Exercise Sheet 12b - Solutions
Predicate Logic – Natural Deduction & Semantics

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1. Here is a constructive Natural Deduction proof of $(S_1) \rightarrow (S_2) \rightarrow \perp$

$$
\dfrac{\dfrac{\dfrac{\overline{S_1}^{\,1} \qquad \dfrac{\dfrac{\overline{\forall y.x < y}^{\,3}}{x < x}\,[\forall E] \qquad \dfrac{\overline{S_2}^{\,2}}{\neg x < x}\,{\scriptstyle [\forall E][\neg E]}}{\perp}}{\perp}\;3\;[\exists E]}{(S_2) \rightarrow \perp}\;2\;[\rightarrow I]}{(S_1) \rightarrow (S_2) \rightarrow \perp}\;1\;[\rightarrow I]
$$

2. Here is a constructive Natural Deduction proof of $(S_2) \rightarrow (S_3) \rightarrow \forall x.x \leq x$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{S_3}^{\,2}}{\forall y.x < y \lor y \leq x}\,[\forall E]}{x < x \lor x \leq x}\,[\forall E] \qquad \dfrac{\dfrac{\dfrac{\overline{x < x}^{\,3} \qquad \dfrac{\overline{S_2}^{\,1}}{\neg x < x}\,{\scriptstyle [\forall E][\neg E]}}{\perp}}{x \leq x}\,[\perp E]}{x < x \rightarrow x \leq x}\;3\;[\rightarrow I] \qquad \dfrac{\overline{x \leq x}^{\,4}}{x \leq x \rightarrow x \leq x}\;4\;[\rightarrow I]}{x \leq x}\,[\lor E]}{\forall x.x \leq x}\,[\forall I]}{(S_3) \rightarrow \forall x.x \leq x}\;2\;[\rightarrow I]}{(S_2) \rightarrow (S_3) \rightarrow \forall x.x \leq x}\;1\;[\rightarrow I]
$$

3. For example, the following models are models of S_3:
   - $M_1 = \langle \mathbb{N}, \langle 0, \langle n \rangle \mapsto n+1 \rangle, \langle \{\langle n,m \rangle \mid n < m\}, \{\langle n,m \rangle \mid n \leq m\} \rangle \rangle$
   - $M_1' = \langle \mathbb{N}, \langle 0, \langle n \rangle \mapsto n+1 \rangle, \langle \{\langle n,m \rangle \mid n \leq m\}, \{\langle n,m \rangle \mid n < m\} \rangle \rangle$
   - $M_1'' = \langle \mathbb{N}, \langle 0, \langle n \rangle \mapsto n \rangle, \langle \{\langle n,m \rangle \mid \text{True}\}, \emptyset \rangle \rangle$

4. For example, the following models are not models of S_3, i.e., are models of $\neg S_3$:
   - $M_2 = \langle \mathbb{N}, \langle 0, \langle n \rangle \mapsto n+1 \rangle, \langle \{\langle n,m \rangle \mid n < m\}, \{\langle n,m \rangle \mid n < m\} \rangle \rangle$
   - $M_2' = \langle \mathbb{N}, \langle 0, \langle n \rangle \mapsto n \rangle, \langle \{\langle n,m \rangle \mid n < m\}, \emptyset \rangle \rangle$
   - $M_2'' = \langle \mathbb{N}, \langle 0, \langle n \rangle \mapsto n \rangle, \langle \emptyset, \emptyset \rangle \rangle$