Lecture 2

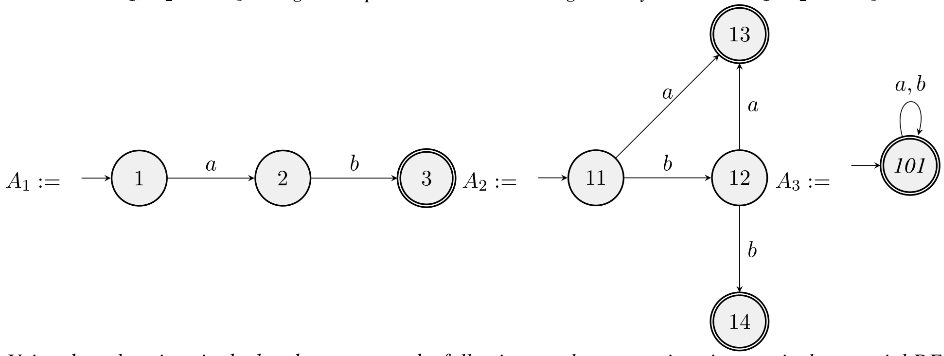
Exercise Sheet 2

Exercise 1 *Show by induction on natural numbers that, for any natural number n, the sum of the first n positive squares is equal to $\frac{n(n+1)(2n+1)}{6}$.*

Exercise 2 *Consider the regular expression $(a^*(b|c))|(b^*b)$. In a step-by-step fashion, convert each sub-expression of the regular expression into an ϵNFA. Then, use the prior epsilon absorption algorithm to obtain a partial DFA that is equivalent to the overall regular expression.*

Exercise 3 *Show that every number $n > 1$ has a prime factor, by course-of-values induction.*

Exercise 4 *Let E_1, E_2 and E_3 be regular expressions that are recognised by automata A_1, A_2 and A_3.*



Using the rules given in the handout, convert the following regular expressions into equivalent partial DFAs:

(i) $E_3|E_1$

(ii) $E_1(E_2|E_3)$

(iii) $E_3{E_3}^*$

Exercise 5 *Andy has a grid of $2^n \times 2^n$ squares that are all white, except for one, which is red. A* triomino *is an L-shaped piece covering 3 squares. Show, by induction on natural numbers, that Andy can cover the white part of the grid by triominoes.*

Sheet 2:

(1) $n=1$:

$$\sum_{i=1}^{1} i^2 = 1^2 \to LHS$$

$$\frac{1(1+1)(2(1)+1)}{6} = \frac{1(2)(3)}{6} = \frac{6}{6} = 1 \to RHS$$

LHS = RHS ∴ base case holds

Assume for $n=k$:

$$\sum_{i=1}^{k} i = \frac{k(k+1)(2k+1)}{6}$$

RTS:

$$\sum_{i=1}^{k+1} i = \frac{1}{6}(k+1)\left((k+2)(2k+3)\right)$$

$$\frac{1}{6}(k+1)(2k^2+7k+6)$$

Prove for $n=k+1$:

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + \frac{6[(k+1)^2]}{6}$$

$$= \frac{1}{6}(k+1)\left[k(2k+1) + 6(k+1)\right]$$

$$\frac{1}{6}(k+1)\left[2k^2 + k + 6k + 6\right]$$

$$= \boxed{\frac{1}{6}(k+1)\left[2k^2 + 7k + 6\right]}$$


2
[a*]
[b*]
[b|c]
Absorbing εs:

(3)
- Assume upto $n = k$, n has a prime factor
- Looking at $n = k+1$

  If $n = k+1$ is prime:
  It's a prime factor itself

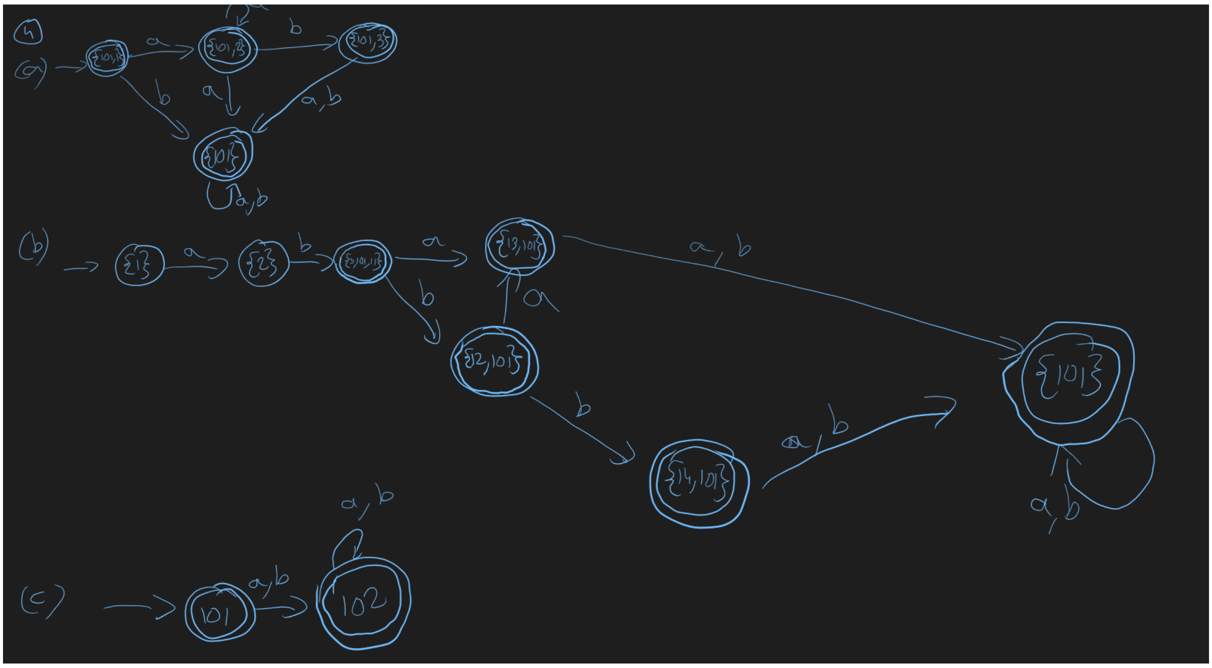
  If $n = k+1$ is not prime:

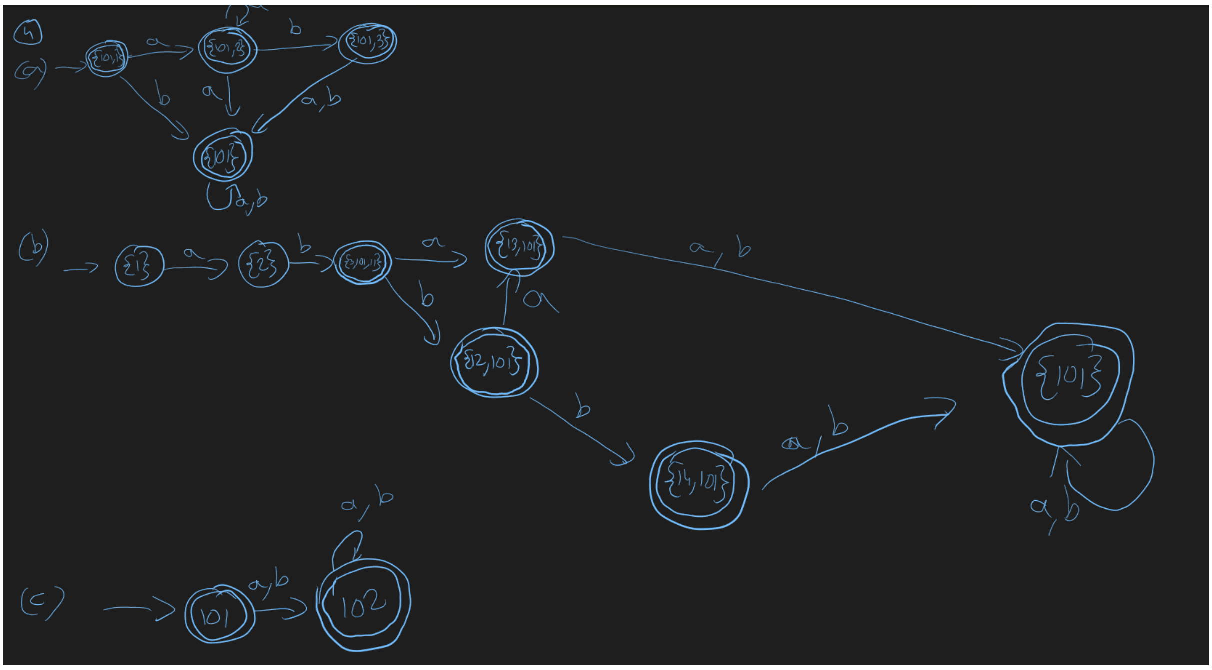
  $n = a \times b$

  where

  $1 < a < n$ & $1 < b < n$

  by inductive hypotheses
  both a & b have prime factors $\therefore$
  $a \times b$ has prime factors
  $\therefore$ n has prime factors


(a)
{101}
a
b
(b)
{1}
{2}
{12,101}
{13,101}
{14,101}
{101}
a, b
(c)
101
102
a, b


(a)
{101}
a
b
(b)
{1}
{2}
{12,101}
{13,101}
{14,101}
{101}
a, b
(c)
101
102
a, b

(5) $n=1$:

Grid is 2×2

So the trinomio covers 3/4 tiles
One tile is red so it can cover all
the white tiles $\therefore$ base case holds.

$n=k$:

Assume trinomio can cover all white
tiles for a grid $2^k \times 2^k$

For $n=k+1$:

$$2^{k+1} \times 2^{k+1}$$

$$= 2\cdot 2^k \times 2\cdot 2^k$$

$$= 2(2^k \times 2^k)$$

$$= [2^k \times 2^k] + [2^k + 2^k]$$

2 $[2^k \times 2^k]$ grids

So it can be covered by
trinomioes

Exercise Sheet 2

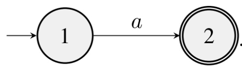
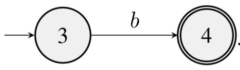
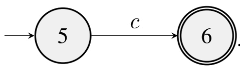
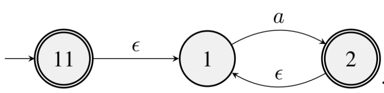
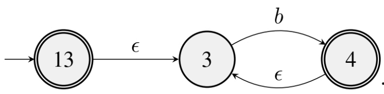
Exercise 1 *Show by induction on natural numbers that, for any natural number n, the sum of the first n positive squares is equal to $\frac{n(n+1)(2n+1)}{6}$.*

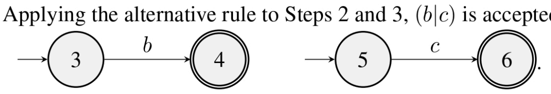
Solution The base case is when $n := 0$, the sum of the first 0 positive squares is 0, and also $\frac{0\times(0+1)\times(2\times0+2)}{6} = 0$; so the sum is given by the formula.

Now the inductive case. By fully expanding the formula, we want to show that the sum of the first $n+1$ positive squares is $\frac{(n+1)(n+2)(2(n+1)+1)}{6} = \frac{2n^3+9n^2+13n+6}{6}$. Let k be the sum of the first n positive squares, therefore the sum of the first $n+1$ positive squares is $k + (n+1)^2 = k + n^2 + 2n + 1$. The inductive hypothesis says that $k = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3+3n^2+n}{6}$. Therefore $k+n^2+2n+1 = \frac{2n^3+3n^2+n}{6} + n^2+2n+1 = \frac{2n^3+3n^2+n+6n^2+12n+6}{6} = \frac{2n^3+9n^2+13n+6}{6}$, which is what we wanted.

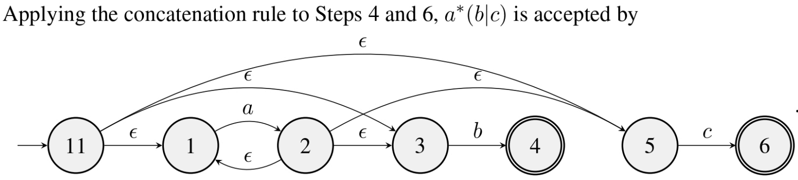
Exercise 2 *Consider the regular expression $(a^*(b|c))|(b^*b)$. In a step-by-step fashion, convert each sub-expression of the regular expression into an ϵNFA. Then, use the prior epsilon absorption algorithm to obtain a partial DFA that is equivalent to the overall regular expression.*

Solution

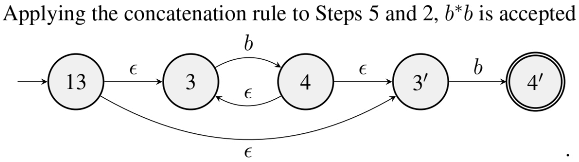
1. By the single-character rule, a is accepted by .
2. By the single-character rule, b is accepted by .
3. By the single-character rule, c is accepted by .
4. Applying the star rule to Step 1, a^* is accepted by .
5. Applying the star rule to Step 2, b^* is accepted by .
6. Applying the alternative rule to Steps 2 and 3, $(b|c)$ is accepted by

.

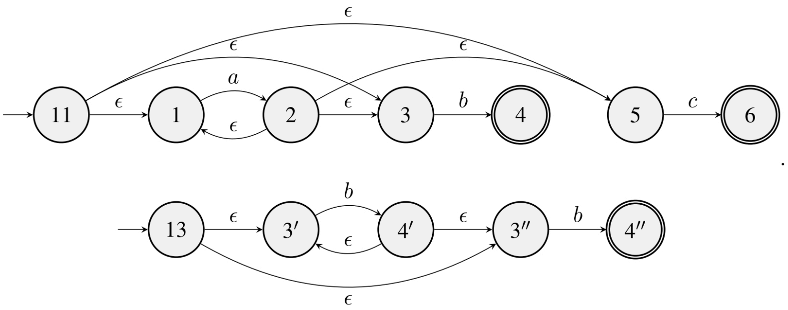
7. Applying the concatenation rule to Steps 4 and 6, $a^*(b|c)$ is accepted by

.

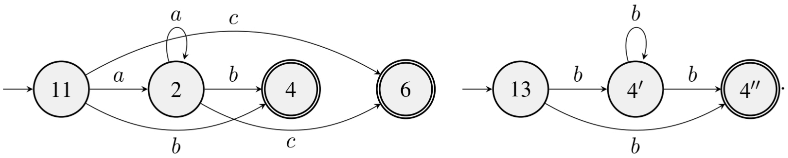
8. Applying the concatenation rule to Steps 5 and 2, b^*b is accepted by

.

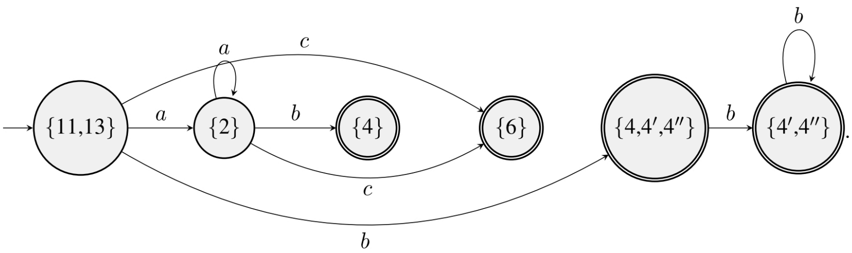
9. Applying the alternative rule to Steps 7 and 8, $(a^*(b|c))|(b^*b)$ is accepted by



10. After using the prior epsilon absorption algorithm, we have



11. After determinising, we have

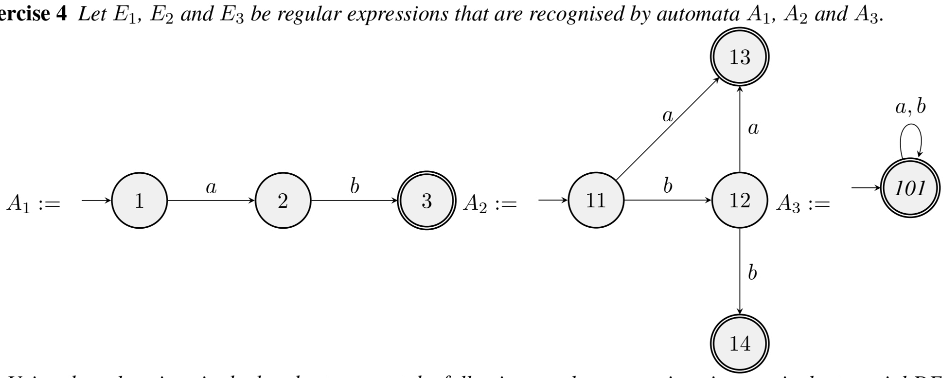


Exercise 3 *Show that every number $n > 1$ has a prime factor, by course-of-values induction.*

Solution If n is prime, then it's a prime factor of itself. Otherwise n can be expressed as $a \times b$, where a and b are natural numbers and $1 < a < n$. By the inductive hypothesis, a has a prime factor, which is also a prime factor of n.

Here's another proof. (You might take the view that it's just the same proof written a little differently.) Let $n_0 \stackrel{\text{def}}{=} n$. If n_0 isn't prime then let n_1 be a factor such that $1 < n_1 < n_0$. If n_1 isn't prime, then let n_2 be a factor such that $1 < n_2 < n_1$. And so on. If this process continues forever, we obtain an infinite sequence of natural numbers $n_0 > n_1 > \cdots$, which is absurd. So we must eventually reach a prime factor.

Exercise 4 *Let E_1, E_2 and E_3 be regular expressions that are recognised by automata A_1, A_2 and A_3.*



Using the rules given in the handout, convert the following regular expressions into equivalent partial DFAs:

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(i) $E_3|E_1$

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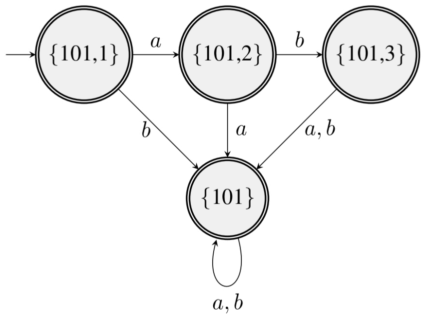
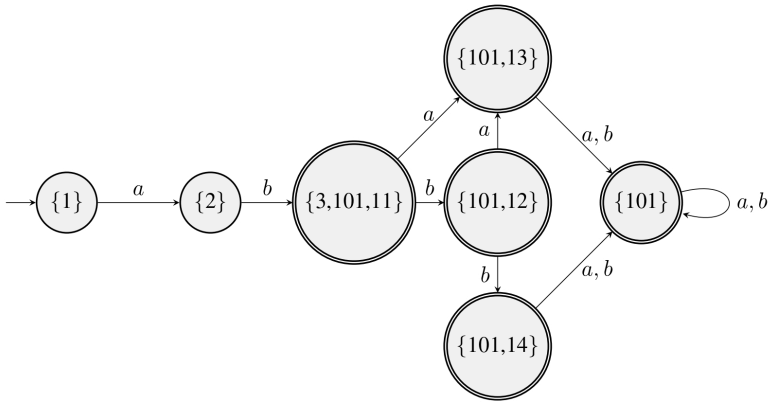
Solution

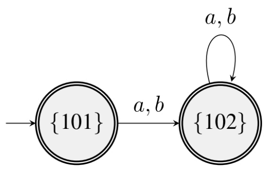
(i)



(ii)



(iii)



Exercise 5 *Andy has a grid of* $2^n \times 2^n$ *squares that are all white, except for one, which is red. A* triomino *is an L-shaped piece covering 3 squares. Show, by induction on natural numbers, that Andy can cover the white part of the grid by triominoes.*

Solution We proceed by induction on n. Base case: If $n = 0$, there is just the red square, so there's nothing to do. Inductive step: Suppose the result is true for n; we shall prove it for $n + 1$. Given a grid of size $2^{n+1} \times 2^{n+1}$, divide it into four subgrids of size $2^n \times 2^n$. Each subgrid contains one of the grid's four central squares, and we paint each of them blue, except for the one that's in the same subgrid as the red square. Now each of the quarters contains a single non-white square, so, by the inductive hypothesis, it can be covered by triominoes. Finally place a triomino over the three blue squares.