Maths Exercise Sheet 6

Here are six revision exercises written in the style of questions on the class test, to help give you an idea of what to expect. An indicative mark breakdown is included for each question.

1. [**Numbers & Induction**]

   Let us define a sequence of natural numbers $a_0, a_1, a_2, \ldots$ by

   - $a_0 = 0$,
   - $a_1 = 5$, and
   - $a_{n+2} = 2a_{n+1} + a_n$ for $n \in \mathbb{N}$.

   (a) Compute the first six values $a_0, \ldots, a_5$ **[2 marks]**

   (b) Use induction to show that $\gcd(a_n, a_{n+1}) = 5$ for all $n \in \mathbb{N}$. You can use the facts that $\gcd(x, y) = \gcd(x, y - x)$ and $\gcd(x, y) = \gcd(y, x)$ for all $x, y \in \mathbb{Z}$. **[4 marks]**

   (c) For which $i \in \{0, 1, 2, 3\}$ does there exist $n \in \mathbb{N}$ with $a_n \equiv i \pmod 4$? For each i that does arise you should given an example of such an a_n. For each i that does not arise you should give at least an informal explanation of your reasoning (there is no need to give a formal proof by induction). **[4 marks]**

2. [**Numbers & Functions**]

   Consider the function $f : \mathbb{Z} \to \{0, 1, 2, \ldots, 233\}$ defined by $f(x) = (55x) \,\texttt{mod}\, 234$.

   (a) Give an explicit example to show that f is not injective. **[2 marks]**

   (b) Perform Euclid's algorithm on 234 and 55. Detail your process including giving the values of the sequence $r_0, r_1, \ldots$. **[4 marks]**

   (c) Is f surjective? Explain your reasoning. **[4 marks]**

3. [**Relations & Sets**]

   Consider the relation R on $\{0, 1, 2, 3, 4, 5\}$ where

   $$R = \{(0,0), (0,1), (0,3), (0,5), (1,5), (3,5), (2,5)\}.$$

   In each case, explain your reasoning.

   (a) Is R reflexive? **[1 mark]**

   (b) Is R symmetric? **[1 mark]**

   (c) Is R antisymmetric? **[2 marks]**

   (d) Is R transitive? **[3 marks]**

   (e) Is the set $\{A \in P(\mathbb{N} \times \mathbb{N}) \mid R \subseteq A\}$ countable or uncountable? **[3 marks]**

4. [**Numbers & Induction**]

   Let us define a sequence of natural numbers $b_0, b_1, b_2, \ldots$ by

   - $b_0 = 10$,
   - $b_{2n+1} = 3b_n + 1$, and
   - $b_{2n+2} = b_{n+1}$

   for all $n \in \mathbb{N}$.

   (a) Use induction to prove that $b_n \equiv 1 \pmod 3$ for all $n \in \mathbb{N}$. [**5 marks**]

   (b) What are the possible values of $b_n \,\texttt{mod}\, 5$? Prove your answer. [Hint: you might try to guess the set S of possible values first by calculating some of the sequence, and then prove $b_n \,\texttt{mod}\, 5 \in S$ by induction on $n \in \mathbb{N}$ to show that this is all possible values.] [**5 marks**]

5. [**Real Numbers & Functions**]

   (a) Explain why $0 < \frac{\sqrt{5}}{3} < 1$. [**2 marks**]

   (b) Find an integer $m \in \mathbb{Z}$ such that

   $$\frac{m}{2^2} < \frac{\sqrt{5}}{3} < \frac{m+1}{2^2}.$$

   Prove your answer correct or make it clear from your method. [**3 marks**]

   (c) Consider the expression "$x \mapsto x\sqrt{5}$". Explain why this can be used to define a function $\mathbb{Q} \to \mathbb{R}$ or $\mathbb{R} \to \mathbb{R}$, but not $\mathbb{Q} \to \mathbb{Q}$. [**2 marks**]

   (d) For each of the two possibilities $\mathbb{Q} \to \mathbb{R}$ and $\mathbb{R} \to \mathbb{R}$, explain whether $x \mapsto x\sqrt{5}$ is injective and/or surjective. [**3 marks**]

6. [**Numbers**]

   (a) Perform Euclid's algorithm on 35 and 54. Detail your process including giving the values of the sequence $r_0, r_1, \ldots$. [**4 marks**]

   (b) Find an integer $x \in \mathbb{Z}$ such that both $x \equiv 2 \pmod{35}$ and $x \equiv 3 \pmod{54}$. [**4 marks**]

   (c) Hence describe the full set of x such that $x \equiv 2 \pmod{35}$ and $x \equiv 3 \pmod{54}$. [**2 marks**]