Models

A model in predicate logic is something that gives a meaning to a statement

It contains 3 things:

- a Domain
- The meanings of the functional symbols
- The meanings of the predicate symbols

The functional symbols & predicate symbols are given in the question:

e.g

Exercise Sheet 10b
Predicate Logic – Natural Deduction & Semantics

Consider the following signature:

- Function symbols: `zero` (arity 0); `succ` (arity 1)
- Predicate symbols: $<$ (arity 2); $\leq$ (arity 2)

We will use infix notation for the binary symbols $<$ and $\leq$. For simplicity we write 0 for `zero`, 1 for `succ(zero)`, 2 for `succ(succ(zero))`, etc. Consider the following formulas that capture properties of the above symbols:

- let S_1 be $\forall x.\exists y.x < y$
- let S_2 be $\forall x.\forall y.x < y \rightarrow \mathtt{succ}(x) \leq y$
- let S_3 be $\exists x.1 \leq x$

1. Provide a constructive Natural Deduction proof of $(S_1) \rightarrow \neg\exists x.\forall y.\neg x < y$
2. Provide a Constructive Natural Deduction proof of $(S_1) \rightarrow (S_2) \rightarrow S_3$
3. Provide a model M_1 such that $\models_{M_1} S_1$
4. Provide a model M_2 such that $\models_{M_2} \neg S_1$

We give a model with respect to some predicate logic formula

e.g Q3 asks for a model M_1

such that $\models_{M_1} (\forall x. \exists y. x<y)$ **(Q3)**

so we are giving a model for $(\forall x. \exists y. x<y)$

Again, 3 parts:

- a Domain
- The meanings of the functional symbols
- The meanings of the predicate symbols

$$\langle \mathbb{N}, \underbrace{\langle 0, +1 \rangle}_{\text{Meanings of the functional symbols}}, \langle \underbrace{\{\langle a,b\rangle \mid a<b\}, \{\langle a,b\rangle \mid a \leq b\}}_{\text{Meanings of the predicate symbols}} \rangle \rangle$$

The domain we are giving is $\mathbb{N}$, the natural numbers.

You are free to choose the Domain (e.g. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{B}$, etc.)

Booleans

It's important to note the meanings must be given in the same order as the question:

$\langle 0, +1 \rangle$ refers to 0 & succ

$\{\langle a,b \rangle \mid a < b\}$ refers to $<$

$\{\langle a,b \rangle \mid a \leq b\}$ refers to $\leq$

What the model does is give an actual meaning to 0, succ, $<$ and $\leq$

$\langle \mathbb{N}, \langle 0, +1 \rangle, \langle \{\langle x,y \rangle \mid x < y\}, \{\langle x,y \rangle \mid x \leq y\} \rangle \rangle$

This model for $\forall x. \exists y. x < y$ says that for all x, there exists a y such that $x < y$.

Here that means that there exists a pair $\langle x,y \rangle \in \{\langle a,b \rangle \mid a < b\}$ as that is the meaning we assigned to $<$

We can test with $x=0$ & $y=1$

is $\langle 0,1\rangle$ a member of $\{\langle a,b\rangle \mid a<b\}$

i.e. $\langle 0,1\rangle \in \{\langle a,b\rangle \mid a<b\}$

This is true

$\therefore$ the model holds

For Q4:

We are asked to give a model such that $\models_{M_2} \neg(\forall x.\exists y. x<y)$

so the part inside the brackets must be False

so:

$\langle \mathbb{N}, \langle 0, +1\rangle, \langle \emptyset, \{\langle x,y\rangle \mid x\leq y\}\rangle\rangle$

This is saying **not** for all x, there exists a y such that $x<y$.

Here that means that there exists a pair $<x,y> \in \emptyset$ - the Empty set

So now the part inside the brackets evaluates to False & the $\neg$ outside changes it to a True