Natural Deduction Calculus of Predicate Logic

Till Rampe, Dave Sima

December 9, 2024

A common mistake involving $\exists$-elimination

Exercise 1

Give a constructive natural deduction proof of the following formula.

$$(\exists x.p(x)) \to \exists x.p(x) \lor q(x)$$

Wrong attempt.

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\exists x.p(x)} \qquad \overline{p(x)}}{p(x)}\;2\;[\exists E]}{p(x) \lor q(x)}\;[\lor IL]}{\exists x.p(x) \lor q(x)}\;[\exists I]}{(\exists x.p(x)) \to \exists x.p(x) \lor q(x)}\;1\;[\to I]$$

$\square$

Do ∃-elimination as early as possible

The problem in the previous attempt is that x is free in $p(x)$. Hence, the ∃-elimination in the top most step is not valid. Instead, perform ∃-elimination when the x in $p(x)$ was still bound by a quantifier.

$$
\dfrac{
  \dfrac{}{\exists x.p(x)}\;1
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{}{p(x)}\;2
    }{p(x) \vee q(x)}\;[\vee IL]
  }{\exists x.p(x) \vee q(x)}\;[\exists I]
}{
  \dfrac{\exists x.p(x) \vee q(x)}{(\exists x.p(x)) \rightarrow \exists x.p(x) \vee q(x)}\;1\;[\rightarrow I]
}\;2\;[\exists E]
$$

When we do ∃-elimination (which step is it?), x is not free in $\exists x.p(x) \vee q(x)$. Hence, we can introduce $p(x)$ as an assumption.

Practice

Exercise 2

Give a constructive natural deduction proof of the following formula.

$$\neg(\exists x.p(x)) \to \forall y.\neg p(y)$$

Exercise 2 Solution

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p(y)}\,2}{\exists x.p(x)}\,[\exists I] \qquad \dfrac{}{\neg\exists x.p(x)}\,1}{\bot}\,[\neg E]}{\neg p(y)}\,2\ [\neg I]}{\dfrac{\neg p(y)}{\forall y.\neg p(y)}\,[\forall I]}
$$

$$
\dfrac{\forall y.\neg p(y)}{\neg(\exists x.p(x)) \rightarrow \forall y.\neg p(y)}\,1\ [\rightarrow I]
$$

More Practice

Exercise 3

Give a constructive natural deduction proof of the following formula.

$$(\exists x.\forall y.p(x,y)) \to (\forall x.\neg p(x,x)) \to \bot$$

Exercise 3 Solution

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\exists x.\forall y.p(x,y)}\;1 \qquad \dfrac{\dfrac{\dfrac{}{\forall y.p(x,y)}\;3}{p(x,x)}\;[\forall E] \quad \dfrac{\dfrac{}{\forall x.\neg p(x,x)}\;2}{\neg p(x,x)}\;[\forall E]}{\bot}\;[\neg E]}{\bot}\;3\;[\exists E]}{(\forall x.\neg p(x,x)) \to \bot}\;2\;[\to I]}{(\exists x.\forall y.p(x,y)) \to (\forall x.\neg p(x,x)) \to \bot}\;1[\to I]
$$