Exercise Sheet 12e - Solutions
Predicate Logic – Natural Deduction & Semantics

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1. Here is a constructive Natural Deduction proof of $(S_1) \to \forall\neg x < 0$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{S_1}{\scriptstyle 1} \qquad \dfrac{\dfrac{}{x<0}{\scriptstyle 2}}{\exists x.x<0}{\scriptstyle [\exists I]}}{\bot}{\scriptstyle [\neg E]}}{\neg x<0}{\scriptstyle 2\ [\neg I]}}{\forall x.\neg x<0}{\scriptstyle [\forall I]}}{(S_1) \to \forall x.\neg x<0}{\scriptstyle 1\ [\to I]}$$

2. Here is a constructive Natural Deduction proof of $(S_1) \to (S_2) \to \exists x.0 < x$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{S_2}{\scriptstyle 2}}{\exists y.0<y \lor y<0}{\scriptstyle [\forall E]} \qquad \dfrac{\dfrac{\dfrac{}{0<y \lor y<0}{\scriptstyle 3} \quad \dfrac{\dfrac{}{0<y}{\scriptstyle 4}}{0<y \to 0<y}{\scriptstyle 4\ [\to I]} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{S_1}{\scriptstyle 1} \quad \dfrac{\dfrac{}{y<0}{\scriptstyle 5}}{\exists x.x<0}{\scriptstyle [\exists I]}}{\bot}{\scriptstyle [\neg E]}}{0<y}{\scriptstyle [\bot E]}}{y<0 \to 0<y}{\scriptstyle 5\ [\to I]}}{0<y}{\scriptstyle [\lor E]}}{\exists x.0<x}{\scriptstyle [\exists I]}}{\exists x.0<x}{\scriptstyle 3\ [\exists E]}}{(S_2) \to \exists x.0<x}{\scriptstyle 2\ [\to I]}}{(S_1) \to (S_2) \to \exists x.0<x}{\scriptstyle 1\ [\to I]}$$

3. For example, the following models are models of S_2:

   - $M_1 = \langle \mathbb{N}, \langle 0, \langle n \rangle \mapsto n \rangle, \langle \{\langle n, m \rangle \mid \text{True}\}, \emptyset \rangle \rangle$
   - $M_1' = \langle \mathbb{N}, \langle 0, \langle n \rangle \mapsto n \rangle, \langle \{\langle n, m \rangle \mid n < m\}, \emptyset \rangle \rangle$
   - $M_1'' = \langle \mathbb{N}, \langle 0, \langle n \rangle \mapsto n \rangle, \langle \{\langle n, m \rangle \mid n > m\}, \emptyset \rangle \rangle$
   - $M_1''' = \langle \{0\}, \langle 0, \langle n \rangle \mapsto n \rangle, \langle \{\langle n, m \rangle \mid \text{True}\}, \emptyset \rangle \rangle$

4. For example, the following models are not models of S_2, i.e., are models of $\neg S_2$:

   - $M_2 = \langle \mathbb{N}, \langle 0, \langle n \rangle \mapsto n \rangle, \langle \emptyset, \emptyset \rangle \rangle$
   - $M_2' = \langle \mathbb{N}, \langle 0, \langle n \rangle \mapsto n \rangle, \langle \{\langle 0, 1 \rangle\}, \emptyset \rangle \rangle$