Exercise 1

In each of the following examples find all possible real values of x.

1. $|-3x+7| = 5$;
2. $|2x+1| < 7$;
3. $||x+2|-4| = 3$;
4. $||2x-3|-5| < 4$;
5. $|x^2-4| = 1$;
6. $|x^2-7x+9| > 3$;
7. $|2x+3|+|x-1| = 4$;
8. $|2x+1|+|x-3| > 4$.

Exercise 1: Solution

The possible values of x in each example are:

1. $x \in \{\frac{2}{3}, 4\}$;
2. $x \in (-4, 3)$;
3. $x \in \{-9, -3, 5, -1\}$;
4. $x \in (-3, 1) \cup (2, 6)$;
5. $x \in \{-\sqrt{5}, -\sqrt{3}, \sqrt{3}, \sqrt{5}\}$;
6. $x \in (-\infty, 1) \cup (3, 4) \cup (6, +\infty)$;
7. $x \in \{-2, 0\}$;
8. $x \in (-\infty, -\frac{2}{3}) \cup (0, +\infty)$.

Minkowski sum

Definition 1 (Minkowski sum)

The *Minkowski sum* of sets $A, B \subset \mathbb{R}$, is another set defined as

$$A \oplus B := \{a + b : a \in A, b \in B\}.$$

Some properties of the Minkowski sum. Try to prove them!

1. $|A \oplus B| \leq |A|\,|B|$;
2. $A \oplus \{0\} = A$ (neutral element);
3. $A \oplus B = B \oplus A$;
4. $A \oplus (B \oplus C) = (A \oplus B) \oplus C$;
5. $A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C)$;
6. $[a_1, a_2] \oplus [b_1, b_2] = [a_1 + b_1, a_2 + b_2]$, for $a_1, a_2, b_1, b_2 \in \mathbb{R}$;
7. if $A \subseteq B$ and $C \subseteq D$, then $(A \oplus C) \subseteq (B \oplus D)$;
8. $k(A \oplus B) = kA \oplus kB$, for $k \in \mathbb{R}$ (where $kA := \{ka : a \in A\}$).

Exercise 2

In each of the following examples, given sets A, B find $A \oplus B$:

1. $A = \{1, 2\}$, $B = \{-2, 4\}$;
2. $A = \{0, 3\}$, $B = \{4, 7\}$;
3. $A = [1, 3]$, $B = [4, 5]$;
4. $A = [-3, -1)$, $B = (1, 3]$;
5. $A = (-1, 0]$, $B = [2, +\infty)$;
6. $A = (3, +\infty)$, $B = (-\infty, 0)$;
7. $A = [0, 1]$, $B = (3, 5) \cup (7, 9)$;
8. $A = [2, 4]$, $B = [3, 5] \cup [7, 9)$;
9. $A = (-3, -2) \cup (1, 2)$, $B = (1, 2) \cup (5, 7)$;
10. $A = [0, 2) \cup (4, 5)$, $B = [-4, -3] \cup (-1, 0]$.

Exercise 2: Solution

The solutions for each example are:

1. $\{-1, 0, 5, 6\}$;
2. $\{4, 7, 10\}$;
3. $[5, 8]$;
4. $(-2, 2)$;
5. $(1, +\infty)$;
6. $(-\infty, +\infty)$;
7. $(3, 6) \cup (7, 10)$;
8. $[5, 13)$;
9. $(-2, 0) \cup (2, 5) \cup (6, 9)$;
10. $[-4, -1) \cup (-1, 2) \cup (3, 5)$.

Exercise 3

The following claims of the Minkowski sum do *not* hold. For each of them, find one example that makes it true, and one example that makes it false.

1. $A \oplus A = 2A$;
2. $A \oplus (B \cap C) = (A \oplus B) \cap (A \oplus C)$;
3. $A \oplus (B \setminus C) = (A \setminus B) \oplus (A \setminus C)$.

Exercise 3: Solution

One possibility for each example is:

1. True: $A = [1, 2]$, False: $A = \{1, 2\}$;
2. True: $A = [0, 1]$, $B = [2, 3]$, $C = [5, 6]$; False: $A = [0, 2]$, $B = [1, 2]$, $C = [3, 4]$;
3. True: $A = \{0\}$, $B = \{2, 3\}$, $C = \{3\}$; False: $A = \{0, 1\}$, $B = \{2, 3, 4\}$, $C = \{3, 4\}$.

Exercise 4

Prove the following using the triangle inequality (i.e. $\forall x, y \in \mathbb{R}$, $|x + y| \leq |x| + |y|$):

1. for all $x, y \in \mathbb{R}$: $||x| - |y|| \leq |x - y|$;
2. for all $x, y \in \mathbb{R}$: $|x^2 - y^2| \leq |x - y|(|x| + |y|)$;
3. for all $x, y, z \in \mathbb{R}$: $|x - y| + |y - z| + |x - z| \geq 2|x - z|$;
4. for all $x_1, x_2, \ldots, x_n \in \mathbb{R}$ (where $n \in \mathbb{N}$): $|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$.

Exercise 4: Solution

We only give a hint for each example:

1. Write $x = x - y + y$;
2. Factor $x^2 - y^2 = (x + y)(x - y)$;
3. Write $x - z = x - y + y - z$;
4. Use proof by induction.

Exercise 5

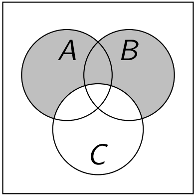
In a Venn diagram of 3 sets A, B, C, sketch the area that corresponds to the following sets:

1. $(A \cup B) \backslash C$;
2. $(A \cap B) \backslash C$;
3. $A \cap (B \cap C)$;
4. $A \backslash (B \cap C)$;
5. $A \triangle (B \setminus C)$;
6. $(A \cap B) \setminus (A \cup C)$.

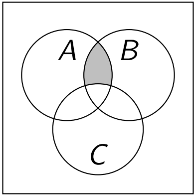
Exercise 5: Solution

The requested Venn diagrams are the following:

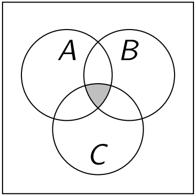
1. $(A \cup B) \setminus C$;



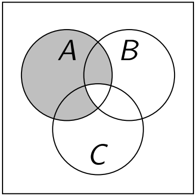
2. $(A \cap B) \setminus C$;



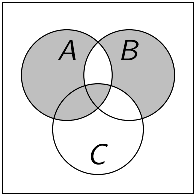
3. $A \cap (B \cap C)$;



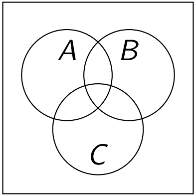
4. $A \setminus (B \cap C)$;



5. $A \triangle (B \setminus C)$;

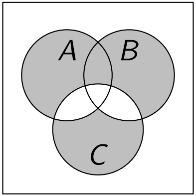


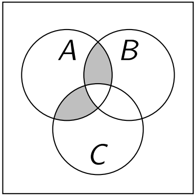
6. $(A \cap B) \setminus (A \cup C)$.

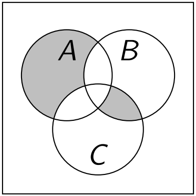


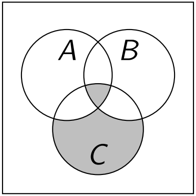
Exercise 6

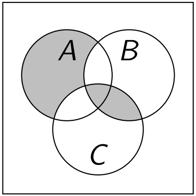
For each of the following Venn diagrams, express the sketched areas using the sets A, B, C and the operators $\cup, \cap, \setminus, \triangle$.

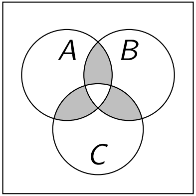
1. 

2. 

3. 

4. 

5. 

6. 

Exercise 6: Solution

One possibility for each example is:

1. $(A \cup B) \triangle C$;
2. $(A \cap B) \triangle (A \cap C)$;
3. $A \triangle (B \cup C)$;
4. $C \backslash (A \triangle B)$;
5. $(A \setminus (B \cup C)) \cup (B \cap C)$;
6. $(A \cap (B \cup C)) \triangle (B \cap C)$.