Exercise Sheet for Maths Material Covered in Week 11
Eigenvalues and Eigenvectors

(1) **(feedback)**
Let $R_z(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$ be the rotation matrix around the z-axis.

(a) Specify a corresponding rotation matrix around the x-axis.

(b) Consider now the case of a $\varphi = \pi/2$ rotation for both, $X = R_x(\pi/2)$ and $Z = R_z(\pi/2)$

(c) Calculate the eigenvalues of the matrices X, Z, XZ, ZX

(d) Calculate the determinant for these matrices.

(e) Comment on your results.

(2) Find the characteristic equation, and the eigenvalues, for the following matrices:
$A = \begin{pmatrix} -1 & 6 \\ 0 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & -2 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \frac{1}{7} \end{pmatrix}$

(3) Suppose that the characteristic polynomial of some matrix A is found to be

$$p(\alpha) = (\alpha - 1)(\alpha - 3)^3(\alpha - 4)^2.$$

(a) What is the size of A?

(b) Is A invertible? If not, justify why. If yes, find an expression for the inverse A^{-1}.
(Hint: you may have to make some assumptions.)

In each part, answer the question and explain your reasoning.

(4) Consider the matrices Let $A = \begin{pmatrix} a & b \\ 1-a & 1-b \end{pmatrix}$ and $B = \begin{pmatrix} a & c & e \\ b & d & f \\ 1-a-b & 1-c-d & 1-e-f \end{pmatrix}$

(a) Calculate the eigenvalues of A, and at least one eigenvector of B.

(b) If you find that one of the eigenvalues is 1, calculate the correspondent eigenvector.

(5) Prove: If α is an eigenvalue of A with eigenvector $\vec{x}$, then $1/\alpha$ is an eigenvalue of A^{-1} (provided that exists), and $\vec{x}$ is the corresponding eigenvector.