MLFCS

11: Elementary Matrices + Invertible Matrix Theorem

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Eigenvectors and Eigenvalues; Diagonalization

- Def.: $v \neq 0$ and λ are eigenvalues of $A \iff Av = \lambda v$
- Find λ_i as roots from equating the characteristic polnomial to zero: $\det(A - \lambda I_n) = 0$
- Find v_i (for each λ_i) by Gaussian Elimination

Coordinate Transformation by the Eigenvectors

- For an invertible matrix A, define $T := (v_1\, v_2 \ldots v_n)$ as the matrix formed by the eigenvectors v_i as **columns**
- Define a diagonal matrix D with the eigenvalues λ_i as the diagonal elements

$$\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

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$R = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$ $\quad D = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} = \begin{pmatrix} c+is & 0 \\ 0 & c-is \end{pmatrix}$, $T = \frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$

$TDT^{-1} = \ldots = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} = R$

Why does that work? Let $\alpha_1 \ldots \alpha_n$ be EV to EV $\vec{t}_1 \ldots \vec{t}_n$

$$AT = \begin{pmatrix} \overline{\quad A \quad} \end{pmatrix} \begin{pmatrix} | & | & & | \\ t_1 & t_2 & \ldots & t_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ (A\vec{t}_1) & (At_2) & \ldots & (At_n) \\ | & | & & | \end{pmatrix}$$

$\underline{A\vec{t}_i = \alpha_i \vec{t}_i}$

$$= \begin{pmatrix} | & | & & | \\ (\alpha_1 t_1) & (\alpha_2 t_2) & \ldots & (\alpha_n t_n) \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ \vec{t}_1 & t_2 & \ldots & t_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} \alpha_1 & & & 0 \\ & \alpha_2 & & \\ & & \ddots & \\ 0 & & & \alpha_n \end{pmatrix} = TD$$

If T is invertible, we have $T^{-1}AT = D$

and $A = TDT^{-1}$

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Coordinate Transformation by the Eigenvectors

- For an invertible matrix A, define $T := (v_1 v_2 \ldots v_n)$ as the matrix formed by the eigenvectors v_i as **columns**
- Define a diagonal matrix D with the eigenvalues λ_i as the diagonal elements
- Then $TDT^{-1} = A$, as well as $T^{-1}AT = D$
- Hence, A has an equivalent diagonal matrix D, in a transformed space
  Therefore this procedure is often called Diagonalization

Finding the Inverse via Gaussian Elimination

Recall: We can use Gaussian Elimination (row operations) to solve $A\vec{x} = \vec{e}_i$ for any i). $\left(A \begin{pmatrix}\vdots\end{pmatrix}\begin{pmatrix}\vdots\end{pmatrix}\cdots\begin{pmatrix}\vdots\end{pmatrix}\right) = \left(A \;\; \mathbb{1}\right)$

– We do this by "augmenting" the $n \times n$ matrix A by a rhs vector.

– Row operations are solely based on matrix A, the rhs is just enjoying the same row operations.

– Why not add more rows, namely **all** n unit vectors on the rhs. Actually, we just add the whole $n \times n$ unity matrix there...

– on the other hand, we simultaneously solve n equations $A\vec{x} = \vec{e}_i$

– We start with a $n \times 2n$ matrix composed of A (left) and a unit matrix (right).

– If A is invertible, we obtain the unix matrix on the left and A^{-1} on the right. $\rightarrow \left(\mathbb{1} \;\; A^{-1}\right)$

Elementary Matrices

Each Gaussian Operation is equivalent by multiplying both sides (from the left) with an Elementary Matrix!
Swapping 2 rows:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 3 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$$

Adding 2 times R2 to R1: (whereby R2 stays unchanged)

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 3 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 8 & 3 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$

Multiplying row 2 by a:

$$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 3 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 3a & a & 0 \end{pmatrix}$$

Okay...? Let’s go!

Elementary Matrices

$$\begin{pmatrix} 1/a & 0 \\ -1/a & 1/c \end{pmatrix} \begin{pmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b/a & 1/a & 0 \\ 0 & (ad-bc)/ac & -1/a & 1/c \end{pmatrix}$$

Next operations

$$\begin{pmatrix} 1 & 0 \\ 0 & ac/(ad-bc) \end{pmatrix} \begin{pmatrix} 1 & b/a & 1/a & 0 \\ 0 & (ad-bc)/ac & -1/a & 1/c \end{pmatrix} =?$$

Work out your steps! The inverse is composed by a product of several elementary matrices
Arriving at:

$$\begin{pmatrix} d/(ad-bc) & -b/(ad-bc) \\ -c/(ad-bc) & a/(ad-bc) \end{pmatrix} \begin{pmatrix} 1 & b/a & 1/a & 0 \\ 0 & (ad-bc)/ac & -1/a & 1/c \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & d/(ad-bc) & -b/(ad-bc) \\ 0 & 1 & -c/(ad-bc) & a/(ad-bc) \end{pmatrix}$$

We recognize that the right half provides the inverse

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Invertible Matrix Theorem

THEOREM. For a $n \times n$ square matrix, equivalently:

(a) $\boxed{A \text{ is invertible}}$
(b) $Ax = 0$ has only the trivial solution
(c) The reduced row echelon form of A is the identity matrix I_n
(c2) A is row-equivalent to the $n \times n$ identity matrix I_n.
(c3) A has n pivot positions
(d) A is expressible as a product of elementary matrices
(e) $Ax = b$ is consistent for every $n \times 1$ matrix b
(f) $Ax = b$ has exactly one solution for every $n \times 1$ matrix b
(g) $\boxed{\det(A) \neq 0}$
(h) The column vectors of A are linearly independent
(i) The row vectors of A are linearly independent
(j) The column vectors of A span $\mathbb{R}^n$
(k) The row vectors of A span $\mathbb{R}^n$
(l) The column vectors of A form a basis for $\mathbb{R}^n$
(m) The row vectors of A form a basis for $\mathbb{R}^n$
(n) A has rank n (rank := number of lin. indep. rows)

Invertible Matrix Theorem (cont'd)

Theorem. For a $n \times n$ square matrix, equivalently:

(a) $\boxed{A \text{ is invertible}}$

(b) $Ax = 0$ has only the trivial solution

(d) A is expressible as a product of elementary matrices

(f) $Ax = b$ has exactly one solution for every $n \times 1$ matrix b

(g) $\boxed{\det(A) \neq 0}$

(h) The column (or row) vectors of A are linearly independent

(n) A has rank n (rank := number of lin. indep. rows)

(o) A has nullity 0 (nullity := $n-$ rank)

(p) The orthogonal complement of the null space of A is $\mathbb{R}^n$

(q) The orthogonal complement of the row space of A is 0

(r) The kernel of T_A is 0

(s) The range of T_A is $\mathbb{R}^n$

(t) T_A is one-to-one

(u) 0 fails to be an eigenvalue of A

(u2) A has n nonzero eigenvalues

(v) The transposed matrix A^T is invertible

Invertible Matrix Theorem (selection)

THEOREM. For a $n \times n$ square matrix, equivalently:

(a) A is invertible

(b) $Ax = 0$ has only the trivial solution

(c) The reduced row echelon form of A is the identity matrix I_n

(d) A is expressible as a product of elementary matrices

(f) $Ax = b$ has exactly one solution for every $n \times 1$ matrix b

(g) $\det(A) \neq 0$

(h) The column (or row) vectors of A are linearly independent

(n) A has rank n

(t) The mapping $x \to Ax$ is one-to-one

(u) 0 fails to be an eigenvalue of A

(u2) A has n nonzero eigenvalues

(v) The transposed matrix A^T is invertible

Linear Algebra: wrap-up!

- We solved systems of linear equations via Gaussian Elimination.

  These could be inconsistent (0=1, no solutions), have a unique solution, or a k-dimensional subspace of (infinitely many) solutions
- Vector space (linear space). Linear independence of vectors, span, basis, inner product, orthogonality...
- Inverse of a matrix. Determinant of a matrix.
- Eigenvalues and Eigenvectors.
  Symmetric matrix $\rightarrow$ real-valued eigenvalues
  Asymmetric matrix: expect complex eigenvalues! e.g.: rotation matrices!
- Coordinate transform (via eigenvectors) to coordinates where the matrix is diagonal
- Finding the Inverse via Gaussian Elimination: each step corresponds to a elementary matrix
- Invertible Matrix Theorem: many useful equivalent conditions!