Exercise Sheet 11b - Solutions
Predicate Logic – Natural Deduction & Semantics

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1. Here is a constructive Natural Deduction proof of $(S_1) \rightarrow \forall x.\neg 0 \leq \texttt{succ}(x)$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{^{2}}
\UnaryInfC{$0 \leq \texttt{succ}(x)$}
\RightLabel{$[\exists I]$}
\UnaryInfC{$\exists x.0 \leq x$}
\AxiomC{}
\RightLabel{^{1}}
\UnaryInfC{$\neg \exists x.0 \leq x$}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{$2\ [\neg I]$}
\UnaryInfC{$\neg 0 \leq \texttt{succ}(x)$}
\RightLabel{$[\forall I]$}
\UnaryInfC{$\forall x.\neg 0 \leq \texttt{succ}(x)$}
\RightLabel{$1\ [\rightarrow I]$}
\UnaryInfC{$(S_1) \rightarrow \forall x.\neg 0 \leq \texttt{succ}(x)$}
\end{prooftree}
$$

2. Here is a constructive Natural Deduction proof of $(S_1) \rightarrow (S_2) \rightarrow S_3$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{^{3}}
\UnaryInfC{$\exists x.1 < x$}
\AxiomC{}
\RightLabel{^{2}}
\UnaryInfC{S_2}
\RightLabel{$[\forall E]$}
\UnaryInfC{$\forall y.1 < y \rightarrow 0 \leq \texttt{succ}(y)$}
\RightLabel{$[\forall E]$}
\UnaryInfC{$1 < x \rightarrow 0 \leq \texttt{succ}(x)$}
\AxiomC{}
\RightLabel{^{4}}
\UnaryInfC{$1 < x$}
\RightLabel{$[\rightarrow E]$}
\BinaryInfC{$0 \leq \texttt{succ}(x)$}
\RightLabel{$[\exists I]$}
\UnaryInfC{$\exists x.0 \leq x$}
\AxiomC{}
\RightLabel{^{1}}
\UnaryInfC{S_1}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{$4\ [\exists E]$}
\BinaryInfC{$\bot$}
\RightLabel{$3\ [\neg I]$}
\UnaryInfC{S_3}
\RightLabel{$2\ [\rightarrow I]$}
\UnaryInfC{$(S_2) \rightarrow S_3$}
\RightLabel{$1\ [\rightarrow I]$}
\UnaryInfC{$(S_1) \rightarrow (S_2) \rightarrow S_3$}
\end{prooftree}
$$

3. For example, the following models are models of $\exists x.\exists y.x \leq y \wedge \neg y < x$:
   - $M_1 = \langle \mathbb{N}, \langle 0, \langle n \rangle \mapsto n+1 \rangle, \langle \{\langle n, m \rangle \mid n < m\}, \{\langle n, m \rangle \mid n \leq m\} \rangle \rangle$
   - and $M_1' = \langle \mathbb{N}, \langle 0, \langle n \rangle \mapsto n \rangle, \langle \emptyset, \{\langle n, m \rangle \mid n \leq m\} \rangle \rangle$
   - and $M_1'' = \langle \mathbb{N}, \langle 0, \langle n \rangle \mapsto n \rangle, \langle \emptyset, \{\langle n, m \rangle \mid \text{True}\} \rangle \rangle$

4. For example, the following models are models of $\neg \exists x.\exists y.x \leq y \wedge \neg y < x$:
   - $M_2 = \langle \mathbb{N}, \langle 0, \langle n \rangle \mapsto n \rangle, \langle \{\langle n, m \rangle \mid m < n\}, \{\langle n, m \rangle \mid n < m\} \rangle \rangle$
   - and $M_2' = \langle \mathbb{N}, \langle 0, \langle n \rangle \mapsto n \rangle, \langle \{\langle n, m \rangle \mid n < m\}, \emptyset \rangle \rangle$
   - and $M_2'' = \langle \mathbb{N}, \langle 0, \langle n \rangle \mapsto n \rangle, \langle \emptyset, \emptyset \rangle \rangle$