Classical Reasoning

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The proof systems we have seen so far are sometimes called **constructive** or **intuitionistic**, i.e., **proofs** can be viewed as **programs**:

- A proof of $A \wedge B$ can be viewed as a **pair** of a proof of A and a proof of B
- A proof of $A \to B$ can be viewed as a **procedure** which transforms evidence for A into evidence for B
- A proof of $A \vee B$ is either a proof of A or a proof of B, which indicates which one it is

There are other proof systems, called **classical**, which

- rely on Boolean truth values
- introduce additional reasoning principles

Classical Reasoning: Proof by Contradiction

A typical classical reasoning principal is the **"proof by contradiction"** proof technique

Example: Euclid's proof of infinitude of primes

- Assume the negation: Suppose there are only finitely many primes, say $p_1, p_2, \ldots, p_r$
- Consider the number $n = (p_1 \times p_2 \times \ldots \times p_r) + 1$
- Then n cannot be a prime (by assumption)
- But none of the primes $p_1, p_2, \ldots, p_r$ can divide n
- Contradiction

Proof by Contradiction:

- If $\neg A \to \perp$ then A
- That is, $\neg\neg A \vdash A$

Classical vs. Intutionistic Reasoning in Natural Deduction

Two more (equivalent) assumptions/rules

Law of Excluded Middle (LEM)

- For each A, we can always prove one of A or $\neg A$
- i.e., $\vdash A \vee \neg A$
- E.g., we can assume every even natural number > 2 is the sum of two primes, or not, without knowing which one is true

Double Negation Elimination (DNE)

- $\neg\neg A \vdash A$
- Equivalently, $(\neg A) \to \bot \vdash A$
- "proof by contradiction"

In constructive proof you can assume A and not A which creates an explosion which allows anything to be concluded but you CAN'T say simply not A implies false which implies anything which you can in classical

Can we deduce A and $\neg\neg A$ from each other?
That is, are they equivalent?

One direction is easy: $A \vdash \neg\neg A$

Here is the proof:

$$\dfrac{\dfrac{A \qquad \overline{\neg A}^{\,1}}{\bot}\,[\neg E]}{\neg\neg A}\,1\,[\neg I]$$

Can we show the other direction, i.e., $\neg\neg A \vdash A$?

Not using the current set of inference rules we have!

Two more (equivalent) assumptions/rules

As rules:

$$\dfrac{}{A \vee \neg A}\,[LEM] \qquad\qquad \dfrac{\neg\neg A}{A}\,[DNE]$$

Classical reasoning allows using these two rules

We so far have not used them, and were therefore using what is called **constructive** or **intuitionistic** logic

LEM implies DNE

Assuming $A \vee \neg A$, infer $\neg\neg A \vdash A$

Here is a proof:

2 : B

$$\dfrac{A \vee \neg A \qquad \dfrac{\overline{A}^{\,1}}{A \to A}\,1\,[\to I] \qquad \dfrac{\dfrac{\dfrac{\overline{\neg A}^{\,2} \qquad \neg\neg A}{\bot}\,[\neg E]}{A}\,[\bot E]}{\neg A \to A}\,2\,[\to I]}{A}\,[\vee E]$$

DNE implies LEM

Assuming $\neg\neg A \vdash A$, infer $\vdash A \vee \neg A$

Here is a proof:

$$\dfrac{}{\neg(A \vee \neg A)}\,1 \qquad \dfrac{\overline{A}^{\,2}}{A \vee \neg A}\,[\vee I_L]$$

DNE implies LEM

Assuming $\neg\neg A \vdash A$, infer $\vdash A \lor \neg A$

Here is a proof:

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(A\lor\neg A)}\,1 \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\neg(A\lor\neg A)}\,1 \quad \dfrac{\dfrac{}{A}\,2}{A\lor\neg A}\,[\lor I_L]}{\bot}\,[\neg E]}{\neg A}\,2\,[\neg I]}{A\lor\neg A}\,[\lor I_R]}{\bot}\,[\neg E]}{\neg\neg(A\lor\neg A)}\,1\,[\neg I]}{A\lor\neg A}\,[DNE]
$$

Provide a classical Natural Deduction proof of $(A \to B) \lor (B \to A)$

$$
\dfrac{\dfrac{}{A\lor\neg A}\,[LEM] \quad \dfrac{\dfrac{\dfrac{\dfrac{}{A}\,1}{B\to A}\,2\,[\to I]}{(A\to B)\lor(B\to A)}\,[\lor I_R]}{A\to(A\to B)\lor(B\to A)}\,1\,[\to I] \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg A}\,3 \quad \dfrac{}{A}\,4}{\bot}\,[\neg E]}{B}\,[\bot E]}{A\to B}\,4\,[\to I]}{(A\to B)\lor(B\to A)}\,[\lor I_L]}{\neg A\to(A\to B)\lor(B\to A)}\,3\,[\to I]}{(A\to B)\lor(B\to A)}\,[\lor E]
$$

Provide a classical Natural Deduction proof of
$(\neg B \to \neg A) \to (A \to B)$
Here is a proof:

$$
\begin{prooftree}
\AxiomC{}\RightLabel{$[LEM]$}\UnaryInfC{$B \vee \neg B$}
\AxiomC{}\RightLabel{3}\UnaryInfC{B}\RightLabel{3 $[\to I]$}\UnaryInfC{$B \to B$}
\AxiomC{}\RightLabel{1}\UnaryInfC{$\neg B \to \neg A$}
\AxiomC{}\RightLabel{4}\UnaryInfC{$\neg B$}
\RightLabel{$[\to E]$}\BinaryInfC{$\neg A$}
\AxiomC{}\RightLabel{2}\UnaryInfC{A}
\RightLabel{$[\neg E]$}\BinaryInfC{$\perp$}
\RightLabel{$[\perp E]$}\UnaryInfC{B}
\RightLabel{4 $[\to I]$}\UnaryInfC{$\neg B \to B$}
\RightLabel{$[\vee E]$}\TrinaryInfC{B}
\RightLabel{2 $[\to I]$}\UnaryInfC{$A \to B$}
\RightLabel{1 $[\to I]$}\UnaryInfC{$(\neg B \to \neg A) \to (A \to B)$}
\end{prooftree}
$$

Contrapositive

Given an implication $A \to B$, the formula $\neg B \to \neg A$ is called the **"contrapositive"**

Can we prove that an implication implies its contrapositive?

$$A \to B \vdash \neg B \to \neg A$$

Here is a proof (intuitionistic):

$$
\begin{prooftree}
\AxiomC{$A \to B$}
\AxiomC{}\RightLabel{2}\UnaryInfC{A}
\RightLabel{$[\to E]$}\BinaryInfC{B}
\AxiomC{}\RightLabel{1}\UnaryInfC{$\neg B$}
\RightLabel{$[\neg E]$}\BinaryInfC{$\perp$}
\RightLabel{2 $[\neg I]$}\UnaryInfC{$\neg A$}
\RightLabel{1 $[\to I]$}\UnaryInfC{$\neg B \to \neg A$}
\end{prooftree}
$$

Contrapositive

Given an implication $A \to B$, the formula $\neg B \to \neg A$ is called the **"contrapositive"**

Can we prove that an implication follows from its contrapositive?
$\neg B \to \neg A \vdash A \to B$

Here is a proof (classical):

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A}^{\,1} \qquad \dfrac{\neg B \to \neg A \qquad \overline{\neg B}^{\,2}}{\neg A}\,[\to E]}{\bot}\,[\neg E]}{\neg\neg B}\,2\,[\neg I]}{B}\,[DNE]}{A \to B}\,1\,[\to I]$$

We used DNE, and hence this proof uses classical reasoning!