- A matrix M acts as a **linear transformation** that maps a standard basis $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ to a new set of vectors $M\vec{e}_1, M\vec{e}_2, \ldots, M\vec{e}_n$.
- This transformation can change the coordinates or scale of vectors.
- The new set of vectors might not always form a basis (if, for example, they become linearly dependent), but that topic is mentioned as something to explore later.

2. Example 1: Rotational Matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

- This matrix **rotates** the unit vectors cyclically:
  - $x \to y \to z \to x$
- In other words, this transformation changes:
  - $(1,0,0) \to (0,1,0)$
  - $(0,1,0) \to (0,0,1)$
  - $(0,0,1) \to (1,0,0)$

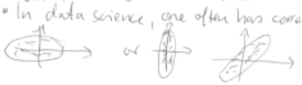
- This is called a **cyclic permutation** of the components, and it rotates around the [111] axis, meaning equal contribution of x, y, z.

3. Example 2: Scaling Matrix

$$\begin{pmatrix} 2.54 & 0 & 0 \\ 0 & 2.54 & 0 \\ 0 & 0 & 2.54 \end{pmatrix}$$

- This matrix **rescales** all the axes by a factor of 2.54:
  - Converting between **centimeters (cm)** and **inches (in)**.
- The multiplication by 2.54 increases each axis proportionally.
- Since volume is calculated as the product of lengths along all three axes, the volume scales by:
$$(2.54)^3$$
  - Numerically, the volume increases by this factor.
  - Physically, the **actual volume** remains the same, but its representation in a different unit system (inches vs. centimeters) changes.

- One might wish to scale units differently for the coordinates, like km and miles in x and y, but m or yard for z (land maps: mountain height vs horizontal distances)
$\begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 5 \end{pmatrix} \leftarrow$ we get a matrix like this
- In data science, one often has correlated data to approximate:



but we might get the same data in different coordinate systems.

$\boxed{\text{Can we "extract" some properties from a given matrix that describes "what it does"?}}$

Definition of eigenvalue and eigenvector

Let A be a $n \times n$ matrix with entries from $\mathbb{R}$.

Definition: Given a real $n \times n$ matrix A, a vector $x \in \mathbb{R}^n$ (with $x \neq \vec{0}$) is an eigenvector of A with eigenvalue λ if $\boxed{Ax = \lambda x}$

- A is a $(n \times n)$ matrix and x is $(n \times 1)$ so Ax is $(n \times 1)$
- Note that in general Ax need not be a multiple of x at all!

Example: $A = \begin{pmatrix} 1 & 4 \\ 3 & 6 \end{pmatrix}$ and $x = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

Then $Ax = \begin{pmatrix} 2 \\ 6 \end{pmatrix} \neq \lambda \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ for any $\lambda \in \mathbb{R}$

Easy to see that Ax is not x multiplied by any constant!

Example: $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ Then

ááá

- A is a $(n \times n)$ matrix and x is $(n \times 1)$ so Ax is $(n \times 1)$
- Note that in general Ax need not be a multiple of x at all!

Example: $A = \begin{pmatrix} 1 & 4 \\ 3 & 6 \end{pmatrix}$ and $x = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

Then $Ax = \begin{pmatrix} 2 \\ 6 \end{pmatrix} \neq \lambda \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ for any $\lambda \in \mathbb{R}$

Easy to see that Ax is not x multiplied by any constant!

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Example: $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ Then

$Bx = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

So $\lambda = -1$ is one eigenvalue.

Can you find the second one, and its eigenvector?

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Definition of eigenvalue and eigenvector

Let F be a field, $n \in \mathbb{N}$, and F^n a n-dimensional vector space with linear mappings (in the vector space) represented by matrices $M \in F^{n \times n}$.

Definition : $\lambda_i \in F$ is called an eigenvalue of M

to the eigenvector $\vec{x}_i \in F^n, \vec{x}_i \neq \vec{0}$, if $\boxed{M\vec{x}_i = \lambda_i \vec{x}_i}$

- For λ_j, we can expect up to n different solutions
- For each eigenvalue λ_i, we can solve for $\vec{x}_i$.
- We obtain up to n pairs (eigenvalue,eigenvector) λ_i, $\vec{x}_i$.
- Some k eigenvalues may be identical ("degenerate"), then these form a k-dimensional subspace

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How to find eigenvalues & eigenvectors of A

Let $A = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$

We want to find an $(n \times 1)$ vector x such that there exists constants λ satisfying $Ax = \lambda x$

$Ax = \lambda x = \lambda I x$ where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Therefore, $(A - \lambda I)x = \vec{0}$

- Let $B = (A - \lambda I)$
- If B has an inverse, say B^{-1}, then we can multiply above equation on both sides by it to get $\vec{0} = B^{-1}\vec{0} = B^{-1}(Bx) = (B^{-1}B)x = Ix = x$
- So, the only solution is $x = \vec{0}$ if $(A - \lambda I)$ has an inverse
- Need to see when $(A - \lambda I)$ does not have an inverse

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Finding eigenvalues (cont'd)

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We define $\det(A) = ad - bc$

More complicated definition of determinant for $(n \times n)$ matrices when $n \geq 3$

For any $n \times n$ matrix A, it has an inverse if and only if $\det(A) \neq 0$

Example from previous slide: $A = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$

- Want to see for which constants λ_i does $(A - \lambda_i I)$ not have an inverse
- That is, for which constants λ_i is $\det(A - \lambda_i I) = 0$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow A - \lambda I = \begin{pmatrix} 1 - \lambda_i & -1 \\ 2 & -1 - \lambda_i \end{pmatrix}$

$0 = \det(A - \lambda_i I) = (1 - \lambda_i)(-1 - \lambda_i) - (-2) = \lambda_i^2 + 1$

If the matrix A is real and symmetric, then all eigenvalues are real numbers

Eigenvalues can be complex even if the matrix A has real entries ($\forall_{i,j}\ a_{ij} \in \mathbb{R}$)

- You will see examples of this next semester(s) in AI and Data Science
- Many other examples in Physics, Robotics, Control, Signal Processing, ...

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Finding eigenvectors from eigenvalues

If we know an eigenvalue λ_i for a $(n \times n)$ matrix A, then we can compute the eigenvector corresponding to it by solving the equation $Ax = \lambda_i x$

- Solve using Gaussian elimination

(but we first have to find the eigenvalues!)

Two steps for finding eigenvalues and eigenvectors of a $(n \times n)$ matrix A:

- First, find all eigenvalues by solving $\det(A - \lambda_i I) = 0$ *(characteristic equation)*
  Note: $\det(A - \lambda_i I) = 0$ is a polynomial of degree n *(characteristic polynomial)*
- Then, for each eigenvalue λ_i, find the corresponding eigenvector x by solving $Ax = \lambda_i x$ using Gaussian elimination

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Eigenvalues of a non-symmetric matrix

2D $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $b \neq c$

Example: $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Then

$0 \stackrel{!}{=} \det(M - \lambda \mathbb{1}) = \det\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = \det\begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix}$

$= (-\lambda)^2 - (-1) \cdot 1 = \lambda^2 + 1$

Reminder: $0 = \lambda^2 - 2$ has no rational $(\in \mathbb{Q})$ solution

Complex Numbers

"Define" i as solution of $i^2 = -1$

and "write" $i = \sqrt{-1}$ (also $-i$ then is a solution)

Here, we get $\lambda_1 = i = \sqrt{-1}$, $\lambda_2 = -i = -\sqrt{-1}$.

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Complex Numbers (Nutshell version)

For any complex number $z \in \mathbb{C}$, there are $x, y \in \mathbb{R}$ such that $z = x + iy$

real part ↑ imaginary part

$\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \underset{\substack{x = (z + z^*)/2 \\ y = (z - z^*)/2i}}{\overset{z := x + iy}{\rightleftarrows}} z \in \mathbb{C}$

$x = Re(z)$

Def. complex

i : imaginary unit

Complex Numbers (Nutshell version)

For any complex number $z \in \mathbb{C}$, there are $x, y \in \mathbb{R}$ such that $z = x + iy$

real part ↗ ↑ imaginary part

$$\binom{x}{y} \in \mathbb{R}^2 \underset{\substack{x = (z+z^*)/2 \\ y = (z-z^*)/2i}}{\overset{z:x+iy}{\rightleftarrows}} z \in \mathbb{C}$$

$x = \mathrm{Re}(z)$

$y = \mathrm{Im}(z)$

Def. complex conjugate $z^* = \bar{z} = x - iy$

i: imaginary unit and solution of $i^2 = -1$

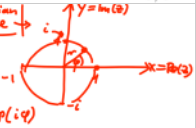
We need to multiply complex numbers!

$(x+iy)(a+ib) = xa + iya + ixb + (i)^2 yb =$

$= (xa - yb) + i(ya + xb)$

Using polar coordinates

Gaussian plane →



$x = r \cdot \cos(\varphi)$

$y = r \cdot \sin(\varphi)$

$z = x + iy = r \cdot (\cos\varphi + i \sin\varphi)$

$= r \cdot e^{i \cdot \varphi} = r \cdot \exp(i\varphi)$

Norm $\|z\|^2 = x^2 + y^2$ $\quad (x+iy)^2 = x^2 - y^2 + 2ixy$

$= (x - iy)(x + iy)$

$= z^* \cdot z \quad \rightarrow \|z\| := \sqrt{z^* \cdot z}$

Need inner product to be defined as

$\langle \vec{z}_1, \vec{z}_2 \rangle = (\vec{z}_1^{\,T})^* (\vec{z}_2) =$

Eigenvalues (& vectors) of a rotation matrix

$R = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$ with $c^2 + s^2 = 1$, $(\cos\varphi)^2 + (\sin\varphi)^2 = 1$, $\quad c = \cos\varphi$, $s = \sin\varphi$

$0 \stackrel{!}{=} \det \begin{vmatrix} c-\alpha & -s \\ s & c-\alpha \end{vmatrix} = (c-\alpha)^2 + s^2 = (\alpha - c)^2 - (is)^2$

$$\left.\begin{matrix} \alpha - c = is \\ \alpha - c = -is \end{matrix}\right\} \Leftrightarrow (\alpha - c)^2 = (is)^2$$

$\Rightarrow \alpha_1 = c + is$ and $\alpha_2 = c - is$ or $\alpha_{1,2} = c \pm is$

Eigenvectors & coordinate transform

$R = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \quad \alpha_{1,2} = c \pm is \quad v_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$

Now define $D := \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$ and $T = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$

$T^{-1} \stackrel{!}{=} \frac{1}{2i}\begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$ → Simplifying 1/i using the complex conjugate and rationalisation rules

Then $T^{-1} = \frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$ $\quad$ (Ex: verify $TT^{-1} = \mathbb{1}$)

$TDT^{-1} = T \cdot \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} \cdot T^{-1} = \ldots = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} = R$

$TD\underbrace{T^{-1}T}_{\mathbb{1}} = RT$ $\quad$ (Even if T not invertible, $TD = RT$ would hold)

Motivation, cont'd

If for some $\vec{x}_i$ the matrix M only stretches $\vec{x}_i$ by some factor λ_i, this means that

$$\boxed{M \vec{x}_i = \lambda_i \vec{x}_i} \quad \text{Eigenvalue \& Eigenvector equation}$$

What does that mean?

! We need to exclude the zero vector $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$M\vec{x}_i = \lambda_i \mathbb{1} \vec{x}_i = \lambda_i \vec{x}_i \Leftrightarrow (M - \lambda_i \cdot \mathbb{1}) = \vec{0}$

Recall: For $A\vec{x} = \vec{0}$, we have 2 cases: $\begin{cases} \text{a unique solution } \vec{x}_0 = \vec{0} \\ \text{if } \vec{x}_i \neq \vec{0}, \text{ for any } \lambda \quad A(\lambda\vec{x}) = \lambda(A\vec{x}) = \vec{0} \end{cases}$

Hence, $\vec{x}_i$ is a nontrivial ($\vec{x}_i \neq \vec{0}$) solution if and only if (iff)

$\det(M - \lambda_i \cdot \mathbb{1}) = 0$ for some $\lambda_i \in \mathbb{F}$

$M = \mathbb{1}_3$ Then $\det(M - \lambda_i \mathbb{1}) = 0$

$= \left| \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix} - \begin{pmatrix} \lambda_i & & 0 \\ & \lambda_i & \\ 0 & & \lambda_i \end{pmatrix} \right| = \begin{vmatrix} 1-\lambda_i & & \\ & 1-\lambda_i & \\ & & 1-\lambda_i \end{vmatrix}$

$= (1 - \lambda_i)^3 \stackrel{!}{=} 0 \quad \lambda_1 = 1 = \lambda_2 = \lambda_3$

For a diagonal matrix

$\begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix} \rightarrow \lambda_1 = d_1, \lambda_2 = d_2 \ldots \lambda_n = d_n.$

Subspace $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \square & \\ 0 & & \end{pmatrix} \quad \lambda_1 = 1, \lambda_2 \ldots$ from $\square$

Some more examples....

2D case $\quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{R}$

$0 \stackrel{!}{=} \det(M - \lambda \mathbb{1}) = \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = \left| \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} \right|$

$= (a - \lambda)(d - \lambda) - b \cdot c = \lambda^2 - (a+d)\lambda + \underbrace{(a \cdot d - bc)}_{q}$

$= \lambda^2 + p\lambda + q$

$\lambda \quad = \frac{a+d}{\ldots} \pm \sqrt{(a+d)^2 \ldots}$

Some more examples....

2D case $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{R}$

$$0 \overset{!}{=} \det(M - \lambda \mathbb{1}) = \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = \left| \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \right|$$

$$= (a-\lambda)(d-\lambda) - b \cdot c = \lambda^2 - (a+d)\lambda + \underbrace{(a \cdot d - bc)}_{q}$$

$$= \lambda^2 + p\lambda + q \qquad p = -(a+d)$$

$$\lambda_{1,2} = \frac{a+d}{2} \pm \sqrt{\left(\frac{a+d}{2}\right)^2 - (ad-bc)}$$

For $b=c$: $\sqrt{\dots} = \frac{1}{2}\sqrt{(a+d)^2 - 4ad + 4b^2} = \frac{1}{2}\sqrt{(a-d)^2 + 4b^2}$

For $b=c$: $\lambda_{1,2} = \frac{a+d}{2} \pm \frac{1}{2}\sqrt{\underbrace{(a-d)^2}_{\geq 0} + \underbrace{4b^2}_{\geq 0}} \in \mathbb{R}$

Theorem: The eigenvalues of a real symmetric matrix are real

$A = A^T$