Mathematical and Logical Foundations of Computer Science

Predicate Logic (Equivalences continued)

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(some slides were adapted from Rajesh Chitnis' slides)

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Where are we?

- Symbolic logic
- Propositional logic
- **Predicate logic**

Today

Equivalences:

- in Natural Deduction
- rewriting using "known" equivalences
- using semantics

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- in Natural Deduction
- rewriting using “known” equivalences
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Further reading:

- Chapter 8 of
  http://leanprover.github.io/logic_and_proof/

Recap: Syntax

The syntax of predicate logic is defined by the following grammar:

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\begin{array}{lcl}
t & ::= & x \mid f(t, \ldots, t) \\
P & ::= & p(t, \ldots, t) \mid \neg P \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid \forall x.P \mid \exists x.P
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where:

- x ranges over variables
- f ranges over function symbols
- $f(t_1, \ldots, t_n)$ is a well-formed term only if f has arity n
- p ranges over predicate symbols
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The scope of a quantifier extends as far right as possible. E.g., $P \wedge \forall x.p(x) \vee q(x)$ is read as $P \wedge \forall x.(p(x) \vee q(x))$

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The additional **conditions** ensure that **free variables do not get captured**.

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The additional **conditions** ensure that **free variables do not get captured**.

These conditions can always be met by silently renaming bound variables before substituting.

Recap: ∀ & ∃ elimination and introduction rules

Natural Deduction rules for quantifiers:

$$\frac{P[x\backslash y]}{\forall x.P}\ [\forall I] \qquad \frac{\forall x.P}{P[x\backslash t]}\ [\forall E] \qquad \frac{P[x\backslash t]}{\exists x.P}\ [\exists I] \qquad \cfrac{\exists x.P \qquad \begin{array}{c} \overline{P[x\backslash y]}^{\,1} \\ \vdots \\ Q \end{array}}{Q}\ 1\ [\exists E]$$

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Condition:

- for $[\forall I]$: y must not be free in any not-yet-discharged hypothesis or in $\forall x.P$
- for $[\forall E]$: $\texttt{fv}(t)$ must not clash with $\texttt{bv}(P)$
- for $[\exists I]$: $\texttt{fv}(t)$ must not clash with $\texttt{bv}(P)$
- for $[\exists E]$: y must not be free in Q or in not-yet-discharged hypotheses or in $\exists x.P$

Recap: Example of a proof

here is a proof of $(\forall z.p(z)) \to \forall x.p(x) \vee q(x)$.

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$$\dfrac{\dfrac{\dfrac{\overline{\forall z.p(z)}^{\;1}}{}}{\forall x.p(x) \lor q(x)}}{(\forall z.p(z)) \to \forall x.p(x) \lor q(x)}\ 1\ [\to I]$$

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Models: a model provides the interpretation of all symbols

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Given a **signature** $\langle\langle f_1^{k_1}, \ldots, f_n^{k_n}\rangle, \langle p_1^{j_1}, \ldots, p_m^{j_m}\rangle\rangle$

- of function symbols f_i of arity k_i, for $1 \leqslant i \leqslant n$
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a **model** is a structure $\langle D, \langle \mathcal{F}_{f_1}, \ldots, \mathcal{F}_{f_n}\rangle, \langle \mathcal{R}_{p_1}, \ldots, \mathcal{R}_{p_m}\rangle\rangle$

- of a non-empty domain D
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Variable valuations:

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- i.e., a mapping of the form $x_1 \mapsto d_1, \ldots, x_n \mapsto d_n$

Recap: Semantics of Predicate Logic

Given a **model** M with domain D and a **variable valuation** v:

- $[\![t]\!]_v^M$ gives meaning to the term t w.r.t. M and v
- $\models_{M,v} P$ gives meaning to the formula P w.r.t. M and v

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Meaning of terms:

- $[\![x]\!]_v^M = v(x)$
- $[\![f(t_1, \ldots, t_n)]\!]_v^M = \mathcal{F}_f(\langle [\![t_1]\!]_v^M, \ldots, [\![t_n]\!]_v^M \rangle)$

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Meaning of formulas:

- $\models_{M,v} p(t_1, \ldots, t_n)$ iff $\langle [\![t_1]\!]_v^M, \ldots, [\![t_n]\!]_v^M \rangle \in \mathcal{R}_p$
- $\models_{M,v} \neg P$ iff $\neg\models_{M,v} P$
- $\models_{M,v} P \land Q$ iff $\models_{M,v} P$ and $\models_{M,v} Q$
- $\models_{M,v} P \lor Q$ iff $\models_{M,v} P$ or $\models_{M,v} Q$
- $\models_{M,v} P \rightarrow Q$ iff $\models_{M,v} Q$ whenever $\models_{M,v} P$
- $\models_{M,v} \forall x.P$ iff for every $d \in D$ we have $\models_{M,(v,x\mapsto d)} P$
- $\models_{M,v} \exists x.P$ iff there exists a $d \in D$ such that $\models_{M,(v,x\mapsto d)} P$

Recap: Logical equivalences for Propositional Logic

The same equivalences hold as in Propositional Logic:

- De Morgan's law (I): $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$
- De Morgan's law (II): $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$
- Implication elimination: $(A \rightarrow B) \leftrightarrow (\neg A \vee B)$
- Commutativity of $\wedge$: $(A \wedge B) \leftrightarrow (B \wedge A)$
- Commutativity of $\vee$: $(A \vee B) \leftrightarrow (B \vee A)$
- Associativity of $\wedge$: $((A \wedge B) \wedge C) \leftrightarrow (A \wedge (B \wedge C))$
- Associativity of $\vee$: $((A \vee B) \vee C) \leftrightarrow (A \vee (B \vee C))$
- Distributivity of $\wedge$ over $\vee$: $(A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$
- Distributivity of $\vee$ over $\wedge$: $(A \vee (B \wedge C)) \leftrightarrow ((A \vee B) \wedge (A \vee C))$
- Double negation elimination: $(\neg\neg A) \leftrightarrow A$
- Idempotence: $(A \wedge A) \leftrightarrow A$ and $(A \vee A) \leftrightarrow A$

Recap: Logical Equivalences

In addition, the following hold (some hold only classically):

- $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$
- $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$
- $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$
- $(\neg\exists x.A) \leftrightarrow (\forall x.\neg A)$
- $(\forall x.A) \leftrightarrow A$ if $x \notin \mathtt{fv}(A)$
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Recap: Logical Equivalences

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Semantical equivalence: two formulas P and Q are equivalent if for all models M and valuations v, $\models_{M,v} P$ iff $\models_{M,v} Q$

Logical Equivalences

As before to prove a logical equivalence $A \leftrightarrow B$, we will prove:

- that we can derive B form A
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- $(\exists x.A) \leftrightarrow A$ if $x \notin \mathtt{fv}(A)$
- $(\forall x.A \vee B) \leftrightarrow ((\forall x.A) \vee B)$ if $x \notin \mathtt{fv}(B)$
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We will use the following result:

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We will use the following result:

Lemma (L1): if $x \notin \mathtt{fv}(A)$ then $A[x\backslash t] = A$

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Prove $(\forall x.A) \leftrightarrow A$ if $x \notin \texttt{fv}(A)$ in Natural Deduction

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$$\dfrac{}{\forall x.A}$$

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$$\dfrac{A[x\backslash y]}{\forall x.A}\ [\forall I]$$

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Prove $(\forall x.A) \leftrightarrow A$ if $x \notin \mathtt{fv}(A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

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- pick y such that it does not occur in A
- by L1, because $x \notin \mathtt{fv}(A)$ then $A[x\backslash y] = A$

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$$\dfrac{A}{\forall x.A}\ [\forall I]$$

- pick y such that it does not occur in A
- by L1, because $x \notin \mathtt{fv}(A)$ then $A[x\backslash y] = A$

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{}{A}$$

Logical Equivalences

Prove $(\forall x.A) \leftrightarrow A$ if $x \notin \mathtt{fv}(A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{A}{\forall x.A}\ [\forall I]$$

- pick y such that it does not occur in A
- by L1, because $x \notin \mathtt{fv}(A)$ then $A[x\backslash y] = A$

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{}{A[x\backslash y]}$$

- by L1, because $x \notin \mathtt{fv}(A)$ then $A[x\backslash y] = A$
- pick y such that it does not occur in A

Logical Equivalences

Prove $(\forall x.A) \leftrightarrow A$ if $x \notin \mathtt{fv}(A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\frac{A}{\forall x.A}\ [\forall I]$$

- pick y such that it does not occur in A
- by L1, because $x \notin \mathtt{fv}(A)$ then $A[x\backslash y] = A$

Here is a proof of the left-to-right implication (constructive):

$$\frac{\forall x.A}{A[x\backslash y]}\ [\forall E]$$

- by L1, because $x \notin \mathtt{fv}(A)$ then $A[x\backslash y] = A$
- pick y such that it does not occur in A

Logical Equivalences

Prove $(\exists x.A) \leftrightarrow A$ if $x \notin \mathtt{fv}(A)$ in Natural Deduction

Logical Equivalences

Prove $(\exists x.A) \leftrightarrow A$ if $x \notin \mathtt{fv}(A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\overline{}$$

Logical Equivalences

Prove $(\exists x.A) \leftrightarrow A$ if $x \notin \texttt{fv}(A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\frac{}{\exists x.A}$$

Logical Equivalences

Prove $(\exists x.A) \leftrightarrow A$ if $x \notin \mathtt{fv}(A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\frac{A[x\backslash y]}{\exists x.A}\ [\exists I]$$

- pick y such that it does not occur in A

Logical Equivalences

Prove $(\exists x.A) \leftrightarrow A$ if $x \notin \texttt{fv}(A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\frac{A}{\exists x.A}\ [\exists I]$$

- pick y such that it does not occur in A
- by L1, because $x \notin \texttt{fv}(A)$ then $A[x\backslash y] = A$

Logical Equivalences

Prove $(\exists x.A) \leftrightarrow A$ if $x \notin \mathtt{fv}(A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\frac{A}{\exists x.A}\ [\exists I]$$

- pick y such that it does not occur in A
- by L1, because $x \notin \mathtt{fv}(A)$ then $A[x \backslash y] = A$

Here is a proof of the left-to-right implication (constructive):

$$\frac{}{A}$$

Logical Equivalences

Prove $(\exists x.A) \leftrightarrow A$ if $x \notin \mathtt{fv}(A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\frac{A}{\exists x.A}\ [\exists I]$$

- pick y such that it does not occur in A
- by L1, because $x \notin \mathtt{fv}(A)$ then $A[x\backslash y] = A$

Here is a proof of the left-to-right implication (constructive):

$$\frac{\exists x.A \quad A}{A}$$

- by L1, because $x \notin \mathtt{fv}(A)$ then $A[x\backslash y] = A$
- pick y such that it does not occur in A

Logical Equivalences

Prove $(\exists x.A) \leftrightarrow A$ if $x \notin \mathtt{fv}(A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\frac{A}{\exists x.A}\ [\exists I]$$

- pick y such that it does not occur in A
- by L1, because $x \notin \mathtt{fv}(A)$ then $A[x\backslash y] = A$

Here is a proof of the left-to-right implication (constructive):

$$\frac{\exists x.A \quad \dfrac{}{A[x\backslash y]}\,1}{A}\ 1\ [\exists E]$$

- by L1, because $x \notin \mathtt{fv}(A)$ then $A[x\backslash y] = A$
- pick y such that it does not occur in A

Logical Equivalences

Prove that $(\forall x.A \vee B) \leftrightarrow ((\forall x.A) \vee B)$ if $x \notin \mathtt{fv}(B)$ in Natural Deduction

Logical Equivalences

Prove that $(\forall x.A \vee B) \leftrightarrow ((\forall x.A) \vee B)$ if $x \notin \mathtt{fv}(B)$ in Natural Deduction

Here is a proof of the left-to-right implication (classical):

$$\dfrac{\dfrac{}{B \vee \neg B}\,[LEM] \qquad \dfrac{\dfrac{\dfrac{}{B}\,1}{(\forall x.A) \vee B}\,[\vee I_R]}{B \to (\forall x.A) \vee B}\,1\ [\to I] \qquad \dfrac{\dfrac{\Pi}{(\forall x.A) \vee B}\,[\vee I_L]}{\neg B \to (\forall x.A) \vee B}\,2\ [\to I]}{(\forall x.A) \vee B}\,[\vee E]$$

where Π is:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x.A \vee B}}{A[x\backslash y] \vee B}\,[\forall E] \qquad \dfrac{\dfrac{}{A[x\backslash y]}\,3}{A[x\backslash y] \to A[x\backslash y]}\,3\ [\to I] \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\neg B}\,2 \quad \dfrac{}{B}\,4}{\bot}\,[\neg E]}{A[x\backslash y]}\,[\bot E]}{B \to A[x\backslash y]}\,4\ [\to I]}{A[x\backslash y]}\,[\vee E]}{\forall x.A}\,[\forall I]$$

- pick y such that it does not occur in A or B
- by L1, because $x \notin \mathtt{fv}(B)$ then $B[x\backslash y] = B$

Logical Equivalences

Prove that $(\forall x.A \vee B) \leftrightarrow ((\forall x.A) \vee B)$ if $x \notin \mathtt{fv}(B)$ in Natural Deduction

Logical Equivalences

Prove that $(\forall x.A \lor B) \leftrightarrow ((\forall x.A) \lor B)$ if $x \notin \mathtt{fv}(B)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{(\forall x.A) \lor B \qquad \dfrac{\dfrac{\dfrac{\overline{\forall x.A}^{\,1}}{A[x\backslash y]}\,[\forall E]}{A[x\backslash y] \lor B}\,[\lor I_L]}{(\forall x.A) \to A[x\backslash y] \lor B}\,1\;[\to I] \qquad \dfrac{\dfrac{\overline{B}^{\,2}}{A[x\backslash y] \lor B}\,[\lor I_R]}{B \to A[x\backslash y] \lor B}\,2\;[\to I]}{\dfrac{A[x\backslash y] \lor B}{\forall x.A \lor B}\,[\forall I]}\,[\lor E]$$

- pick y such that it does not occur in A or B
- by L1, because $x \notin \mathtt{fv}(B)$ then $B[x\backslash y] = B$

Logical Equivalences

Prove that $(\exists x.A \wedge B) \leftrightarrow ((\exists x.A) \wedge B)$ if $x \notin \mathtt{fv}(B)$ in Natural Deduction

Logical Equivalences

Prove that $(\exists x.A \wedge B) \leftrightarrow ((\exists x.A) \wedge B)$ if $x \notin \mathtt{fv}(B)$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\exists x.A \wedge B \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y] \wedge B}\ {}^{1}}{A[x\backslash y]}\ [\wedge E_L]}{\exists x.A}\ [\exists I] \qquad \dfrac{\dfrac{}{A[x\backslash y] \wedge B}\ {}^{1}}{B}\ [\wedge E_R]}{(\exists x.A) \wedge B}\ [\wedge I]}{(\exists x.A) \wedge B}\ 1\ [\exists E]$$

- pick y such that it does not occur in A or B
- by L1, because $x \notin \mathtt{fv}(B)$ then $B[x\backslash y] = B$

Logical Equivalences

Prove that $(\exists x.A \wedge B) \leftrightarrow ((\exists x.A) \wedge B)$ if $x \notin \texttt{fv}(B)$ in Natural Deduction

Logical Equivalences

Prove that $(\exists x.A \land B) \leftrightarrow ((\exists x.A) \land B)$ if $x \notin \texttt{fv}(B)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{(\exists x.A) \land B}{\exists x.A}\,[\land E_L] \qquad \dfrac{\dfrac{\dfrac{}{A[x\backslash y]}\,1 \quad \dfrac{(\exists x.A) \land B}{B}\,[\land E_R]}{A[x\backslash y] \land B}\,[\land I]}{\exists x.A \land B}\,[\exists I]}{\exists x.A \land B}\,1\ [\exists E]$$

- pick y such that it does not occur in A or B
- by L1, because $x \notin \texttt{fv}(B)$ then $B[x\backslash y] = B$

Logical Equivalences

We will now prove the following using the other equivalences:

- $(\forall x.A \to B) \leftrightarrow ((\exists x.A) \to B)$ if $x \notin \mathtt{fv}(B)$
- $(\exists x.A \to B) \leftrightarrow ((\forall x.A) \to B)$ if $x \notin \mathtt{fv}(B)$

Logical Equivalences

We will now prove the following using the other equivalences:

- $(\forall x.A \rightarrow B) \leftrightarrow ((\exists x.A) \rightarrow B)$ if $x \notin \texttt{fv}(B)$
- $(\exists x.A \rightarrow B) \leftrightarrow ((\forall x.A) \rightarrow B)$ if $x \notin \texttt{fv}(B)$

Prove that $(\forall x.A \rightarrow B) \leftrightarrow ((\exists x.A) \rightarrow B)$ if $x \notin \texttt{fv}(B)$ using the other equivalences

Logical Equivalences

We will now prove the following using the other equivalences:

- $(\forall x.A \to B) \leftrightarrow ((\exists x.A) \to B)$ if $x \notin \mathtt{fv}(B)$
- $(\exists x.A \to B) \leftrightarrow ((\forall x.A) \to B)$ if $x \notin \mathtt{fv}(B)$

Prove that $(\forall x.A \to B) \leftrightarrow ((\exists x.A) \to B)$ if $x \notin \mathtt{fv}(B)$ using the other equivalences

- $\forall x.A \to B$

Logical Equivalences

We will now prove the following using the other equivalences:

- $(\forall x.A \rightarrow B) \leftrightarrow ((\exists x.A) \rightarrow B)$ if $x \notin \texttt{fv}(B)$
- $(\exists x.A \rightarrow B) \leftrightarrow ((\forall x.A) \rightarrow B)$ if $x \notin \texttt{fv}(B)$

Prove that $(\forall x.A \rightarrow B) \leftrightarrow ((\exists x.A) \rightarrow B)$ if $x \notin \texttt{fv}(B)$ using the other equivalences

- $\forall x.A \rightarrow B$
- $\leftrightarrow \forall x.\neg A \vee B$ – using implication elimination

Logical Equivalences

We will now prove the following using the other equivalences:

- $(\forall x.A \rightarrow B) \leftrightarrow ((\exists x.A) \rightarrow B)$ if $x \notin \texttt{fv}(B)$
- $(\exists x.A \rightarrow B) \leftrightarrow ((\forall x.A) \rightarrow B)$ if $x \notin \texttt{fv}(B)$

Prove that $(\forall x.A \rightarrow B) \leftrightarrow ((\exists x.A) \rightarrow B)$ if $x \notin \texttt{fv}(B)$ using the other equivalences

- $\forall x.A \rightarrow B$
- $\leftrightarrow \forall x.\neg A \vee B$ – using implication elimination
- $\leftrightarrow (\forall x.\neg A) \vee B$ – using $(\forall x.A \vee B) \leftrightarrow ((\forall x.A) \vee B)$ if $x \notin \texttt{fv}(B)$

Logical Equivalences

We will now prove the following using the other equivalences:

- $(\forall x.A \rightarrow B) \leftrightarrow ((\exists x.A) \rightarrow B)$ if $x \notin \mathtt{fv}(B)$
- $(\exists x.A \rightarrow B) \leftrightarrow ((\forall x.A) \rightarrow B)$ if $x \notin \mathtt{fv}(B)$

Prove that $(\forall x.A \rightarrow B) \leftrightarrow ((\exists x.A) \rightarrow B)$ if $x \notin \mathtt{fv}(B)$ using the other equivalences

- $\forall x.A \rightarrow B$
- $\leftrightarrow \forall x.\neg A \vee B$ – using implication elimination
- $\leftrightarrow (\forall x.\neg A) \vee B$ – using $(\forall x.A \vee B) \leftrightarrow ((\forall x.A) \vee B)$ if $x \notin \mathtt{fv}(B)$
- $\leftrightarrow (\neg\exists x.A) \vee B$ – using $(\neg\exists x.A) \leftrightarrow (\forall x.\neg A)$

Logical Equivalences

We will now prove the following using the other equivalences:

- $(\forall x.A \to B) \leftrightarrow ((\exists x.A) \to B)$ if $x \notin \mathtt{fv}(B)$
- $(\exists x.A \to B) \leftrightarrow ((\forall x.A) \to B)$ if $x \notin \mathtt{fv}(B)$

Prove that $(\forall x.A \to B) \leftrightarrow ((\exists x.A) \to B)$ if $x \notin \mathtt{fv}(B)$ using the other equivalences

- $\forall x.A \to B$
- $\leftrightarrow \forall x.\neg A \vee B$ – using implication elimination
- $\leftrightarrow (\forall x.\neg A) \vee B$ – using $(\forall x.A \vee B) \leftrightarrow ((\forall x.A) \vee B)$ if $x \notin \mathtt{fv}(B)$
- $\leftrightarrow (\neg\exists x.A) \vee B$ – using $(\neg\exists x.A) \leftrightarrow (\forall x.\neg A)$
- $\leftrightarrow (\exists x.A) \to B$ – using implication elimination

Logical Equivalences

Prove that $(\exists x.A \to B) \leftrightarrow ((\forall x.A) \to B)$ if $x \notin \texttt{fv}(B)$ using the other equivalences

Logical Equivalences

Prove that $(\exists x.A \rightarrow B) \leftrightarrow ((\forall x.A) \rightarrow B)$ if $x \notin \mathtt{fv}(B)$ using the other equivalences

- $\exists x.A \rightarrow B$

Logical Equivalences

Prove that $(\exists x.A \rightarrow B) \leftrightarrow ((\forall x.A) \rightarrow B)$ if $x \notin \texttt{fv}(B)$ using the other equivalences

- $\exists x.A \rightarrow B$
- $\leftrightarrow \exists x.\neg A \vee B$ – using implication elimination

Logical Equivalences

Prove that $(\exists x.A \to B) \leftrightarrow ((\forall x.A) \to B)$ if $x \notin \mathtt{fv}(B)$ using the other equivalences

- $\exists x.A \to B$
- $\leftrightarrow \exists x.\neg A \vee B$ – using implication elimination
- $\leftrightarrow (\exists x.\neg A) \vee (\exists x.B)$ – using $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$

Logical Equivalences

Prove that $(\exists x.A \to B) \leftrightarrow ((\forall x.A) \to B)$ if $x \notin \mathtt{fv}(B)$ using the other equivalences

- $\exists x.A \to B$
- $\leftrightarrow \exists x.\neg A \vee B$ – using implication elimination
- $\leftrightarrow (\exists x.\neg A) \vee (\exists x.B)$ – using $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$
- $\leftrightarrow (\exists x.\neg A) \vee B$ – using $(\exists x.A) \leftrightarrow A$ if $x \notin \mathtt{fv}(A)$

Logical Equivalences

Prove that $(\exists x.A \rightarrow B) \leftrightarrow ((\forall x.A) \rightarrow B)$ if $x \notin \mathtt{fv}(B)$ using the other equivalences

- $\exists x.A \rightarrow B$
- $\leftrightarrow \exists x.\neg A \vee B$ – using implication elimination
- $\leftrightarrow (\exists x.\neg A) \vee (\exists x.B)$ – using $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$
- $\leftrightarrow (\exists x.\neg A) \vee B$ – using $(\exists x.A) \leftrightarrow A$ if $x \notin \mathtt{fv}(A)$
- $\leftrightarrow (\neg\forall x.A) \vee B$ – using $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$

Logical Equivalences

Prove that $(\exists x.A \to B) \leftrightarrow ((\forall x.A) \to B)$ if $x \notin \mathtt{fv}(B)$ using the other equivalences

- $\exists x.A \to B$
- $\leftrightarrow \exists x.\neg A \vee B$ – using implication elimination
- $\leftrightarrow (\exists x.\neg A) \vee (\exists x.B)$ – using $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$
- $\leftrightarrow (\exists x.\neg A) \vee B$ – using $(\exists x.A) \leftrightarrow A$ if $x \notin \mathtt{fv}(A)$
- $\leftrightarrow (\neg\forall x.A) \vee B$ – using $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$
- $\leftrightarrow (\forall x.A) \to B$ – using implication elimination

Logical Equivalences

We will now prove the following using semantics:

- $(\forall x.A \rightarrow B) \leftrightarrow (A \rightarrow \forall x.B)$ if $x \notin \mathtt{fv}(A)$
- $(\exists x.A \rightarrow B) \leftrightarrow (A \rightarrow \exists x.B)$ if $x \notin \mathtt{fv}(A)$

Logical Equivalences

We will now prove the following using semantics:

- $(\forall x.A \to B) \leftrightarrow (A \to \forall x.B)$ if $x \notin \mathtt{fv}(A)$
- $(\exists x.A \to B) \leftrightarrow (A \to \exists x.B)$ if $x \notin \mathtt{fv}(A)$

We will use following result:

Logical Equivalences

We will now prove the following using semantics:

- $(\forall x.A \to B) \leftrightarrow (A \to \forall x.B)$ if $x \notin \mathtt{fv}(A)$
- $(\exists x.A \to B) \leftrightarrow (A \to \exists x.B)$ if $x \notin \mathtt{fv}(A)$

We will use following result:

Lemma (L2): if $x \notin \mathtt{fv}(A)$, then $\models_{M,v,x\mapsto d} A$ iff $\models_{M,v} A$

Logical Equivalences

Prove $(\forall x.A \to B) \leftrightarrow (A \to \forall x.B)$ if $x \notin \texttt{fv}(A)$ using the semantics method

Logical Equivalences

Prove $(\forall x.A \to B) \leftrightarrow (A \to \forall x.B)$ if $x \notin \mathtt{fv}(A)$ using the semantics method

Assume $x \notin \mathtt{fv}(A)$, M is a model with domain D and v a valuation

Logical Equivalences

Prove $(\forall x.A \to B) \leftrightarrow (A \to \forall x.B)$ if $x \notin \mathtt{fv}(A)$ using the semantics method

Assume $x \notin \mathtt{fv}(A)$, M is a model with domain D and v a valuation

Left-to-right implication:

Logical Equivalences

Prove $(\forall x.A \to B) \leftrightarrow (A \to \forall x.B)$ if $x \notin \texttt{fv}(A)$ using the semantics method

Assume $x \notin \texttt{fv}(A)$, M is a model with domain D and v a valuation

Left-to-right implication:

- if $\models_{M,v} \forall x.A \to B$ then $\models_{M,v} A \to \forall x.B$

Logical Equivalences

Prove $(\forall x.A \to B) \leftrightarrow (A \to \forall x.B)$ if $x \notin \texttt{fv}(A)$ using the semantics method

Assume $x \notin \texttt{fv}(A)$, M is a model with domain D and v a valuation

Left-to-right implication:

- if $\models_{M,v} \forall x.A \to B$ then $\models_{M,v} A \to \forall x.B$
  - to prove: $\models_{M,v} A \to \forall x.B$, i.e., $\models_{M,v} \forall x.B$ whenever $\models_{M,v} A$

Logical Equivalences

Prove $(\forall x.A \to B) \leftrightarrow (A \to \forall x.B)$ if $x \notin \texttt{fv}(A)$ using the semantics method

Assume $x \notin \texttt{fv}(A)$, M is a model with domain D and v a valuation

Left-to-right implication:

- if $\models_{M,v} \forall x.A \to B$ then $\models_{M,v} A \to \forall x.B$
  - to prove: $\models_{M,v} A \to \forall x.B$, i.e., $\models_{M,v} \forall x.B$ whenever $\models_{M,v} A$
  - assume $\models_{M,v} A$ and prove $\models_{M,v} \forall x.B$, i.e., for all $d \in D$, $\models_{M,v,x\mapsto d} B$

Logical Equivalences

Prove $(\forall x.A \rightarrow B) \leftrightarrow (A \rightarrow \forall x.B)$ if $x \notin \mathtt{fv}(A)$ using the semantics method

Assume $x \notin \mathtt{fv}(A)$, M is a model with domain D and v a valuation

Left-to-right implication:

- if $\models_{M,v} \forall x.A \rightarrow B$ then $\models_{M,v} A \rightarrow \forall x.B$
  - to prove: $\models_{M,v} A \rightarrow \forall x.B$, i.e., $\models_{M,v} \forall x.B$ whenever $\models_{M,v} A$
  - assume $\models_{M,v} A$ and prove $\models_{M,v} \forall x.B$, i.e., for all $d \in D$, $\models_{M,v,x\mapsto d} B$
  - assumption: $\models_{M,v} \forall x.A \rightarrow B$, i.e., for all $e \in D$, $\models_{M,v,x\mapsto e} B$ whenever $\models_{M,v,x\mapsto e} A$

Logical Equivalences

Prove $(\forall x.A \rightarrow B) \leftrightarrow (A \rightarrow \forall x.B)$ if $x \notin \mathtt{fv}(A)$ using the semantics method

Assume $x \notin \mathtt{fv}(A)$, M is a model with domain D and v a valuation

Left-to-right implication:

- if $\models_{M,v} \forall x.A \rightarrow B$ then $\models_{M,v} A \rightarrow \forall x.B$
  - to prove: $\models_{M,v} A \rightarrow \forall x.B$, i.e., $\models_{M,v} \forall x.B$ whenever $\models_{M,v} A$
  - assume $\models_{M,v} A$ and prove $\models_{M,v} \forall x.B$, i.e., for all $d \in D$, $\models_{M,v,x\mapsto d} B$
  - assumption: $\models_{M,v} \forall x.A \rightarrow B$, i.e., for all $e \in D$, $\models_{M,v,x\mapsto e} B$ whenever $\models_{M,v,x\mapsto e} A$
  - because $\models_{M,v} A$ by L2, $\models_{M,v,x\mapsto d} A$

Logical Equivalences

Prove $(\forall x.A \to B) \leftrightarrow (A \to \forall x.B)$ if $x \notin \mathtt{fv}(A)$ using the semantics method

Assume $x \notin \mathtt{fv}(A)$, M is a model with domain D and v a valuation

Left-to-right implication:

- if $\models_{M,v} \forall x.A \to B$ then $\models_{M,v} A \to \forall x.B$
  - to prove: $\models_{M,v} A \to \forall x.B$, i.e., $\models_{M,v} \forall x.B$ whenever $\models_{M,v} A$
  - assume $\models_{M,v} A$ and prove $\models_{M,v} \forall x.B$, i.e., for all $d \in D$, $\models_{M,v,x\mapsto d} B$
  - assumption: $\models_{M,v} \forall x.A \to B$, i.e., for all $e \in D$, $\models_{M,v,x\mapsto e} B$ whenever $\models_{M,v,x\mapsto e} A$
  - because $\models_{M,v} A$ by L2, $\models_{M,v,x\mapsto d} A$
  - instantiating this assumption with d gives us: $\models_{M,v,x\mapsto d} B$ whenever $\models_{M,v,x\mapsto d} A$

Logical Equivalences

Prove $(\forall x.A \to B) \leftrightarrow (A \to \forall x.B)$ if $x \notin \texttt{fv}(A)$ using the semantics method

Assume $x \notin \texttt{fv}(A)$, M is a model with domain D and v a valuation

Left-to-right implication:

- if $\models_{M,v} \forall x.A \to B$ then $\models_{M,v} A \to \forall x.B$
  - to prove: $\models_{M,v} A \to \forall x.B$, i.e., $\models_{M,v} \forall x.B$ whenever $\models_{M,v} A$
  - assume $\models_{M,v} A$ and prove $\models_{M,v} \forall x.B$, i.e., for all $d \in D$, $\models_{M,v,x\mapsto d} B$
  - assumption: $\models_{M,v} \forall x.A \to B$, i.e., for all $e \in D$, $\models_{M,v,x\mapsto e} B$ whenever $\models_{M,v,x\mapsto e} A$
  - because $\models_{M,v} A$ by L2, $\models_{M,v,x\mapsto d} A$
  - instantiating this assumption with d gives us: $\models_{M,v,x\mapsto d} B$ whenever $\models_{M,v,x\mapsto d} A$
  - therefore, because $\models_{M,v,x\mapsto d} A$ is true, $\models_{M,v,x\mapsto d} B$ is also true

Logical Equivalences

Right-to-left implication:

Logical Equivalences

Right-to-left implication:

- if $\models_{M,v} A \rightarrow \forall x.B$ then $\models_{M,v} \forall x.A \rightarrow B$

Logical Equivalences

Right-to-left implication:

- if $\models_{M,v} A \to \forall x.B$ then $\models_{M,v} \forall x.A \to B$
  - to prove: $\models_{M,v} \forall x.A \to B$, i.e., for all $d \in D$, $\models_{M,v,x\mapsto d} B$ whenever $\models_{M,v,x\mapsto d} A$

Logical Equivalences

Right-to-left implication:

- if $\vDash_{M,v} A \to \forall x.B$ then $\vDash_{M,v} \forall x.A \to B$
  - to prove: $\vDash_{M,v} \forall x.A \to B$, i.e., for all $d \in D$, $\vDash_{M,v,x\mapsto d} B$ whenever $\vDash_{M,v,x\mapsto d} A$
  - assume $d \in D$ and $\vDash_{M,v,x\mapsto d} A$, and prove $\vDash_{M,v,x\mapsto d} B$

Logical Equivalences

Right-to-left implication:

- if $\vDash_{M,v} A \to \forall x.B$ then $\vDash_{M,v} \forall x.A \to B$
  - to prove: $\vDash_{M,v} \forall x.A \to B$, i.e., for all $d \in D$, $\vDash_{M,v,x\mapsto d} B$ whenever $\vDash_{M,v,x\mapsto d} A$
  - assume $d \in D$ and $\vDash_{M,v,x\mapsto d} A$, and prove $\vDash_{M,v,x\mapsto d} B$
  - by L2, we can assume $\vDash_{M,v} A$

Logical Equivalences

Right-to-left implication:

- if $\models_{M,v} A \to \forall x.B$ then $\models_{M,v} \forall x.A \to B$
  - to prove: $\models_{M,v} \forall x.A \to B$, i.e., for all $d \in D$, $\models_{M,v,x\mapsto d} B$ whenever $\models_{M,v,x\mapsto d} A$
  - assume $d \in D$ and $\models_{M,v,x\mapsto d} A$, and prove $\models_{M,v,x\mapsto d} B$
  - by L2, we can assume $\models_{M,v} A$
  - assumption: $\models_{M,v} A \to \forall x.B$, i.e., $\models_{M,v} \forall x.B$ whenever $\models_{M,v} A$

Logical Equivalences

Right-to-left implication:

- if $\models_{M,v} A \to \forall x.B$ then $\models_{M,v} \forall x.A \to B$
  - to prove: $\models_{M,v} \forall x.A \to B$, i.e., for all $d \in D$, $\models_{M,v,x\mapsto d} B$ whenever $\models_{M,v,x\mapsto d} A$
  - assume $d \in D$ and $\models_{M,v,x\mapsto d} A$, and prove $\models_{M,v,x\mapsto d} B$
  - by L2, we can assume $\models_{M,v} A$
  - assumption: $\models_{M,v} A \to \forall x.B$, i.e., $\models_{M,v} \forall x.B$ whenever $\models_{M,v} A$
  - because $\models_{M,v} A$, we can assume $\models_{M,v} \forall x.B$, i.e., for all $e \in D$, $\models_{M,v,x\mapsto e} B$

Logical Equivalences

Right-to-left implication:

- if $\vDash_{M,v} A \to \forall x.B$ then $\vDash_{M,v} \forall x.A \to B$
  - to prove: $\vDash_{M,v} \forall x.A \to B$, i.e., for all $d \in D$, $\vDash_{M,v,x\mapsto d} B$ whenever $\vDash_{M,v,x\mapsto d} A$
  - assume $d \in D$ and $\vDash_{M,v,x\mapsto d} A$, and prove $\vDash_{M,v,x\mapsto d} B$
  - by L2, we can assume $\vDash_{M,v} A$
  - assumption: $\vDash_{M,v} A \to \forall x.B$, i.e., $\vDash_{M,v} \forall x.B$ whenever $\vDash_{M,v} A$
  - because $\vDash_{M,v} A$, we can assume $\vDash_{M,v} \forall x.B$, i.e., for all $e \in D$, $\vDash_{M,v,x\mapsto e} B$
  - instantiating this assumption using d, we get to assume $\vDash_{M,v,x\mapsto d} B$, which is what we wanted to prove

Conclusion

What did we cover today?

- Equivalence using Natural Deduction
- Rewriting using “known” equivalences
- Equivalences using semantics

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Further reading:

- Chapter 8 of
  http://leanprover.github.io/logic_and_proof/

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Next time?

- Theorem Proving