Exercise Sheet 10b - Solutions
Predicate Logic – Natural Deduction & Semantics

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1. Here is a constructive Natural Deduction proof of $(S_1) \rightarrow \neg\exists x.\forall y.\neg x < y$

$$
\dfrac{\dfrac{\dfrac{\overline{\exists x.\forall y.\neg x < y}^{\,2} \qquad \dfrac{\dfrac{\overline{S_1}^{\,1}}{\exists y.x < y}\,{\scriptstyle 4\ [\forall E]} \qquad \dfrac{\overline{x < y}^{\,4} \qquad \dfrac{\overline{\forall y.\neg x < y}^{\,3}}{\neg x < y}\,{\scriptstyle [\forall E]}}{\bot}\,{\scriptstyle [\neg E]}}{\bot}\,{\scriptstyle 4\ [\exists E]}}{\bot}\,{\scriptstyle 3\ [\exists E]}}{\neg\exists x.\forall y.\neg x < y}\,{\scriptstyle 2\ [\neg I]}}{(S_1) \rightarrow \neg\exists x.\forall y.\neg x < y}\,{\scriptstyle 1\ [\rightarrow I]}
$$

2. Here is a constructive Natural Deduction proof of $(S_1) \rightarrow (S_2) \rightarrow S_3$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overline{S_1}^{\,1}}{\exists y.0 < y}\,{\scriptstyle [\forall E]} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overline{S_2}^{\,2}}{\forall y.0 < y \rightarrow 1 \leq y}\,{\scriptstyle [\forall E]}}{0 < y \rightarrow 1 \leq y}\,{\scriptstyle [\forall E]} \qquad \overline{0 < y}^{\,3}}{1 \leq y}\,{\scriptstyle [\rightarrow E]}}{S_3}\,{\scriptstyle [\exists I]}}{S_3}\,{\scriptstyle 3\ [\exists E]}}{(S_2) \rightarrow S_3}\,{\scriptstyle 2\ [\rightarrow I]}}{(S_1) \rightarrow (S_2) \rightarrow S_3}\,{\scriptstyle 1\ [\rightarrow I]}
$$

3. For example, the models
   - $M_1 = \langle \mathbb{N}, \langle 0, \langle n \rangle \mapsto n+1 \rangle, \langle \{\langle n, m \rangle \mid n < m\}, \{\langle n, m \rangle \mid n \leq m\} \rangle \rangle$
   - and $M_1' = \langle \mathbb{N}, \langle 0, \langle n \rangle \mapsto n \rangle, \langle \{\langle n, m \rangle \mid n \leq m\}, \emptyset \rangle \rangle$
   - and $M_1'' = \langle \mathbb{N}, \langle 0, \langle n \rangle \mapsto n \rangle, \langle \{\langle n, m \rangle \mid \text{True}\}, \emptyset \rangle \rangle$

   are models of S_1;

4. For example, the models
   - $M_2 = \langle \mathbb{N}, \langle 0, \langle n \rangle \mapsto n \rangle, \langle \{\langle n, m \rangle \mid n > m\}, \emptyset \rangle \rangle$
   - and $M_2' = \langle \mathbb{N}, \langle 0, \langle n \rangle \mapsto n \rangle, \langle \emptyset, \emptyset \rangle \rangle$
   - and $M_2'' = \langle \{0\}, \langle 0, \langle n \rangle \mapsto n \rangle, \langle \{\langle n, m \rangle \mid n < m\}, \emptyset \rangle \rangle$

   are models of $\neg S_1$.