Maths Exercise Sheet 5
Sets and Cardinalities (and revision)

You do not have to answer all questions, but try to answer as many as you can.
Questions marked with one or more asterisks go beyond the scope of the course in difficulty or topic.

Note that answers to Question 3 may be submitted for feedback.

1. How many surjections are there from $\{0, 1, 2\}$ to $\{0, 1\}$?

2. Which of the following sets are countable and which are uncountable?
(a) $\{2^n \mid n \in \mathbb{N}\}$ (b) $\{n \in \mathbb{Z} \mid n \equiv 3 \pmod 7\}$ (c) $\mathbb{N} \times \mathbb{R}$ (d) $\{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ (e) $\mathbb{R}^{\mathbb{R}}$ (f) $\{A \in P(\mathbb{N}) \mid A \text{ is finite}\}$ **(g) $\{A \in P(\mathbb{Z}) \mid \forall x, y \in A.\, x - y \in A\}$.

3. **(Feedback)** For both of the following sets, find whether they are countable or uncountable.
[You may quote results from lectures without proof. As always, explain your reasoning carefully.]
(a) $\{A \subseteq \mathbb{N} \mid 0 \notin A\}$ (b) $\{f : \mathbb{N} \to \mathbb{N} \mid \forall n \in \mathbb{N}. f(n + 1) = f(n) + 1\}$

4. (a) Let X be any set. Show that $A \mapsto X \backslash A$ is a bijection $P(X) \to P(X)$.
(b) Prove that the symmetric difference $\triangle$ is an associative operation $P(X)^2 \to P(X)$. Is it commutative? Does it have a neutral element? If $\triangle$ were 'addition', would it have negatives?

5. Show that $(m, n) \mapsto 2^m(2n + 1) - 1$ gives a bijection $\mathbb{N}^2 \to \mathbb{N}$.

6. Give an explicit injection $\{0, 1\}^* \to \{0, 1\}^{\mathbb{N}}$. Is there an injection the other way round?

7. Consider the relation R on $P(\mathbb{N})$ given by $R = \{(A, B) \in P(\mathbb{N})^2 \mid A \triangle B \text{ is finite}\}$. Show that R is an equivalence relation. ** Is $P(\mathbb{N})/R$ countable or uncountable?

8. (a) Construct an explicit bijection between $\{0, 1\}^{\mathbb{N}}$ and $\{0, 1, 2, 3\}^{\mathbb{N}}$.
(b) Deduce that there exists a bijection between $\{0, 1\}^{\mathbb{N}}$ and $\{0, 1, 2\}^{\mathbb{N}}$.

9. Consider binary operations $\oplus, \otimes : (\mathbb{R}^2)^2 \to \mathbb{R}^2$ on $\mathbb{R}^2$ given by
$$(a, b) \oplus (c, d) = (a + c, b + d) \qquad (a, b) \otimes (c, d) = (ac - bd, ad + bc).$$
Show that these operations make $\mathbb{R}^2$ a field. [Optional hint: The neutral element for $\oplus$ will be $(0, 0)$. The reciprocal of a non-zero (a, b) will be $(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2})$.]

10. (a) Explain how you know there exists a bijection $\mathbb{Z}_{mn} \to \mathbb{Z}_m \times \mathbb{Z}_n$ for any $m, n \in \mathbb{N}$, $m, n > 0$.
****(b) When m and n are coprime, show that $[N]_{mn} \mapsto ([N]_m, [N]_n)$ is such a bijection.

**11. A function $f : \mathbb{N} \to \mathbb{N}$ is decreasing if $f(n+1) \leq f(n)$ for all n. Is the set of decreasing functions $\mathbb{N} \to \mathbb{N}$ countable or uncountable? What about increasing functions?

$*^{10}$12. Define a relation $\sim$ on $\mathbb{Q}^{\mathbb{N}}$ by $(a_n) \sim (b_n)$ iff
$$\forall C \in \mathbb{N}.\, \exists N \in \mathbb{N}.\, \forall m, n \in \mathbb{N}.\, m, n > N \implies |a_m - b_n| < \frac{1}{C + 1}.$$
Show that the situation of Sheet 4 Q5 applies. Do you recognize $\mathbb{Q}^{\mathbb{N}}/{\sim}$?