Maths Exercise Sheet 4
Reals (and revision)

You do not have to answer all questions, but try to answer as many as you can.
Questions marked with one or more asterisks go beyond the scope of the course in difficulty or topic.

1. Which of the following functions are injective/surjective/bijective?
   (a) $n \mapsto -n : \mathbb{Z} \to \mathbb{Z}$.
   (b) $x \mapsto x^2 : [0, \infty) \cap \mathbb{Q} \to [0, \infty) \cap \mathbb{Q}$.
   (c) $x \mapsto x^2 : [0, \infty) \to [0, \infty)$.
   (d) $x \mapsto \frac{1}{x} : (0, \infty) \to \mathbb{R}$.
   (e) $x \mapsto x^7 - x : \mathbb{R} \to \mathbb{R}$.
   * (f) $x \mapsto \log(|x|) : \mathbb{R}\backslash\{0\} \to \mathbb{R}$.
   * (g) $x \mapsto \sin x : [-t, t] \to [-1, 1]$ (determine for each $t \in [0, \infty)$)

2. Consider the following functions $f, g : \mathbb{R} \to \mathbb{R}$ given by
$$f(x) = 2x + 1 \qquad g(x) = x^2 - 2.$$
Compute formulas for the composites $f \circ g$ and $g \circ f$.

3. Define two sequences (p_n), (q_n) of natural numbers by
$$p_0 = 1 \qquad q_0 = 0 \qquad p_{n+1} = p_n + 2q_n \qquad q_{n+1} = p_n + q_n$$
   (a) Prove by induction that $p_n^2 - 2q_n^2 = (-1)^n$.
   (b) Deduce that $|\sqrt{2} - \frac{p_n}{q_n}| < \frac{1}{2q_n^2}$ for $n > 0$.

4. Find $n \in \mathbb{N}$ such that $|\frac{n}{8} - \sqrt{5}| < \frac{1}{16}$. Now write down the first 6 binary digits of $\sqrt{5}$.
[You should not use the square root function on a calculator.]

5. Let X be a set and let $R \subseteq X \times X$ be a relation on X. Suppose that R is symmetric and transitive. Consider the subset $D \subseteq X$ given by
$$D = \{x \in X \mid x \; R \; x\}.$$
   (a) Show that $R \subseteq D \times D$.
   (b) As a relation on D, show that R is an equivalence relation.

* 6. Show that the set
$$A = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$$
is closed under $+, -, \times, \div$ in $\mathbb{R}$.
[‘Closed under an operation $\odot$’ means that, whenever $x, y \in A$ then also $x \odot y \in A$.]

** 7. A compression algorithm is *lossless* if the original data can be exactly reconstructed. A new lossless compression algorithm promises that no compressed file is larger than the original. Prove that in this case the ‘compression’ algorithm actually leaves every file exactly the same size. [You may model a ‘file’ as a vector $v \in \{0, 1\}^N$ where $N \in \mathbb{N}$ is considered to be the *size* of the file.]

*** 8. Let p be a prime number greater than 2. Show that
$$|\{x^2 \mid x \in \mathbb{Z}_p\}| = \frac{1}{2}(p + 1).$$