Semantics

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Interpretation of Predicate & Function Symbols

Semantics: Assigning meaning/interpretations to formulas

Earlier in the module: a **particular semantics** for propositional logic

- Each proposition has a meaning (**a truth value**) of **T** or **F**
- Used truth tables to check **semantic validity**

We now **extend** this particular semantics to predicate logic

- Propositional logic constructs are interpreted similarly
- In addition, we need to interpret
  - **predicate & function symbols**
  - **quantifiers**

Predicate symbols: for example, given the domain $\mathbb{N}$ and a unary predicate symbol `even`, what is the meaning of `even`?

- to state that a number is $0, 2, 4, \ldots$?
- is it always obvious?
- what if we had a predicate symbol `small`?
- what does that mean?

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Interpretation of Predicate & Function Symbols

Given a domain D and a predicate symbol p of arity n

- p is interpreted by a n-ary relation $\mathcal{R}_p$
- of the form $\{\langle d_1^1, \ldots, d_n^1\rangle, \langle d_1^2, \ldots, d_n^2\rangle, \ldots\}$
- where each d_j^i is in D
- we write: $\mathcal{R}_p \in 2^{D^n}$ or $\mathcal{R}_p \subseteq D^n$

For example

- a meaningful interpretation for `even` would be
  - $\{\langle 0\rangle, \langle 2\rangle, \langle 4\rangle, \ldots\}$
- a meaningful interpretation for `odd` would be
  - $\{\langle 1\rangle, \langle 3\rangle, \langle 5\rangle, \ldots\}$
- a meaningful interpretation for `prime` would be
  - $\{\langle 2\rangle, \langle 3\rangle, \langle 5\rangle, \ldots\}$

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Interpretation of Predicate & Function Symbols

Function symbols: for example, given the domain $\mathbb{N}$ and a binary function symbol `add`, what is the meaning of `add`?

- is it addition?
- is it always obvious?
- what if we had a binary function symbol `combine`?
- what does that mean?

Interpretation of Predicate & Function Symbols

Function symbols: for example, given the domain $\mathbb{N}$ and a binary function symbol `add`, what is the meaning of `add`?

- is it addition?
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Given a domain D and a function symbol f of arity n

- f is interpreted by a function $\mathcal{F}_f$ from D^n to D
- we write: $\mathcal{F}_f \in D^n \to D$

For example

- a meaningful interpretation for `add` would be
  - $+$ (formally: $\langle n, m \rangle \mapsto n + m$)
- a meaningful interpretation for `mult` would be
  - $\times$ (formally: $\langle n, m \rangle \mapsto n \times m$)

Interpretation of Predicate & Function Symbols

WARNING ⚠: sometimes for convenience we will use the same symbol for a function symbol and its interpretation

For example:

1. we have used 0 in our examples as a **constant symbol**, which has no meaning on its own
2. this constant symbol would be interpreted by the natural number 0, which is an **object of the domain** $\mathbb{N}$

Even though we used the same symbols, these symbols stand for different entities:

1. a **constant symbol**
2. an **object of the domain**

If we want to distinguish them, we might use:

1. $\overline{0}$ or `zero` for the **constant symbol**
2. 0 for the **object of the domain**

Models

Models: a model provides the interpretation of all symbols

Given a **signature** $\langle\langle f_1^{k_1}, \dots, f_n^{k_n}\rangle, \langle p_1^{j_1}, \dots, p_m^{j_m}\rangle\rangle$

- of function symbols f_i of arity k_i, for $1 \leqslant i \leqslant n$
- of predicate symbols p_i of arity j_i, for $1 \leqslant i \leqslant m$

a **model** is a structure $\langle D, \langle \mathcal{F}_{f_1}, \dots, \mathcal{F}_{f_n}\rangle, \langle \mathcal{R}_{p_1}, \dots, \mathcal{R}_{p_m}\rangle\rangle$

- of a non-empty domain D
- interpretations $\mathcal{F}_{f_i}$ for function symbols f_i ($\in D^{k_i} \to D$)
- interpretations $\mathcal{R}_{p_i}$ for predicate symbols p_i ($\subseteq D^{j_i}$)

Models of predicate logic replace **truth assignments** for propositional logic

For example:

- we might interpret the signature $\langle\langle \mathtt{add}\rangle, \langle \mathtt{even}\rangle\rangle$
  - where $\mathtt{add}$ is a binary function symbol
  - and $\mathtt{even}$ is a unary predicate symbol
- by the model $\langle \mathbb{N}, \langle\langle +\rangle, \langle\{\langle 0\rangle, \langle 2\rangle, \langle 4\rangle, \dots\}\rangle\rangle\rangle$

Models

A **model** assigns meaning to function and predicate symbols

Variable valuations: In addition, we need to assign meaning to **variables**:

- this is done using a partial function v
- that maps variables to D
- i.e., a mapping of the form $x_1 \mapsto d_1, \dots, x_n \mapsto d_n$
- which maps each x_i to d_i, i.e., to $v(x_i)$
- $\mathtt{dom}(v) = \{x_1, \dots, x_n\}$
- let $\cdot$ be the empty mapping
- we write $v, x \mapsto d$ for the mapping that
  - maps x to d
  - and maps each $y \in \mathtt{dom}(v)$ such that $x \neq y$ to $v(y)$

For example

- $(x_1 \mapsto d_1), x_2 \mapsto d_2$ maps x_1 to $?d_1$ and x_2 to $?d_2$
- $(x_1 \mapsto d_1, x_2 \mapsto d_2), x_1 \mapsto d_3$ maps x_1 to $?d_3$ and x_2 to $?d_2$

Semantics of Predicate Logic

For example:

- consider the signature $\langle\langle \mathtt{zero}, \mathtt{succ}, \mathtt{add}\rangle, \langle \mathtt{lt}, \mathtt{ge}\rangle\rangle$
- the model M:
  $\langle \mathbb{N}, \langle 0, +1, +\rangle, \langle\{\langle 0,1\rangle, \langle 0,2\rangle, \langle 1,2\rangle, \dots\}, \{\langle 0,0\rangle, \langle 1,1\rangle, \langle 1,0\rangle, \dots\}\rangle\rangle$
- we write $+1$ for the function that given a number increments it by 1
- $+(n,m)$ stands for $n+m$

What is $\models_{M,\cdot} \forall x.\forall y.\mathtt{lt}(x,y) \to \mathtt{ge}(y,x)$?

- iff for all $n, m \in \mathbb{N}$, $\models_{M,x\mapsto n,y\mapsto m} \mathtt{lt}(x,y) \to \mathtt{ge}(y,x)$
- iff for all $n, m \in \mathbb{N}$, $\models_{M,x\mapsto n,y\mapsto m} \mathtt{ge}(y,x)$ whenever $\models_{M,x\mapsto n,y\mapsto m} \mathtt{lt}(x,y)$
- iff for all $n, m \in \mathbb{N}$,
  $\langle [\![y]\!]^M_{x\mapsto n,y\mapsto m}, [\![x]\!]^M_{x\mapsto n,y\mapsto m}\rangle \in \{\langle 0,0\rangle, \langle 1,1\rangle, \langle 1,0\rangle, \dots\}$ whenever
  $\langle [\![x]\!]^M_{x\mapsto n,y\mapsto m}, [\![y]\!]^M_{x\mapsto n,y\mapsto m}\rangle \in \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 1,2\rangle, \dots\}$

- iff for all $n, m \in \mathbb{N}$, $\langle [\![y]\!]^M_{x\mapsto n, y\mapsto m}, [\![x]\!]^M_{x\mapsto n, y\mapsto m} \rangle \in \{\langle 0,0\rangle, \langle 1,1\rangle, \langle 1,0\rangle, \ldots\}$ whenever $\langle [\![x]\!]^M_{x\mapsto n, y\mapsto m}, [\![y]\!]^M_{x\mapsto n, y\mapsto m} \rangle \in \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 1,2\rangle, \ldots\}$
- iff for all $n, m \in \mathbb{N}$, $\langle m, n\rangle \in \{\langle 0,0\rangle, \langle 1,1\rangle, \langle 1,0\rangle, \ldots\}$ whenever $\langle n, m\rangle \in \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 1,2\rangle, \ldots\}$
- iff True

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Satisfiability & Validity

We write $\models_M P$ for $\models_{M,\cdot} P$

Truth: P is **true** in the model M if $\models_M P$

We also say that M is a model of P

Satisfiability: P is **satisfiable** if there is a model M such that P is true in M, i.e., $\models_M P$

Validity: P is **valid** if for all model M, P is true in M

Example: $\models_{M,\cdot} \forall x.\mathtt{even}(x) \rightarrow \neg\mathtt{odd}(x)$ is satisfiable (see above) but not valid because not true for example in the model $\langle \mathbb{N}, \langle 0, +1, +\rangle, \langle \{\langle 0\rangle, \langle 2\rangle, \langle 4\rangle, \ldots\}, \{\langle 0\rangle, \langle 2\rangle, \langle 4\rangle, \ldots\}\rangle\rangle$

Decidability: Validity is not decidable for predicate logic, i.e., there is no algorithm that given a formula P either returns "yes" if P is valid, and otherwise returns "no", while it is decidable for propositional logic

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Semantics of Predicate Logic

Given a **model** M with domain D and a **variable valuation** v, to assign **meaning** to Predicate Logic formulas, we define two operations:

- $[\![t]\!]^M_v$, which gives meaning to the term t w.r.t. M and v
- $\models_{M,v} P$, which gives meaning to the formula P w.r.t. M and v

Meaning of terms:

- $[\![x]\!]^M_v = v(x)$
- $[\![f(t_1, \ldots, t_n)]\!]^M_v = \mathcal{F}_f(\langle [\![t_1]\!]^M_v, \ldots, [\![t_n]\!]^M_v \rangle)$

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- $\models_{M,v} P$, which gives meaning to the formula P w.r.t. M and v

Meaning of formulas:

- $\models_{M,v} p(t_1, \ldots, t_n)$ iff $\langle [\![t_1]\!]_v^M, \ldots, [\![t_n]\!]_v^M \rangle \in \mathcal{R}_p$
- $\models_{M,v} \neg P$ iff $\neg \models_{M,v} P$
- $\models_{M,v} P \wedge Q$ iff $\models_{M,v} P$ and $\models_{M,v} Q$
- $\models_{M,v} P \vee Q$ iff $\models_{M,v} P$ or $\models_{M,v} Q$
- $\models_{M,v} P \rightarrow Q$ iff $\models_{M,v} Q$ whenever $\models_{M,v} P$
- $\models_{M,v} \forall x.P$ iff for every $d \in D$ we have $\models_{M,(v,x\mapsto d)} P$
- $\models_{M,v} \exists x.P$ iff there exists a $d \in D$ such that $\models_{M,(v,x\mapsto d)} P$

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