MAT 216

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Sec: 07

Assignment 02

Ans to the question no 1(a)

$$A = \begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ -1 & 1 & 3 & -1 & 0 \\ -2 & 1 & 4 & -1 & 3 \\ 3 & -1 & -5 & 1 & -6 \end{pmatrix} \xRightarrow[\substack{r_3+2r_1 \\ r_4-3r_1}]{r_2+r_1} \begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 3 & 5 \\ 0 & -1 & -2 & -5 & -9 \end{pmatrix}$$

$$\xRightarrow[r_4+r_3]{r_3-r_2} \begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & -4 & -8 \end{pmatrix} \xRightarrow{r_4+2r_3} \begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

A has three pivots meaning three linearly independent column.

$\therefore$ rank $(A) = 3$

We know,

rank (A) + nullity (A) = ~~number~~ number of column (A)

$\Rightarrow$ nullity$(A) = 5 - 3 = 2$

$\therefore$ dimension $\big($nullspace$(A)\big) = 2$

Now, B is a 5×5 matrix for which $AB = 0$. Now let's consider the columns of B as c_1, c_2, c_3, c_4, c_5

$\because AB = 0$

$\Rightarrow A(c_1 \; c_2 \; c_3 \; c_4 \; c_5) = 0$ [where 0 is a 4×5 matrix]

$\Rightarrow (Ac_1 \; Ac_2 \; Ac_3 \; A_4 \; Ac_5) = 0$

$\therefore AC_1 = AC_2 = AC_3 = AC_4 = AC_5 = 0$

$\therefore C_i \in$ nullspace (A)

As ~~dim~~ $\dim$ (nullspace(A)) $= 2$, B can be a matrix which has 2 linearly independent column.

Now. $AC = 0$ $[C \in \text{nullspace}(A)]$

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0$$

$\therefore$ from rref of A x_5 and x_3 are free variables.

$\therefore x_5 = a \quad x_3 = b$

from RREF,

i) $2x_4 + 4x_5 = 0$

$\Rightarrow x_4 = -2x_5 = -2a$

ii) $x_2 + 2x_3 + x_4 + x_5 = 0$

$\Rightarrow x_2 = -2x_3 - x_4 - x_5 = -2b + 2a - a = -2b + a$

iii) $x_1 - x_3 + 2x_4 + x_5 = 0$

$\Rightarrow x_1 = x_3 - 2x_4 - x_5 = b + 4a - a = b + 3a$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} b+3a \\ -2b+a \\ b \\ -2a \\ a \end{pmatrix} = b \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + a \begin{pmatrix} 3 \\ 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

$\therefore$ nullspace is spanned by $\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}$

$\therefore$ two linearly indepent column are $\begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}$

$$\therefore B = \begin{pmatrix} -1 & 3 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Ans to the question no 1(b)

From 1(a) we got,

$\text{rank}(A) = 3$

$\dim(\text{nullspace}) = 2$

Here C is a 5×5 matrix for which $AC = 0$. Now let's consider the column of B as c_1, c_2, c_3, c_4, c_5.

$\therefore AC = 0$

$\Rightarrow A(c_1\ c_2\ c_3\ c_4\ c_5) = 0$

$\Rightarrow (Ac_1\ Ac_2\ Ac_3\ Ac_4\ Ac_5) = 0$

$\therefore Ac_1 = Ac_2 = Ac_3 = Ac_4 = Ac_5 = 0$

$\therefore c_i \in \text{nullspace}(A)$

As $\dim(\text{nullspace}(A)) = 2$, C can have 2 or less than 2 linearly independent column.

$\therefore \text{rank}(C) \leq 2$

Ans to the question no 2(a)

Given RREF form of matrix A,

$$A_{RREF} = \begin{pmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -5 & 0 & -3 \\ 0 & 0 & 0 & 1 & 6 \end{pmatrix}$$

Here second, first and fourth column has pivots. Therefore these three columns are linearly independent column. Then the other two columns, column three and column five are linear combination of column column ~~to~~ one, two and four. as they are linearly dependent.

In the rref form

$$2(\text{column }1) + (-5)(\text{column }2) = \text{column }3$$

$$-2(\text{column }1) + (-3)(\text{column }2) + 6(\text{column }4) = \text{column }5$$

Here, Given column of A,

$$\text{column }1 = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \quad \text{column }2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \quad \text{column }4 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

$$\therefore \text{column }3 = 2\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + (-5)\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 6 \end{pmatrix} + \begin{pmatrix} 0 \\ 5 \\ -5 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$$\therefore \text{column }5 = -2\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + (-3)\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + 6\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 2 \\ -6 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix} + \begin{pmatrix} 6 \\ -12 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -7 \\ -9 \end{pmatrix}$$

$$\therefore A = \begin{pmatrix} 1 & 0 & 2 & 1 & 4 \\ -1 & -1 & 3 & -2 & -7 \\ 3 & 1 & 1 & 0 & -9 \end{pmatrix}$$

<u>Ans to the question no 3(a)</u>

Given system of equations:

$$x + 2y - z = -1$$

$$2x + 2y + z = 1$$

$$5x + 5y - 2z = -1$$

This can be represented as:

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 1 \\ 5 & 5 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

Now,

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 1 \\ 5 & 5 & -2 \end{bmatrix} \underset{r_3-5r_1}{\overset{r_2-2r_1}{\Longrightarrow}} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 3 \\ 0 & -5 & 3 \end{bmatrix} \underset{(-1)r_3}{\overset{r_2 \div (-2)}{\Longrightarrow}} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 5 & -3 \end{bmatrix} \overset{r_3 - 5r_2}{\Longrightarrow} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & \frac{9}{2} \end{bmatrix}$$

$$\underset{r_3 \times \frac{2}{9}}{\Longrightarrow} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

Again,

$$\begin{bmatrix}-1\\1\\-1\end{bmatrix} \xRightarrow[r_3-5r_1]{r_2-2r_1} \begin{bmatrix}-1\\3\\4\end{bmatrix} \xRightarrow[(-1)r_3]{r_2\div(-2)} \begin{bmatrix}-1\\-\frac{3}{2}\\-4\end{bmatrix} \xRightarrow{r_3-5r_2} \begin{bmatrix}-1\\-\frac{3}{2}\\\frac{7}{2}\end{bmatrix} \xRightarrow{r_3\times\frac{2}{9}} \begin{bmatrix}-1\\-\frac{3}{2}\\\frac{7}{9}\end{bmatrix}$$

$$\therefore \begin{bmatrix}1 & 2 & -1\\0 & 1 & -\frac{3}{2}\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-1\\-\frac{3}{2}\\\frac{7}{9}\end{bmatrix}$$

$$\therefore z = \frac{7}{9}$$

$$\therefore y - \frac{3}{2}z = -\frac{3}{2}$$

$$\Rightarrow y = -\frac{3}{2} + \left(\frac{3}{2}\times\frac{7}{9}\right) = -\frac{3}{2} + \frac{7}{6} = \frac{-2}{6} = \frac{-1}{3}$$

$$\therefore x + 2y - z = -1$$

$$\Rightarrow x = -1 + \frac{2}{3} + \frac{7}{9} = \frac{-9+6+7}{9} = \frac{4}{9}$$

$\therefore$ solution of the system:

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}\frac{4}{9}\\-\frac{1}{3}\\\frac{7}{9}\end{bmatrix}$$

Ans to the question no 3(b)

The given system:

$$x_1 + x_2 - 3x_3 + x_4 = 0$$

$$x_1 + x_2 + x_3 - x_4 = 0$$

$$x_1 + x_2 + x_3 = 0$$

This can be represented as:

$$\begin{bmatrix} 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 0 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 1 & -3 & 1 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 2 & -1 \end{bmatrix} \xrightarrow{R_3 - \frac{R_2}{2}} \begin{bmatrix} 1 & 1 & -3 & 1 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & 1 & -3 & 1 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ x_2 and x_4 are free variables.

Let, $x_2 = p$ and $x_4 = q$

$$\therefore 4x_3 - 2x_4 = 0$$

$$\Rightarrow x_3 = \frac{q}{2}$$

$$\therefore x_1 + x_2 - 3x_3 + x_4 = 0$$

$$\Rightarrow x_1 = -p + \frac{q}{2} - q = -p - \frac{q}{2}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -p - \frac{q}{2} \\ p \\ \frac{q}{2} \\ q \end{bmatrix} = \begin{bmatrix} -p \\ p \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{q}{2} \\ 0 \\ \frac{q}{2} \\ q \end{bmatrix} = p \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{q}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ basis for solution set are $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

Ans to the question no 4(a)

We know, for a consistent system of linear equation there can be two outcomes:

i) one solution

ii) infinite solution.

If a system has more than one solution there has to be infinite ~~sto~~ solution for that system.

Therefore, we can't find a system of linear equation that has exactly two real solution. If that system has two solutions, there has to be infinite solutions.

Ans to the question no 4(b)

Let us consider solution set of a linear system

$Ax = b$ is B

Now, b_1 and b_2 are two stolution of the solution set B

$\therefore Ab_1 = b$ and $Ab_2 = b$

Again, $y = tb_1 + (1-t)b_2$

$Ay = A(tb_1 + (1-t)b_2)$

$Ay = t(Ab_1) + (1-t)Ab_2 \quad [t \in [0,1]]$

$\Rightarrow Ay = tb + (1-t)b$

$= b$

$\therefore y \in S$

$\Rightarrow tb_1 + (1-t)b_2 \in S$

$\therefore$ Solution set of any linear equation is convex