Mathematical and Logical Foundations of Computer Science
Logic Tips

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1 Propositional Logic

Natural Deduction Proofs

1. When working backwards, most of the time:

   Start with introducing the $\to$, $\wedge$, and $\neg$ symbols, i.e., with $[\to I]$, $[\neg I]$, $[\wedge I]$:

   $$\dfrac{\dfrac{B \quad A}{B \wedge A}\ [\wedge I]}{A \wedge B \to B \wedge A}\ 1\ [\to I]$$

   Pause, think, and start using the elimination rules, i.e. $[\to E]$, $[\neg E]$, $[\wedge E_L]$, $[\wedge E_R]$:

   $$\dfrac{\dfrac{\dfrac{A \wedge B}{B}\ [\wedge E_R] \quad \dfrac{A \wedge B}{A}\ [\wedge E_L]}{B \wedge A}\ [\wedge I]}{A \wedge B \to B \wedge A}\ 1\ [\to I]$$

   And finally cancel all your leaves:

   $$\dfrac{\dfrac{\dfrac{\overline{A \wedge B}^{\,1}}{B}\ [\wedge E_R] \quad \dfrac{\overline{A \wedge B}^{\,1}}{A}\ [\wedge E_L]}{B \wedge A}\ [\wedge I]}{A \wedge B \to B \wedge A}\ 1\ [\to I]$$

2. With $\vee$s, the pattern is typically different:

   Start with your eliminations:

   $$\dfrac{\dfrac{A \vee B \quad A \to B \vee A \quad B \to B \vee A}{B \vee A}\ [\vee E]}{A \vee B \to B \vee A}\ 1\ [\to I]$$

   and then do your introductions:

   $$\dfrac{\dfrac{\overline{A \vee B}^{\,1} \quad \dfrac{\dfrac{\overline{A}^{\,2}}{B \vee A}\ [\vee I_R]}{A \to B \vee A}\ 2\ [\to I] \quad \dfrac{\dfrac{\overline{B}^{\,3}}{B \vee A}\ [\vee I_L]}{B \to B \vee A}\ 3\ [\to I]}{B \vee A}\ [\vee E]}{A \vee B \to B \vee A}\ 1\ [\to I]$$

3. Typical pattern to prove $A \to A$:

   $$\dfrac{\overline{A}^{\,1}}{A \to A}\ 1\ [\to I]$$

4. Typical classical pattern using $[LEM]$:

   $$\dfrac{\overline{A \vee \neg A}\ [LEM] \quad A \to C \quad \neg A \to C}{C}\ [\vee E]$$

5. Typical classical pattern using $[DNE]$:

   $$\dfrac{\dfrac{\bot}{\neg\neg A}\ 1\ [\neg I]}{A}\ [DNE]$$

6. Typical pattern given an hypothesis of the form $A \to B \vee C$. When proving some formula D with an hypothesis of the form $A \to B \vee C$, see if you can prove A, in which case consider eliminating $B \vee C$ right away since you know you can get it from the implication:

   $$\dfrac{\dfrac{A \to B \vee C \quad A}{B \vee C}\ [\to E] \quad B \to D \quad C \to D}{D}\ [\vee E]$$

Semantics

1. Lay out your truth tables as follows, i.e., A's column has 4 **T**s followed by 4 **F**s, then B's column has 2 **T**s followed by 2 **F**s followed by 2 **T**s followed by 2 **F**s, etc.:

A	B	C	$\cdots$
T	**T**	**T**	
T	**T**	**F**	
T	**F**	**T**	
T	**F**	**F**	
F	**T**	**T**	
F	**T**	**F**	
F	**F**	**T**	
F	**F**	**F**	

2. If you know that a formula is provable using a Natural Deduction proof, then by soundness you know that it is semantically valid, and therefore satisfiable and not falsifiable.