Exercise Sheet 7 - Solutions
Predicate Logic – Syntax

1.
   - $\exists y.\exists z.\mathit{Student}(y, x) \wedge (\mathit{Module}(z, \text{Math}) \vee \mathit{Module}(z, \text{OOP})) \wedge \mathit{Enroll}(y, z)$
   - $\exists u.\exists v.\exists w.\mathit{Student}(u, x) \wedge \mathit{Student}(v, y) \wedge \mathit{Enroll}(u, w) \wedge \mathit{Enroll}(v, w)$

2. For example, consider the signature that contains the two infix binary function symbols $\wedge$ and $\vee$, the unary function symbol $\neg$, and two nullary function symbols (constants) $\top$ and $\bot$, which stand for true and false, respectively. Our signature also contains the infix binary predicate symbol $=$ to state equalities between Boolean expressions. We can state the de Morgan's laws as follows:
   - $\forall x.\forall y.\neg(x \wedge y) = (\neg x) \vee (\neg y)$
   - $\forall x.\forall y.\neg(x \vee y) = (\neg x) \wedge (\neg y)$

3. For example, let the domain be the set of propositional logic formulas. Let the signature be such that it includes at least the infix binary predicate symbol $\iff$, which stands for the logical equivalence relation on propositions. We can capture that $\iff$ is an equivalence relation as follows:
   - reflexive: $\forall x.x \iff x$
   - symmetric: $\forall x.\forall y.(x \iff y) \rightarrow (y \iff x)$
   - transitive: $\forall x.\forall y.\forall z.(x \iff y) \rightarrow (y \iff z) \rightarrow (x \iff z)$

4. Let the domain be D. Let the signature be
   - function symbols: $0, 1$ (arity 0), $-$ (arity 1), $+, \times$ (arity 2)
   - predicate symbols: $=$ (arity 2)

   We will allow using infix notation for function and predicate symbols. The ring laws can be specified as follows:
   - $\forall x.x + 0 = x$
   - $\forall x.\forall y.(x + y) = (y + x)$
   - $\forall x.x \times 1 = x$
   - $\forall x.\forall y.x \times y = y \times x$
   - $\forall x.x + (-x) = 0$
   - $\forall x.\forall y.\forall z.(x + y) + z = x + (y + z)$
   - $\forall x.\forall y.\forall z.(x \times y) \times z = x \times (y \times z)$
   - $\forall x.\forall y.\forall z.x \times (y + z) = (x \times y) + (x \times z)$

5.
   - $\forall x.\forall y.x > y \rightarrow x \geq (y + 1)$
   - $\forall x.\mathtt{prime}(x) \rightarrow x > 1 \wedge \neg\exists y.\exists z.y > 1 \wedge z > 1 \wedge x = y \times z$

6. Here is a constructive Natural Deduction proof of

$$(A \land \neg B) \to (A \to C) \to (D \to B) \to (\neg D \land C)$$

$$
\begin{prooftree}
\AxiomC{$\overline{D \to B}^{\,3}$}
\AxiomC{$\overline{D}^{\,4}$}
\RightLabel{$[\to E]$}
\BinaryInfC{B}
\AxiomC{$\overline{A \land \neg B}^{\,1}$}
\RightLabel{$[\land E_R]$}
\UnaryInfC{$\neg B$}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{$4\ [\neg I]$}
\UnaryInfC{$\neg D$}
\AxiomC{$\overline{A \to C}^{\,2}$}
\AxiomC{$\overline{A \land \neg B}^{\,1}$}
\RightLabel{$[\land E_L]$}
\UnaryInfC{A}
\RightLabel{$[\to E]$}
\BinaryInfC{C}
\RightLabel{$[\land I]$}
\BinaryInfC{$\neg D \land C$}
\RightLabel{$3\ [\to I]$}
\UnaryInfC{$(D \to B) \to (\neg D \land C)$}
\RightLabel{$2\ [\to I]$}
\UnaryInfC{$(A \to C) \to (D \to B) \to (\neg D \land C)$}
\RightLabel{$1\ [\to I]$}
\UnaryInfC{$(A \land \neg B) \to (A \to C) \to (D \to B) \to (\neg D \land C)$}
\end{prooftree}
$$