Logic Session - Mon 30ᵗʰ Oct - W6

(Difficulty in yellow)

Q1: Constructive: $((\neg\neg A \lor B) \to (\neg D \lor C)) \to (\neg D \to C) \to A \to C$ — Moderate

Q2: Classical: $(\neg A \to B \lor C) \to \neg B \to \neg C \to A$ — Moderate

Q3: Constructive: $(A \land B \to \neg C) \to C \to A \to \neg B$ — Easy

Q4: Classical: $((A \land \neg B) \to \neg C) \to C \to A \to B$ — Easy

Q5: Constructive: $(A \to B \lor C) \to (B \to D) \to (C \to B \lor D) \to \neg D \to \neg A$ — Hard!

Q6: Constructive: $(D \to A \lor C) \to (A \land B \to C) \to D \land B \to C$ — Moderate

Q7: Constructive: $(C \to B \lor A) \to (B \to A) \to ((C \to A) \lor C)$ — Moderate

Q8: Classical: $(C \to B) \to (B \to A) \to (\neg C \to A) \to A$ — Moderate

Q9: Classical: $(C \to A) \to (A \to B) \to (\neg B \to C) \to B$ — Moderate

Q10: Constructive: $A \to (B \to D) \to (\neg B \to C) \to (A \lor C)$ — Easy

Q11: Constructive: $(D \to A \lor C) \to (A \land B \to C) \to (D \land B) \to C$ — Moderate

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Q1: Constructive: $((\neg\neg A \lor B) \to (\neg D \lor C)) \to (\neg D \to C) \to A \to C$ — Moderate

$$\frac{\overline{\neg A}^5 \quad \overline{A}^3}{\dfrac{\dfrac{\dfrac{\bot}{\neg\neg A}5[\neg I]}{\neg\neg A \lor B}[\lor I_L]}{\neg D \lor C}[\neg E]}$$

$$\frac{(\neg\neg A \lor B) \to (\neg D \lor C)}{\quad}1$$

$$\frac{\neg D \lor C \qquad \neg D \to C}{2} \qquad \frac{\overline{C}^4}{C \to C}4[\to I]$$

$$\frac{C}{A \to C}3[\to I]$$

$$\frac{}{(\neg D \to C) \to A \to C}2[\to I]$$

$$\frac{}{((\neg\neg A \lor B) \to (\neg D \lor C)) \to (\neg D \to C) \to A \to C}1[\to I]$$

1. $(\neg\neg A \lor B) \to (\neg D \lor C)$
2. $(\neg D \to C)$
3. A
4. C
5. ¬A

Q2: Classical: $(\neg A \to B \lor C) \to \neg B \to \neg C \to A$ — Moderate

$$\cfrac{\cfrac{}{\neg A \to B \lor C}^1 \quad \cfrac{}{\neg A}^4}{B \lor C} \qquad \cfrac{\cfrac{\cfrac{}{\neg B}^2 \quad \cfrac{}{B}^5}{\bot}[\neg E]}{B \to \bot}5[\to I] \qquad \cfrac{\cfrac{\cfrac{}{\neg C}^3 \quad \cfrac{}{C}^6}{\bot}[\neg E]}{C \to \bot}6[\to I]$$

$$\cfrac{\cfrac{\cfrac{\bot}{\neg\neg A}4[\neg I]}{A}DNE}{\neg C \to A}3[\to I]$$

$$\cfrac{[\lor E]}{}$$

$$\cfrac{\neg B \to \neg C \to A}{(\neg A \to B \lor C) \to \neg B \to \neg C \to A}2\{\to I\}, 1[\to I]$$

Q3: Constructive: $(A \land B \to \neg C) \to C \to A \to \neg B$ — Easy

$$\cfrac{\cfrac{}{\overline{C}}^2 \quad \cfrac{\cfrac{}{A \land B \to \neg C}^1 \quad \cfrac{\cfrac{}{\overline{A}}^3 \quad \cfrac{}{\overline{B}}^4}{A \land B}[\land I]}{\neg C}[\to E]}{\bot}[\neg E]$$

$$\cfrac{\cfrac{\cfrac{\bot}{\neg B}4[\neg I]}{A \to \neg B}3[\to I]}{\cfrac{C \to A \to \neg B}{(A \land B \to \neg C) \to C \to A \to \neg B}1[\to I]}2[\to I]$$

Q4: Classical: $((A \land \neg B) \to \neg C) \to C \to A \to B$ — Easy

$$\cfrac{\cfrac{}{(A \land \neg B) \to \neg C}^1 \quad \cfrac{\cfrac{}{\overline{A}}^3 \quad \cfrac{}{\neg B}^4}{A \land \neg B}[\land I]}{\neg C}[\to E] \qquad \cfrac{}{C}^2$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\bot}{\neg\neg B}4[\neg I]}{B}[DNE]}{A \to B}3[\to I]}{\cfrac{C \to A \to B}{((A \land \neg B) \to \neg C) \to C \to A \to B}1[\to I]}2[\to I]$$

Q5: Constructive: $(A \to B \lor C) \to (B \to D) \to (C \to B \lor D) \to \neg D \to \neg A$ — Hard!

$$\cfrac{\cfrac{\cfrac{\overline{A \to B \lor C}^{1} \quad \overline{A}^{5}}{B \lor C}[\to E] \quad \cfrac{\cfrac{\cfrac{\cfrac{\overline{\neg D}^{4} \quad \cfrac{\overline{B \to D}^{2} \quad \overline{B}^{9}}{D}}{\bot}}{C} 9.[\to I]}{B \to C} \quad \cfrac{\overline{C}^{8}}{C \to C}8[\to I]}{\quad} \quad \cfrac{\cfrac{\overline{\neg D}^{4} \quad \cfrac{\overline{B \to D}^{2} \quad \overline{B}^{6}}{D}[\to E]}{\bot}}{B \to \bot}6[\to I] \quad \cfrac{\cfrac{\overline{\neg D}^{4} \quad \overline{D}^{7}}{\bot}[\to E]}{D \to \bot}7[\to I]}{\cfrac{\cfrac{\cfrac{\overline{C \to B \lor D}^{3} \quad C}{B \lor D}}{\bot}}{\cfrac{\neg A}{\cfrac{\neg D \to \neg A}{\cfrac{(C \to B \lor D) \to \neg D \to \neg A}{\cfrac{(B \to D) \to (C \to B \lor D) \to \neg D \to \neg A}{(A \to B \lor C) \to (B \to D) \to (C \to B \lor D) \to \neg D \to \neg A}1[\to I]}2[\to I]}3[\to I]}4[\to I]}5[\neg I]}}$$

- $\overline{A \to B \lor C}^{1}$, $\overline{A}^{5}$, $[\to E]$ giving $B \lor C$
- $\overline{C \to B \lor D}^{3}$
- $B \lor C$ and $C \to B \lor D$ give $B \lor D$
- $\overline{B \to D}^{2}$, $\overline{B}^{9}$ give D; with $\overline{\neg D}^{4}$ give $\bot$, then C, then $B \to C$ (9, $[\to I]$)
- $\overline{C}^{8}$, $C \to C$ (8, $[\to I]$)
- $\overline{B \to D}^{2}$, $\overline{B}^{6}$, $[\to E]$ give D; with $\overline{\neg D}^{4}$ give $\bot$; $B \to \bot$ (6, $[\to I]$)
- $\overline{\neg D}^{4}$, $\overline{D}^{7}$, $[\to E]$ give $\bot$; $D \to \bot$ (7, $[\to I]$)
- $\bot$, $\neg A$ (5, $[\neg I]$), $\neg D \to \neg A$ (4, $[\to I]$)
- $(C \to B \lor D) \to \neg D \to \neg A$ (3, $[\to I]$)
- $(B \to D) \to (C \to B \lor D) \to \neg D \to \neg A$ (2, $[\to I]$)
- $(A \to B \lor C) \to (B \to D) \to (C \to B \lor D) \to \neg D \to \neg A$ (1, $[\to I]$)

Alternatively:

$$\cfrac{\cfrac{\overline{A \to B \lor C}^{1} \quad \overline{A}^{5}}{B \lor C} \quad \cfrac{\cfrac{\cfrac{\overline{C \to B \lor D}^{3} \quad \overline{C}^{6}}{B \lor D} \quad \overline{B \to D}^{2} \quad \cfrac{\cfrac{\overline{\neg D}^{7}}{D \to D}7[\to I]}{}[\lor E]}{\cfrac{D}{C \to D}6[\to I]}[\lor E]}{\cfrac{\overline{B \to D}^{2}}{B \to D}}}{\cfrac{\cfrac{\cfrac{D \quad \overline{\neg D}^{4}}{\bot}[\to E]}{\cfrac{\neg A}{\cfrac{\neg D \to \neg A}{\cfrac{(C \to B \lor D) \to \neg D \to \neg A}{\cfrac{(B \to D) \to (C \to B \lor D) \to \neg D \to \neg A}{(A \to B \lor C) \to (B \to D) \to (C \to B \lor D) \to \neg D \to \neg A}1[\to I]}2[\to I]}3[\to I]}4[\to I]}5[\neg I]}}[\lor E]}$$

(Proof tree transcription — lower "Alternatively" section)

Discharge labels as shown: $5[\neg I]$, $4[\to I]$, $3[\to I]$, $2[\to I]$, $1[\to I]$, $6[\to I]$, $7[\to I]$, $[\lor E]$, $[\to E]$.

(Credit to Emil for this proof)

Q6: Constructive: $(D \to A \vee C) \to (A \wedge B \to C) \to D \wedge B \to C$ — Moderate

$$
\cfrac{
  \cfrac{
    \cfrac{\overline{D \to A \vee C}^{\,1} \quad \cfrac{\overline{D \wedge B}^{\,3}}{D}[\wedge E_L]}{A \vee C}[\to E]
    \qquad
    \cfrac{\cfrac{\overline{A \wedge B \to C}^{\,2} \quad \cfrac{\overline{A}^{\,5} \quad \cfrac{\overline{D \wedge B}^{\,3}}{B}[\wedge E_R]}{A \wedge B}[\wedge I]}{C}[\to E]}{A \to C}\,5[\to I]
    \qquad
    \cfrac{\overline{C}^{\,4}}{C \to C}\,4[\to I]
  }{C}[\vee E]
}{
  \cfrac{
    \cfrac{C}{D \wedge B \to C}\,3[\to I]
  }{
    \cfrac{(A \wedge B \to C) \to D \wedge B \to C}{(D \to A \vee C) \to (A \wedge B \to C) \to D \wedge B \to C}\,1[\to I]
  }\,2[\to I]
}
$$

Q7: Constructive: $(C \to B \vee A) \to (B \to A) \to ((C \to A) \vee C)$ — Moderate

$$
\cfrac{
  \cfrac{\overline{C \to B \vee A}^{\,1} \quad \overline{C}^{\,3}}{B \vee A}[\to E]
  \qquad
  \overline{B \to A}^{\,2}
  \qquad
  \cfrac{\overline{A}^{\,4}}{A \to A}\,4[\to I]
}{
  \cfrac{
    \cfrac{\cfrac{A}{C \to A}\,3[\to I]}{\left((C \to A) \vee C\right)}[\vee I_L]
  }{
    \cfrac{(B \to A) \to \left((C \to A) \vee C\right)}{(C \to B \vee A) \to (B \to A) \to \left((C \to A) \vee C\right)}\,1[\to I]
  }\,2[\to I]
}[\vee E]
$$

Q8: Classical: $(C \to B) \to (B \to A) \to (\neg C \to A) \to A$ — Moderate

$$
\cfrac{
  \cfrac{}{C \vee \neg C}[LEM]
  \quad
  \cfrac{\cfrac{\cfrac{\overline{C \to B}^{\,1} \quad \overline{C}^{\,5}}{B}[\to E]}{B \vee \neg C}[\vee I_L]}{C \to B \vee \neg C}\,5[\to I]
  \quad
  \cfrac{\cfrac{\overline{\neg C}^{\,4}}{B \vee \neg C}[\vee I_R]}{\neg C \to B \vee \neg C}\,4[\to I]
}{
  \cfrac{
    \cfrac{B \vee \neg C \quad \overline{\neg C \to A}^{\,3} \quad \overline{B \to A}^{\,2}}{A}[\vee E]
  }{
    \cfrac{
      \cfrac{A}{(\neg C \to A) \to A}\,3[\to I]
    }{
      \cfrac{(B \to A) \to (\neg C \to A) \to A}{(C \to B) \to (B \to A) \to (\neg C \to A) \to A}\,1
    }\,2
  }[\vee E]
}
$$

Alternatively

$$
\cfrac{
  \cfrac{}{C \vee \neg C}[LEM]
  \quad
  \cfrac{\overline{B \to A}^{\,2} \quad \cfrac{\overline{C \to B}^{\,1} \quad \overline{C}^{\,4}}{B}[\to E]}{\cfrac{A}{C \to A}\,4[\to I]}
  \quad
  \overline{\neg C \to A}^{\,3}
}{
  \cfrac{
    \cfrac{A}{(\neg C \to A) \to A}\,3[\to I]
  }{
    \cfrac{(B \to A) \to (\neg C \to A) \to A}{(C \to B) \to (B \to A) \to (\neg C \to A) \to A}\,1
  }\,2
}[\vee E]
$$

Q9: Classical: $(C \to A) \to (A \to B) \to (\neg B \to C) \to B$ — Moderate

$$\cfrac{\cfrac{\cfrac{\overline{A \to B}^{\,2} \quad \cfrac{\cfrac{\overline{C \to A}^{\,1} \quad \cfrac{\overline{\neg B \to C}^{\,3} \quad \overline{\neg B}^{\,4}}{C}\,[\to E]}{A}\,[\to E]}{B}\,[\to E] \quad \overline{\neg B}^{\,4}}{\cfrac{\cfrac{\cfrac{\bot}{\neg\neg B}\,[\neg I]^{4}}{B}\,[DNE]}{\cfrac{(\neg B \to C) \to B}{\cfrac{(A \to B) \to (\neg B \to C) \to B}{(C \to A) \to (A \to B) \to (\neg B \to C) \to B}\,1\,[\to I]}\,2\,[\to I]}\,3\,[\to I]}}{}}$$

(Derivation tree as drawn: $[\neg E]$, $4[\neg I]$, $[DNE]$, $3[\to I]$, $2[\to I]$, $1[\to I]$)

Q10: Constructive: $A \to (B \to D) \to (\neg B \to C) \to (A \lor C)$ — Easy

$$\cfrac{\cfrac{\cfrac{\cfrac{\overline{A}^{\,1}}{(A \lor C)}\,[\lor I_{L}]}{(\neg B \to C) \to (A \lor C)}\,3\,[\to I]}{(B \to D) \to (\neg B \to C) \to (A \lor C)}\,2\,[\to I]}{A \to (B \to D) \to (\neg B \to C) \to (A \lor C)}\,1\,[\to I]$$

Q11: Constructive: $(D \to A \lor C) \to (A \land B \to C) \to (D \land B) \to C$ — Moderate

$$\cfrac{\cfrac{\cfrac{\cfrac{\overline{D \to A \lor C}^{\,1} \quad \cfrac{\overline{D \land B}^{\,3}}{D}\,[\land E_{L}]}{A \lor C}\,[\to E] \quad \cfrac{\overline{A \land B \to C}^{\,2} \quad \cfrac{\overline{A}^{\,4} \quad \cfrac{\overline{D \land B}^{\,3}}{B}\,[\land E_{R}]}{A \land B}\,[\land I]}{\cfrac{C}{A \to C}\,4[\to I]}\,[\to E] \quad \cfrac{\overline{C}^{\,7}}{C \to C}\,7[\to I]}{C}\,[\lor E]}{\cfrac{C}{(D \land B) \to C}\,3[\to I]}}{\cfrac{(A \land B \to C) \to (D \land B) \to C}{(D \to A \lor C) \to (A \land B \to C) \to (D \land B) \to C}\,1[\to I]}\,2[\to I]$$