Exercise Sheet 10
Predicate Logic – Semantics

Note that you can submit question 5 for feedback.

1. Consider the following:

   - The signature $\langle\langle \texttt{zero}, \texttt{succ}\rangle, \langle \texttt{even}, \texttt{odd}, >\rangle\rangle$
   - The model M: $\langle \mathbb{N}, \langle 0, +1\rangle, \langle \{\langle 0\rangle, \langle 2\rangle, \langle 4\rangle, \dots\}, \{\langle 1\rangle, \langle 3\rangle, \langle 5\rangle, \dots\}, \{\langle 1, 0\rangle, \langle 2, 0\rangle, \langle 2, 1\rangle, \dots\}\rangle\rangle$
   - Formally, because `zero` has arity 0, its interpretation is a function from an empty tuple to 0, written $\langle\rangle \mapsto 0$, which we often write as 0 for simplicity.
   - We write $+1$ for the function that given a number increments it by 1. Formally, this function is $\langle x\rangle \mapsto x{+}1$.
   - The interpretation of `even` can also be written as follows: $\{\langle x\rangle \mid x \text{ is even}\}$.
   - The interpretation of `odd` can also be written as follows: $\{\langle x\rangle \mid x \text{ is odd}\}$.
   - The interpretation of $>$ can also be written as follows: $\{\langle x, y\rangle \mid x > y\}$.

   Prove that $\vDash_{M,\cdot} \forall x.\texttt{even}(x) \to \exists y.\texttt{odd}(y) \land y > x$. Detail your answer as we did in the lectures.

2. Consider the above signature and model M. Prove that $\neg \vDash_{M,\cdot} \forall x.\texttt{even}(x) \to \texttt{even}(\texttt{succ}(x))$. Detail your answer as we did in the lectures.

3. Consider the above signature. Provide a model M' such that $\vDash_{M',\cdot} \forall x.\texttt{even}(x) \to \texttt{even}(\texttt{succ}(x))$.

4. Consider the signature that does not contain any function symbols, and that only contains the two unary predicate symbols p and q. Using the semantical approach, prove that $(\forall x.p(x) \land q(x)) \to \forall x.p(x)$.

5. **[feedback]** Consider the following signature:

   - function symbols: `zero` of arity 0 and `succ` of arity 1
   - predicate symbols: `even` of arity 1 and `odd` of arity 1

   Provide:

   - a model M_1 such that $\vDash_{M_1} \exists x.\texttt{odd}(x) \land \neg\texttt{even}(x)$
   - a model M_2 such that $\vDash_{M_2} \neg(\exists x.\texttt{odd}(x) \land \neg\texttt{even}(x))$