Equivalences

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Logical equivalences

Let $A \leftrightarrow B$ be defined as $(A \to B) \wedge (B \to A)$

- it means that A and B are logically equivalent
- A and B have the same semantics
- A is provable if and only if B is provable
- this is called a "bi-implication"
- read as "A if and only if B"

De Morgan's Laws (I): Negation of OR

Show the logical equivalence $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$ in Natural Deduction

We first prove the left-to-right implication:
$\neg(A \vee B) \vdash (\neg A \wedge \neg B)$

Here is a proof:

$$\dfrac{\dfrac{\dfrac{\neg(A \vee B) \quad \dfrac{\overline{A}^{\,1}}{A \vee B}\,[\vee I_L]}{\bot}\,[\neg E]}{\neg A}\,1\,[\neg I] \quad \dfrac{\dfrac{\neg(A \vee B) \quad \dfrac{\overline{B}^{\,2}}{A \vee B}\,[\vee I_R]}{\bot}\,[\neg E]}{\neg B}\,2\,[\neg I]}{\neg A \wedge \neg B}\,[\wedge I]$$

Proof only uses intuitionistic rules!

We now prove the right-to-left implication:
$(\neg A \wedge \neg B) \vdash \neg(A \vee B)$

Here is a proof:

$$\dfrac{\overline{A \vee B}^{\,1} \quad \dfrac{\dfrac{\overline{A}^{\,2} \quad \dfrac{\neg A \wedge \neg B}{\neg A}\,[\wedge E]}{\bot}\,[\neg E]}{A \to \bot}\,2\,[\to I] \quad \dfrac{\dfrac{\overline{B}^{\,3} \quad \dfrac{\neg A \wedge \neg B}{\neg B}\,[\wedge E]}{\bot}\,[\neg E]}{B \to \bot}\,3\,[\to I]}{\quad}\,[\vee E]$$

We now prove the right-to-left implication:
$(\neg A \wedge \neg B) \vdash \neg(A \vee B)$

Here is a proof:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$A \vee B$}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{A}
\AxiomC{$\neg A \wedge \neg B$}
\RightLabel{$[\wedge E]$}
\UnaryInfC{$\neg A$}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{2 $[\to I]$}
\UnaryInfC{$A \to \bot$}
\AxiomC{}
\RightLabel{3}
\UnaryInfC{B}
\AxiomC{$\neg A \wedge \neg B$}
\RightLabel{$[\wedge E]$}
\UnaryInfC{$\neg B$}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{3 $[\to I]$}
\UnaryInfC{$B \to \bot$}
\RightLabel{$[\vee E]$}
\TrinaryInfC{$\bot$}
\RightLabel{1 $[\neg I]$}
\UnaryInfC{$\neg(A \vee B)$}
\end{prooftree}
$$

Again, we only used intuitionistic rules!

Logical equivalences

Many equivalences hold, some constructively, but some only classically. **Try and prove the following equivalences**:

- De Morgan's law (I): $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$
- De Morgan's law (II): $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$
- implication elimination: $(A \to B) \leftrightarrow (\neg A \vee B)$
- Commutativity of $\wedge$: $(A \wedge B) \leftrightarrow (B \wedge A)$
- Commutativity of $\vee$: $(A \vee B) \leftrightarrow (B \vee A)$
- Associativity of $\wedge$: $((A \wedge B) \wedge C) \leftrightarrow (A \wedge (B \wedge C))$
- Associativity of $\vee$: $((A \vee B) \vee C) \leftrightarrow (A \vee (B \vee C))$
- Distributivity of $\wedge$ over $\vee$: $(A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$
- Distributivity of $\vee$ over $\wedge$: $(A \vee (B \wedge C)) \leftrightarrow ((A \vee B) \wedge (A \vee C))$
- Double negation elimination: $(\neg\neg A) \leftrightarrow A$
- Idempotence: $(A \wedge A) \leftrightarrow A$ and $(A \vee A) \leftrightarrow A$

De Morgan's Laws (II): Negation of AND

Show the logical equivalence $\neg(A \wedge B) \leftrightarrow \neg A \vee \neg B$ in Natural Deduction

We first prove the right-to-left implication: $\neg A \vee \neg B \vdash \neg(A \wedge B)$

Here is a proof:

$$
\begin{prooftree}
\AxiomC{$\neg A \vee \neg B$}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{$\neg A$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$A \wedge B$}
\RightLabel{$[\wedge E_L]$}
\UnaryInfC{A}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{2 $[\rightarrow I]$}
\UnaryInfC{$\neg A \rightarrow \bot$}
\AxiomC{}
\RightLabel{3}
\UnaryInfC{$\neg B$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$A \wedge B$}
\RightLabel{$[\wedge E_R]$}
\UnaryInfC{B}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{3 $[\rightarrow I]$}
\UnaryInfC{$\neg B \rightarrow \bot$}
\RightLabel{$[\vee E]$}
\TrinaryInfC{$\bot$}
\RightLabel{1 $[\neg I]$}
\UnaryInfC{$\neg(A \wedge B)$}
\end{prooftree}
$$

Proof uses intuitionistic rules!

We now prove the left-to-right implication: $\neg(A \wedge B) \vdash \neg A \vee \neg B$

Here is a proof (classical—we use DNE thrice):

$$
\begin{prooftree}
\AxiomC{$\neg(A \wedge B)$}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{$\neg A$}
\RightLabel{$[\vee I_L]$}
\UnaryInfC{$\neg A \vee \neg B$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\neg(\neg A \vee \neg B)$}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{2 $[\neg I]$}
\UnaryInfC{$\neg\neg A$}
\RightLabel{$[DNE]$}
\UnaryInfC{A}
\AxiomC{}
\RightLabel{3}
\UnaryInfC{$\neg B$}
\RightLabel{$[\vee I_R]$}
\UnaryInfC{$\neg A \vee \neg B$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\neg(\neg A \vee \neg B)$}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{3 $[\neg I]$}
\UnaryInfC{$\neg\neg B$}
\RightLabel{$[DNE]$}
\UnaryInfC{B}
\RightLabel{$[\wedge I]$}
\BinaryInfC{$A \wedge B$}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{1 $[\neg I]$}
\UnaryInfC{$\neg\neg(\neg A \vee \neg B)$}
\RightLabel{$[DNE]$}
\UnaryInfC{$\neg A \vee \neg B$}
\end{prooftree}
$$

Logical equivalences

As our Natural Deduction equivalence proofs will all be as follows:

$$\dfrac{\dfrac{\dfrac{\overline{A}^{\,1}}{\vdots}\quad B}{A \to B}\,1\,[\to I] \qquad \dfrac{\dfrac{\overline{B}^{\,2}}{\vdots}\quad A}{B \to A}\,2\,[\to I]}{A \leftrightarrow B}\,[\wedge I]$$

then, we will focus on proving

- $A \vdash B$ (left-to-right implication)
- $B \vdash A$ (right-to-left implication)

Expressing $\to$ using $\neg$ and $\vee$

Show the logical equivalence: $A \to B \leftrightarrow \neg A \vee B$

We first prove the left-to-right implication $A \to B \vdash \neg A \vee B$

Here is a proof (classical—it uses LEM):

$$\dfrac{\dfrac{}{A \vee \neg A}\,[LEM] \qquad \dfrac{\dfrac{\dfrac{\overline{A}^{\,1} \quad A \to B}{B}\,[\to E]}{\neg A \vee B}\,[\vee I_R]}{A \to (\neg A \vee B)}\,1\,[\to I] \qquad \dfrac{\dfrac{\overline{\neg A}^{\,2}}{\neg A \vee B}\,[\vee I_L]}{\neg A \to (\neg A \vee B)}\,2\,[\to I]}{\neg A \vee B}\,[\vee E]$$

The other direction holds intuitionistically (next slide)

We now prove the right-to-left implication $\neg A \vee B \vdash A \to B$

Here is a proof (intuitionistic):

$$
\begin{prooftree}
\AxiomC{$\neg A \vee B$}
\AxiomC{$\overline{\neg A}^{\,2}$}
\AxiomC{$\overline{A}^{\,1}$}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{$[\bot E]$}
\UnaryInfC{B}
\RightLabel{$2\ [\to I]$}
\UnaryInfC{$\neg A \to B$}
\AxiomC{$\overline{B}^{\,3}$}
\RightLabel{$3\ [\to I]$}
\UnaryInfC{$B \to B$}
\RightLabel{$[\vee E]$}
\TrinaryInfC{B}
\RightLabel{$1\ [\to I]$}
\UnaryInfC{$A \to B$}
\end{prooftree}
$$