Countable set

Definition 1 (Countable set)

Let A be a set. Then A is *countable* iff it is finite, or there exists a bijection $A \to \mathbb{N}$. It is *uncountable*, iff it is not countable.

Theorem 2 (Countability of subset)

If A is countable and $|B| \leq k|A|$, for some $k \in \mathbb{N}$, then B is countable.

Theorem 3 (Countabilty of Cartesian products)

If A and B are countable, then $A \times B$ is countable.

Theorem 4 (Countability of union)

Let $\{A_n : n \in \mathbb{N}\}$ be a countable family of countable sets. Then, the set $\bigcup_{n \in \mathbb{N}} A_n$ is countable.

Bijection as relation

Theorem 5 (Bijection is an equivalence relation)

Let $\mathcal{A}$ be any family of sets and define a relation $R \subset \mathcal{A} \times \mathcal{A}$ such that, for all $A, B \in \mathcal{A}$, ARB iff there exists a bijection $A \to B$. Then, R is an equivalence relation. That is, R is

1. *reflexive: for all $A \in \mathcal{A}$, ARA;*
2. *symmetric: for all $A, B \in \mathcal{A}$, $ARB \to BRA$;*
3. *transitive: for all $A, B, C \in \mathcal{A}$, $ARB \wedge BRC \to ARC$.*

Proof.

Homework! $\square$

Corollary 6

If B is countable, and there exists a bijection $A \to B$, then A is countable.

Exercise 1

In each of the following examples, let A be an infinite set with the given properties. Prove that A is countable:

1. $A = \{0, 1\}^*$ is the set of all finite binary strings;
2. $A \subset \{0, 1\}^{\mathbb{N}}$ is the set of all infinite binary streams that have finitely many 1's;
3. $A \subset \mathbb{R}$ contains only isolated points, i.e. for all $x \in A$, there exist $a, b \in \mathbb{Q}$, with $a < b$, such that $(a, b) \cap A = \{x\}$;
4. $A \subset \mathbb{C}$ is the set of all algebraic numbers, i.e. the roots of non-zero polynomials of finite degree, in one variable with integer coefficients. Recall that every non-zero polynomial of degree n has exactly n roots (although some may be repeated and some may be complex numbers).

Exercise 1: Solution

We only give a hint for each example:

1. Let A_n be the set of binary strings of length n. Show that, for all $n \in \mathbb{N}$, A_n is finite;
2. Let A_n be the set of infinite binary streams, where the final 1 is at the n-th position. Show that, for all $n \in \mathbb{N}$, A_n is finite;
3. Each $x \in A$ can be associated with a unique pair of rational numbers, and $\mathbb{Q}^2$ is countable;
4. Let P_n be the set of non-zero polynomials of degree n, in one variable, with integer coefficients. Show that for all $n \in \mathbb{N}$, P_n is countable. Let R_n be the set of roots of all polynomials in P_n. Show that R_n is countable.