Mathematical and Logical Foundations of Computer Science

Predicate Logic (Equivalences)

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(some slides were adapted from Rajesh Chitnis' slides)

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Where are we?

- Symbolic logic
- Propositional logic
- **Predicate logic**

Today

Equivalences:

- in Natural Deduction
- using semantics

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- in Natural Deduction
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Further reading:

- Chapter 8 of
  http://leanprover.github.io/logic_and_proof/

Recap: Syntax

The syntax of predicate logic is defined by the following grammar:

$$
\begin{array}{lcl}
t & ::= & x \mid f(t, \ldots, t) \\
P & ::= & p(t, \ldots, t) \mid \neg P \mid P \wedge P \mid P \vee P \mid P \rightarrow P \mid \forall x.P \mid \exists x.P
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where:

- x ranges over variables
- f ranges over function symbols
- $f(t_1, \ldots, t_n)$ is a well-formed term only if f has arity n
- p ranges over predicate symbols
- $p(t_1, \ldots, t_n)$ is a well-formed formula only if p has arity n

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The scope of a quantifier extends as far right as possible. E.g., $P \wedge \forall x.p(x) \vee q(x)$ is read as $P \wedge \forall x.(p(x) \vee q(x))$

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The additional **conditions** ensure that **free variables do not get captured**.

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The additional **conditions** ensure that **free variables do not get captured**.

These conditions can always be met by silently renaming bound variables before substituting.

Recap: ∀ & ∃ elimination and introduction rules

Natural Deduction rules for quantifiers:

$$\frac{P[x\backslash y]}{\forall x.P}\ [\forall I] \qquad \frac{\forall x.P}{P[x\backslash t]}\ [\forall E] \qquad \frac{P[x\backslash t]}{\exists x.P}\ [\exists I] \qquad \frac{\exists x.P \qquad \begin{matrix}\overline{P[x\backslash y]}\ ^{1} \\ \vdots \\ Q\end{matrix}}{Q}\ 1\ [\exists E]$$

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Condition:

- for $[\forall I]$: y must not be free in any not-yet-discharged hypothesis or in $\forall x.P$
- for $[\forall E]$: $\mathtt{fv}(t)$ must not clash with $\mathtt{bv}(P)$
- for $[\exists I]$: $\mathtt{fv}(t)$ must not clash with $\mathtt{bv}(P)$
- for $[\exists E]$: y must not be free in Q or in not-yet-discharged hypotheses or in $\exists x.P$

Recap: Example of a proof

here is a proof of $(\forall z.p(z)) \to \forall x.p(x) \lor q(x)$.

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here is a proof of $(\forall z.p(z)) \rightarrow \forall x.p(x) \vee q(x)$.

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$$\dfrac{\overline{\forall z.p(z)}^{\;1}}{}$$

$$\dfrac{\dfrac{\overline{p(y) \vee q(y)}}{\forall x.p(x) \vee q(x)}\;[\forall I]}{(\forall z.p(z)) \rightarrow \forall x.p(x) \vee q(x)}\;1\;[\rightarrow I]$$

Conditions:

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- y does not clash with bound variables in $p(z)$

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Models: a model provides the interpretation of all symbols

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Given a **signature** $\langle\langle f_1^{k_1}, \ldots, f_n^{k_n}\rangle, \langle p_1^{j_1}, \ldots, p_m^{j_m}\rangle\rangle$

- of function symbols f_i of arity k_i, for $1 \leqslant i \leqslant n$
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a **model** is a structure $\langle D, \langle \mathcal{F}_{f_1}, \ldots, \mathcal{F}_{f_n}\rangle, \langle \mathcal{R}_{p_1}, \ldots, \mathcal{R}_{p_m}\rangle\rangle$

- of a non-empty domain D
- interpretations $\mathcal{F}_{f_i}$ for function symbols f_i
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Models of predicate logic replace **truth assignments** for propositional logic

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Models of predicate logic replace **truth assignments** for propositional logic

Variable valuations:

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- that map variables to D
- i.e., a mapping of the form $x_1 \mapsto d_1, \ldots, x_n \mapsto d_n$

Recap: Semantics of Predicate Logic

Given a **model** M with domain D and a **variable valuation** v:

- $[\![t]\!]_v^M$ gives meaning to the term t w.r.t. M and v
- $\models_{M,v} P$ gives meaning to the formula P w.r.t. M and v

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Meaning of terms:

- $[\![x]\!]_v^M = v(x)$
- $[\![f(t_1, \ldots, t_n)]\!]_v^M = \mathcal{F}_f(\langle [\![t_1]\!]_v^M, \ldots, [\![t_n]\!]_v^M \rangle)$

Recap: Semantics of Predicate Logic

Given a **model** M with domain D and a **variable valuation** v:

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Meaning of formulas:

- $\vDash_{M,v} p(t_1, \ldots, t_n)$ iff $\langle [\![t_1]\!]_v^M, \ldots, [\![t_n]\!]_v^M \rangle \in \mathcal{R}_p$
- $\vDash_{M,v} \neg P$ iff $\neg \vDash_{M,v} P$
- $\vDash_{M,v} P \wedge Q$ iff $\vDash_{M,v} P$ and $\vDash_{M,v} Q$
- $\vDash_{M,v} P \vee Q$ iff $\vDash_{M,v} P$ or $\vDash_{M,v} Q$
- $\vDash_{M,v} P \rightarrow Q$ iff $\vDash_{M,v} Q$ whenever $\vDash_{M,v} P$
- $\vDash_{M,v} \forall x.P$ iff for every $d \in D$ we have $\vDash_{M,(v,x\mapsto d)} P$
- $\vDash_{M,v} \exists x.P$ iff there exists a $d \in D$ such that $\vDash_{M,(v,x\mapsto d)} P$

Recap: Logical equivalences for Propositional Logic

The same equivalences hold as in Propositional Logic:

- De Morgan's law (I): $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$
- De Morgan's law (II): $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$
- Implication elimination: $(A \rightarrow B) \leftrightarrow (\neg A \vee B)$
- Commutativity of $\wedge$: $(A \wedge B) \leftrightarrow (B \wedge A)$
- Commutativity of $\vee$: $(A \vee B) \leftrightarrow (B \vee A)$
- Associativity of $\wedge$: $((A \wedge B) \wedge C) \leftrightarrow (A \wedge (B \wedge C))$
- Associativity of $\vee$: $((A \vee B) \vee C) \leftrightarrow (A \vee (B \vee C))$
- Distributivity of $\wedge$ over $\vee$: $(A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$
- Distributivity of $\vee$ over $\wedge$: $(A \vee (B \wedge C)) \leftrightarrow ((A \vee B) \wedge (A \vee C))$
- Double negation elimination: $(\neg\neg A) \leftrightarrow A$
- Idempotence: $(A \wedge A) \leftrightarrow A$ and $(A \vee A) \leftrightarrow A$

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- $(\forall x.A) \leftrightarrow A$ if $x \notin \mathtt{fv}(A)$

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- $(\forall x.A \vee B) \leftrightarrow ((\forall x.A) \vee B)$ if $x \notin \mathtt{fv}(B)$
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- $(\forall x.A \rightarrow B) \leftrightarrow ((\exists x.A) \rightarrow B)$ if $x \notin \mathtt{fv}(B)$
- $(\exists x.A \rightarrow B) \leftrightarrow ((\forall x.A) \rightarrow B)$ if $x \notin \mathtt{fv}(B)$

Logical Equivalences

In addition, the following hold (some hold only classically):

- $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$
- $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$
- $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$
- $(\neg\exists x.A) \leftrightarrow (\forall x.\neg A)$
- $(\forall x.A) \leftrightarrow A$ if $x \notin \mathtt{fv}(A)$
- $(\exists x.A) \leftrightarrow A$ if $x \notin \mathtt{fv}(A)$
- $(\forall x.A \vee B) \leftrightarrow ((\forall x.A) \vee B)$ if $x \notin \mathtt{fv}(B)$
- $(\exists x.A \wedge B) \leftrightarrow ((\exists x.A) \wedge B)$ if $x \notin \mathtt{fv}(B)$
- $(\forall x.A \rightarrow B) \leftrightarrow ((\exists x.A) \rightarrow B)$ if $x \notin \mathtt{fv}(B)$
- $(\exists x.A \rightarrow B) \leftrightarrow ((\forall x.A) \rightarrow B)$ if $x \notin \mathtt{fv}(B)$
- $(\forall x.A \rightarrow B) \leftrightarrow (A \rightarrow \forall x.B)$ if $x \notin \mathtt{fv}(A)$

Logical Equivalences

In addition, the following hold (some hold only classically):

- $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$
- $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$
- $(\neg \forall x.A) \leftrightarrow (\exists x.\neg A)$
- $(\neg \exists x.A) \leftrightarrow (\forall x.\neg A)$
- $(\forall x.A) \leftrightarrow A$ if $x \notin \texttt{fv}(A)$
- $(\exists x.A) \leftrightarrow A$ if $x \notin \texttt{fv}(A)$
- $(\forall x.A \vee B) \leftrightarrow ((\forall x.A) \vee B)$ if $x \notin \texttt{fv}(B)$
- $(\exists x.A \wedge B) \leftrightarrow ((\exists x.A) \wedge B)$ if $x \notin \texttt{fv}(B)$
- $(\forall x.A \rightarrow B) \leftrightarrow ((\exists x.A) \rightarrow B)$ if $x \notin \texttt{fv}(B)$
- $(\exists x.A \rightarrow B) \leftrightarrow ((\forall x.A) \rightarrow B)$ if $x \notin \texttt{fv}(B)$
- $(\forall x.A \rightarrow B) \leftrightarrow (A \rightarrow \forall x.B)$ if $x \notin \texttt{fv}(A)$
- $(\exists x.A \rightarrow B) \leftrightarrow (A \rightarrow \exists x.B)$ if $x \notin \texttt{fv}(A)$

Logical Equivalences

As before to prove a logical equivalence $A \leftrightarrow B$, we will prove:

- that we can derive B form A
- that we can derive A form B

Logical Equivalences

As before to prove a logical equivalence $A \leftrightarrow B$, we will prove:

- that we can derive B form A
- that we can derive A form B

We will prove:

- $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$
- $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$
- $(\neg \forall x.A) \leftrightarrow (\exists x.\neg A)$
- $(\neg \exists x.A) \leftrightarrow (\forall x.\neg A)$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{}}{}}{} \qquad \dfrac{\dfrac{\dfrac{}{}}{}}{}}{(\forall x.A) \wedge (\forall x.B)}$$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \land B) \leftrightarrow ((\forall x.A) \land (\forall x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\dfrac{}{\forall x.A} \qquad \dfrac{}{\forall x.B}}{(\forall x.A) \land (\forall x.B)}\ [\land I]$$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \land B) \leftrightarrow ((\forall x.A) \land (\forall x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\dfrac{A[x\backslash y]}{\forall x.A}\;[\forall I] \qquad\qquad \dfrac{}{\forall x.B}}{(\forall x.A) \land (\forall x.B)}\;[\land I]$$

- pick y such that it does not occur in A or B
- y must not be free in $\forall x.A \land B$ or in $\forall x.A$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y] \wedge B[x\backslash y]}}{A[x\backslash y]}\,[\wedge E_L]}{\forall x.A}\,[\forall I] \qquad \dfrac{\dfrac{\dfrac{}{\quad}}{\quad}}{\forall x.B}}{(\forall x.A) \wedge (\forall x.B)}\,[\wedge I]$$

- pick y such that it does not occur in A or B
- y must not be free in $\forall x.A \wedge B$ or in $\forall x.A$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \land B) \leftrightarrow ((\forall x.A) \land (\forall x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x.A \land B}{A[x\backslash y] \land B[x\backslash y]}\,[\forall E]}{A[x\backslash y]}\,[\land E_L]}{\forall x.A}\,[\forall I] \qquad \dfrac{\dfrac{\dfrac{\qquad\qquad}{\qquad\qquad}}{\qquad}}{\forall x.B}}{(\forall x.A) \land (\forall x.B)}\,[\land I]$$

- pick y such that it does not occur in A or B
- y must not be free in $\forall x.A \land B$ or in $\forall x.A$
- y must not clash with $\texttt{bv}(A \land B)$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x.A \wedge B}{A[x\backslash y] \wedge B[x\backslash y]}\,[\forall E]}{A[x\backslash y]}\,[\wedge E_L]}{\forall x.A}\,[\forall I] \qquad \dfrac{\dfrac{\dfrac{\qquad}{\qquad}}{B[x\backslash y]}}{\forall x.B}\,[\forall I]}{(\forall x.A) \wedge (\forall x.B)}\,[\wedge I]$$

- pick y such that it does not occur in A or B
- y must not be free in $\forall x.A \wedge B$ or in $\forall x.A$
- y must not clash with $\mathtt{bv}(A \wedge B)$
- y must not be free in $\forall x.A \wedge B$ or in $\forall x.B$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \land B) \leftrightarrow ((\forall x.A) \land (\forall x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x.A \land B}{A[x\backslash y] \land B[x\backslash y]}[\forall E]}{A[x\backslash y]}[\land E_L]}{\forall x.A}[\forall I] \quad \dfrac{\dfrac{\dfrac{}{A[x\backslash y] \land B[x\backslash y]}}{B[x\backslash y]}[\land E_R]}{\forall x.B}[\forall I]}{(\forall x.A) \land (\forall x.B)}[\land I]$$

- pick y such that it does not occur in A or B
- y must not be free in $\forall x.A \land B$ or in $\forall x.A$
- y must not clash with $\texttt{bv}(A \land B)$
- y must not be free in $\forall x.A \land B$ or in $\forall x.B$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\forall x.A \wedge B}{A[x\backslash y] \wedge B[x\backslash y]}\ [\forall E]}{A[x\backslash y]}\ [\wedge E_L]}{\forall x.A}\ [\forall I]
\qquad
\dfrac{\dfrac{\dfrac{\forall x.A \wedge B}{A[x\backslash y] \wedge B[x\backslash y]}\ [\forall E]}{B[x\backslash y]}\ [\wedge E_R]}{\forall x.B}\ [\forall I]
}{(\forall x.A) \wedge (\forall x.B)}\ [\wedge I]
$$

- pick y such that it does not occur in A or B
- y must not be free in $\forall x.A \wedge B$ or in $\forall x.A$
- y must not clash with $\mathtt{bv}(A \wedge B)$
- y must not be free in $\forall x.A \wedge B$ or in $\forall x.B$
- y must not clash with $\mathtt{bv}(A \wedge B)$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\forall x.A \wedge B$$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{A[x\backslash y] \wedge B[x\backslash y]}{\forall x.A \wedge B}\ [\forall I]$$

- pick y such that it does not occur in A or B
- y must not be free in $(\forall x.A) \wedge (\forall x.B)$ or in $\forall x.A \wedge B$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\qquad\qquad}}{A[x\backslash y]} \qquad \dfrac{\dfrac{}{\qquad\qquad}}{B[x\backslash y]}}{A[x\backslash y] \wedge B[x\backslash y]}\,[\wedge I]}{\forall x.A \wedge B}\,[\forall I]$$

- pick y such that it does not occur in A or B
- y must not be free in $(\forall x.A) \wedge (\forall x.B)$ or in $\forall x.A \wedge B$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{\dfrac{\overline{}}{\forall x.A}}{A[x\backslash y]}\,[\forall E] \qquad \dfrac{\overline{}}{B[x\backslash y]}}{\dfrac{A[x\backslash y] \wedge B[x\backslash y]}{\forall x.A \wedge B}\,[\forall I]}\,[\wedge I]$$

- pick y such that it does not occur in A or B
- y must not be free in $(\forall x.A) \wedge (\forall x.B)$ or in $\forall x.A \wedge B$
- y must not clash with $\texttt{bv}(A)$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \land B) \leftrightarrow ((\forall x.A) \land (\forall x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{\dfrac{\dfrac{(\forall x.A) \land (\forall x.B)}{\forall x.A}\,[\land E_L]}{A[x\backslash y]}\,[\forall E] \qquad \dfrac{\dfrac{\qquad}{\qquad}}{B[x\backslash y]}}{A[x\backslash y] \land B[x\backslash y]}\,[\land I]}{\forall x.A \land B}\,[\forall I]$$

- pick y such that it does not occur in A or B
- y must not be free in $(\forall x.A) \land (\forall x.B)$ or in $\forall x.A \land B$
- y must not clash with $\texttt{bv}(A)$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \land B) \leftrightarrow ((\forall x.A) \land (\forall x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$
\dfrac{\dfrac{\dfrac{(\forall x.A) \land (\forall x.B)}{\forall x.A}\,[\land E_L]}{A[x\backslash y]}\,[\forall E] \qquad \dfrac{\dfrac{}{\forall x.B}}{B[x\backslash y]}\,[\forall E]}{\dfrac{A[x\backslash y] \land B[x\backslash y]}{\forall x.A \land B}\,[\forall I]}\,[\land I]
$$

- pick y such that it does not occur in A or B
- y must not be free in $(\forall x.A) \land (\forall x.B)$ or in $\forall x.A \land B$
- y must not clash with $\mathtt{bv}(A)$
- y must not clash with $\mathtt{bv}(B)$

Logical Equivalences

Prove the logical equivalence $(\forall x.A \wedge B) \leftrightarrow ((\forall x.A) \wedge (\forall x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$
\dfrac{\dfrac{\dfrac{\dfrac{(\forall x.A) \wedge (\forall x.B)}{\forall x.A}\,[\wedge E_L]}{A[x\backslash y]}\,[\forall E] \qquad \dfrac{\dfrac{(\forall x.A) \wedge (\forall x.B)}{\forall x.B}\,[\wedge E_R]}{B[x\backslash y]}\,[\forall E]}{A[x\backslash y] \wedge B[x\backslash y]}\,[\wedge I]}{\forall x.A \wedge B}\,[\forall I]
$$

- pick y such that it does not occur in A or B
- y must not be free in $(\forall x.A) \wedge (\forall x.B)$ or in $\forall x.A \wedge B$
- y must not clash with $\texttt{bv}(A)$
- y must not clash with $\texttt{bv}(B)$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$(\exists x.A) \vee (\exists x.B)$$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \lor B) \leftrightarrow ((\exists x.A) \lor (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\exists x.A \lor B \qquad (\exists x.A) \lor (\exists x.B)}{(\exists x.A) \lor (\exists x.B)}\ 1\ [\exists E]$$

- pick y such that it does not occur in A or B
- 1: $A[x\backslash y] \lor B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\exists x.A \vee B \qquad \dfrac{\dfrac{}{A[x\backslash y] \vee B[x\backslash y]} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{}}{}}{}}{A[x\backslash y] \rightarrow (\exists x.A) \vee (\exists x.B)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{}}{}}{}}{B[x\backslash y] \rightarrow (\exists x.A) \vee (\exists x.B)}}{(\exists x.A) \vee (\exists x.B)}\,[\vee E]}{(\exists x.A) \vee (\exists x.B)}\;1\;[\exists E]$$

- pick y such that it does not occur in A or B
- 1: $A[x\backslash y] \vee B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \lor B) \leftrightarrow ((\exists x.A) \lor (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\exists x.A \lor B \qquad \dfrac{\dfrac{}{A[x\backslash y] \lor B[x\backslash y]}\,1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{}}{}}{}}{A[x\backslash y] \to (\exists x.A) \lor (\exists x.B)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{}}{}}{}}{B[x\backslash y] \to (\exists x.A) \lor (\exists x.B)}}{(\exists x.A) \lor (\exists x.B)}\,[\lor E]}{(\exists x.A) \lor (\exists x.B)}\,1\ [\exists E]$$

- pick y such that it does not occur in A or B
- 1: $A[x\backslash y] \lor B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\exists x.A \vee B \qquad \dfrac{\dfrac{}{A[x\backslash y] \vee B[x\backslash y]}\ 1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{}}{}}{(\exists x.A) \vee (\exists x.B)}}{A[x\backslash y] \rightarrow (\exists x.A) \vee (\exists x.B)}\ 2\ [\rightarrow I] \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{}}{}}{}}{B[x\backslash y] \rightarrow (\exists x.A) \vee (\exists x.B)}}{(\exists x.A) \vee (\exists x.B)}\ [\vee E]}{(\exists x.A) \vee (\exists x.B)}\ 1\ [\exists E]$$

- pick y such that it does not occur in A or B
- 1: $A[x\backslash y] \vee B[x\backslash y]$
- 2: $A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$
\begin{prooftree}
\AxiomC{$\exists x.A \vee B$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$A[x\backslash y] \vee B[x\backslash y]$}
\AxiomC{}
\UnaryInfC{}
\UnaryInfC{$\exists x.A$}
\RightLabel{$[\vee I_L]$}
\UnaryInfC{$(\exists x.A) \vee (\exists x.B)$}
\RightLabel{2 $[\to I]$}
\UnaryInfC{$A[x\backslash y] \to (\exists x.A) \vee (\exists x.B)$}
\AxiomC{}
\UnaryInfC{}
\UnaryInfC{}
\UnaryInfC{$B[x\backslash y] \to (\exists x.A) \vee (\exists x.B)$}
\RightLabel{$[\vee E]$}
\TrinaryInfC{$(\exists x.A) \vee (\exists x.B)$}
\RightLabel{1 $[\exists E]$}
\BinaryInfC{$(\exists x.A) \vee (\exists x.B)$}
\end{prooftree}
$$

- pick y such that it does not occur in A or B
- 1: $A[x\backslash y] \vee B[x\backslash y]$
- 2: $A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$
\dfrac{\exists x.A \vee B \qquad \dfrac{\dfrac{}{A[x\backslash y] \vee B[x\backslash y]}\ 1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y]}}{\exists x.A}\ [\exists I]}{(\exists x.A) \vee (\exists x.B)}\ [\vee I_L]}{A[x\backslash y] \rightarrow (\exists x.A) \vee (\exists x.B)}\ 2\ [\rightarrow I] \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{}}{}}{}}{B[x\backslash y] \rightarrow (\exists x.A) \vee (\exists x.B)}}{(\exists x.A) \vee (\exists x.B)}\ [\vee E]}{(\exists x.A) \vee (\exists x.B)}\ 1\ [\exists E]
$$

- pick y such that it does not occur in A or B
- 1: $A[x\backslash y] \vee B[x\backslash y]$
- 2: $A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \lor B) \leftrightarrow ((\exists x.A) \lor (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\exists x.A \lor B \qquad \dfrac{\dfrac{}{A[x\backslash y] \lor B[x\backslash y]}\,1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y]}\,2}{\exists x.A}\,[\exists I]}{(\exists x.A) \lor (\exists x.B)}\,[\lor I_L]}{A[x\backslash y] \to (\exists x.A) \lor (\exists x.B)}\,2\ [\to I] \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\quad}}{\quad}}{\quad}}{B[x\backslash y] \to (\exists x.A) \lor (\exists x.B)}}{(\exists x.A) \lor (\exists x.B)}\,[\lor E]}{(\exists x.A) \lor (\exists x.B)}\,1\ [\exists E]$$

- pick y such that it does not occur in A or B
- 1: $A[x\backslash y] \lor B[x\backslash y]$
- 2: $A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$
\dfrac{\exists x.A \vee B \qquad \dfrac{\dfrac{}{A[x\backslash y] \vee B[x\backslash y]}\,1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y]}\,2}{\exists x.A}\,[\exists I]}{(\exists x.A) \vee (\exists x.B)}\,[\vee I_L]}{A[x\backslash y] \to (\exists x.A) \vee (\exists x.B)}\,2\ [\to I] \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\qquad}}{\qquad}}{(\exists x.A) \vee (\exists x.B)}}{B[x\backslash y] \to (\exists x.A) \vee (\exists x.B)}\,3\ [\to I]}{(\exists x.A) \vee (\exists x.B)}\,[\vee E]}{(\exists x.A) \vee (\exists x.B)}\,1\ [\exists E]
$$

- pick y such that it does not occur in A or B
- 1: $A[x\backslash y] \vee B[x\backslash y]$
- 2: $A[x\backslash y]$
- 3: $B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$
\frac{\exists x.A \vee B \qquad \dfrac{\dfrac{}{A[x\backslash y] \vee B[x\backslash y]}\,1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y]}\,2}{\exists x.A}\,[\exists I]}{(\exists x.A) \vee (\exists x.B)}\,[\vee I_L]}{A[x\backslash y] \to (\exists x.A) \vee (\exists x.B)}\,2\ [\to I] \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{}}{\exists x.B}}{(\exists x.A) \vee (\exists x.B)}\,[\vee I_R]}{B[x\backslash y] \to (\exists x.A) \vee (\exists x.B)}\,3\ [\to I]}{(\exists x.A) \vee (\exists x.B)}\,[\vee E]}{(\exists x.A) \vee (\exists x.B)}\,1\ [\exists E]
$$

- pick y such that it does not occur in A or B
- 1: $A[x\backslash y] \vee B[x\backslash y]$
- 2: $A[x\backslash y]$
- 3: $B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$
\dfrac{\exists x.A \vee B \qquad \dfrac{\dfrac{}{A[x\backslash y] \vee B[x\backslash y]}\,1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A[x\backslash y]}\,2}{\exists x.A}\,[\exists I]}{(\exists x.A) \vee (\exists x.B)}\,[\vee I_L]}{A[x\backslash y] \to (\exists x.A) \vee (\exists x.B)}\,2\;[\to I] \qquad \dfrac{\dfrac{\dfrac{B[x\backslash y]}{\exists x.B}\,[\exists I]}{(\exists x.A) \vee (\exists x.B)}\,[\vee I_R]}{B[x\backslash y] \to (\exists x.A) \vee (\exists x.B)}\,3\;[\to I]}{(\exists x.A) \vee (\exists x.B)}\,[\vee E]}{(\exists x.A) \vee (\exists x.B)}\,1\;[\exists E]
$$

- pick y such that it does not occur in A or B
- 1: $A[x\backslash y] \vee B[x\backslash y]$
- 2: $A[x\backslash y]$
- 3: $B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$
\begin{prooftree}
\AxiomC{$\exists x.A \vee B$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$A[x\backslash y] \vee B[x\backslash y]$}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{$A[x\backslash y]$}
\RightLabel{$[\exists I]$}
\UnaryInfC{$\exists x.A$}
\RightLabel{$[\vee I_L]$}
\UnaryInfC{$(\exists x.A) \vee (\exists x.B)$}
\RightLabel{2 $[\rightarrow I]$}
\UnaryInfC{$A[x\backslash y] \rightarrow (\exists x.A) \vee (\exists x.B)$}
\AxiomC{}
\RightLabel{3}
\UnaryInfC{$B[x\backslash y]$}
\RightLabel{$[\exists I]$}
\UnaryInfC{$\exists x.B$}
\RightLabel{$[\vee I_R]$}
\UnaryInfC{$(\exists x.A) \vee (\exists x.B)$}
\RightLabel{3 $[\rightarrow I]$}
\UnaryInfC{$B[x\backslash y] \rightarrow (\exists x.A) \vee (\exists x.B)$}
\RightLabel{$[\vee E]$}
\TrinaryInfC{$(\exists x.A) \vee (\exists x.B)$}
\RightLabel{1 $[\exists E]$}
\BinaryInfC{$(\exists x.A) \vee (\exists x.B)$}
\end{prooftree}
$$

- pick y such that it does not occur in A or B
- 1: $A[x\backslash y] \vee B[x\backslash y]$
- 2: $A[x\backslash y]$
- 3: $B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\exists x.A \vee B$$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{(\exists x.A) \vee (\exists x.B) \qquad \dfrac{}{\exists x.A \to \exists x.A \vee B} \qquad \dfrac{}{\exists x.B \to \exists x.A \vee B}}{\exists x.A \vee B}\ [\vee E]$$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{(\exists x.A) \vee (\exists x.B) \qquad \dfrac{\exists x.A \vee B}{\exists x.A \to \exists x.A \vee B}\ 1\ [\to I] \qquad \dfrac{}{\exists x.B \to \exists x.A \vee B}}{\exists x.A \vee B}\ [\vee E]$$

- 1: $\exists x.A$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{(\exists x.A) \vee (\exists x.B) \qquad \dfrac{\dfrac{\exists x.A \qquad \exists x.A \vee B}{\exists x.A \vee B}\ 2\ [\exists E]}{\exists x.A \to \exists x.A \vee B}\ 1\ [\to I] \qquad \exists x.B \to \exists x.A \vee B}{\exists x.A \vee B}\ [\vee E]$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{(\exists x.A) \vee (\exists x.B) \qquad \dfrac{\dfrac{\dfrac{}{\exists x.A}\,1 \qquad \dfrac{\dfrac{\dfrac{}{}}{}}{\exists x.A \vee B}}{\exists x.A \vee B}\,2\ [\exists E]}{\exists x.A \to \exists x.A \vee B}\,1\ [\to I] \qquad \dfrac{\dfrac{\dfrac{}{} \qquad \dfrac{\dfrac{}{}}{}}{}}{\exists x.B \to \exists x.A \vee B}}{\exists x.A \vee B}\,[\vee E]$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$
\dfrac{(\exists x.A) \vee (\exists x.B) \qquad \dfrac{\dfrac{\dfrac{}{\exists x.A}\,1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{}}{}}{A[x\backslash y] \vee B[x\backslash y]}}{\exists x.A \vee B}\,[\exists I]}{\exists x.A \vee B}\,2\ [\exists E]}{\exists x.A \rightarrow \exists x.A \vee B}\,1\ [\rightarrow I] \qquad \dfrac{\dfrac{\dfrac{}{} \qquad \dfrac{\dfrac{\dfrac{}{}}{}}{}}{}}{\exists x.B \rightarrow \exists x.A \vee B}}{\exists x.A \vee B}\,[\vee E]
$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$
\dfrac{(\exists x.A) \vee (\exists x.B) \qquad \dfrac{\dfrac{\dfrac{}{\exists x.A}\,1 \qquad \dfrac{\dfrac{\dfrac{}{A[x\backslash y]}}{A[x\backslash y] \vee B[x\backslash y]}\,[\vee I_L]}{\exists x.A \vee B}\,[\exists I]}{\exists x.A \vee B}\,2\ [\exists E]}{\exists x.A \rightarrow \exists x.A \vee B}\,1\ [\rightarrow I] \qquad \dfrac{\dfrac{\qquad \quad \dfrac{\dfrac{\dfrac{}{\qquad}}{\qquad\qquad}}{\qquad\qquad}}{\qquad\qquad\qquad}}{\exists x.B \rightarrow \exists x.A \vee B}}{\exists x.A \vee B}\,[\vee E]
$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$
\dfrac{(\exists x.A) \vee (\exists x.B) \qquad \dfrac{\dfrac{\dfrac{}{\exists x.A}\,1 \qquad \dfrac{\dfrac{\dfrac{}{A[x\backslash y]}\,2}{A[x\backslash y] \vee B[x\backslash y]}\,[\vee I_L]}{\exists x.A \vee B}\,[\exists I]}{\exists x.A \vee B}\,2\ [\exists E]}{\exists x.A \rightarrow \exists x.A \vee B}\,1\ [\rightarrow I] \qquad \exists x.B \rightarrow \exists x.A \vee B}{\exists x.A \vee B}\,[\vee E]
$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$
\begin{prooftree}
\AxiomC{$(\exists x.A) \vee (\exists x.B)$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\exists x.A$}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{$A[x\backslash y]$}
\RightLabel{$[\vee I_L]$}
\UnaryInfC{$A[x\backslash y] \vee B[x\backslash y]$}
\RightLabel{$[\exists I]$}
\UnaryInfC{$\exists x.A \vee B$}
\RightLabel{2 $[\exists E]$}
\BinaryInfC{$\exists x.A \vee B$}
\RightLabel{1 $[\rightarrow I]$}
\UnaryInfC{$\exists x.A \rightarrow \exists x.A \vee B$}
\AxiomC{}
\UnaryInfC{}
\AxiomC{}
\UnaryInfC{}
\UnaryInfC{}
\UnaryInfC{}
\BinaryInfC{$\exists x.A \vee B$}
\RightLabel{3 $[\rightarrow I]$}
\UnaryInfC{$\exists x.B \rightarrow \exists x.A \vee B$}
\RightLabel{$[\vee E]$}
\TrinaryInfC{$\exists x.A \vee B$}
\end{prooftree}
$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$
- 3: $\exists x.B$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{(\exists x.A) \vee (\exists x.B) \qquad \dfrac{\dfrac{\dfrac{}{\exists x.A}\,1 \quad \dfrac{\dfrac{\dfrac{}{A[x\backslash y]}\,2}{A[x\backslash y] \vee B[x\backslash y]}\,[\vee I_L]}{\exists x.A \vee B}\,[\exists I]}{\exists x.A \vee B}\,2\ [\exists E]}{\exists x.A \to \exists x.A \vee B}\,1\ [\to I] \qquad \dfrac{\dfrac{\dfrac{}{\exists x.B} \quad \dfrac{\dfrac{\dfrac{}{\qquad}}{\qquad}}{\exists x.A \vee B}}{\exists x.A \vee B}\,4\ [\exists E]}{\exists x.B \to \exists x.A \vee B}\,3\ [\to I]}{\exists x.A \vee B}\,[\vee E]$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$
- 3: $\exists x.B$
- 4: $B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$
\begin{prooftree}
\AxiomC{$(\exists x.A) \vee (\exists x.B)$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\exists x.A$}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{$A[x\backslash y]$}
\RightLabel{$[\vee I_L]$}
\UnaryInfC{$A[x\backslash y] \vee B[x\backslash y]$}
\RightLabel{$[\exists I]$}
\UnaryInfC{$\exists x.A \vee B$}
\RightLabel{2 $[\exists E]$}
\BinaryInfC{$\exists x.A \vee B$}
\RightLabel{1 $[\to I]$}
\UnaryInfC{$\exists x.A \to \exists x.A \vee B$}
\AxiomC{}
\RightLabel{3}
\UnaryInfC{$\exists x.B$}
\AxiomC{}
\UnaryInfC{}
\UnaryInfC{}
\UnaryInfC{$\exists x.A \vee B$}
\RightLabel{4 $[\exists E]$}
\BinaryInfC{$\exists x.A \vee B$}
\RightLabel{3 $[\to I]$}
\UnaryInfC{$\exists x.B \to \exists x.A \vee B$}
\RightLabel{$[\vee E]$}
\TrinaryInfC{$\exists x.A \vee B$}
\end{prooftree}
$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$
- 3: $\exists x.B$
- 4: $B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \lor B) \leftrightarrow ((\exists x.A) \lor (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$
\begin{prooftree}
\AxiomC{$(\exists x.A) \lor (\exists x.B)$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\exists x.A$}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{$A[x\backslash y]$}
\RightLabel{$[\lor I_L]$}
\UnaryInfC{$A[x\backslash y] \lor B[x\backslash y]$}
\RightLabel{$[\exists I]$}
\UnaryInfC{$\exists x.A \lor B$}
\RightLabel{2 $[\exists E]$}
\BinaryInfC{$\exists x.A \lor B$}
\RightLabel{1 $[\rightarrow I]$}
\UnaryInfC{$\exists x.A \rightarrow \exists x.A \lor B$}
\AxiomC{}
\RightLabel{3}
\UnaryInfC{$\exists x.B$}
\AxiomC{}
\UnaryInfC{}
\UnaryInfC{$A[x\backslash y] \lor B[x\backslash y]$}
\RightLabel{$[\exists I]$}
\UnaryInfC{$\exists x.A \lor B$}
\RightLabel{4 $[\exists E]$}
\BinaryInfC{$\exists x.A \lor B$}
\RightLabel{3 $[\rightarrow I]$}
\UnaryInfC{$\exists x.B \rightarrow \exists x.A \lor B$}
\RightLabel{$[\lor E]$}
\TrinaryInfC{$\exists x.A \lor B$}
\end{prooftree}
$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$
- 3: $\exists x.B$
- 4: $B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \vee B) \leftrightarrow ((\exists x.A) \vee (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{(\exists x.A) \vee (\exists x.B) \qquad \dfrac{\dfrac{\dfrac{}{\exists x.A}\,1 \quad \dfrac{\dfrac{\dfrac{}{A[x\backslash y]}\,2}{A[x\backslash y] \vee B[x\backslash y]}\,[\vee I_L]}{\exists x.A \vee B}\,[\exists I]}{\exists x.A \vee B}\,2\ [\exists E]}{\exists x.A \to \exists x.A \vee B}\,1\ [\to I] \qquad \dfrac{\dfrac{\dfrac{}{\exists x.B}\,3 \quad \dfrac{\dfrac{\dfrac{}{B[x\backslash y]}}{A[x\backslash y] \vee B[x\backslash y]}\,[\vee I_R]}{\exists x.A \vee B}\,[\exists I]}{\exists x.A \vee B}\,4\ [\exists E]}{\exists x.B \to \exists x.A \vee B}\,3\ [\to I]}{\exists x.A \vee B}\,[\vee E]$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$
- 3: $\exists x.B$
- 4: $B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\exists x.A \lor B) \leftrightarrow ((\exists x.A) \lor (\exists x.B))$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$
\dfrac{(\exists x.A) \lor (\exists x.B) \qquad \dfrac{\dfrac{\dfrac{}{\exists x.A}\,1 \qquad \dfrac{\dfrac{\dfrac{}{A[x\backslash y]}\,2}{A[x\backslash y] \lor B[x\backslash y]}\,[\lor I_L]}{\exists x.A \lor B}\,[\exists I]}{\exists x.A \lor B}\,2\ [\exists E]}{\exists x.A \to \exists x.A \lor B}\,1\ [\to I] \qquad \dfrac{\dfrac{\dfrac{}{\exists x.B}\,3 \qquad \dfrac{\dfrac{\dfrac{}{B[x\backslash y]}\,4}{A[x\backslash y] \lor B[x\backslash y]}\,[\lor I_R]}{\exists x.A \lor B}\,[\exists I]}{\exists x.A \lor B}\,4\ [\exists E]}{\exists x.B \to \exists x.A \lor B}\,3\ [\to I]}{\exists x.A \lor B}\,[\lor E]
$$

- 1: $\exists x.A$
- pick y such that it does not occur in A or B
- 2: $A[x\backslash y]$
- 3: $\exists x.B$
- 4: $B[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (classical):

$$\exists x.\neg A$$

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (classical):

$$\dfrac{\neg\neg(\exists x.\neg A)}{\exists x.\neg A}\ [DNE]$$

- 1: $\neg(\exists x.\neg A)$

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (classical):

$$\dfrac{\dfrac{\bot}{\neg\neg(\exists x.\neg A)}\ 1\ [\neg I]}{\exists x.\neg A}\ [DNE]$$

- 1: $\neg(\exists x.\neg A)$

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (classical):

$$\dfrac{\dfrac{\dfrac{\neg\forall x.A \qquad \dfrac{}{\forall x.A}}{\bot}\,[\neg E]}{\neg\neg(\exists x.\neg A)}\,1\ [\neg I]}{\exists x.\neg A}\,[DNE]$$

- 1: $\neg(\exists x.\neg A)$

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (classical):

$$
\dfrac{\dfrac{\dfrac{\neg\forall x.A \qquad \dfrac{A[x\backslash y]}{\forall x.A}\,[\forall I]}{\bot}\,[\neg E]}{\neg\neg(\exists x.\neg A)}\,1\ [\neg I]}{\exists x.\neg A}\,[DNE]
$$

- 1: $\neg(\exists x.\neg A)$
- pick y such that it does not occur in A

Logical Equivalences

Prove the logical equivalence $(\neg \forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (classical):

$$\dfrac{\dfrac{\neg\forall x.A \qquad \dfrac{\dfrac{\overline{\neg\neg A[x\backslash y]}}{A[x\backslash y]}\,[DNE]}{\forall x.A}\,[\forall I]}{\bot}\,[\neg E]}{\dfrac{\neg\neg(\exists x.\neg A)}{\exists x.\neg A}\,[DNE]}\;1\;[\neg I]$$

- 1: $\neg(\exists x.\neg A)$
- pick y such that it does not occur in A

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (classical):

$$
\dfrac{\dfrac{\dfrac{\neg\forall x.A \qquad \dfrac{\dfrac{\dfrac{\bot}{\neg\neg A[x\backslash y]}\ 2\ [\neg I]}{A[x\backslash y]}\ [DNE]}{\forall x.A}\ [\forall I]}{\bot}\ [\neg E]}{\neg\neg(\exists x.\neg A)}\ 1\ [\neg I]}{\exists x.\neg A}\ [DNE]
$$

- 1: $\neg(\exists x.\neg A)$
- pick y such that it does not occur in A
- 2: $\neg A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (classical):

$$\dfrac{\dfrac{\dfrac{\neg\forall x.A \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(\exists x.\neg A)} \qquad \dfrac{\dfrac{}{}}{\exists x.\neg A}}{\bot}\,[\neg E]}{\neg\neg A[x\backslash y]}\,2\ [\neg I]}{A[x\backslash y]}\,[DNE]}{\forall x.A}\,[\forall I]}{\bot}\,[\neg E]}{\neg\neg(\exists x.\neg A)}\,1\ [\neg I]}{\exists x.\neg A}\,[DNE]$$

- 1: $\neg(\exists x.\neg A)$
- pick y such that it does not occur in A
- 2: $\neg A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (classical):

$$
\dfrac{\dfrac{\dfrac{\neg\forall x.A \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(\exists x.\neg A)}\,1 \qquad \dfrac{\dfrac{}{}}{\exists x.\neg A}}{\bot}\,[\neg E]}{\neg\neg A[x\backslash y]}\,2\ [\neg I]}{A[x\backslash y]}\,[DNE]}{\forall x.A}\,[\forall I]}{\bot}\,[\neg E]}{\neg\neg(\exists x.\neg A)}\,1\ [\neg I]}{\exists x.\neg A}\,[DNE]
$$

- 1: $\neg(\exists x.\neg A)$
- pick y such that it does not occur in A
- 2: $\neg A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (classical):

$$
\begin{prooftree}
\AxiomC{$\neg\forall x.A$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\neg(\exists x.\neg A)$}
\AxiomC{}
\UnaryInfC{$\neg A[x\backslash y]$}
\RightLabel{$[\exists I]$}
\UnaryInfC{$\exists x.\neg A$}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{$2\ [\neg I]$}
\UnaryInfC{$\neg\neg A[x\backslash y]$}
\RightLabel{$[DNE]$}
\UnaryInfC{$A[x\backslash y]$}
\RightLabel{$[\forall I]$}
\UnaryInfC{$\forall x.A$}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{$1\ [\neg I]$}
\UnaryInfC{$\neg\neg(\exists x.\neg A)$}
\RightLabel{$[DNE]$}
\UnaryInfC{$\exists x.\neg A$}
\end{prooftree}
$$

- 1: $\neg(\exists x.\neg A)$
- pick y such that it does not occur in A
- 2: $\neg A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (classical):

$$
\dfrac{\dfrac{\dfrac{\neg\forall x.A \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overline{\neg(\exists x.\neg A)}^{\,1} \qquad \dfrac{\overline{\neg A[x\backslash y]}^{\,2}}{\exists x.\neg A}\,[\exists I]}{\bot}\,[\neg E]}{\neg\neg A[x\backslash y]}\,2\ [\neg I]}{A[x\backslash y]}\,[DNE]}{\forall x.A}\,[\forall I]}{\bot}\,[\neg E]}{\neg\neg(\exists x.\neg A)}\,1\ [\neg I]}{\exists x.\neg A}\,[DNE]
$$

- 1: $\neg(\exists x.\neg A)$
- pick y such that it does not occur in A
- 2: $\neg A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{}{\neg\forall x.A}$$

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\bot}{\neg\forall x.A}\ 1\ [\neg I]$$

- 1: $\forall x.A$

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{\exists x.\neg A \qquad \dfrac{\dfrac{}{} \qquad \dfrac{\dfrac{}{}}{}}{\bot}}{\bot}\;2\;[\exists E]}{\neg\forall x.A}\;1\;[\neg I]$$

- 1: $\forall x.A$
- pick y such that it does not occur in A
- 2: $\neg A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{\exists x.\neg A \qquad \dfrac{\dfrac{}{\neg A[x\backslash y]} \qquad \dfrac{\dfrac{}{}}{A[x\backslash y]}}{\bot}\ [\neg E]}{\bot}\ 2\ [\exists E]}{\neg\forall x.A}\ 1\ [\neg I]$$

- 1: $\forall x.A$
- pick y such that it does not occur in A
- 2: $\neg A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{\exists x.\neg A \qquad \dfrac{\dfrac{}{\neg A[x\backslash y]}\ 2 \qquad \dfrac{\dfrac{}{}}{A[x\backslash y]}}{\bot}\ [\neg E]}{\bot}\ 2\ [\exists E]}{\neg\forall x.A}\ 1\ [\neg I]$$

- 1: $\forall x.A$
- pick y such that it does not occur in A
- 2: $\neg A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\cfrac{\cfrac{\exists x.\neg A \qquad \cfrac{\cfrac{}{\neg A[x\backslash y]}\;2 \qquad \cfrac{\cfrac{}{\forall x.A}}{A[x\backslash y]}\;[\forall E]}{\bot}\;[\neg E]}{\bot}\;2\ [\exists E]}{\neg\forall x.A}\;1\ [\neg I]$$

- 1: $\forall x.A$
- pick y such that it does not occur in A
- 2: $\neg A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\neg\forall x.A) \leftrightarrow (\exists x.\neg A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{\exists x.\neg A \qquad \dfrac{\dfrac{}{\neg A[x\backslash y]}\;2 \qquad \dfrac{\dfrac{}{\forall x.A}\;1}{A[x\backslash y]}\;[\forall E]}{\bot}\;[\neg E]}{\bot}\;2\ [\exists E]}{\neg\forall x.A}\;1\ [\neg I]$$

- 1: $\forall x.A$
- pick y such that it does not occur in A
- 2: $\neg A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\neg\exists x.A) \leftrightarrow (\forall x.\neg A)$ in Natural Deduction

Logical Equivalences

Prove the logical equivalence $(\neg\exists x.A) \leftrightarrow (\forall x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\forall x.\neg A$$

Logical Equivalences

Prove the logical equivalence $(\neg\exists x.A) \leftrightarrow (\forall x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\overline{\neg A[x\backslash y]}}{\forall x.\neg A}\ [\forall I]$$

- pick y such that it does not occur in A

Logical Equivalences

Prove the logical equivalence $(\neg\exists x.A) \leftrightarrow (\forall x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\dfrac{\perp}{\neg A[x\backslash y]}\ 1\ [\neg I]}{\forall x.\neg A}\ [\forall I]$$

- pick y such that it does not occur in A
- 1: $A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\neg\exists x.A) \leftrightarrow (\forall x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\dfrac{\dfrac{\neg\exists x.A \qquad \dfrac{\overline{}}{\exists x.A}}{\bot}\ [\neg E]}{\neg A[x\backslash y]}\ 1\ [\neg I]}{\forall x.\neg A}\ [\forall I]$$

- pick y such that it does not occur in A
- 1: $A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\neg\exists x.A) \leftrightarrow (\forall x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\cfrac{\cfrac{\cfrac{\neg\exists x.A \qquad \cfrac{\overline{A[x\backslash y]}}{\exists x.A}\;[\exists I]}{\bot}\;[\neg E]}{\neg A[x\backslash y]}\;1\ [\neg I]}{\forall x.\neg A}\;[\forall I]$$

- pick y such that it does not occur in A
- 1: $A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\neg\exists x.A) \leftrightarrow (\forall x.\neg A)$ in Natural Deduction

Here is a proof of the left-to-right implication (constructive):

$$\dfrac{\dfrac{\neg\exists x.A \qquad \dfrac{\dfrac{}{A[x\backslash y]}\ 1}{\exists x.A}\ [\exists I]}{\bot}\ [\neg E]}{\dfrac{\neg A[x\backslash y]}{\forall x.\neg A}\ [\forall I]}\ 1\ [\neg I]$$

- pick y such that it does not occur in A
- 1: $A[x\backslash y]$

Logical Equivalences

Prove the logical equivalence $(\neg\exists x.A) \leftrightarrow (\forall x.\neg A)$ in Natural Deduction

Logical Equivalences

Prove the logical equivalence $(\neg\exists x.A) \leftrightarrow (\forall x.\neg A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\dfrac{\dfrac{}{} \qquad \dfrac{\dfrac{}{} \qquad \dfrac{}{}}{}}{}}{\neg\exists x.A}$$

Logical Equivalences

Prove the logical equivalence $(\neg\exists x.A) \leftrightarrow (\forall x.\neg A)$ in Natural Deduction

Here is a proof of the right-to-left implication (constructive):

$$\dfrac{\perp}{\neg\exists x.A}\ 1\ [\neg I]$$

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- 1: $\exists x.A$
- pick y such that it does not occur in A
- 2: $A[x\backslash y]$

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\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\exists x.A$}
\AxiomC{$\forall x.\neg A$}
\RightLabel{$[\forall E]$}
\UnaryInfC{$\neg A[x\backslash y]$}
\AxiomC{}
\UnaryInfC{$A[x\backslash y]$}
\RightLabel{$[\neg E]$}
\BinaryInfC{$\bot$}
\RightLabel{2 $[\exists E]$}
\BinaryInfC{$\bot$}
\RightLabel{1 $[\neg I]$}
\UnaryInfC{$\neg \exists x.A$}
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Logical Equivalences

As before: if ($P \leftrightarrow Q$ or $Q \leftrightarrow P$) and P occurs in A, then replacing P by Q in A leads to a formula B, such that $A \leftrightarrow B$

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Semantical equivalence: two formulas P and Q are equivalent if for all models M and valuations v, $\models_{M,v} P$ iff $\models_{M,v} Q$

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- if $\vDash_{M,v} \neg\exists x.A$ then $\vDash_{M,v} \forall x.\neg A$
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  - assumption: $\vDash_{M,v} \neg\exists x.A$, i.e., it is not the case that there exists a $e \in D$ such that $\vDash_{M,v,x\mapsto e} A$
  - contradiction! there is one: take $e = d$
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  - assume that there exists a $e \in D$ such that $\vDash_{M,v,x\mapsto e} A$, and prove a contradiction
  - assumption: $\vDash_{M,v} \forall x.\neg A$, i.e., for every $d \in D$ it is not the case that $\vDash_{M,v,x\mapsto d} A$
  - therefore, instantiating this assumption with e: it is not the case that $\vDash_{M,v,x\mapsto e} A$

Logical Equivalences

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  - contradiction!

Conclusion

What did we cover today?

- Equivalence using Natural Deduction
- Equivalences using semantics

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Further reading:

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Next time?

- Predicate Logic – Equivalences