Exercise Sheet 11
Predicate Logic

Consider the following signature:

- Function symbols: `zero` (arity 0); `succ` (arity 1)
- Predicate symbols: $<$ (arity 2); $\leq$ (arity 2)

We will use infix notation for the binary symbols $<$ and $\leq$. For simplicity we write 0 for `zero`, 1 for `succ(zero)`, 2 for `succ(succ(zero))`, etc. Consider the following formulas that capture properties of the above symbols:

- let S_1 be $\forall x.\neg 0 \leq x$
- let S_2 be $\forall x.\forall y.x < y \rightarrow x \leq \texttt{succ}(y)$
- let S_3 be $\neg\exists x.x < 0$
- let S_4 be $\forall x.\forall y.\texttt{succ}(x) \leq y \rightarrow x < y$

1. Provide a constructive Natural Deduction proof of $(S_1) \rightarrow (S_2) \rightarrow \forall x.\neg 0 < x$
2. Provide a constructive Natural Deduction proof of $(S_3) \rightarrow (S_4) \rightarrow \forall x.\neg\texttt{succ}(x) \leq 0$
3. Provide a constructive Natural Deduction proof of $(S_3) \rightarrow (S_4) \rightarrow \neg\exists x.\texttt{succ}(x) \leq 0$
4. Provide a model M_1 such that $\models_{M_1} \exists x.\exists y.x < y \wedge \neg x \leq y$
5. Provide a model M_2 such that $\models_{M_2} \neg\exists x.\exists y.x < y \wedge \neg x \leq y$

Natural Deduction Calculus of Predicate Logic

Till Rampe, Dave Sima

December 9, 2024

A common mistake involving $\exists$-elimination

Exercise 1

Give a constructive natural deduction proof of the following formula.

$$(\exists x.p(x)) \to \exists x.p(x) \vee q(x)$$

Wrong attempt.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\exists x.p(x)} \quad \dfrac{}{p(x)}}{p(x)}\ 2\ [\exists E]}{p(x) \vee q(x)}\ [\vee IL]}{\exists x.p(x) \vee q(x)}\ [\exists I]}{(\exists x.p(x)) \to \exists x.p(x) \vee q(x)}\ 1\ [\to I]$$

$\square$

Do ∃-elimination as early as possible

The problem in the previous attempt is that x is free in $p(x)$. Hence, the ∃-elimination in the top most step is not valid. Instead, perform ∃-elimination when the x in $p(x)$ was still bound by a quantifier.

$$
\dfrac{\dfrac{\dfrac{}{\exists x.p(x)}\;1 \qquad \dfrac{\dfrac{\dfrac{}{p(x)}\;2}{p(x)\vee q(x)}\;[\vee IL]}{\exists x.p(x)\vee q(x)}\;[\exists I]}{\exists x.p(x)\vee q(x)}\;2\;[\exists E]}{(\exists x.p(x)) \to \exists x.p(x)\vee q(x)}\;1\;[\to I]
$$

When we do ∃-elimination (which step is it?), x is not free in $\exists x.p(x) \vee q(x)$. Hence, we can introduce $p(x)$ as an assumption.

Practice

Exercise 2

Give a constructive natural deduction proof of the following formula.

$$\neg(\exists x.p(x)) \to \forall y.\neg p(y)$$

Exercise 2 Solution

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p(y)}\ 2}{\exists x.p(x)}\ [\exists I] \qquad \dfrac{}{\neg\exists x.p(x)}\ 1}{\perp}\ [\neg E]}{\neg p(y)}\ 2\ [\neg I]}{\forall y.\neg p(y)}\ [\forall I]}{\neg(\exists x.p(x)) \to \forall y.\neg p(y)}\ 1\ [\to I]$$

More Practice

Exercise 3

Give a constructive natural deduction proof of the following formula.

$$(\exists x . \forall y . p(x, y)) \to (\forall x . \neg p(x, x)) \to \bot$$

Exercise 3 Solution

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\exists x.\forall y.p(x,y)}\;1 \qquad \dfrac{\dfrac{\dfrac{}{\forall y.p(x,y)}\;3}{p(x,x)}\;[\forall E] \qquad \dfrac{\dfrac{}{\forall x.\neg p(x,x)}\;2}{\neg p(x,x)}\;[\forall E]}{\bot}\;[\neg E]}{\bot}\;3\ [\exists E]}{(\forall x.\neg p(x,x)) \to \bot}\;2\ [\to I]}{(\exists x.\forall y.p(x,y)) \to (\forall x.\neg p(x,x)) \to \bot}\;1[\to I]$$

Mathematical and Logical Foundations of Computer Science

Predicate Logic (Equivalences continued)

Vincent Rahli

(some slides were adapted from Rajesh Chitnis' slides)

University of Birmingham

Where are we?

- Symbolic logic
- Propositional logic
- **Predicate logic**

Today

Equivalences:

- in Natural Deduction
- rewriting using “known” equivalences
- using semantics

Today

Equivalences:

- in Natural Deduction
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Further reading:

- Chapter 8 of
  http://leanprover.github.io/logic_and_proof/

Recap: Syntax

The syntax of predicate logic is defined by the following grammar:

$$
\begin{array}{lcl}
t & ::= & x \mid f(t, \ldots, t) \\
P & ::= & p(t, \ldots, t) \mid \neg P \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid \forall x.P \mid \exists x.P
\end{array}
$$

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where:

- x ranges over variables
- f ranges over function symbols
- $f(t_1, \ldots, t_n)$ is a well-formed term only if f has arity n
- p ranges over predicate symbols
- $p(t_1, \ldots, t_n)$ is a well-formed formula only if p has arity n

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The pair of a collection of function symbols, and a collection of predicate symbols, along with their arities, is called a **signature**.

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The pair of a collection of function symbols, and a collection of predicate symbols, along with their arities, is called a **signature**.

The scope of a quantifier extends as far right as possible. E.g., $P \wedge \forall x.p(x) \vee q(x)$ is read as $P \wedge \forall x.(p(x) \vee q(x))$

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$P[x\backslash t]$ substitute all the free occurrences of x in P with t.

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x[y\backslash t] & = & x \\
(f(t_1,\ldots,t_n))[x\backslash t] & = & f(t_1[x\backslash t],\ldots,t_n[x\backslash t]) \\
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The additional **conditions** ensure that **free variables do not get captured**.

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The additional **conditions** ensure that **free variables do not get captured**.

These conditions can always be met by silently renaming bound variables before substituting.

Recap: ∀ & ∃ elimination and introduction rules

Natural Deduction rules for quantifiers:

$$\frac{P[x\backslash y]}{\forall x.P}\ [\forall I] \qquad \frac{\forall x.P}{P[x\backslash t]}\ [\forall E] \qquad \frac{P[x\backslash t]}{\exists x.P}\ [\exists I] \qquad \frac{\exists x.P \qquad \begin{array}{c}\overline{P[x\backslash y]}^{\,1} \\ \vdots \\ Q\end{array}}{Q}\ 1\ [\exists E]$$

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Condition:

- for $[\forall I]$: y must not be free in any not-yet-discharged hypothesis or in $\forall x.P$
- for $[\forall E]$: $\texttt{fv}(t)$ must not clash with $\texttt{bv}(P)$
- for $[\exists I]$: $\texttt{fv}(t)$ must not clash with $\texttt{bv}(P)$
- for $[\exists E]$: y must not be free in Q or in not-yet-discharged hypotheses or in $\exists x.P$

Recap: Example of a proof

here is a proof of $(\forall z.p(z)) \rightarrow \forall x.p(x) \vee q(x)$.

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$$\dfrac{}{\forall z.p(z)}\ 1$$

$$\dfrac{\dfrac{\dfrac{}{p(y) \vee q(y)}}{\forall x.p(x) \vee q(x)}\ [\forall I]}{(\forall z.p(z)) \to \forall x.p(x) \vee q(x)}\ 1\ [\to I]$$

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- y does not clash with bound variables in $p(z)$

Recap: Models

Models: a model provides the interpretation of all symbols

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Given a **signature** $\langle\langle f_1^{k_1}, \ldots, f_n^{k_n}\rangle, \langle p_1^{j_1}, \ldots, p_m^{j_m}\rangle\rangle$

- of function symbols f_i of arity k_i, for $1 \leqslant i \leqslant n$
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a **model** is a structure $\langle D, \langle \mathcal{F}_{f_1}, \ldots, \mathcal{F}_{f_n}\rangle, \langle \mathcal{R}_{p_1}, \ldots, \mathcal{R}_{p_m}\rangle\rangle$

- of a non-empty domain D
- interpretations $\mathcal{F}_{f_i}$ for function symbols f_i
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Models of predicate logic replace **truth assignments** for propositional logic

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- that map variables to D

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Models of predicate logic replace **truth assignments** for propositional logic

Variable valuations:

- a partial function v
- that map variables to D
- i.e., a mapping of the form $x_1 \mapsto d_1, \ldots, x_n \mapsto d_n$

Recap: Semantics of Predicate Logic

Given a **model** M with domain D and a **variable valuation** v:

- $[\![t]\!]_v^M$ gives meaning to the term t w.r.t. M and v
- $\models_{M,v} P$ gives meaning to the formula P w.r.t. M and v

Recap: Semantics of Predicate Logic

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Meaning of terms:

- $[\![x]\!]_v^M = v(x)$
- $[\![f(t_1, \ldots, t_n)]\!]_v^M = \mathcal{F}_f(\langle [\![t_1]\!]_v^M, \ldots, [\![t_n]\!]_v^M \rangle)$

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Meaning of formulas:

- $\models_{M,v} p(t_1, \ldots, t_n)$ iff $\langle [\![t_1]\!]_v^M, \ldots, [\![t_n]\!]_v^M \rangle \in \mathcal{R}_p$
- $\models_{M,v} \neg P$ iff $\neg \models_{M,v} P$
- $\models_{M,v} P \wedge Q$ iff $\models_{M,v} P$ and $\models_{M,v} Q$
- $\models_{M,v} P \vee Q$ iff $\models_{M,v} P$ or $\models_{M,v} Q$
- $\models_{M,v} P \rightarrow Q$ iff $\models_{M,v} Q$ whenever $\models_{M,v} P$
- $\models_{M,v} \forall x.P$ iff for every $d \in D$ we have $\models_{M,(v,x \mapsto d)} P$
- $\models_{M,v} \exists x.P$ iff there exists a $d \in D$ such that $\models_{M,(v,x \mapsto d)} P$

Recap: Logical equivalences for Propositional Logic

The same equivalences hold as in Propositional Logic:

- De Morgan's law (I): $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$
- De Morgan's law (II): $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$
- Implication elimination: $(A \to B) \leftrightarrow (\neg A \vee B)$
- Commutativity of $\wedge$: $(A \wedge B) \leftrightarrow (B \wedge A)$
- Commutativity of $\vee$: $(A \vee B) \leftrightarrow (B \vee A)$
- Associativity of $\wedge$: $((A \wedge B) \wedge C) \leftrightarrow (A \wedge (B \wedge C))$
- Associativity of $\vee$: $((A \vee B) \vee C) \leftrightarrow (A \vee (B \vee C))$
- Distributivity of $\wedge$ over $\vee$: $(A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$
- Distributivity of $\vee$ over $\wedge$: $(A \vee (B \wedge C)) \leftrightarrow ((A \vee B) \wedge (A \vee C))$
- Double negation elimination: $(\neg\neg A) \leftrightarrow A$
- Idempotence: $(A \wedge A) \leftrightarrow A$ and $(A \vee A) \leftrightarrow A$

Recap: Logical Equivalences

In addition, the following hold (some hold only classically):

- $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$
- $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$
- $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$
- $(\neg\exists x.A) \leftrightarrow (\forall x.\neg A)$
- $(\forall x.A) \leftrightarrow A$ if $x \notin \texttt{fv}(A)$
- $(\exists x.A) \leftrightarrow A$ if $x \notin \texttt{fv}(A)$
- $(\forall x.A \vee B) \leftrightarrow ((\forall x.A) \vee B)$ if $x \notin \texttt{fv}(B)$
- $(\exists x.A \wedge B) \leftrightarrow ((\exists x.A) \wedge B)$ if $x \notin \texttt{fv}(B)$
- $(\forall x.A \rightarrow B) \leftrightarrow ((\exists x.A) \rightarrow B)$ if $x \notin \texttt{fv}(B)$
- $(\exists x.A \rightarrow B) \leftrightarrow ((\forall x.A) \rightarrow B)$ if $x \notin \texttt{fv}(B)$
- $(\forall x.A \rightarrow B) \leftrightarrow (A \rightarrow \forall x.B)$ if $x \notin \texttt{fv}(A)$
- $(\exists x.A \rightarrow B) \leftrightarrow (A \rightarrow \exists x.B)$ if $x \notin \texttt{fv}(A)$

Recap: Logical Equivalences

As before: if ($P \leftrightarrow Q$ or $Q \leftrightarrow P$) and P occurs in A, then replacing P by Q in A leads to a formula B, such that $A \leftrightarrow B$

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Also,

Semantical equivalence: two formulas P and Q are equivalent if for all models M and valuations v, $\models_{M,v} P$ iff $\models_{M,v} Q$

Logical Equivalences

As before to prove a logical equivalence $A \leftrightarrow B$, we will prove:

- that we can derive B form A
- that we can derive A form B

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- $(\exists x.A) \leftrightarrow A$ if $x \notin \mathtt{fv}(A)$
- $(\forall x.A \lor B) \leftrightarrow ((\forall x.A) \lor B)$ if $x \notin \mathtt{fv}(B)$
- $(\exists x.A \land B) \leftrightarrow ((\exists x.A) \land B)$ if $x \notin \mathtt{fv}(B)$

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- $(\forall x.A \vee B) \leftrightarrow ((\forall x.A) \vee B)$ if $x \notin \mathtt{fv}(B)$
- $(\exists x.A \wedge B) \leftrightarrow ((\exists x.A) \wedge B)$ if $x \notin \mathtt{fv}(B)$

We will use the following result:

Logical Equivalences

As before to prove a logical equivalence $A \leftrightarrow B$, we will prove:

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We will start by proving:

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- $(\exists x.A \wedge B) \leftrightarrow ((\exists x.A) \wedge B)$ if $x \notin \texttt{fv}(B)$

We will use the following result:

Lemma (L1): if $x \notin \texttt{fv}(A)$ then $A[x \backslash t] = A$

Logical Equivalences

Prove $(\forall x.A) \leftrightarrow A$ if $x \notin \mathtt{fv}(A)$ in Natural Deduction

Logical Equivalences

Prove $(\forall x.A) \leftrightarrow A$ if $x \notin \mathtt{fv}(A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\overline{}$$

Logical Equivalences

Prove $(\forall x.A) \leftrightarrow A$ if $x \notin \mathtt{fv}(A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\frac{}{\forall x.A}$$

Logical Equivalences

Prove $(\forall x.A) \leftrightarrow A$ if $x \notin \mathtt{fv}(A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\frac{A[x\backslash y]}{\forall x.A}\ [\forall I]$$

- pick y such that it does not occur in A

Logical Equivalences

Prove $(\forall x.A) \leftrightarrow A$ if $x \notin \mathtt{fv}(A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{A}{\forall x.A}\ [\forall I]$$

- pick y such that it does not occur in A
- by L1, because $x \notin \mathtt{fv}(A)$ then $A[x\backslash y] = A$

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Here is a proof of the right-to-left implication (constructive):

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Prove $(\forall x.A) \leftrightarrow A$ if $x \notin \texttt{fv}(A)$ in Natural Deduction

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- pick y such that it does not occur in A
- by L1, because $x \notin \texttt{fv}(A)$ then $A[x\backslash y] = A$

Here is a proof of the left-to-right implication (constructive):

$$\frac{\forall x.A}{A[x\backslash y]}\ [\forall E]$$

- by L1, because $x \notin \texttt{fv}(A)$ then $A[x\backslash y] = A$
- pick y such that it does not occur in A

Logical Equivalences

Prove $(\exists x.A) \leftrightarrow A$ if $x \notin \texttt{fv}(A)$ in Natural Deduction

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Logical Equivalences

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Logical Equivalences

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Here is a proof of the right-to-left implication (constructive):

$$\frac{A[x\backslash y]}{\exists x.A}\ [\exists I]$$

- pick y such that it does not occur in A

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Logical Equivalences

Prove $(\exists x.A) \leftrightarrow A$ if $x \notin \mathtt{fv}(A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

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- by L1, because $x \notin \mathtt{fv}(A)$ then $A[x\backslash y] = A$

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\exists x.A \quad A}{A}$$

- by L1, because $x \notin \mathtt{fv}(A)$ then $A[x\backslash y] = A$
- pick y such that it does not occur in A

Logical Equivalences

Prove $(\exists x.A) \leftrightarrow A$ if $x \notin \mathtt{fv}(A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{A}{\exists x.A}\ [\exists I]$$

- pick y such that it does not occur in A
- by L1, because $x \notin \mathtt{fv}(A)$ then $A[x\backslash y] = A$

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\exists x.A \quad \dfrac{}{A[x\backslash y]}\ 1}{A}\ 1\ [\exists E]$$

- by L1, because $x \notin \mathtt{fv}(A)$ then $A[x\backslash y] = A$
- pick y such that it does not occur in A

Logical Equivalences

Prove that $(\forall x.A \vee B) \leftrightarrow ((\forall x.A) \vee B)$ if $x \notin \mathtt{fv}(B)$ in Natural Deduction

Logical Equivalences

Prove that $(\forall x.A \vee B) \leftrightarrow ((\forall x.A) \vee B)$ if $x \notin \texttt{fv}(B)$ in Natural Deduction

Here is a proof of the left-to-right implication (classical):

$$\dfrac{\dfrac{}{B \vee \neg B}\,[LEM] \qquad \dfrac{\dfrac{\dfrac{}{B}\,1}{(\forall x.A) \vee B}\,[\vee I_R]}{B \to (\forall x.A) \vee B}\,1\ [\to I] \qquad \dfrac{\dfrac{\Pi}{(\forall x.A) \vee B}\,[\vee I_L]}{\neg B \to (\forall x.A) \vee B}\,2\ [\to I]}{(\forall x.A) \vee B}\,[\vee E]$$

where Π is:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x.A \vee B}}{A[x\backslash y] \vee B}\,[\forall E] \qquad \dfrac{\dfrac{}{A[x\backslash y]}\,3}{A[x\backslash y] \to A[x\backslash y]}\,3\ [\to I] \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\neg B}\,2 \quad \dfrac{}{B}\,4}{\bot}\,[\neg E]}{A[x\backslash y]}\,[\bot E]}{B \to A[x\backslash y]}\,4\ [\to I]}{A[x\backslash y]}\,[\vee E]}{\forall x.A}\,[\forall I]$$

- pick y such that it does not occur in A or B
- by L1, because $x \notin \texttt{fv}(B)$ then $B[x\backslash y] = B$

Logical Equivalences

Prove that $(\forall x.A \vee B) \leftrightarrow ((\forall x.A) \vee B)$ if $x \notin \mathtt{fv}(B)$ in Natural Deduction

Logical Equivalences

Prove that $(\forall x.A \lor B) \leftrightarrow ((\forall x.A) \lor B)$ if $x \notin \mathtt{fv}(B)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$
\dfrac{(\forall x.A) \lor B \quad \dfrac{\dfrac{\dfrac{\overline{\forall x.A}^{\,1}}{A[x\backslash y]}\,[\forall E]}{A[x\backslash y] \lor B}\,[\lor I_L]}{(\forall x.A) \to A[x\backslash y] \lor B}\,1\,[\to I] \quad \dfrac{\dfrac{\overline{B}^{\,2}}{A[x\backslash y] \lor B}\,[\lor I_R]}{B \to A[x\backslash y] \lor B}\,2\,[\to I]}{\dfrac{A[x\backslash y] \lor B}{\forall x.A \lor B}\,[\forall I]}\,[\lor E]
$$

- pick y such that it does not occur in A or B
- by L1, because $x \notin \mathtt{fv}(B)$ then $B[x\backslash y] = B$

Logical Equivalences

Prove that $(\exists x.A \wedge B) \leftrightarrow ((\exists x.A) \wedge B)$ if $x \notin \mathtt{fv}(B)$ in Natural Deduction

Logical Equivalences

Prove that $(\exists x.A \wedge B) \leftrightarrow ((\exists x.A) \wedge B)$ if $x \notin \texttt{fv}(B)$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\exists x.A \wedge B \qquad \dfrac{\dfrac{\dfrac{\overline{A[x\backslash y] \wedge B}^{\,1}}{A[x\backslash y]}\,[\wedge E_L]}{\exists x.A}\,[\exists I] \qquad \dfrac{\overline{A[x\backslash y] \wedge B}^{\,1}}{B}\,[\wedge E_R]}{(\exists x.A) \wedge B}\,[\wedge I]}{(\exists x.A) \wedge B}\ 1\ [\exists E]$$

- pick y such that it does not occur in A or B
- by L1, because $x \notin \texttt{fv}(B)$ then $B[x\backslash y] = B$

Logical Equivalences

Prove that $(\exists x.A \wedge B) \leftrightarrow ((\exists x.A) \wedge B)$ if $x \notin \texttt{fv}(B)$ in Natural Deduction

Logical Equivalences

Prove that $(\exists x.A \wedge B) \leftrightarrow ((\exists x.A) \wedge B)$ if $x \notin \mathtt{fv}(B)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$
\dfrac{
\dfrac{(\exists x.A) \wedge B}{\exists x.A}\;[\wedge E_L]
\qquad
\dfrac{
\dfrac{
\dfrac{}{A[x\backslash y]}\;1
\quad
\dfrac{(\exists x.A) \wedge B}{B}\;[\wedge E_R]
}{A[x\backslash y] \wedge B}\;[\wedge I]
}{\exists x.A \wedge B}\;[\exists I]
}{\exists x.A \wedge B}\;1\;[\exists E]
$$

- pick y such that it does not occur in A or B
- by L1, because $x \notin \mathtt{fv}(B)$ then $B[x\backslash y] = B$

Logical Equivalences

We will now prove the following using the other equivalences:

- $(\forall x.A \rightarrow B) \leftrightarrow ((\exists x.A) \rightarrow B)$ if $x \notin \mathtt{fv}(B)$
- $(\exists x.A \rightarrow B) \leftrightarrow ((\forall x.A) \rightarrow B)$ if $x \notin \mathtt{fv}(B)$

Logical Equivalences

We will now prove the following using the other equivalences:

- $(\forall x.A \to B) \leftrightarrow ((\exists x.A) \to B)$ if $x \notin \texttt{fv}(B)$
- $(\exists x.A \to B) \leftrightarrow ((\forall x.A) \to B)$ if $x \notin \texttt{fv}(B)$

Prove that $(\forall x.A \to B) \leftrightarrow ((\exists x.A) \to B)$ if $x \notin \texttt{fv}(B)$ using the other equivalences

Logical Equivalences

We will now prove the following using the other equivalences:

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Prove that $(\forall x.A \rightarrow B) \leftrightarrow ((\exists x.A) \rightarrow B)$ if $x \notin \mathtt{fv}(B)$ using the other equivalences

- $\forall x.A \rightarrow B$

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Prove that $(\forall x.A \to B) \leftrightarrow ((\exists x.A) \to B)$ if $x \notin \mathtt{fv}(B)$ using the other equivalences

- $\forall x.A \to B$
- $\leftrightarrow \forall x.\neg A \vee B$ – using implication elimination

Logical Equivalences

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Prove that $(\forall x.A \rightarrow B) \leftrightarrow ((\exists x.A) \rightarrow B)$ if $x \notin \mathtt{fv}(B)$ using the other equivalences

- $\forall x.A \rightarrow B$
- $\leftrightarrow \forall x.\neg A \vee B$ – using implication elimination
- $\leftrightarrow (\forall x.\neg A) \vee B$ – using $(\forall x.A \vee B) \leftrightarrow ((\forall x.A) \vee B)$ if $x \notin \mathtt{fv}(B)$

Logical Equivalences

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Prove that $(\forall x.A \rightarrow B) \leftrightarrow ((\exists x.A) \rightarrow B)$ if $x \notin \texttt{fv}(B)$ using the other equivalences

- $\forall x.A \rightarrow B$
- $\leftrightarrow \forall x.\neg A \vee B$ – using implication elimination
- $\leftrightarrow (\forall x.\neg A) \vee B$ – using $(\forall x.A \vee B) \leftrightarrow ((\forall x.A) \vee B)$ if $x \notin \texttt{fv}(B)$
- $\leftrightarrow (\neg\exists x.A) \vee B$ – using $(\neg\exists x.A) \leftrightarrow (\forall x.\neg A)$

Logical Equivalences

We will now prove the following using the other equivalences:

- $(\forall x.A \rightarrow B) \leftrightarrow ((\exists x.A) \rightarrow B)$ if $x \notin \texttt{fv}(B)$
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Prove that $(\forall x.A \rightarrow B) \leftrightarrow ((\exists x.A) \rightarrow B)$ if $x \notin \texttt{fv}(B)$ using the other equivalences

- $\forall x.A \rightarrow B$
- $\leftrightarrow \forall x.\neg A \vee B$ – using implication elimination
- $\leftrightarrow (\forall x.\neg A) \vee B$ – using $(\forall x.A \vee B) \leftrightarrow ((\forall x.A) \vee B)$ if $x \notin \texttt{fv}(B)$
- $\leftrightarrow (\neg\exists x.A) \vee B$ – using $(\neg\exists x.A) \leftrightarrow (\forall x.\neg A)$
- $\leftrightarrow (\exists x.A) \rightarrow B$ – using implication elimination

Logical Equivalences

Prove that $(\exists x.A \rightarrow B) \leftrightarrow ((\forall x.A) \rightarrow B)$ if $x \notin \mathtt{fv}(B)$ using the other equivalences

Logical Equivalences

Prove that $(\exists x.A \to B) \leftrightarrow ((\forall x.A) \to B)$ if $x \notin \mathtt{fv}(B)$ using the other equivalences

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Logical Equivalences

Prove that $(\exists x.A \to B) \leftrightarrow ((\forall x.A) \to B)$ if $x \notin \texttt{fv}(B)$ using the other equivalences

- $\exists x.A \to B$
- $\leftrightarrow \exists x.\neg A \vee B$ – using implication elimination

Logical Equivalences

Prove that $(\exists x.A \rightarrow B) \leftrightarrow ((\forall x.A) \rightarrow B)$ if $x \notin \mathtt{fv}(B)$ using the other equivalences

- $\exists x.A \rightarrow B$
- $\leftrightarrow \exists x.\neg A \vee B$ – using implication elimination
- $\leftrightarrow (\exists x.\neg A) \vee (\exists x.B)$ – using $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$

Logical Equivalences

Prove that $(\exists x.A \to B) \leftrightarrow ((\forall x.A) \to B)$ if $x \notin \texttt{fv}(B)$ using the other equivalences

- $\exists x.A \to B$
- $\leftrightarrow \exists x.\neg A \vee B$ – using implication elimination
- $\leftrightarrow (\exists x.\neg A) \vee (\exists x.B)$ – using $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$
- $\leftrightarrow (\exists x.\neg A) \vee B$ – using $(\exists x.A) \leftrightarrow A$ if $x \notin \texttt{fv}(A)$

Logical Equivalences

Prove that $(\exists x.A \rightarrow B) \leftrightarrow ((\forall x.A) \rightarrow B)$ if $x \notin \mathtt{fv}(B)$ using the other equivalences

- $\exists x.A \rightarrow B$
- $\leftrightarrow \exists x.\neg A \vee B$ – using implication elimination
- $\leftrightarrow (\exists x.\neg A) \vee (\exists x.B)$ – using $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$
- $\leftrightarrow (\exists x.\neg A) \vee B$ – using $(\exists x.A) \leftrightarrow A$ if $x \notin \mathtt{fv}(A)$
- $\leftrightarrow (\neg\forall x.A) \vee B$ – using $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$

Logical Equivalences

Prove that $(\exists x.A \to B) \leftrightarrow ((\forall x.A) \to B)$ if $x \notin \mathtt{fv}(B)$ using the other equivalences

- $\exists x.A \to B$
- $\leftrightarrow \exists x.\neg A \vee B$ – using implication elimination
- $\leftrightarrow (\exists x.\neg A) \vee (\exists x.B)$ – using $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$
- $\leftrightarrow (\exists x.\neg A) \vee B$ – using $(\exists x.A) \leftrightarrow A$ if $x \notin \mathtt{fv}(A)$
- $\leftrightarrow (\neg \forall x.A) \vee B$ – using $(\neg \forall x.A) \leftrightarrow (\exists x.\neg A)$
- $\leftrightarrow (\forall x.A) \to B$ – using implication elimination

Logical Equivalences

We will now prove the following using semantics:

- $(\forall x.A \to B) \leftrightarrow (A \to \forall x.B)$ if $x \notin \mathtt{fv}(A)$
- $(\exists x.A \to B) \leftrightarrow (A \to \exists x.B)$ if $x \notin \mathtt{fv}(A)$

Logical Equivalences

We will now prove the following using semantics:

- $(\forall x.A \to B) \leftrightarrow (A \to \forall x.B)$ if $x \notin \texttt{fv}(A)$
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We will use following result:

Logical Equivalences

We will now prove the following using semantics:

- $(\forall x.A \to B) \leftrightarrow (A \to \forall x.B)$ if $x \notin \mathtt{fv}(A)$
- $(\exists x.A \to B) \leftrightarrow (A \to \exists x.B)$ if $x \notin \mathtt{fv}(A)$

We will use following result:

Lemma (L2): if $x \notin \mathtt{fv}(A)$, then $\models_{M,v,x\mapsto d} A$ iff $\models_{M,v} A$

Logical Equivalences

Prove $(\forall x.A \to B) \leftrightarrow (A \to \forall x.B)$ if $x \notin \mathtt{fv}(A)$ using the semantics method

Logical Equivalences

Prove $(\forall x.A \to B) \leftrightarrow (A \to \forall x.B)$ if $x \notin \mathtt{fv}(A)$ using the semantics method

Assume $x \notin \mathtt{fv}(A)$, M is a model with domain D and v a valuation

Logical Equivalences

Prove $(\forall x.A \to B) \leftrightarrow (A \to \forall x.B)$ if $x \notin \mathtt{fv}(A)$ using the semantics method

Assume $x \notin \mathtt{fv}(A)$, M is a model with domain D and v a valuation

Left-to-right implication:

Logical Equivalences

Prove $(\forall x.A \to B) \leftrightarrow (A \to \forall x.B)$ if $x \notin \texttt{fv}(A)$ using the semantics method

Assume $x \notin \texttt{fv}(A)$, M is a model with domain D and v a valuation

Left-to-right implication:

- if $\models_{M,v} \forall x.A \to B$ then $\models_{M,v} A \to \forall x.B$

Logical Equivalences

Prove $(\forall x.A \rightarrow B) \leftrightarrow (A \rightarrow \forall x.B)$ if $x \notin \mathtt{fv}(A)$ using the semantics method

Assume $x \notin \mathtt{fv}(A)$, M is a model with domain D and v a valuation

Left-to-right implication:

- if $\models_{M,v} \forall x.A \rightarrow B$ then $\models_{M,v} A \rightarrow \forall x.B$
  - to prove: $\models_{M,v} A \rightarrow \forall x.B$, i.e., $\models_{M,v} \forall x.B$ whenever $\models_{M,v} A$

Logical Equivalences

Prove $(\forall x.A \to B) \leftrightarrow (A \to \forall x.B)$ if $x \notin \mathtt{fv}(A)$ using the semantics method

Assume $x \notin \mathtt{fv}(A)$, M is a model with domain D and v a valuation

Left-to-right implication:

- if $\models_{M,v} \forall x.A \to B$ then $\models_{M,v} A \to \forall x.B$
  - to prove: $\models_{M,v} A \to \forall x.B$, i.e., $\models_{M,v} \forall x.B$ whenever $\models_{M,v} A$
  - assume $\models_{M,v} A$ and prove $\models_{M,v} \forall x.B$, i.e., for all $d \in D$, $\models_{M,v,x \mapsto d} B$

Logical Equivalences

Prove $(\forall x.A \to B) \leftrightarrow (A \to \forall x.B)$ if $x \notin \mathtt{fv}(A)$ using the semantics method

Assume $x \notin \mathtt{fv}(A)$, M is a model with domain D and v a valuation

Left-to-right implication:

- if $\models_{M,v} \forall x.A \to B$ then $\models_{M,v} A \to \forall x.B$
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  - assumption: $\models_{M,v} \forall x.A \to B$, i.e., for all $e \in D$, $\models_{M,v,x\mapsto e} B$ whenever $\models_{M,v,x\mapsto e} A$

Logical Equivalences

Prove $(\forall x.A \to B) \leftrightarrow (A \to \forall x.B)$ if $x \notin \mathtt{fv}(A)$ using the semantics method

Assume $x \notin \mathtt{fv}(A)$, M is a model with domain D and v a valuation

Left-to-right implication:

- if $\models_{M,v} \forall x.A \to B$ then $\models_{M,v} A \to \forall x.B$
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  - assume $\models_{M,v} A$ and prove $\models_{M,v} \forall x.B$, i.e., for all $d \in D$, $\models_{M,v,x\mapsto d} B$
  - assumption: $\models_{M,v} \forall x.A \to B$, i.e., for all $e \in D$, $\models_{M,v,x\mapsto e} B$ whenever $\models_{M,v,x\mapsto e} A$
  - because $\models_{M,v} A$ by L2, $\models_{M,v,x\mapsto d} A$

Logical Equivalences

Prove $(\forall x.A \to B) \leftrightarrow (A \to \forall x.B)$ if $x \notin \mathtt{fv}(A)$ using the semantics method

Assume $x \notin \mathtt{fv}(A)$, M is a model with domain D and v a valuation

Left-to-right implication:

- if $\models_{M,v} \forall x.A \to B$ then $\models_{M,v} A \to \forall x.B$
  - to prove: $\models_{M,v} A \to \forall x.B$, i.e., $\models_{M,v} \forall x.B$ whenever $\models_{M,v} A$
  - assume $\models_{M,v} A$ and prove $\models_{M,v} \forall x.B$, i.e., for all $d \in D$, $\models_{M,v,x\mapsto d} B$
  - assumption: $\models_{M,v} \forall x.A \to B$, i.e., for all $e \in D$, $\models_{M,v,x\mapsto e} B$ whenever $\models_{M,v,x\mapsto e} A$
  - because $\models_{M,v} A$ by L2, $\models_{M,v,x\mapsto d} A$
  - instantiating this assumption with d gives us: $\models_{M,v,x\mapsto d} B$ whenever $\models_{M,v,x\mapsto d} A$

Logical Equivalences

Prove $(\forall x.A \rightarrow B) \leftrightarrow (A \rightarrow \forall x.B)$ if $x \notin \mathtt{fv}(A)$ using the semantics method

Assume $x \notin \mathtt{fv}(A)$, M is a model with domain D and v a valuation

Left-to-right implication:

- if $\models_{M,v} \forall x.A \rightarrow B$ then $\models_{M,v} A \rightarrow \forall x.B$
  - to prove: $\models_{M,v} A \rightarrow \forall x.B$, i.e., $\models_{M,v} \forall x.B$ whenever $\models_{M,v} A$
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  - instantiating this assumption with d gives us: $\models_{M,v,x\mapsto d} B$ whenever $\models_{M,v,x\mapsto d} A$
  - therefore, because $\models_{M,v,x\mapsto d} A$ is true, $\models_{M,v,x\mapsto d} B$ is also true

Logical Equivalences

Right-to-left implication:

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Logical Equivalences

Right-to-left implication:

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  - assume $d \in D$ and $\vDash_{M,v,x\mapsto d} A$, and prove $\vDash_{M,v,x\mapsto d} B$
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  - because $\models_{M,v} A$, we can assume $\models_{M,v} \forall x.B$, i.e., for all $e \in D$, $\models_{M,v,x\mapsto e} B$
  - instantiating this assumption using d, we get to assume $\models_{M,v,x\mapsto d} B$, which is what we wanted to prove

Conclusion

What did we cover today?

- Equivalence using Natural Deduction
- Rewriting using “known” equivalences
- Equivalences using semantics

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Next time?

- Theorem Proving

Mathematical and Logical Foundations of Computer Science

Predicate Logic (Equivalences)

Vincent Rahli

(some slides were adapted from Rajesh Chitnis' slides)

University of Birmingham

Where are we?

- Symbolic logic
- Propositional logic
- **Predicate logic**

Today

Equivalences:

- in Natural Deduction
- using semantics

Today

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- in Natural Deduction
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Recap: Syntax

The syntax of predicate logic is defined by the following grammar:

$$\begin{array}{lcl} t & ::= & x \mid f(t, \ldots, t) \\ P & ::= & p(t, \ldots, t) \mid \neg P \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid \forall x.P \mid \exists x.P \end{array}$$

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where:

- x ranges over variables
- f ranges over function symbols
- $f(t_1, \ldots, t_n)$ is a well-formed term only if f has arity n
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The pair of a collection of function symbols, and a collection of predicate symbols, along with their arities, is called a **signature**.

The scope of a quantifier extends as far right as possible. E.g., $P \wedge \forall x.p(x) \vee q(x)$ is read as $P \wedge \forall x.(p(x) \vee q(x))$

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The additional **conditions** ensure that **free variables do not get captured**.

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$$

The additional **conditions** ensure that **free variables do not get captured**.

These conditions can always be met by silently renaming bound variables before substituting.

Recap: ∀ & ∃ elimination and introduction rules

Natural Deduction rules for quantifiers:

$$\frac{P[x\backslash y]}{\forall x.P}\ [\forall I] \qquad \frac{\forall x.P}{P[x\backslash t]}\ [\forall E] \qquad \frac{P[x\backslash t]}{\exists x.P}\ [\exists I] \qquad \frac{\exists x.P \qquad \begin{array}{c} \overline{P[x\backslash y]}^{\,1} \\ \vdots \\ Q \end{array}}{Q}\ 1\ [\exists E]$$

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Condition:

- for $[\forall I]$: y must not be free in any not-yet-discharged hypothesis or in $\forall x.P$
- for $[\forall E]$: $\texttt{fv}(t)$ must not clash with $\texttt{bv}(P)$
- for $[\exists I]$: $\texttt{fv}(t)$ must not clash with $\texttt{bv}(P)$
- for $[\exists E]$: y must not be free in Q or in not-yet-discharged hypotheses or in $\exists x.P$

Recap: Example of a proof

here is a proof of $(\forall z.p(z)) \to \forall x.p(x) \vee q(x)$.

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Recap: Example of a proof

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here is a proof of $(\forall z.p(z)) \to \forall x.p(x) \vee q(x)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\forall z.p(z)}\,1}{}\qquad\dfrac{}{p(y) \vee q(y)}}{\forall x.p(x) \vee q(x)}\,[\forall I]}{(\forall z.p(z)) \to \forall x.p(x) \vee q(x)}\,1\ [\to I]$$

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Conditions:

- y does not occur free in not-yet-discharged hypotheses or in $\forall x.p(x) \vee q(x)$
- y does not clash with bound variables in $p(z)$

Recap: Models

Models: a model provides the interpretation of all symbols

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Given a **signature** $\langle\langle f_1^{k_1}, \ldots, f_n^{k_n}\rangle, \langle p_1^{j_1}, \ldots, p_m^{j_m}\rangle\rangle$

- of function symbols f_i of arity k_i, for $1 \leqslant i \leqslant n$
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a **model** is a structure $\langle D, \langle \mathcal{F}_{f_1}, \ldots, \mathcal{F}_{f_n}\rangle, \langle \mathcal{R}_{p_1}, \ldots, \mathcal{R}_{p_m}\rangle\rangle$

- of a non-empty domain D
- interpretations $\mathcal{F}_{f_i}$ for function symbols f_i
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Models of predicate logic replace **truth assignments** for propositional logic

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Models of predicate logic replace **truth assignments** for propositional logic

Variable valuations:

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- that map variables to D
- i.e., a mapping of the form $x_1 \mapsto d_1, \ldots, x_n \mapsto d_n$

Recap: Semantics of Predicate Logic

Given a **model** M with domain D and a **variable valuation** v:

- $[\![t]\!]_v^M$ gives meaning to the term t w.r.t. M and v
- $\models_{M,v} P$ gives meaning to the formula P w.r.t. M and v

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Meaning of terms:

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- $[\![f(t_1, \ldots, t_n)]\!]_v^M = \mathcal{F}_f(\langle [\![t_1]\!]_v^M, \ldots, [\![t_n]\!]_v^M \rangle)$

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Meaning of formulas:

- $\models_{M,v} p(t_1, \dots, t_n)$ iff $\langle [\![t_1]\!]_v^M, \dots, [\![t_n]\!]_v^M \rangle \in \mathcal{R}_p$
- $\models_{M,v} \neg P$ iff $\neg \models_{M,v} P$
- $\models_{M,v} P \wedge Q$ iff $\models_{M,v} P$ and $\models_{M,v} Q$
- $\models_{M,v} P \vee Q$ iff $\models_{M,v} P$ or $\models_{M,v} Q$
- $\models_{M,v} P \rightarrow Q$ iff $\models_{M,v} Q$ whenever $\models_{M,v} P$
- $\models_{M,v} \forall x.P$ iff for every $d \in D$ we have $\models_{M,(v,x\mapsto d)} P$
- $\models_{M,v} \exists x.P$ iff there exists a $d \in D$ such that $\models_{M,(v,x\mapsto d)} P$

Recap: Logical equivalences for Propositional Logic

The same equivalences hold as in Propositional Logic:

- De Morgan's law (I): $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$
- De Morgan's law (II): $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$
- Implication elimination: $(A \rightarrow B) \leftrightarrow (\neg A \vee B)$
- Commutativity of $\wedge$: $(A \wedge B) \leftrightarrow (B \wedge A)$
- Commutativity of $\vee$: $(A \vee B) \leftrightarrow (B \vee A)$
- Associativity of $\wedge$: $((A \wedge B) \wedge C) \leftrightarrow (A \wedge (B \wedge C))$
- Associativity of $\vee$: $((A \vee B) \vee C) \leftrightarrow (A \vee (B \vee C))$
- Distributivity of $\wedge$ over $\vee$: $(A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$
- Distributivity of $\vee$ over $\wedge$: $(A \vee (B \wedge C)) \leftrightarrow ((A \vee B) \wedge (A \vee C))$
- Double negation elimination: $(\neg\neg A) \leftrightarrow A$
- Idempotence: $(A \wedge A) \leftrightarrow A$ and $(A \vee A) \leftrightarrow A$

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In addition, the following hold (some hold only classically):

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As before to prove a logical equivalence $A \leftrightarrow B$, we will prove:

- that we can derive B form A
- that we can derive A form B

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Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\dfrac{\dfrac{\overline{\quad\quad\quad\quad}}{\quad\quad\quad\quad}}{\quad\quad} \qquad \dfrac{\dfrac{\overline{\quad\quad\quad\quad}}{\quad\quad\quad\quad}}{\quad\quad}}{(\forall x.A) \wedge (\forall x.B)}$$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

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$$\dfrac{\dfrac{\dfrac{\dfrac{}{}}{}}{\forall x.A} \qquad \dfrac{\dfrac{\dfrac{}{}}{}}{\forall x.B}}{(\forall x.A) \wedge (\forall x.B)}\ [\wedge I]$$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\dfrac{\dfrac{\dfrac{\quad}{\quad}}{A[x\backslash y]}}{\forall x.A}\,[\forall I] \qquad \dfrac{\dfrac{\dfrac{\quad}{\quad}}{\quad}}{\forall x.B}}{(\forall x.A) \wedge (\forall x.B)}\,[\wedge I]$$

- pick y such that it does not occur in A or B
- y must not be free in $\forall x.A \wedge B$ or in $\forall x.A$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y] \wedge B[x\backslash y]}}{A[x\backslash y]}[\wedge E_L]}{\forall x.A}[\forall I] \qquad \dfrac{\dfrac{\dfrac{}{}}{}}{\forall x.B}}{(\forall x.A) \wedge (\forall x.B)}[\wedge I]$$

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Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x.A \wedge B}{A[x\backslash y] \wedge B[x\backslash y]}\,[\forall E]}{A[x\backslash y]}\,[\wedge E_L]}{\forall x.A}\,[\forall I] \qquad \dfrac{\dfrac{\dfrac{}{}}{}}{\forall x.B}}{(\forall x.A) \wedge (\forall x.B)}\,[\wedge I]$$

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Here is a proof of the right-to-left implication (constructive):

$$\forall x.A \wedge B$$

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Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{A[x\backslash y] \wedge B[x\backslash y]}{\forall x.A \wedge B}\ [\forall I]}{}$$

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Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x.A}}{A[x\backslash y]}\,[\forall E] \qquad \dfrac{\dfrac{}{}}{B[x\backslash y]}}{A[x\backslash y] \land B[x\backslash y]}\,[\land I]}{\forall x.A \land B}\,[\forall I]$$

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Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

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$$
\dfrac{\dfrac{\dfrac{\dfrac{(\forall x.A) \wedge (\forall x.B)}{\forall x.A}\,[\wedge E_L]}{A[x\backslash y]}\,[\forall E] \qquad \dfrac{\dfrac{\quad}{\quad}}{B[x\backslash y]}}{A[x\backslash y] \wedge B[x\backslash y]}\,[\wedge I]}{\forall x.A \wedge B}\,[\forall I]
$$

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$$
\dfrac{
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}{A[x\backslash y]}\,[\forall E]
\qquad
\dfrac{
\dfrac{(\forall x.A) \wedge (\forall x.B)}{\forall x.B}\,[\wedge E_R]
}{B[x\backslash y]}\,[\forall E]
}{
\dfrac{A[x\backslash y] \wedge B[x\backslash y]}{\forall x.A \wedge B}\,[\forall I]
}\,[\wedge I]
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Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\dfrac{\dfrac{}{} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{}}{}}{}}{} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{}}{}}{}}{}}{}}{(\exists x.A) \vee (\exists x.B)}$$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\exists x.A \vee B \qquad (\exists x.A) \vee (\exists x.B)}{(\exists x.A) \vee (\exists x.B)}\ 1\ [\exists E]$$

- pick y such that it does not occur in A or B
- 1: $A[x\backslash y] \vee B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \lor B) \leftrightarrow ((\exists x.A) \lor (\exists x.B))$ in Natural Deduction

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$$\dfrac{\exists x.A \lor B \qquad \dfrac{\overline{A[x\backslash y] \lor B[x\backslash y]} \qquad \overline{A[x\backslash y] \to (\exists x.A) \lor (\exists x.B)} \qquad \overline{B[x\backslash y] \to (\exists x.A) \lor (\exists x.B)}}{(\exists x.A) \lor (\exists x.B)}\,[\lor E]}{(\exists x.A) \lor (\exists x.B)}\; 1\ [\exists E]$$

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$$

- pick y such that it does not occur in A or B
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Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\exists x.A \lor B \qquad \dfrac{\dfrac{}{A[x\backslash y] \lor B[x\backslash y]}\;1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y]}}{\exists x.A}\;[\exists I]}{(\exists x.A) \lor (\exists x.B)}\;[\lor I_L]}{A[x\backslash y] \to (\exists x.A) \lor (\exists x.B)}\;2\;[\to I] \qquad \dfrac{\dfrac{\dfrac{}{}}{}}{B[x\backslash y] \to (\exists x.A) \lor (\exists x.B)}}{(\exists x.A) \lor (\exists x.B)}\;[\lor E]}{(\exists x.A) \lor (\exists x.B)}\;1\;[\exists E]$$

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- 2: $A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$
\dfrac{\exists x.A \vee B \qquad \dfrac{\dfrac{}{A[x\backslash y] \vee B[x\backslash y]}\,1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y]}\,2}{\exists x.A}\,[\exists I]}{(\exists x.A) \vee (\exists x.B)}\,[\vee I_L]}{A[x\backslash y] \rightarrow (\exists x.A) \vee (\exists x.B)}\,2\ [\rightarrow I] \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{}}{}}{}}{B[x\backslash y] \rightarrow (\exists x.A) \vee (\exists x.B)}}{(\exists x.A) \vee (\exists x.B)}\,[\vee E]}{(\exists x.A) \vee (\exists x.B)}\,1\ [\exists E]
$$

- pick y such that it does not occur in A or B
- 1: $A[x\backslash y] \vee B[x\backslash y]$
- 2: $A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\exists x.A \vee B \qquad \dfrac{\dfrac{}{A[x\backslash y] \vee B[x\backslash y]}\,1 \quad \dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y]}\,2}{\exists x.A}\,[\exists I]}{(\exists x.A) \vee (\exists x.B)}\,[\vee I_L]}{A[x\backslash y] \rightarrow (\exists x.A) \vee (\exists x.B)}\,2\;[\rightarrow I] \quad \dfrac{\dfrac{\dfrac{\dfrac{}{}}{}}{(\exists x.A) \vee (\exists x.B)}}{B[x\backslash y] \rightarrow (\exists x.A) \vee (\exists x.B)}\,3\;[\rightarrow I]}{(\exists x.A) \vee (\exists x.B)}\,[\vee E]}{(\exists x.A) \vee (\exists x.B)}\,1\;[\exists E]$$

- pick y such that it does not occur in A or B
- 1: $A[x\backslash y] \vee B[x\backslash y]$
- 2: $A[x\backslash y]$
- 3: $B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$
\begin{prooftree}
\AxiomC{$\exists x.A \vee B$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$A[x\backslash y] \vee B[x\backslash y]$}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{$A[x\backslash y]$}
\RightLabel{$[\exists I]$}
\UnaryInfC{$\exists x.A$}
\RightLabel{$[\vee I_L]$}
\UnaryInfC{$(\exists x.A) \vee (\exists x.B)$}
\RightLabel{2 $[\rightarrow I]$}
\UnaryInfC{$A[x\backslash y] \rightarrow (\exists x.A) \vee (\exists x.B)$}
\AxiomC{}
\UnaryInfC{}
\UnaryInfC{$\exists x.B$}
\RightLabel{$[\vee I_R]$}
\UnaryInfC{$(\exists x.A) \vee (\exists x.B)$}
\RightLabel{3 $[\rightarrow I]$}
\UnaryInfC{$B[x\backslash y] \rightarrow (\exists x.A) \vee (\exists x.B)$}
\RightLabel{$[\vee E]$}
\TrinaryInfC{$(\exists x.A) \vee (\exists x.B)$}
\RightLabel{1 $[\exists E]$}
\BinaryInfC{$(\exists x.A) \vee (\exists x.B)$}
\end{prooftree}
$$

- pick y such that it does not occur in A or B
- 1: $A[x\backslash y] \vee B[x\backslash y]$
- 2: $A[x\backslash y]$
- 3: $B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \lor B) \leftrightarrow ((\exists x.A) \lor (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\exists x.A \lor B \qquad \dfrac{\dfrac{}{A[x\backslash y] \lor B[x\backslash y]}\;1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y]}\;2}{\exists x.A}\;[\exists I]}{(\exists x.A) \lor (\exists x.B)}\;[\lor I_L]}{A[x\backslash y] \to (\exists x.A) \lor (\exists x.B)}\;2\ [\to I] \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{B[x\backslash y]}}{\exists x.B}\;[\exists I]}{(\exists x.A) \lor (\exists x.B)}\;[\lor I_R]}{B[x\backslash y] \to (\exists x.A) \lor (\exists x.B)}\;3\ [\to I]}{(\exists x.A) \lor (\exists x.B)}\;[\lor E]}{(\exists x.A) \lor (\exists x.B)}\;1\ [\exists E]$$

- pick y such that it does not occur in A or B
- 1: $A[x\backslash y] \lor B[x\backslash y]$
- 2: $A[x\backslash y]$
- 3: $B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \lor B) \leftrightarrow ((\exists x.A) \lor (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\exists x.A \lor B \qquad \dfrac{\dfrac{}{A[x\backslash y] \lor B[x\backslash y]}\,1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y]}\,2}{\exists x.A}\,[\exists I]}{(\exists x.A) \lor (\exists x.B)}\,[\lor I_L]}{A[x\backslash y] \rightarrow (\exists x.A) \lor (\exists x.B)}\,2\ [\rightarrow I] \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{B[x\backslash y]}\,3}{\exists x.B}\,[\exists I]}{(\exists x.A) \lor (\exists x.B)}\,[\lor I_R]}{B[x\backslash y] \rightarrow (\exists x.A) \lor (\exists x.B)}\,3\ [\rightarrow I]}{(\exists x.A) \lor (\exists x.B)}\,[\lor E]}{(\exists x.A) \lor (\exists x.B)}\,1\ [\exists E]$$

- pick y such that it does not occur in A or B
- 1: $A[x\backslash y] \lor B[x\backslash y]$
- 2: $A[x\backslash y]$
- 3: $B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\exists x.A \vee B$$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{(\exists x.A) \vee (\exists x.B) \qquad \exists x.A \to \exists x.A \vee B \qquad \exists x.B \to \exists x.A \vee B}{\exists x.A \vee B}\ [\vee E]$$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\cfrac{(\exists x.A) \vee (\exists x.B) \qquad \cfrac{\exists x.A \vee B}{\exists x.A \to \exists x.A \vee B}\ 1\ [\to I] \qquad \cfrac{}{\exists x.B \to \exists x.A \vee B}}{\exists x.A \vee B}\ [\vee E]$$

- 1: $\exists x.A$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \lor B) \leftrightarrow ((\exists x.A) \lor (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{(\exists x.A) \lor (\exists x.B) \qquad \dfrac{\dfrac{\exists x.A \qquad \exists x.A \lor B}{\exists x.A \lor B}\ 2\ [\exists E]}{\exists x.A \to \exists x.A \lor B}\ 1\ [\to I] \qquad \exists x.B \to \exists x.A \lor B}{\exists x.A \lor B}\ [\lor E]$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{(\exists x.A) \vee (\exists x.B) \qquad \dfrac{\dfrac{\dfrac{}{\exists x.A}\,1 \qquad \dfrac{}{\exists x.A \vee B}}{\exists x.A \vee B}\;2\ [\exists E]}{\exists x.A \to \exists x.A \vee B}\;1\ [\to I] \qquad \dfrac{}{\exists x.B \to \exists x.A \vee B}}{\exists x.A \vee B}\;[\vee E]$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$

Prove the logical equivalence $(\exists x.A \lor B) \leftrightarrow ((\exists x.A) \lor (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{(\exists x.A) \lor (\exists x.B) \qquad \dfrac{\dfrac{\dfrac{}{\exists x.A}\ 1 \qquad \dfrac{\dfrac{\overline{\qquad}}{A[x\backslash y] \lor B[x\backslash y]}}{\exists x.A \lor B}\ [\exists I]}{\exists x.A \lor B}\ 2\ [\exists E]}{\exists x.A \to \exists x.A \lor B}\ 1\ [\to I] \qquad \dfrac{\dfrac{\dfrac{}{} \qquad \dfrac{\dfrac{\overline{\qquad}}{}}{}}{}}{\exists x.B \to \exists x.A \lor B}}{\exists x.A \lor B}\ [\lor E]$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{(\exists x.A) \vee (\exists x.B) \qquad \dfrac{\dfrac{\dfrac{}{\exists x.A}\,1 \qquad \dfrac{\dfrac{\dfrac{}{A[x\backslash y]}}{A[x\backslash y] \vee B[x\backslash y]}\,[\vee I_L]}{\exists x.A \vee B}\,[\exists I]}{\exists x.A \vee B}\,2\ [\exists E]}{\exists x.A \to \exists x.A \vee B}\,1\ [\to I] \qquad \dfrac{\dfrac{\dfrac{}{} \quad \dfrac{\dfrac{\dfrac{}{}}{}}{}}{}}{\exists x.B \to \exists x.A \vee B}}{\exists x.A \vee B}\,[\vee E]$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{(\exists x.A) \vee (\exists x.B) \qquad \dfrac{\dfrac{\dfrac{}{\exists x.A}\,1 \qquad \dfrac{\dfrac{\dfrac{}{A[x\backslash y]}\,2}{A[x\backslash y] \vee B[x\backslash y]}\,[\vee I_L]}{\exists x.A \vee B}\,[\exists I]}{\exists x.A \vee B}\,2\ [\exists E]}{\exists x.A \to \exists x.A \vee B}\,1\ [\to I] \qquad \dfrac{\dfrac{\dfrac{}{\quad} \qquad \dfrac{\dfrac{\dfrac{}{\quad}}{\quad}}{\quad}}{\quad}}{\exists x.B \to \exists x.A \vee B}}{\exists x.A \vee B}\,[\vee E]$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$
\dfrac{(\exists x.A) \vee (\exists x.B) \qquad \dfrac{\dfrac{\dfrac{}{\exists x.A}\,1 \qquad \dfrac{\dfrac{\dfrac{}{A[x\backslash y]}\,2}{A[x\backslash y] \vee B[x\backslash y]}\,[\vee I_L]}{\exists x.A \vee B}\,[\exists I]}{\exists x.A \vee B}\,2\ [\exists E]}{\exists x.A \rightarrow \exists x.A \vee B}\,1\ [\rightarrow I] \qquad \dfrac{\dfrac{\dfrac{}{\quad} \qquad \dfrac{\dfrac{\dfrac{}{\quad}}{\quad}}{\quad}}{\exists x.A \vee B}}{\exists x.B \rightarrow \exists x.A \vee B}\,3\ [\rightarrow I]}{\exists x.A \vee B}\,[\vee E]
$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$
- 3: $\exists x.B$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$
\begin{prooftree}
\AxiomC{$(\exists x.A) \vee (\exists x.B)$}
\AxiomC{}\RightLabel{1}\UnaryInfC{$\exists x.A$}
\AxiomC{}\RightLabel{2}\UnaryInfC{$A[x\backslash y]$}
\RightLabel{$[\vee I_L]$}\UnaryInfC{$A[x\backslash y] \vee B[x\backslash y]$}
\RightLabel{$[\exists I]$}\UnaryInfC{$\exists x.A \vee B$}
\RightLabel{2 $[\exists E]$}\BinaryInfC{$\exists x.A \vee B$}
\RightLabel{1 $[\to I]$}\UnaryInfC{$\exists x.A \to \exists x.A \vee B$}
\AxiomC{}\UnaryInfC{$\exists x.B$}
\AxiomC{}\UnaryInfC{}\UnaryInfC{}\UnaryInfC{$\exists x.A \vee B$}
\RightLabel{4 $[\exists E]$}\BinaryInfC{$\exists x.A \vee B$}
\RightLabel{3 $[\to I]$}\UnaryInfC{$\exists x.B \to \exists x.A \vee B$}
\RightLabel{$[\vee E]$}\TrinaryInfC{$\exists x.A \vee B$}
\end{prooftree}
$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$
- 3: $\exists x.B$
- 4: $B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$
\dfrac{(\exists x.A) \vee (\exists x.B) \qquad \dfrac{\dfrac{\dfrac{}{\exists x.A}\,1 \qquad \dfrac{\dfrac{\dfrac{}{A[x\backslash y]}\,2}{A[x\backslash y] \vee B[x\backslash y]}\,[\vee I_L]}{\exists x.A \vee B}\,[\exists I]}{\exists x.A \vee B}\,2\ [\exists E]}{\exists x.A \rightarrow \exists x.A \vee B}\,1\ [\rightarrow I] \qquad \dfrac{\dfrac{\dfrac{}{\exists x.B}\,3 \qquad \dfrac{\dfrac{\dfrac{}{\quad}}{\quad}}{\exists x.A \vee B}}{\exists x.A \vee B}\,4\ [\exists E]}{\exists x.B \rightarrow \exists x.A \vee B}\,3\ [\rightarrow I]}{\exists x.A \vee B}\,[\vee E]
$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$
- 3: $\exists x.B$
- 4: $B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$
\dfrac{(\exists x.A) \vee (\exists x.B) \qquad \dfrac{\dfrac{\dfrac{}{\exists x.A}\,{}^{1} \quad \dfrac{\dfrac{\dfrac{}{A[x\backslash y]}\,{}^{2}}{A[x\backslash y] \vee B[x\backslash y]}\,[\vee I_L]}{\exists x.A \vee B}\,[\exists I]}{\exists x.A \vee B}\,2\ [\exists E]}{\exists x.A \to \exists x.A \vee B}\,1\ [\to I] \qquad \dfrac{\dfrac{\dfrac{}{\exists x.B}\,{}^{3} \quad \dfrac{\dfrac{\dfrac{}{}}{A[x\backslash y] \vee B[x\backslash y]}}{\exists x.A \vee B}\,[\exists I]}{\exists x.A \vee B}\,4\ [\exists E]}{\exists x.B \to \exists x.A \vee B}\,3\ [\to I]}{\exists x.A \vee B}\,[\vee E]
$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$
- 3: $\exists x.B$
- 4: $B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \lor B) \leftrightarrow ((\exists x.A) \lor (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$
\dfrac{(\exists x.A) \lor (\exists x.B) \qquad \dfrac{\dfrac{\dfrac{}{\exists x.A}\,{\scriptstyle 1} \qquad \dfrac{\dfrac{\dfrac{}{A[x\backslash y]}\,{\scriptstyle 2}}{A[x\backslash y] \lor B[x\backslash y]}\,[\lor I_L]}{\exists x.A \lor B}\,[\exists I]}{\exists x.A \lor B}\,{\scriptstyle 2}\;[\exists E]}{\exists x.A \to \exists x.A \lor B}\,{\scriptstyle 1}\;[\to I] \qquad \dfrac{\dfrac{\dfrac{}{\exists x.B}\,{\scriptstyle 3} \qquad \dfrac{\dfrac{\dfrac{}{B[x\backslash y]}}{A[x\backslash y] \lor B[x\backslash y]}\,[\lor I_R]}{\exists x.A \lor B}\,[\exists I]}{\exists x.A \lor B}\,{\scriptstyle 4}\;[\exists E]}{\exists x.B \to \exists x.A \lor B}\,{\scriptstyle 3}\;[\to I]}{\exists x.A \lor B}\,[\lor E]
$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$
- 3: $\exists x.B$
- 4: $B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \lor B) \leftrightarrow ((\exists x.A) \lor (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$
\dfrac{(\exists x.A) \lor (\exists x.B) \qquad \dfrac{\dfrac{\overline{\exists x.A}^{\,1} \quad \dfrac{\dfrac{\overline{A[x\backslash y]}^{\,2}}{A[x\backslash y] \lor B[x\backslash y]}\,[\lor I_L]}{\exists x.A \lor B}\,[\exists I]}{\exists x.A \lor B}\,2\ [\exists E]}{\exists x.A \to \exists x.A \lor B}\,1\ [\to I] \qquad \dfrac{\dfrac{\overline{\exists x.B}^{\,3} \quad \dfrac{\dfrac{\overline{B[x\backslash y]}^{\,4}}{A[x\backslash y] \lor B[x\backslash y]}\,[\lor I_R]}{\exists x.A \lor B}\,[\exists I]}{\exists x.A \lor B}\,4\ [\exists E]}{\exists x.B \to \exists x.A \lor B}\,3\ [\to I]}{\exists x.A \lor B}\,[\lor E]
$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$
- 3: $\exists x.B$
- 4: $B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Logical Equivalences

Prove the logical equivalence $(\neg \forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (classical):

$$\exists x.\neg A$$

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (classical):

$$\dfrac{\overline{\neg\neg(\exists x.\neg A)}}{\exists x.\neg A}\ [DNE]$$

- 1: $\neg(\exists x.\neg A)$

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (classical):

$$\dfrac{\dfrac{\perp}{\neg\neg(\exists x.\neg A)}\ 1\ [\neg I]}{\exists x.\neg A}\ [DNE]$$

- 1: $\neg(\exists x.\neg A)$

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (classical):

$$
\begin{prooftree}
\AxiomC{$\neg\forall x.A$}
\AxiomC{}
\UnaryInfC{$\forall x.A$}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{$1\ [\neg I]$}
\UnaryInfC{$\neg\neg(\exists x.\neg A)$}
\RightLabel{$[DNE]$}
\UnaryInfC{$\exists x.\neg A$}
\end{prooftree}
$$

- 1: $\neg(\exists x.\neg A)$

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (classical):

$$\dfrac{\dfrac{\dfrac{\neg\forall x.A \qquad \dfrac{A[x\backslash y]}{\forall x.A}\,[\forall I]}{\bot}\,[\neg E]}{\neg\neg(\exists x.\neg A)}\,1\ [\neg I]}{\exists x.\neg A}\,[DNE]$$

- 1: $\neg(\exists x.\neg A)$
- pick y such that it does not occur in A

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (classical):

$$\dfrac{\dfrac{\neg\forall x.A \qquad \dfrac{\dfrac{\overline{\neg\neg A[x\backslash y]}}{A[x\backslash y]}\,[DNE]}{\forall x.A}\,[\forall I]}{\bot}\,[\neg E]}{\dfrac{\neg\neg(\exists x.\neg A)}{\exists x.\neg A}\,[DNE]}\;1\;[\neg I]$$

- 1: $\neg(\exists x.\neg A)$
- pick y such that it does not occur in A

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (classical):

$$
\begin{prooftree}
\AxiomC{$\neg\forall x.A$}
\AxiomC{}
\UnaryInfC{}
\AxiomC{}
\UnaryInfC{}
\UnaryInfC{}
\BinaryInfC{$\bot$}
\RightLabel{2 $[\neg I]$}
\UnaryInfC{$\neg\neg A[x\backslash y]$}
\RightLabel{$[DNE]$}
\UnaryInfC{$A[x\backslash y]$}
\RightLabel{$[\forall I]$}
\UnaryInfC{$\forall x.A$}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{1 $[\neg I]$}
\UnaryInfC{$\neg\neg(\exists x.\neg A)$}
\RightLabel{$[DNE]$}
\UnaryInfC{$\exists x.\neg A$}
\end{prooftree}
$$

- 1: $\neg(\exists x.\neg A)$
- pick y such that it does not occur in A
- 2: $\neg A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (classical):

$$
\dfrac{\neg\forall x.A \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(\exists x.\neg A)} \qquad \dfrac{\dfrac{}{}}{\exists x.\neg A}}{\bot}[\neg E]}{\neg\neg A[x\backslash y]}\,2\ [\neg I]}{A[x\backslash y]}[DNE]}{\forall x.A}[\forall I]}{\dfrac{\dfrac{\bot}{\neg\neg(\exists x.\neg A)}\,1\ [\neg I]}{\exists x.\neg A}[DNE]}[\neg E]
$$

- 1: $\neg(\exists x.\neg A)$
- pick y such that it does not occur in A
- 2: $\neg A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (classical):

$$
\dfrac{\dfrac{\dfrac{\neg\forall x.A \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overline{\neg(\exists x.\neg A)}^{\,1} \qquad \dfrac{\overline{\qquad\qquad}}{\exists x.\neg A}}{\bot}\,[\neg E]}{\neg\neg A[x\backslash y]}\,2\ [\neg I]}{A[x\backslash y]}\,[DNE]}{\forall x.A}\,[\forall I]}{\bot}\,[\neg E]}{\neg\neg(\exists x.\neg A)}\,1\ [\neg I]}{\exists x.\neg A}\,[DNE]
$$

- 1: $\neg(\exists x.\neg A)$
- pick y such that it does not occur in A
- 2: $\neg A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (classical):

$$
\dfrac{\dfrac{\dfrac{\neg\forall x.A \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(\exists x.\neg A)}\,1 \qquad \dfrac{\dfrac{}{\neg A[x\backslash y]}}{\exists x.\neg A}\,[\exists I]}{\bot}\,[\neg E]}{\neg\neg A[x\backslash y]}\,2\ [\neg I]}{A[x\backslash y]}\,[DNE]}{\forall x.A}\,[\forall I]}{\bot}\,[\neg E]}{\neg\neg(\exists x.\neg A)}\,1\ [\neg I]}{\exists x.\neg A}\,[DNE]
$$

- 1: $\neg(\exists x.\neg A)$
- pick y such that it does not occur in A
- 2: $\neg A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (classical):

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg\forall x.A \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(\exists x.\neg A)}\;1 \qquad \dfrac{\dfrac{}{\neg A[x\backslash y]}\;2}{\exists x.\neg A}\;[\exists I]}{\bot}\;[\neg E]}{\neg\neg A[x\backslash y]}\;2\;[\neg I]}{A[x\backslash y]}\;[DNE]}{\forall x.A}\;[\forall I]}{\bot}\;[\neg E]}{\neg\neg(\exists x.\neg A)}\;1\;[\neg I]}{\exists x.\neg A}\;[DNE]$$

- 1: $\neg(\exists x.\neg A)$
- pick y such that it does not occur in A
- 2: $\neg A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\neg \forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\overline{\neg\forall x.A}$$

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\bot}{\neg\forall x.A}\ 1\ [\neg I]$$

- 1: $\forall x.A$

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{\exists x.\neg A \qquad \dfrac{\dfrac{\quad}{\quad} \qquad \dfrac{\dfrac{\quad}{\quad}}{\quad}}{\bot}}{\bot}\ 2\ [\exists E]}{\neg\forall x.A}\ 1\ [\neg I]$$

- 1: $\forall x.A$
- pick y such that it does not occur in A
- 2: $\neg A[x\backslash y]$

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$$\dfrac{\dfrac{\exists x.\neg A \qquad \dfrac{\overline{\neg A[x\backslash y]}^{\,2} \qquad \dfrac{\overline{\forall x.A}^{\,1}}{A[x\backslash y]}\,[\forall E]}{\bot}\,[\neg E]}{\bot}\,2\ [\exists E]}{\neg\forall x.A}\,1\ [\neg I]$$

- 1: $\forall x.A$
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$$
\begin{prooftree}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\exists x.A$}
\AxiomC{$\forall x.\neg A$}
\RightLabel{$[\forall E]$}
\UnaryInfC{$\neg A[x\backslash y]$}
\AxiomC{}
\UnaryInfC{$A[x\backslash y]$}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{2 $[\exists E]$}
\BinaryInfC{$\bot$}
\RightLabel{1 $[\neg I]$}
\UnaryInfC{$\neg\exists x.A$}
\end{prooftree}
$$

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As before: if ($P \leftrightarrow Q$ or $Q \leftrightarrow P$) and P occurs in A, then replacing P by Q in A leads to a formula B, such that $A \leftrightarrow B$

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Logical Equivalences

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Also,

Semantical equivalence: two formulas P and Q are equivalent if for all models M and valuations v, $\models_{M,v} P$ iff $\models_{M,v} Q$

Logical Equivalences

Example: prove $(\neg \exists x.A) \leftrightarrow (\forall x.\neg A)$

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  - assumption: $\vDash_{M,v} \forall x.\neg A$, i.e., for every $d \in D$ it is not the case that $\vDash_{M,v,x\mapsto d} A$
  - therefore, instantiating this assumption with e: it is not the case that $\vDash_{M,v,x\mapsto e} A$

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  - therefore, instantiating this assumption with e: it is not the case that $\models_{M,v,x\mapsto e} A$
  - contradiction!

Conclusion

What did we cover today?

- Equivalence using Natural Deduction
- Equivalences using semantics

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Further reading:

- Chapter 8 of
  http://leanprover.github.io/logic_and_proof/

Conclusion

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- Equivalence using Natural Deduction
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Next time?

- Predicate Logic – Equivalences

Models

A model in predicate logic is something that gives a meaning to a statement

It contains 3 things:

- a Domain
- The meanings of the functional symbols
- The meanings of the predicate symbols

The functional symbols & predicate symbols are given in the question:

e.g

Exercise Sheet 10b
Predicate Logic – Natural Deduction & Semantics

Consider the following signature:

- Function symbols: `zero` (arity 0); `succ` (arity 1)
- Predicate symbols: $<$ (arity 2); $\leq$ (arity 2)

We will use infix notation for the binary symbols $<$ and $\leq$. For simplicity we write 0 for `zero`, 1 for `succ(zero)`, 2 for `succ(succ(zero))`, etc. Consider the following formulas that capture properties of the above symbols:

- let S_1 be $\forall x.\exists y.x < y$
- let S_2 be $\forall x.\forall y.x < y \rightarrow \texttt{succ}(x) \leq y$
- let S_3 be $\exists x.1 \leq x$

1. Provide a constructive Natural Deduction proof of $(S_1) \rightarrow \neg\exists x.\forall y.\neg x < y$
2. Provide a Constructive Natural Deduction proof of $(S_1) \rightarrow (S_2) \rightarrow S_3$
3. Provide a model M_1 such that $\models_{M_1} S_1$
4. Provide a model M_2 such that $\models_{M_2} \neg S_1$

We give a model with respect to some predicate logic formula

e.g Q3 asks for a model $\mathcal{M}_1$ such that $\models_{\mathcal{M}_1} (\forall x. \exists y. x<y)$ **(Q3)**

so we are giving a model for $(\forall x. \exists y. x<y)$

Again, 3 parts:

- a Domain
- The meanings of the functional symbols
- The meanings of the predicate symbols

$$\langle \mathbb{N}, \underbrace{\langle 0, +1 \rangle}_{\text{Meanings of the functional symbols}}, \langle \underbrace{\{\langle a,b\rangle \mid a<b\}, \{\langle a,b\rangle \mid a \leq b\}}_{\text{Meanings of the predicate symbols}} \rangle \rangle$$

The domain we are giving is $\mathbb{N}$, the natural numbers.

You are free to choose the Domain (e.g. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{B}$, etc.)

Booleans

It's important to note the meanings must be given in the same order as the question:

$\langle 0, +1 \rangle$ refers to 0 & succ

$\{\langle a,b \rangle \mid a < b\}$ refers to $<$

$\{\langle a,b \rangle \mid a \leq b\}$ refers to $\leq$

What the model does is give an actual meaning to 0, succ, $<$ and $\leq$

$$\langle \mathbb{N}, \langle 0, +1 \rangle, \langle \{\langle x,y \rangle \mid x < y\}, \{\langle x,y \rangle \mid x \leq y\} \rangle \rangle$$

This model for $\forall x. \exists y. x < y$ says that for all x, there exists a y such that $x < y$.

Here that means that there exists a pair $\langle x,y \rangle \in \{\langle a,b \rangle \mid a < b\}$ as that is the meaning we assigned to $<$

We can test with $x=0$ & $y=1$

is $\langle 0,1\rangle$ a member of $\{\langle a,b\rangle \mid a<b\}$

i.e. $\langle 0,1\rangle \in \{\langle a,b\rangle \mid a<b\}$

This is true

$\therefore$ the model holds

For Q4:

We are asked to give a model such that $\models_{M_2} \neg(\forall x.\exists y. x<y)$

so the part inside the brackets must be False

so:

$\langle \mathbb{N}, \langle 0, +1\rangle, \langle \emptyset, \{\langle x,y\rangle \mid x\leq y\}\rangle\rangle$

This is saying **not** for all x, there exists a y such that $x<y$.

Here that means that there exists a pair $\langle x,y \rangle \in \emptyset$ - the Empty set

So now the part inside the brackets evaluates to False & the $\neg$ outside changes it to a True

Mathematical and Logical Foundations of Computer Science

Predicate Logic (Semantics)

Vincent Rahli

(some slides were adapted from Rajesh Chitnis' slides)

University of Birmingham

Where are we?

- Symbolic logic
- Propositional logic
- **Predicate logic**

Today

- Semantics of Predicate Logic
- Models
- Variable valuations
- Satisfiability & validity

Further reading:

- Chapter 10 of
  http://leanprover.github.io/logic_and_proof/

Recap: Syntax

The syntax of predicate logic is defined by the following grammar:

$$
\begin{array}{lcl}
t & ::= & x \mid f(t, \ldots, t) \\
P & ::= & p(t, \ldots, t) \mid \neg P \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid \forall x.P \mid \exists x.P
\end{array}
$$

where:

- x ranges of variables
- f ranges over function symbols
- $f(t_1, \ldots, t_n)$ is a well-formed term only if f has arity n
- p ranges over predicate symbols
- $p(t_1, \ldots, t_n)$ is a well-formed formula only if p has arity n

The pair of a collection of function symbols, and a collection of predicate symbols, along with their arities, is called a **signature**.

The scope of a quantifier extends as far right as possible. E.g., $P \wedge \forall x.p(x) \vee q(x)$ is read as $P \wedge \forall x.(p(x) \vee q(x))$

Recap: Substitution

Substitution is defined recursively on terms and formulas:
$P[x\backslash t]$ substitute all the free occurrences of x in P with t.

$$
\begin{array}{lcl}
x[x\backslash t] & = & t \\
x[y\backslash t] & = & x \\
(f(t_1,\ldots,t_n))[x\backslash t] & = & f(t_1[x\backslash t],\ldots,t_n[x\backslash t]) \\
(p(t_1,\ldots,t_n))[x\backslash t] & = & p(t_1[x\backslash t],\ldots,t_n[x\backslash t]) \\
\hline
(\neg P)[x\backslash t] & = & \neg P[x\backslash t] \\
(P_1 \wedge P_2)[x\backslash t] & = & P_1[x\backslash t] \wedge P_2[x\backslash t] \\
(P_1 \vee P_2)[x\backslash t] & = & P_1[x\backslash t] \vee P_2[x\backslash t] \\
(P_1 \rightarrow P_2)[x\backslash t] & = & P_1[x\backslash t] \rightarrow P_2[x\backslash t] \\
\hline
(\forall x.P)[x\backslash t] & = & \forall x.P \\
(\exists x.P)[x\backslash t] & = & \exists x.P \\
(\forall y.P)[x\backslash t] & = & \forall y.P[x\backslash t], \text{ if } y \notin \mathtt{fv}(t) \\
(\exists y.P)[x\backslash t] & = & \exists y.P[x\backslash t], \text{ if } y \notin \mathtt{fv}(t)
\end{array}
$$

The additional **conditions** ensure that **free variables do not get captured**.

These conditions can always be met by silently renaming bound variables before substituting.

Recap: $\forall$ & $\exists$ elimination and introduction rules

Natural Deduction rules for quantifiers:

$$\dfrac{P[x\backslash y]}{\forall x.P}\ [\forall I] \qquad \dfrac{\forall x.P}{P[x\backslash t]}\ [\forall E] \qquad \dfrac{P[x\backslash t]}{\exists x.P}\ [\exists I] \qquad \dfrac{\exists x.P \qquad \begin{array}{c}\overline{P[x\backslash y]}^{\,1}\\ \vdots \\ Q\end{array}}{Q}\ 1\ [\exists E]$$

Condition:

- for $[\forall I]$: y must not be free in any not-yet-discharged hypothesis or in $\forall x.P$
- for $[\forall E]$: $\texttt{fv}(t)$ must not clash with $\texttt{bv}(P)$
- for $[\exists I]$: $\texttt{fv}(t)$ must not clash with $\texttt{bv}(P)$
- for $[\exists E]$: y must not be free in Q or in not-yet-discharged hypotheses or in $\exists x.P$

Recap: Example of a simple proof

here is a proof of $(\forall z.p(z)) \to \forall x.p(x) \vee q(x)$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overline{\forall z.p(z)}\ {}^{1}}{p(y)}\ [\forall E]}{p(y) \vee q(y)}\ [\vee I_L]}{\forall x.p(x) \vee q(x)}\ [\forall I]}{(\forall z.p(z)) \to \forall x.p(x) \vee q(x)}\ 1\ [\to I]
$$

Conditions:

- y does not occur free in not-yet-discharged hypotheses or in $\forall x.p(x) \vee q(x)$
- y does not clash with bound variables in $p(z)$

Interpretation of Predicate & Function Symbols

Semantics: Assigning meaning/interpretations to formulas

Earlier in the module: a **particular semantics** for propositional logic

- Each proposition has a meaning (**a truth value**) of **T** or **F**
- Used truth tables to check **semantic validity**

We now **extend** this particular semantics to predicate logic

- Propositional logic constructs are interpreted similarly
- In addition, we need to interpret
  - **predicate & function symbols**
  - **quantifiers**

Predicate symbols: for example, given the domain $\mathbb{N}$ and a unary predicate symbol `even`, what is the meaning of `even`?

- to state that a number is $0, 2, 4, \ldots$?
- is it always obvious?
- what if we had a predicate symbol `small`?
- what does that mean?

Interpretation of Predicate & Function Symbols

Given a domain D and a predicate symbol p of arity n

- p is interpreted by a n-ary relation $\mathcal{R}_p$
- of the form $\{\langle d_1^1, \ldots, d_n^1\rangle, \langle d_1^2, \ldots, d_n^2\rangle, \ldots\}$
- where each d_j^i is in D
- we write: $\mathcal{R}_p \in 2^{D^n}$ or $\mathcal{R}_p \subseteq D^n$

For example

- a meaningful interpretation for `even` would be
  - $\{\langle 0\rangle, \langle 2\rangle, \langle 4\rangle, \ldots\}$
- a meaningful interpretation for `odd` would be
  - $\{\langle 1\rangle, \langle 3\rangle, \langle 5\rangle, \ldots\}$
- a meaningful interpretation for `prime` would be
  - $\{\langle 2\rangle, \langle 3\rangle, \langle 5\rangle, \ldots\}$

Interpretation of Predicate & Function Symbols

Function symbols: for example, given the domain $\mathbb{N}$ and a binary function symbol `add`, what is the meaning of `add`?

- is it addition?
- is it always obvious?
- what if we had a binary function symbol `combine`?
- what does that mean?

Given a domain D and a function symbol f of arity n

- f is interpreted by a function $\mathcal{F}_f$ from D^n to D
- we write: $\mathcal{F}_f \in D^n \to D$

For example

- a meaningful interpretation for `add` would be
  - $+$ (formally: $\langle n, m \rangle \mapsto n + m$)
- a meaningful interpretation for `mult` would be
  - $\times$ (formally: $\langle n, m \rangle \mapsto n \times m$)

Interpretation of Predicate & Function Symbols

WARNING ⚠: sometimes for convenience we will use the same symbol for a function symbol and its interpretation

For example:

1. we have used 0 in our examples as a **constant symbol**, which has no meaning on its own
2. this constant symbol would be interpreted by the natural number 0, which is an **object of the domain** $\mathbb{N}$

Even though we used the same symbols, these symbols stand for different entities:

1. a **constant symbol**
2. an **object of the domain**

If we want to distinguish them, we might use:

1. $\overline{0}$ or `zero` for the **constant symbol**
2. 0 for the **object of the domain**

Models

Models: a model provides the interpretation of all symbols

Given a **signature** $\langle\langle f_1^{k_1}, \ldots, f_n^{k_n}\rangle, \langle p_1^{j_1}, \ldots, p_m^{j_m}\rangle\rangle$

- of function symbols f_i of arity k_i, for $1 \leqslant i \leqslant n$
- of predicate symbols p_i of arity j_i, for $1 \leqslant i \leqslant m$

a **model** is a structure $\langle D, \langle \mathcal{F}_{f_1}, \ldots, \mathcal{F}_{f_n}\rangle, \langle \mathcal{R}_{p_1}, \ldots, \mathcal{R}_{p_m}\rangle\rangle$

- of a non-empty domain D
- interpretations $\mathcal{F}_{f_i}$ for function symbols f_i ($\in D^{k_i} \rightarrow D$)
- interpretations $\mathcal{R}_{p_i}$ for predicate symbols p_i ($\subseteq D^{j_i}$)

Models of predicate logic replace **truth assignments** for propositional logic

For example:

- we might interpret the signature $\langle\langle \mathtt{add}\rangle, \langle \mathtt{even}\rangle\rangle$
  - where `add` is a binary function symbol
  - and `even` is a unary predicate symbol
- by the model $\langle \mathbb{N}, \langle\langle + \rangle, \langle \{\langle 0\rangle, \langle 2\rangle, \langle 4\rangle, \ldots\}\rangle\rangle\rangle$

Models

A **model** assigns meaning to function and predicate symbols

Variable valuations: In addition, we need to assign meaning to **variables**:

- this is done using a partial function v
- that maps variables to D
- i.e., a mapping of the form $x_1 \mapsto d_1, \ldots, x_n \mapsto d_n$
- which maps each x_i to d_i, i.e., to $v(x_i)$
- $\mathtt{dom}(v) = \{x_1, \ldots, x_n\}$
- let $\cdot$ be the empty mapping
- we write $v, x \mapsto d$ for the mapping that
  - maps x to d
  - and maps each $y \in \mathtt{dom}(v)$ such that $x \neq y$ to $v(y)$

For example

- $(x_1 \mapsto d_1), x_2 \mapsto d_2$ maps x_1 to ?d_1 and x_2 to ?d_2
- $(x_1 \mapsto d_1, x_2 \mapsto d_2), x_1 \mapsto d_3$ maps x_1 to ?d_3 and x_2 to ?d_2

Semantics of Predicate Logic

Given a **model** M with domain D and a **variable valuation** v, to assign **meaning** to Predicate Logic formulas, we define two operations:

- $[\![t]\!]_v^M$, which gives meaning to the term t w.r.t. M and v
- $\models_{M,v} P$, which gives meaning to the formula P w.r.t. M and v

Meaning of terms:

- $[\![x]\!]_v^M = v(x)$
- $[\![f(t_1, \ldots, t_n)]\!]_v^M = \mathcal{F}_f(\langle [\![t_1]\!]_v^M, \ldots, [\![t_n]\!]_v^M \rangle)$

Semantics of Predicate Logic

Given a **model** M with domain D and a **variable valuation** v, to assign **meaning** to Predicate Logic formulas, we define two operations:

- $[\![t]\!]_v^M$, which gives meaning to the term t w.r.t. M and v
- $\models_{M,v} P$, which gives meaning to the formula P w.r.t. M and v

Meaning of formulas:

- $\models_{M,v} p(t_1, \ldots, t_n)$ iff $\langle [\![t_1]\!]_v^M, \ldots, [\![t_n]\!]_v^M \rangle \in \mathcal{R}_p$
- $\models_{M,v} \neg P$ iff $\neg \models_{M,v} P$
- $\models_{M,v} P \wedge Q$ iff $\models_{M,v} P$ and $\models_{M,v} Q$
- $\models_{M,v} P \vee Q$ iff $\models_{M,v} P$ or $\models_{M,v} Q$
- $\models_{M,v} P \rightarrow Q$ iff $\models_{M,v} Q$ whenever $\models_{M,v} P$
- $\models_{M,v} \forall x.P$ iff for every $d \in D$ we have $\models_{M,(v,x \mapsto d)} P$
- $\models_{M,v} \exists x.P$ iff there exists a $d \in D$ such that $\models_{M,(v,x \mapsto d)} P$

Semantics of Predicate Logic

For example:

- consider the signature $\langle\langle\texttt{zero}, \texttt{succ}, \texttt{add}\rangle, \langle\texttt{even}, \texttt{odd}\rangle\rangle$
- the model M: $\langle\mathbb{N}, \langle 0, +1, +\rangle, \langle\{\langle 0\rangle, \langle 2\rangle, \langle 4\rangle, \ldots\}, \{\langle 1\rangle, \langle 3\rangle, \langle 5\rangle, \ldots\}\rangle\rangle$
- we write $+1$ for the function that given a number increments it by 1
- $+(n, m)$ stands for $n + m$

What is $\models_{M,\cdot} \texttt{even}(\texttt{succ}(\texttt{zero})) \vee \texttt{odd}(\texttt{succ}(\texttt{zero}))$?

- iff $\models_{M,\cdot} \texttt{even}(\texttt{succ}(\texttt{zero}))$ or $\models_{M,\cdot} \texttt{odd}(\texttt{succ}(\texttt{zero}))$
- iff $\langle[\![\texttt{succ}(\texttt{zero})]\!]^{M}_{\cdot}\rangle \in \{\langle 0\rangle, \langle 2\rangle, \langle 4\rangle, \ldots\}$ or $\langle[\![\texttt{succ}(\texttt{zero})]\!]^{M}_{\cdot}\rangle \in \{\langle 1\rangle, \langle 3\rangle, \langle 5\rangle, \ldots\}$
- iff $\langle 1\rangle \in \{\langle 0\rangle, \langle 2\rangle, \langle 4\rangle, \ldots\}$ or $\langle 1\rangle \in \{\langle 1\rangle, \langle 3\rangle, \langle 5\rangle, \ldots\}$
- iff True

Semantics of Predicate Logic

For example:

- consider the signature $\langle\langle \mathtt{zero}, \mathtt{succ}, \mathtt{add}\rangle, \langle \mathtt{even}, \mathtt{odd}\rangle\rangle$
- the model M: $\langle \mathbb{N}, \langle 0, +1, +\rangle, \langle \{\langle 0\rangle, \langle 2\rangle, \langle 4\rangle, \dots\}, \{\langle 1\rangle, \langle 3\rangle, \langle 5\rangle, \dots\}\rangle\rangle$
- we write $+1$ for the function that given a number increments it by 1
- $+(n, m)$ stands for $n + m$

What is $\vDash_{M,\cdot} \forall x.\mathtt{even}(x)$?

- iff for all $n \in \mathbb{N}$, $\vDash_{M,x\mapsto n} \mathtt{even}(x)$
- iff for all $n \in \mathbb{N}$, $\langle [\![x]\!]^M_{x\mapsto n}\rangle \in \{\langle 0\rangle, \langle 2\rangle, \langle 4\rangle, \dots\}$
- iff for all $n \in \mathbb{N}$, $\langle n\rangle \in \{\langle 0\rangle, \langle 2\rangle, \langle 4\rangle, \dots\}$
- iff False, because $1 \notin \{0, 2, 4, \dots\}$

Semantics of Predicate Logic

For example:

- consider the signature $\langle\langle \texttt{zero}, \texttt{succ}, \texttt{add}\rangle, \langle \texttt{even}, \texttt{odd}\rangle\rangle$
- the model M: $\langle \mathbb{N}, \langle 0, +1, +\rangle, \langle \{\langle 0\rangle, \langle 2\rangle, \langle 4\rangle, \dots\}, \{\langle 1\rangle, \langle 3\rangle, \langle 5\rangle, \dots\}\rangle\rangle$
- we write $+1$ for the function that given a number increments it by 1
- $+(n, m)$ stands for $n + m$

What is $\models_{M,\cdot} \forall x.\texttt{even}(x) \rightarrow \neg\texttt{odd}(x)$?

- iff for all $n \in \mathbb{N}$, $\models_{M,x\mapsto n} \texttt{even}(x) \rightarrow \neg\texttt{odd}(x)$
- iff for all $n \in \mathbb{N}$, $\models_{M,x\mapsto n} \neg\texttt{odd}(x)$ whenever $\models_{M,x\mapsto n} \texttt{even}(x)$
- iff for all $n \in \mathbb{N}$, $\neg \models_{M,x\mapsto n} \texttt{odd}(x)$ whenever $\models_{M,x\mapsto n} \texttt{even}(x)$
- iff for all $n \in \mathbb{N}$, $\langle [\![x]\!]^{M}_{x\mapsto n}\rangle \notin \{\langle 1\rangle, \langle 3\rangle, \langle 5\rangle, \dots\}$ whenever $\langle [\![x]\!]^{M}_{x\mapsto n}\rangle \in \{\langle 0\rangle, \langle 2\rangle, \langle 4\rangle, \dots\}$
- iff for all $n \in \mathbb{N}$, $\langle n\rangle \notin \{\langle 1\rangle, \langle 3\rangle, \langle 5\rangle, \dots\}$ whenever $\langle n\rangle \in \{\langle 0\rangle, \langle 2\rangle, \langle 4\rangle, \dots\}$
- iff for all $n \in \mathbb{N}$, $n \notin \{1, 3, 5, \dots\}$ whenever $n \in \{0, 2, 4, \dots\}$
- iff True

Semantics of Predicate Logic

For example:

- consider the signature $\langle\langle \texttt{zero}, \texttt{succ}, \texttt{add}\rangle, \langle \texttt{lt}, \texttt{ge}\rangle\rangle$
- the model M:
  $\langle \mathbb{N}, \langle 0, +1, +\rangle, \langle \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 1,2\rangle, \dots\}, \{\langle 0,0\rangle, \langle 1,1\rangle, \langle 1,0\rangle, \dots\}\rangle\rangle$
- we write $+1$ for the function that given a number increments it by 1
- $+(n,m)$ stands for $n+m$

What is $\models_{M,\cdot} \forall x.\forall y.\texttt{lt}(x,y) \rightarrow \texttt{ge}(y,x)$?

- iff for all $n, m \in \mathbb{N}$, $\models_{M,x\mapsto n,y\mapsto m} \texttt{lt}(x,y) \rightarrow \texttt{ge}(y,x)$
- iff for all $n, m \in \mathbb{N}$, $\models_{M,x\mapsto n,y\mapsto m} \texttt{ge}(y,x)$ whenever $\models_{M,x\mapsto n,y\mapsto m} \texttt{lt}(x,y)$
- iff for all $n, m \in \mathbb{N}$,
  $\langle [\![y]\!]^M_{x\mapsto n,y\mapsto m}, [\![x]\!]^M_{x\mapsto n,y\mapsto m}\rangle \in \{\langle 0,0\rangle, \langle 1,1\rangle, \langle 1,0\rangle, \dots\}$ whenever
  $\langle [\![x]\!]^M_{x\mapsto n,y\mapsto m}, [\![y]\!]^M_{x\mapsto n,y\mapsto m}\rangle \in \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 1,2\rangle, \dots\}$
- iff for all $n, m \in \mathbb{N}$, $\langle m, n\rangle \in \{\langle 0,0\rangle, \langle 1,1\rangle, \langle 1,0\rangle, \dots\}$ whenever $\langle n, m\rangle \in \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 1,2\rangle, \dots\}$
- iff True

Satisfiability & Validity

We write $\models_M P$ for $\models_{M,\cdot} P$

Truth: P is **true** in the model M if $\models_M P$

We also say that M is a model of P

Satisfiability: P is **satisfiable** if there is a model M such that P is true in M, i.e., $\models_M P$

Validity: P is **valid** if for all model M, P is true in M

Example: $\models_{M,\cdot} \forall x.\mathtt{even}(x) \rightarrow \neg\mathtt{odd}(x)$ is satisfiable (see above) but not valid because not true for example in the model $\langle \mathbb{N}, \langle 0, +1, + \rangle, \langle \{\langle 0 \rangle, \langle 2 \rangle, \langle 4 \rangle, \ldots\}, \{\langle 0 \rangle, \langle 2 \rangle, \langle 4 \rangle, \ldots\} \rangle \rangle$

Decidability: Validity is <u>not decidable for predicate logic</u>, i.e., there is no algorithm that given a formula P either returns "yes" if P is valid, and otherwise returns "no", while it is <u>decidable for propositional logic</u>

Recap: Soundness & Completeness

Given a deduction system such as Natural deduction, a formula is said to be **provable** if there is a proof of it in that deduction system

- This is a **syntactic** notion
- it asserts the existence of a syntactic object: a proof
- typically written $\vdash A$

A formula A is **valid** if for all model M, A is true in M, i.e., $\models_M P$

- it is a **semantic** notion
- it is checked w.r.t. valuations/models that give meaning to formulas
- written $\models A$

Soundness: a deduction system is sound w.r.t. a semantics if every provable formula is valid

- i.e., if $\vdash A$ then $\models A$

Completeness: a deduction system is complete w.r.t. a semantics if every valid formula is provable

- i.e., if $\models A$ then $\vdash A$

Soundness & Completeness

Natural Deduction for Predicate Logic is

- **sound** and
- **complete**

w.r.t. the **model semantics of Predicate Logic**

Proving those properties is done within the **metatheory**

We will not prove them here

Conclusion

What did we cover today?

- Semantics of Predicate Logic
- Models
- Variable valuations
- Satisfiability & validity

Further reading:

- Chapter 10 of
  http://leanprover.github.io/logic_and_proof/

Next time?

- Equivalences in Predicate Logic