Propositional logic syntax

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More examples:

1. You can eat a burger or pasta.
2. You ate a burger.
3. Therefore, you did not eat pasta.

Valid? No Because you could eat both. In propositional logic, or is not exclusive as it is often the case in English.

1. If the control software crashes, then the car's brakes will fail.
2. The car's brakes failed.
3. Therefore, the control software crashed.

valid? No The car's brakes could have failed for another reason.

1. If the control software crashes, then the car's brakes will fail.
2. The control software did not crash.
3. Therefore, the car's brakes did not fail.

valid? No The car's brakes could have failed for another reason.

What are the atomic propositions and connectives?

- The car's brakes failed
  an atomic proposition
- The control software crashed and the car's brakes failed
  a conjunction of 2 atomic propositions
- If the control software crashes, then the car's brakes will fail
  an implication connecting 2 atomic propositions

Example argument

1. If John is at home, then his television is on
2. His television is not on
3. Therefore, John is not at home

Identify atomic propositions:

- p = "John is at home"
- q = "John's television is on"

How do we write this argument in propositional logic?

- Premise 1: $p \rightarrow q$
- Premise 2: $\neg q$
- Conclusion: $\neg p$

Example argument

- Premise 1: $p \rightarrow q$
- Premise 2: $\neg q$
- Conclusion: $\neg p$

Example argument

- Premise 1: $p \to q$
- Premise 2: $\neg q$
- Conclusion: $\neg p$

Notation: written as a **sequent**

- $p \to q, \neg q \vdash \neg p$ turnstile
- i.e., set of premises separated by commas, then a **turnstile** followed by the conclusion.
- Recall that premises and conclusions are both formulas.
- A sequent is **valid** if the argument has been proven, i.e., if the conclusion is true assuming that the premises are true.

Propositional logic is a **symbolic logic** to reason about logical statements called **propositions** that can (in principle) be true or false.

Propositions are built by combining atomic propositions using the and, or, not, and implies logical connectives.

Are these examples of propositions?

- Birmingham is north of London Yes
- Is Birmingham north of London? No
- $8 \times 7 = 42$ Yes
- Every even natural number > 2 is the sum of two primes Yes
- Please mind the gap No

Symbols:

- atomic propositions (true/false atomic statements)
- combined using logical connectives

Atomic propositions (atoms)

- propositions that cannot be broken into smaller parts
- Let $p, q, r, \ldots$ be atomic propositions
- two special atoms: $\top$ stands for True, $\bot$ stands for False

Logical Connectives

- conjunction: $\wedge$ (and)
- disjunction: $\vee$ (or)
- implication: $\rightarrow$ (if .... then / implies)
- negation: $\neg$ (not) — can be defined using $\rightarrow$ and $\bot$

The syntax of propositional logic formulas (called propositions) is defined by the following grammar:

$$P ::= a \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid \neg P$$

where a ranges over **atomic propositions**.

Atomic propositions are formulas.

The syntax of propositional logic formulas (called propositions) is defined by the following grammar:

$$P ::= a \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid \neg P$$

where a ranges over **atomic propositions**.

Atomic propositions are formulas.

If P and Q are formulas, then

- $P \wedge Q$ is a formula
- $P \vee Q$ is a formula
- $P \rightarrow Q$ is a formula
- $\neg P$ is a formula

Those are called **compound formulas**.

Example of a compound formula: $\neg p \wedge q \wedge q \wedge \neg r$.
Conjunction: $P \wedge Q$, i.e., P and Q

- true if both individual propositions P and Q are true

Disjunction: $P \vee Q$, i.e., P or Q

- true if one or both individual propositions P and Q are true
- also sometimes called "inclusive or"
- Note: Or in English is often an "exclusive or" (i.e. where one or the other is true, but not both)
- e.g., "Your mark will be pass or fail"
- but logical disjunction is always defined as above

Parse Trees

Parentheses help clarify how formulas are derived given the propositional logic's grammar:

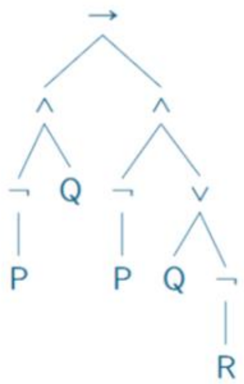
$$P ::= a \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid \neg P$$

The parse tree for $(P \wedge Q) \vee R$ is:

while the parse tree for $P \wedge (Q \vee R)$ is:

Leaves are atomic propositions and the other nodes are connectives.

What it the parse tree for: $(\neg P \wedge Q) \rightarrow (\neg P \wedge (Q \vee \neg R))$?



Scope and Main connective

Scope of a connective

- The connective itself, plus what it connects
- That is, the sub-tree of the parse tree rooted at the connective
- The scope of $\wedge$ in $(P \wedge Q) \vee R$ is $P \wedge Q$

Main connective of a formula

- The connective whose scope is the whole formula
- That is, the root node of the parse tree
- The main connective of $(P \wedge Q) \vee R$ is $\vee$

We want to formalise such statements and arguments.

We will take a **symbolic approach**.

It will allow us proving the (in)validity of statements generally.

Advantages of formal symbolic language over natural languages are:

- **unambiguous**
- **more concise**

Connectives - informal semantics

Implication: $P \to Q$, i.e., P implies Q

- means: if P is true then Q must be true too
- if P is false, we can conclude nothing about Q
- P is the antecedent, Q is the consequent

Negation: $\neg P$, i.e., not P

- it can be defined as $P \to \perp$
- if P is true, then $\perp$ (False)
- true iff P is false

$P \wedge Q \vee R$

- Is this a well-formed formula? Yes
- what does it mean?
- $(P \wedge Q) \vee R$?
- $P \wedge (Q \vee R)$?
- We don't know.

In general use parentheses to avoid ambiguities.
Use either $(P \wedge Q) \vee R$ or $P \wedge (Q \vee R)$.

Precedence: in decreasing order of precedence $\neg, \wedge, \vee, \rightarrow$.
For example, $\neg P \vee Q$ means $(\neg P) \vee Q$.

Associativity: all operators are right associative
For example, $P \vee Q \vee R$ means $P \vee (Q \vee R)$.

However use parentheses around compound formulas for clarity.