Exercise Sheet 4 - Solutions
Propositional Logic – Constructive & Classical Reasoning

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1. Here is a proof of $\neg\neg\neg A \to \neg A$

$$
\dfrac{\dfrac{\overline{\neg\neg\neg A}^{\,1} \qquad \dfrac{\dfrac{\overline{A}^{\,2} \quad \overline{\neg A}^{\,3}}{\bot}\,{\scriptstyle[\neg E]}}{\neg\neg A}\,{\scriptstyle 3\ [\neg I]}}{\dfrac{\bot}{\neg A}\,{\scriptstyle 2\ [\neg I]}}\,{\scriptstyle[\neg E]}}{\neg\neg\neg A \to \neg A}\,{\scriptstyle 1\ [\to I]}
$$

2. Here is a proof of $(A \vee \neg A) \to (\neg\neg A \to A)$:

$$
\dfrac{\dfrac{\dfrac{\overline{A \vee \neg A}^{\,1} \qquad \dfrac{\overline{A}^{\,3}}{A \to A}\,{\scriptstyle 3\ [\to I]} \qquad \dfrac{\dfrac{\dfrac{\overline{\neg A}^{\,4} \quad \overline{\neg\neg A}^{\,2}}{\bot}\,{\scriptstyle[\neg E]}}{A}\,{\scriptstyle[\bot E]}}{\neg A \to A}\,{\scriptstyle 4\ [\to I]}}{A}\,{\scriptstyle[\vee E]}}{\neg\neg A \to A}\,{\scriptstyle 2\ [\to I]}}{(A \vee \neg A) \to (\neg\neg A \to A)}\,{\scriptstyle 1\ [\to I]}
$$

3. Here is a proof that $((P \to \bot) \to P) \to P$ implies $\neg\neg P \to P$:

$$
\dfrac{\dfrac{\dfrac{\overline{((P \to\bot) \to P) \to P}^{\,1} \qquad \dfrac{\dfrac{\dfrac{\overline{\neg\neg P}^{\,2} \qquad \dfrac{\dfrac{\overline{P}^{\,4} \quad \overline{P \to\bot}^{\,3}}{\bot}\,{\scriptstyle[\to E]}}{\neg P}\,{\scriptstyle 4\ [\neg I]}}{\bot}\,{\scriptstyle[\neg E]}}{P}\,{\scriptstyle[\bot E]}}{(P \to\bot) \to P}\,{\scriptstyle 3\ [\to I]}}{P}\,{\scriptstyle[\to E]}}{\neg\neg P \to P}\,{\scriptstyle 2\ [\to I]}}{(((P \to\bot) \to P) \to P) \to \neg\neg P \to P}\,{\scriptstyle 1\ [\to I]}
$$

Here is a proof that $\neg\neg P \to P$ implies $((P \to\bot) \to P) \to P$:

$$
\dfrac{\dfrac{\dfrac{\overline{\neg\neg P \to P}^{\,1} \qquad \dfrac{\dfrac{\dfrac{\overline{(P \to\bot) \to P}^{\,2} \qquad \dfrac{\dfrac{\overline{P}^{\,4} \quad \overline{\neg P}^{\,3}}{\bot}\,{\scriptstyle[\neg E]}}{P \to\bot}\,{\scriptstyle 4\ [\to I]}}{P}\,{\scriptstyle[\to E]} \qquad \overline{\neg P}^{\,3}}{\bot}\,{\scriptstyle[\neg E]}}{\neg\neg P}\,{\scriptstyle 3\ [\neg I]}}{P}\,{\scriptstyle[\to E]}}{((P \to\bot) \to P) \to P}\,{\scriptstyle 2\ [\to I]}}{(\neg\neg P \to P) \to ((P \to\bot) \to P) \to P}\,{\scriptstyle 1\ [\to I]}
$$

4. Here is a classical Natural Deduction proof of $((P \to Q) \to P) \to P$:

$$
\begin{prooftree}
\AxiomC{}\RightLabel{^{2}}\UnaryInfC{$\neg P$}
\AxiomC{}\RightLabel{^{1}}\UnaryInfC{$(P \to Q) \to P$}
\AxiomC{}\RightLabel{^{2}}\UnaryInfC{$\neg P$}
\AxiomC{}\RightLabel{^{3}}\UnaryInfC{P}
\RightLabel{$[\neg E]$}\BinaryInfC{$\bot$}
\RightLabel{$[\bot\ E]$}\UnaryInfC{Q}
\RightLabel{$3\ [\to I]$}\UnaryInfC{$P \to Q$}
\RightLabel{$[\to E]$}\BinaryInfC{P}
\RightLabel{$[\neg E]$}\BinaryInfC{$\bot$}
\RightLabel{$2\ [\neg I]$}\UnaryInfC{$\neg\neg P$}
\RightLabel{$[DNE]$}\UnaryInfC{P}
\RightLabel{$1\ [\to I]$}\UnaryInfC{$((P \to Q) \to P) \to P$}
\end{prooftree}
$$

5. Here is a classical Natural Deduction proof of $\neg(A \wedge B) \to (\neg A \vee \neg B)$:

$$
\begin{prooftree}
\AxiomC{}\RightLabel{$[LEM]$}\UnaryInfC{$A \vee \neg A$}
\AxiomC{}\RightLabel{^{1}}\UnaryInfC{$\neg(A \wedge B)$}
\AxiomC{}\RightLabel{^{2}}\UnaryInfC{A}
\AxiomC{}\RightLabel{^{4}}\UnaryInfC{B}
\RightLabel{$[\wedge I]$}\BinaryInfC{$A \wedge B$}
\RightLabel{$[\neg E]$}\BinaryInfC{$\bot$}
\RightLabel{$4\ [\neg I]$}\UnaryInfC{$\neg B$}
\RightLabel{$[\vee I_R]$}\UnaryInfC{$\neg A \vee \neg B$}
\RightLabel{$2\ [\to I]$}\UnaryInfC{$A \to \neg A \vee \neg B$}
\AxiomC{}\RightLabel{^{3}}\UnaryInfC{$\neg A$}
\RightLabel{$[\vee I_L]$}\UnaryInfC{$\neg A \vee \neg B$}
\RightLabel{$3\ [\to I]$}\UnaryInfC{$\neg A \to \neg A \vee \neg B$}
\RightLabel{$[\vee E]$}\TrinaryInfC{$\neg A \vee \neg B$}
\RightLabel{$1\ [\to I]$}\UnaryInfC{$\neg(A \wedge B) \to (\neg A \vee \neg B)$}
\end{prooftree}
$$

6. Here is a classical Natural Deduction proof of $(\neg B \to A) \to A \vee B$:

$$
\begin{prooftree}
\AxiomC{}\RightLabel{$[LEM]$}\UnaryInfC{$B \vee \neg B$}
\AxiomC{}\RightLabel{^{2}}\UnaryInfC{B}
\RightLabel{$[\vee I_R]$}\UnaryInfC{$A \vee B$}
\RightLabel{$2\ [\to I]$}\UnaryInfC{$B \to A \vee B$}
\AxiomC{}\RightLabel{^{1}}\UnaryInfC{$\neg B \to A$}
\AxiomC{}\RightLabel{^{3}}\UnaryInfC{$\neg B$}
\RightLabel{$[\to E]$}\BinaryInfC{A}
\RightLabel{$[\vee I_L]$}\UnaryInfC{$A \vee B$}
\RightLabel{$3\ [\to I]$}\UnaryInfC{$\neg B \to A \vee B$}
\RightLabel{$[\vee E]$}\TrinaryInfC{$A \vee B$}
\RightLabel{$1\ [\to I]$}\UnaryInfC{$(\neg B \to A) \to A \vee B$}
\end{prooftree}
$$