Exercise Sheet 10b
Predicate Logic – Natural Deduction & Semantics

Consider the following signature:

- Function symbols: `zero` (arity 0); `succ` (arity 1)
- Predicate symbols: $<$ (arity 2); $\leq$ (arity 2)

We will use infix notation for the binary symbols $<$ and $\leq$. For simplicity we write 0 for `zero`, 1 for `succ(zero)`, 2 for `succ(succ(zero))`, etc. Consider the following formulas that capture properties of the above symbols:

- let S_1 be $\forall x.\exists y.x < y$
- let S_2 be $\forall x.\forall y.x < y \to \texttt{succ}(x) \leq y$
- let S_3 be $\exists x.1 \leq x$

1. Provide a constructive Natural Deduction proof of $(S_1) \to \neg\exists x.\forall y.\neg x < y$
2. Provide a Constructive Natural Deduction proof of $(S_1) \to (S_2) \to S_3$
3. Provide a model M_1 such that $\vDash_{M_1} S_1$
4. Provide a model M_2 such that $\vDash_{M_2} \neg S_1$