Exercise Sheet 8
Predicate Logic – Natural Deduction

Consider the following signature:

- Functions: $0, 1, 2, \ldots$ (arity 0); $+, \times, \mathtt{max}, \mathtt{min}$ (arity 2)
- Predicates: p, q, $\mathtt{prime}$, $\mathtt{even}$, $\mathtt{odd}$ (arity 1); r, $=$, $>$, $\geq$ (arity 2)

1. Assuming that the domain is the set of natural numbers, express the following sentence in predicate logic: "The maximum of two numbers is greater than or equal to the minimum of those numbers"

2. Assuming that the domain is the set of natural numbers, express the following sentence in predicate logic: "for all numbers x, there is no number different from x, that makes the maximum and minimum of the two numbers equal"

3. Provide a Natural Deduction proof of

$$(\forall x.p(x) \to q(x)) \to (\exists x.p(x)) \to \exists x.q(x)$$

4. Provide a Natural Deduction proof of

$$(\forall x.p(x) \to \neg\exists y.r(x, y)) \to \neg\exists x.\exists y.p(x) \wedge r(x, y)$$

5. (This is a hard exercise.) Assume that

   - $\forall x.\forall y.x > y \to \mathtt{min}(x, y) = y$
   - $\forall x.\forall y.\neg(x > y) \to \mathtt{min}(x, y) = x$
   - $\forall x.\forall y.\forall z.x = y \to y \geq z \to x \geq z$
   - $\forall x.\forall y.x + y \geq x$
   - $\forall x.\forall y.x + y \geq y$
   - $\forall x.\forall y.x > y \vee \neg(x > y)$

   Prove that $\forall x.\forall y.\forall z.\mathtt{min}(x + z, y + z) \geq z$