Exercise Sheet 11
Predicate Logic

Consider the following signature:

- Function symbols: zero (arity 0); succ (arity 1)

- Predicate symbols: $<$ (arity 2); $\leq$ (arity 2)

We will use infix notation for the binary symbols $<$ and $\leq$. For simplicity we write 0 for zero, 1 for succ(zero), 2 for succ(succ(zero)), etc. Consider the following formulas that capture properties of the above symbols:

- let S_1 be $\forall x.\neg 0 \leq x$

- let S_2 be $\forall x.\forall y.x < y \rightarrow x \leq \mathtt{succ}(y)$

- let S_3 be $\neg\exists x.x < 0$

- let S_4 be $\forall x.\forall y.\mathtt{succ}(x) \leq y \rightarrow x < y$

1. Provide a constructive Natural Deduction proof of $(S_1) \rightarrow (S_2) \rightarrow \forall x.\neg 0 < x$

2. Provide a constructive Natural Deduction proof of $(S_3) \rightarrow (S_4) \rightarrow \forall x.\neg\mathtt{succ}(x) \leq 0$

3. Provide a constructive Natural Deduction proof of $(S_3) \rightarrow (S_4) \rightarrow \neg\exists x.\mathtt{succ}(x) \leq 0$

4. Provide a model M_1 such that $\vDash_{M_1} \exists x.\exists y.x < y \wedge \neg x \leq y$

5. Provide a model M_2 such that $\vDash_{M_2} \neg\exists x.\exists y.x < y \wedge \neg x \leq y$