GAUSS Elimination

$$4x + y = 16$$

$$x + y = 7$$

In linear algebra, we write this as a matrix equation,

$\begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 16 \\ 7 \end{pmatrix}$ or short: $Ax = b$ where $x = (x_1, x_2)$

Indices are useful: x_i gives you the $i - th$ element of vector x

Let A be a $n \times n$ matrix with entries from $\mathbb{R}$.

Definition: Given a real $n \times n$ matrix A, a vector $x \in \mathbb{R}^n$ (with $x \neq \vec{0}$) is an eigenvector of A with eigenvalue λ if $\boxed{Ax = \lambda x}$

- A is a $(n \times n)$ matrix and x is $(n \times 1)$ so Ax is $(n \times 1)$
- Note that in general Ax need not be a multiple of x at all!

Example: $A = \begin{pmatrix} 1 & 4 \\ 3 & 6 \end{pmatrix}$ and $x = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ Then $Ax = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$

Easy to see that Ax is not x multiplied by any constant!

Example: $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ Then

$$Bx = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So $\lambda = -1$ is one eigenvalue.

Can you find the second one, and its eigenvector?

Consider the equation $ax = b$ where x is a variable and a, b are rational numbers. How many solutions does this equation have?

- Exactly one solution
- No solutions
- Infinitely many solutions
- Don't know
- Question is not well-defined

- Question is not well-defined

What if we have only 2 equations?

$$3x + 2y + z = 6$$
$$x + 3y + 2z = 4$$

What if we have 4 equations?

$$3x + 2y + z = 6$$
$$x + 3y + 2z = 4$$
$$2x + 3y + 3z = 9$$
$$x + y + z = 1$$

Each equation imposes a constraint
Each variable gives a "degree of freedom"

A better method is the GAUSS elimination:

- Base case is one variable and one equation, i.e., $ax = b$
- Eliminate any variable to get $(n - 1)$ equations in $(n - 1)$ variables
  - We can choose which one to eliminate!
- Perform this recursively till you reach the base case.
  - Somewhere in the middle you might reach a special case
  - Special cases: No solution, infinitely many solutions, ...
  - Otherwise you will find the unique solution!

Aim is to get the matrix into row echelon form:

$$\left(\begin{array}{cccc|c} \bullet & * & * & * & * \\ 0 & \bullet & * & * & * \\ 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & \bullet & * \end{array}\right)$$

$$\left(\begin{array}{cccc|c} \bullet & * & * & 0 & * \\ 0 & \bullet & * & 0 & * \\ 0 & 0 & \bullet & 0 & * \\ 0 & 0 & 0 & 1 & * \end{array}\right)$$

$$\left(\begin{array}{cccc|c} \bullet & * & 0 & 0 & * \\ 0 & \bullet & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & * \\ 0 & 1 & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \end{array}\right)$$

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<table>
<tr><th>System of equations</th><th>Row operations</th><th>Augmented matrix</th></tr>
<tr><td>$\begin{aligned} 2x + y - z &= 8 \\ -3x - y + 2z &= -11 \\ -2x + y + 2z &= -3 \end{aligned}$</td><td></td><td>$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{array}\right]$</td></tr>
<tr><td>$\begin{aligned} 2x + y - z &= 8 \\ \frac{1}{2}y + \frac{1}{2}z &= 1 \\ 2y + z &= 5 \end{aligned}$</td><td>$\begin{aligned} R_2 + \frac{3}{2}R_1 &\to R_2 \\ R_3 + R_1 &\to R_3 \end{aligned}$</td><td>$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 2 & 1 & 5 \end{array}\right]$</td></tr>
<tr><td>$\begin{aligned} 2x + y - z &= 8 \\ \frac{1}{2}y + \frac{1}{2}z &= 1 \\ -z &= 1 \end{aligned}$</td><td>$R_3 + -4R_2 \to R_3$</td><td>$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right]$</td></tr>
</table>

The matrix is now in row-echelon form. This is also called *triangular form*.

System of equations	Row operations	Augmented matrix
$\begin{aligned} 2x + y &= 7 \\ \frac{1}{2}y &= 3/2 \\ -z &= 1 \end{aligned}$	$\begin{aligned} R_2 + \frac{1}{2}R_3 &\to R_2 \\ R_1 - R_3 &\to R_1 \end{aligned}$	$\left[\begin{array}{ccc\|c} 2 & 1 & 0 & 7 \\ 0 & 1/2 & 0 & 3/2 \\ 0 & 0 & -1 & 1 \end{array}\right]$
$\begin{aligned} 2x + y &= 7 \\ y &= 3 \\ z &= -1 \end{aligned}$	$\begin{aligned} 2R_2 &\to R_2 \\ -R_3 &\to R_3 \end{aligned}$	$\left[\begin{array}{ccc\|c} 2 & 1 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$
$\begin{aligned} x &= 2 \\ y &= 3 \\ z &= -1 \end{aligned}$	$\begin{aligned} R_1 - R_2 &\to R_1 \\ \frac{1}{2}R_1 &\to R_1 \end{aligned}$	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$

The matrix is now in reduced row-echelon form. Reading this matrix tells us that the solutions for this system of equations occur when x = 2, y = 3, and z = -1.

Special cases:

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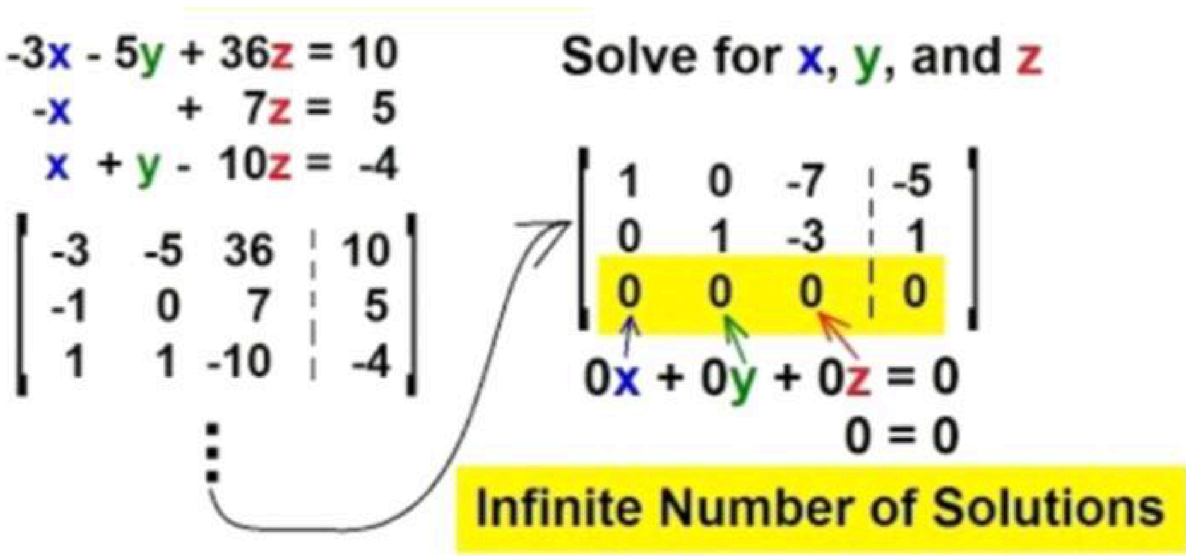
Special cases:

$$\begin{aligned} x + 2y + 3z + 4w &= 5 \\ x + 3y + 5z + 7w &= 11 \\ x \quad\quad - z - 2w &= -6 \end{aligned} \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 7 & 11 \\ 1 & 0 & -1 & -2 & -6 \end{array}\right]$$

$$\downarrow \text{RREF}$$

$$= \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \emptyset \quad \text{no solution} \longleftarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & -7 \\ 0 & 1 & 2 & 3 & 6 \\ \boxed{0 & 0 & 0 & 0 & 1} \end{array}\right]$$

If an entire row on the left is 0 and t he right value is non-zero there are no solutions



If an ENTIRE row is 0 then there are infinite solutions

$$x_1 + 5x_2 - 2x_3 + 3x_4 = -11$$

$$3x_1 - 2x_2 + 7x_3 + x_4 = 5$$

$$-2x_1 - x_2 - x_3 - 2x_4 = 0$$

$$5x_1 + 3x_2 + 4x_3 - x_4 = 13$$