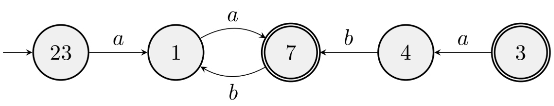
Lecture 3

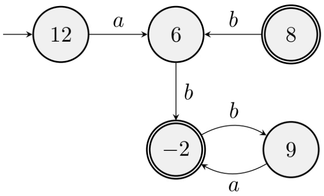
Exercise Sheet 3

Exercise 1. *Check which of the following partial DFAs over the alphabet $\Sigma = \{a, b\}$ are equivalent. If they are not equivalent, you should give a word that's accepted by one but not by the other.*

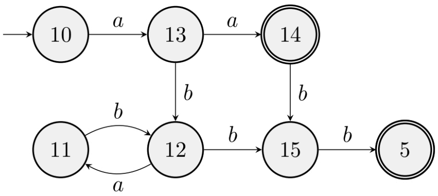
1.



2.

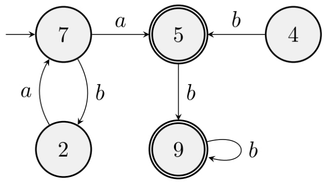


3.



Exercise 2. *For any string $w = w_1 w_2 \dots w_n$, the **reverse of** w, written w^R, is the string w in reverse order, $w_n \dots w_2 w_1$. For any language L, let $L^R = \{w^R \mid w \in L\}$. Show that if L is regular, so is L^R.*[1]

Exercise 3. *Is the following partial DFA minimal? If not, find an equivalent minimal partial DFA.*



Exercise 4. *Give a partial DFA over the alphabet $\Sigma = \{a, b\}$ for words with at least one a, and another partial DFA over the same alphabet for words with at least two characters. By combining these using pairs of states, obtain a partial DFA for words with at least one a and at least two characters.*

Exercise 5. *Consider the following language over the alphabet $\Sigma = \{a, b\}$:*

$$L = \{w \mid w \text{ contains the same number of } a\text{'s and } b\text{'s}\}$$

Show that L is not regular.

[1] Hint. Use Kleene's theorem.

Exercise 6. *Are the following languages over $\Sigma = \{a, b\}$ regular? Why (not)?*

1. $L = \{a^m b^n \mid m > n\}$
2. $L = \{a^m b^n \mid m < n\}$
3. $L = \{w \mid \textit{length}(w) \text{ is a square number}\}$

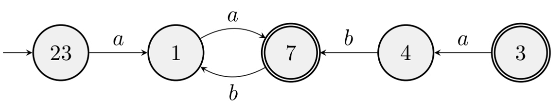
Exercise 7. *Let $\Sigma = \{a, b\}$.*

1. *Let $L_1 = \{a^k u a^k \mid k \geq 1 \text{ and } u \in \Sigma^*\}$. Show that L_1 is regular.*
2. *Let $L_2 = \{a^k b u a^k \mid k \geq 1 \text{ and } u \in \Sigma^*\}$. Show that L_2 is not regular.*

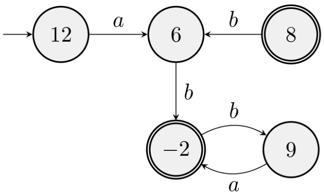
Exercise Sheet 3

Exercise 1. *Check which of the following partial DFAs over the alphabet $\Sigma = \{a, b\}$ are equivalent. If they are not equivalent, you should give a word that's accepted by one but not by the other.*

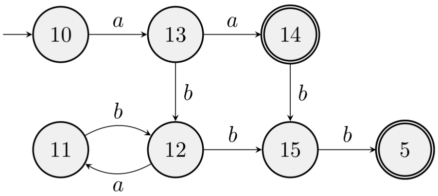
1.



2.



3.



Solution The second partial DFA rejects the word aa, which is accepted by the first and third. Hence the second pratial DFA is inequivalent to the others. The third partial DFA accepts $abbb$, which is rejected by the first. Hence these two partial DFA are inequivalent as well.

Exercise 2. *For any string $w = w_1 w_2 \dots w_n$, the* ***reverse of*** *w, written w^R, is the string w in reverse order, $w_n \dots w_2 w_1$. For any language L, let $L^R = \{w^R \mid w \in L\}$. Show that if L is regular, so is L^R.*[1]

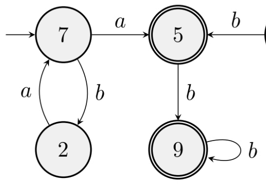
Solution By Kleene's theorem there is a partial DFA A that accepts the language L. Can we, given A, construct a partial DFA that accepts L^R? The answer is to reverse all transitions in A, and switch initial and accepting states. This may result in an NFA, which is nevertheless equivalent to a regular expression.

Alternatively, suppose L is recognized by a regexp E. We construct a new regexp E^R that recognizes L^R, by induction:

1. If E is $\emptyset$ or E is ϵ, we set $E^R = E$.
2. If E is a character x of the alphabet, we set E^R to be x.
3. If E is $E_0|E_1$, we set E^R to be $E_0^R|E_1^R$.
4. If E is E_0E_1, we set E^R to be $E_1^R E_0^R$. (This is the only thing that changes between E and E^R.)
5. If E is $(E_0)^*$, we set E^R to be $(E_0^R)^*$.

[1]Hint. Use Kleene's theorem.

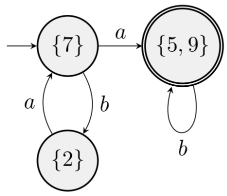
Exercise 3. *Is the following partial DFA minimal? If not, find an equivalent minimal partial DFA.*



Solution

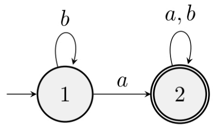
- States 5 and 9 are equivalent: both are accepting words matched by b^* (i.e., $\epsilon, b, bb, \ldots$) and rejecting any word containing an a.
- State 4 is unreachable.
- States 2 and 7 are inequivalent to each other, and to 5 (and hence 9).

Removing the unreachable state 4, and collapsing the equivalent states 5 and 9 gives the following partial DFA:

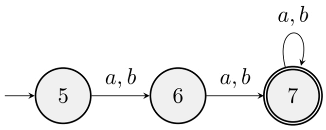


Exercise 4. *Give a partial DFA over the alphabet $\Sigma = \{\mathtt{a}, \mathtt{b}\}$ for words with at least one* $\mathtt{a}$*, and another partial DFA over the same alphabet for words with at least two characters. By combining these using pairs of states, obtain a partial DFA for words with at least one* $\mathtt{a}$ *and at least two characters.*

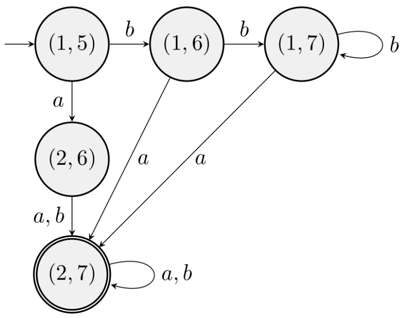
Solution A partial DFA for words with at least one a:



A partial DFA for words with at least two characters:



This yields the following partial DFA for the intersection:



State $(2, 7)$ is accepting because it consists of only accepting states 2 and 7.

Exercise 5. *Consider the following language over the alphabet* $\Sigma = \{a, b\}$*:*

$$L = \{w \mid w \textit{ contains the same number of } a\textit{'s and } b\textit{'s}\}$$

Show that L *is not regular.*

Solution

- Suppose that we are given a DFA D that recognizes L.
- Consider x_n the state of D reached after reading a^n. State x_n accepts the word b^n, but not the word b^m for $m < n$.
- Hence all x_n are inequivalent to x_m for $m < n$.
- Hence the DFA D has infinitely many different states, a contradiction to its assumed finiteness.

Exercise 6. *Are the following languages over $\Sigma = \{a, b\}$ regular? Why (not)?*

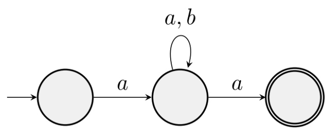
1. $L = \{a^m b^n \mid m > n\}$
2. $L = \{a^m b^n \mid m < n\}$
3. $L = \{w \mid$ *length*(w) *is a square number*$\}$

Exercise 7. *Let $\Sigma = \{a, b\}$.*

1. *Let $L_1 = \{a^k u a^k \mid k \geq 1$ and $u \in \Sigma^*\}$. Show that L_1 is regular.*
2. *Let $L_2 = \{a^k b u a^k \mid k \geq 1$ and $u \in \Sigma^*\}$. Show that L_2 is not regular.*

Solution

1. Note that L_1 is equivalently written as $L_1 = \{ava \mid v \in \Sigma^*\}$. (We need at least one a at the beginning and one at the end, the others are absorbed into v.) A partial DFA for this is easy to build:



2. Suppose that we have a DFA accepting this language. For any $n \in \mathbb{N}$, let x_n be the state reached from the initial state after reading $a^n b$. For $m < n$, if we start at x_m and read in a^m we reach an accepting state, but if we start at x_n and read in a^m we reach a rejecting state, so x_m is not equivalent to x_n. Hence there are infinitely many states, contradicting the assumed finiteness of the DFA.

4