverses the value.

Example:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix},$$

Find the matrix of minors / matrix of cofactors:

For every value in the matrix find the value of its determinant and replace the original value with the value of its determinant

Example:

Example: Find the cofactor matrix of $\mathbf{A}$ given that $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$.

Solution: First find the cofactor of each element.

$$A_{11} = \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = 24 \qquad A_{12} = -\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} = 5 \qquad A_{13} = \begin{vmatrix} 0 & 4 \\ 1 & 0 \end{vmatrix} = -4$$

$$A_{21} = -\begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} = -12 \qquad A_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 3 \qquad A_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2$$

$$A_{31} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -2 \qquad A_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = -5 \qquad A_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 4$$

The cofactor matrix is thus $\begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}$.

The symbol of the matrix then changes from matrix A to matrix C, to represent the that it is the matrix of cofactors

Transpose the matrix:

Transposing a matrix means making the rows, columns and the columns, rows.

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}^T = \begin{bmatrix} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{bmatrix}$$

: The numbers in the highlighted region never change in transposition {The diagonal remains unchanged}

The symbol of the matrix then changes from matrix C to matrix C^T to represent that it has been transposed

Final step. Calculating the inverse value:

The final step is:

Example:

$$M = \begin{pmatrix} 1 & 6 & 3 \\ 2 & 4 & 5 \\ 9 & 7 & 1 \end{pmatrix} \quad M^{-1} = \frac{1}{\det(M)} C^{T}$$

Then obviously the symbol of the matrix changes back finally from C^T to M^{-1}