Natural deduction

06 January 2025 17:19

Natural Deduction

- "natural" style of constructing a proof (like a human would)
- syntactic (rather than semantic) proof method
- proofs are constructed by applying inference rules

Basic idea to prove an argument is valid:

- start with the premises (we can assume these are true)
- repeatedly apply inference rules (which "preserve truth")
- until we have inferred the conclusion

Inference rules are the tools we have/are allowed to use

And-introduction:

$$\frac{A \quad B}{A \wedge B}\ [\wedge I]$$

Notation

- Premise(s) at the top
- Conclusion at the bottom
- Name of the inference rule on the right

Implication-elimination

$$\frac{A \quad A \rightarrow B}{B}\ [\rightarrow E]$$

Notation

- Premise(s) at the top
- Conclusion at the bottom
- Name of the inference rule on the right

False-elimination

$$\frac{\bot}{A}\ [\bot E]$$

True-introduction

$$\frac{}{\top}\ [\top I]$$

Notation

- Premise(s) at the top
- Conclusion at the bottom
- Name of the inference rule on the right

Negation-elimination, i.e., both A and $\neg A$ cannot be true at same time

Formally, want to prove $A, \neg A \vdash \bot$

A **proof** is a tree of instances of inference rules.

Assuming that $\neg A$ is defined as $A \rightarrow \bot$, a proof of the above sequent (or argument) is:

$$\frac{A \quad \neg A}{\bot}\ [\rightarrow E]$$

Notation

- Premise(s) at the top
- Conclusion at the bottom
- Name of the inference rule on the right

Given three hypotheses A, B, C, how can we prove $(A \wedge B) \wedge (A \wedge C)$?

Here is a proof:

$$\dfrac{\dfrac{A \quad B}{A \wedge B}\,[\wedge I] \qquad \dfrac{A \quad C}{A \wedge C}\,[\wedge I]}{(A \wedge B) \wedge (A \wedge C)}\,[\wedge I]$$

The rule used at each step is **and-introduction**, i.e., $\wedge I$

Two key points:

- Can work both forwards and backwards
- Natural doesn't mean there is unique proof

Slightly confusing aspect of natural Deduction

Discharging/cancellation of hypothesis

$$\dfrac{\begin{array}{c}\overline{A}^{\,1} \\ \vdots \\ B\end{array}}{A \to B}\;1\ [\to I]$$

This is the "implication-introduction" rule.

We don't have to make use of A in which case we can just omit it:

$$\dfrac{B}{A \to B}$$

Given the hypothesis A, C how can we prove $B \to ((A \wedge B) \wedge (A \wedge C))$?

Here is a proof:

$$\dfrac{\dfrac{\dfrac{A \quad \overline{B}^{\,1}}{A \wedge B}\,[\wedge I] \qquad \dfrac{A \quad C}{A \wedge C}\,[\wedge I]}{(A \wedge B) \wedge (A \wedge C)}\,[\wedge I]}{B \to ((A \wedge B) \wedge (A \wedge C))}\,1\,[\to I]$$

At this point, we can also cancel another hypothesis, say A

This gives a proof of

$$A \to (B \to ((A \wedge B) \wedge (A \wedge C)))$$

using the hypothesis C only

Given $A \to B$ and $B \to C$, give a proof of $A \to C$

Here is a proof:

$$\dfrac{\dfrac{\dfrac{\overline{A}^{\,1} \qquad A \to B}{B}\,[\to E] \qquad B \to C}{C}\,[\to E]}{A \to C}\,1\,[\to I]$$

Given $\neg A \vee B$ and A, how do we derive B?

Here is a proof:

$$\dfrac{\neg A \vee B \qquad \dfrac{\dfrac{\dfrac{A \quad \overline{\neg A}^{\,1}}{\bot}\,[\neg E]}{B}\,[\bot E]}{\neg A \to B}\,1\,[\to I] \qquad \dfrac{\overline{B}^{\,2}}{B \to B}\,2\,[\to I]}{B}\,[\vee E]$$

Show $(B \wedge A)$ given the hypothesis $(A \wedge B)$

Here is a proof:

$$A \wedge B \qquad\qquad A \wedge B$$

Show $(B \wedge A)$ given the hypothesis $(A \wedge B)$

Here is a proof:

$$\dfrac{\dfrac{A \wedge B}{B}\,[\wedge E_R] \qquad \dfrac{A \wedge B}{A}\,[\wedge E_L]}{B \wedge A}\,[\wedge I]$$

Prove the following:
$R\ , (P \to Q) \wedge (Q \to P)\ , Q \to Z\ , R \to P \vdash Z$

Here is a proof:

$$\dfrac{Q \to Z \qquad \dfrac{\dfrac{R \quad R \to P}{P}\,[\to E] \qquad \dfrac{P \to Q \wedge Q \to P}{P \to Q}\,[\wedge E]}{Q}\,[\to E]}{Z}\,[\to E]$$

Forward & backward reasoning in Natural Deduction

We typically go both **forward and backward** in proofs

Rules for $\to$ (implication)

- implication-introduction

$$\dfrac{\begin{matrix} \dfrac{}{A}\ 1 \\ \vdots \\ B \end{matrix}}{A \to B}\ 1\ [\to I]$$

$$\dfrac{}{A \to B}\ 1\ [\to I]$$

- implication-elimination

$$\dfrac{A \to B \qquad A}{B}\ [\to E]$$

Rules for $\neg$ (not)

- Negation-introduction

$$\begin{array}{c} \overline{A}^{\,1} \\ \vdots \\ \dfrac{\perp}{\neg A}\ 1\ [\neg I] \end{array}$$

- Negation-elimination

$$\dfrac{A \qquad \neg A}{\perp}\ [\neg E]$$

Rules for $\vee$ (or)

- or-introduction (for any formula B)

$$\dfrac{A}{A \vee B}\ [\vee I_L] \qquad \dfrac{A}{B \vee A}\ [\vee I_R]$$

- or-elimination

$$\frac{A \vee B \qquad A \rightarrow C \qquad B \rightarrow C}{C}\ [\vee E]$$

Rules for $\wedge$ (and)

- and-introduction

$$\frac{A \qquad B}{A \wedge B}\ [\wedge I]$$

- and-elimination

$$\frac{A \wedge B}{B}\ [\wedge E_R] \qquad \frac{A \wedge B}{A}\ [\wedge E_L]$$