# Exercise Sheet for Maths Material Covered in Week 8 Vector Spaces: Definition, Examples, Span, Linear (in)dependence

(1) Consider the vector space $\mathbb{Q}^2$ of 2-tuples of rational numbers over the field $Q$ of rational numbers. Show that if $\vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ then $\mathrm{Span}(\vec{u}, \vec{v}) = \mathbb{Q}^2$.

**Comment:** You need to show that $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathrm{Span}(\vec{u}, \vec{v})$ for every $a, b \in \mathbb{Q}$.

(2) Consider the vector space $\mathbb{Q}^3$ of 3-tuples of rational numbers over the field $Q$ of rational numbers. Show that $\vec{w} \notin \mathrm{Span}(\vec{u}, \vec{v})$ where $\vec{w} = \begin{pmatrix} 9 \\ 27 \\ 35 \end{pmatrix}$, $\vec{u} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 2 \\ 6 \\ 8 \end{pmatrix}$.

(3) Let $V$ be a vector space over a field $F$. Let $\vec{u}, \vec{v}$ be any two vectors in $V$. Recall that $\mathrm{Span}(\vec{u}, \vec{v}) = \{r\vec{u} \oplus s\vec{v} \mid r, s \in F\}$. Verify that $\mathrm{Span}(\vec{u}, \vec{v})$ is a vector space over $F$.

**Comment**: First define for $\mathrm{Span}(\vec{u}, \vec{v})$ the two operations of (i) vector addition, and (ii) multiplication of a scalar with a vector. Each of the 8 conditions in the definition of vector space over a field now needs to be verified. You can use any of the 8 conditions available due to $V$ being a vector space over the field $F$.

(4) Consider the vector space $\mathbb{Q}^3$ of 3-tuples of rational numbers over the field $Q$ of rational numbers. Let $\vec{u} = \begin{pmatrix} 9 \\ 17 \\ 135 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 19 \\ 27 \\ 100 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} 28 \\ 44 \\ 235 \end{pmatrix}$. Are the vectors $\vec{u}, \vec{v}, \vec{w}$ linearly independent?
**Comment**: Does the observation $\vec{u} \oplus \vec{v} = \vec{w}$ help you?