Vector Analysis

Objectives of this Chapter:
Mainly a revision of the basic concepts of Vector Analysis (2^{nd} Semester Outcome):

Important Concepts regarding **Vector Field:**

1. A Vector Field may be defined as a **Vector Function of a Position Vector** .

Important Properties of the **Dot Product**:

1. Dot Product is Commutative:

$$\boxed{\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}}$$

2. Obeys the Distributive law:

3. A vector 'dotted' with itself yields its magnitude squared.

$$\boxed{\mathbf{A} \cdot \mathbf{A} = A^2 = |\mathbf{A}|^2}$$

Important Properties of the **Cross Product:**

1. Cross Product is non-Commutative:

$$\mathbf{B} \times \mathbf{A} = -(\mathbf{A} \times \mathbf{B})$$

2.

$$\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z,$$

$$\mathbf{a}_y \times \mathbf{a}_z = \mathbf{a}_x$$

$$\mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y$$

Note the alphabetic symmetry. As long as the three vectors $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ are written in order (and assuming that $\mathbf{a}_x$ follows $\mathbf{a}_z$, like three elephants in a circle holding tails, so that we could also write $\mathbf{a}_y$, $\mathbf{a}_z$, $\mathbf{a}_x$ or $\mathbf{a}_z$, $\mathbf{a}_x$, $\mathbf{a}_y$), then the cross and equal sign may be placed in either of the two vacant spaces. As a matter of fact, it is now simpler to define a right-handed rectangular coordinate system by saying that $\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z$.

A simple example of the use of the cross product may be taken from geometry or trigonometry. To find the area of a parallelogram, the product of the lengths of two adjacent sides is multiplied by the sine of the angle between them. Using vector notation for the two sides, we then may express the (scalar) area as the *magnitude* of $\mathbf{A} \times \mathbf{B}$, or $|\mathbf{A} \times \mathbf{B}|$.

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\mathbf{a}_x + (A_z B_x - A_x B_z)\mathbf{a}_y + (A_x B_y - A_y B_x)\mathbf{a}_z$$

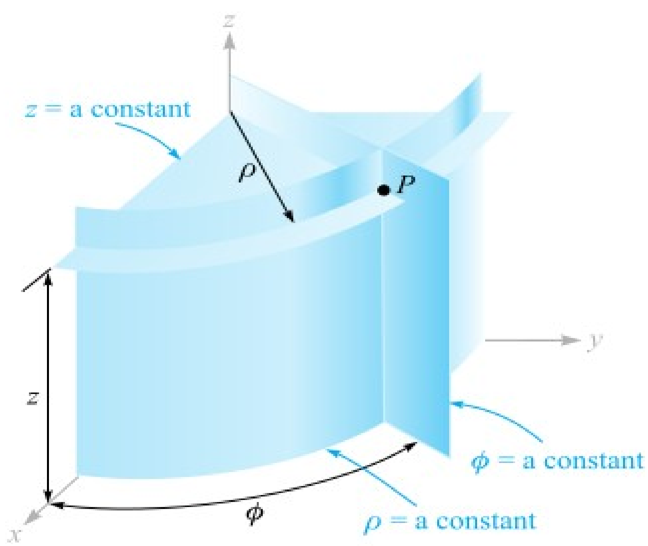
$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Cylindrical Coordinates

The **Circular Cylindrical Coordinate System** is the 3D version of the Polar Coordinate System in analytic geometry.

Just as we assume any point in the rectangular coordinate system as the intersection of 3 planes *(x = const)(|| to yz-plane), (y = const)(|| to xz-plane), (z = const)(|| to xy-plane).* The point in a cylindrical coordinate system is an intersection of the cylindrical surface (ρ = const) , the plane passing through the central axis ($\varphi = const$) and through the plane parallel to the yz-plane ($z = const$)

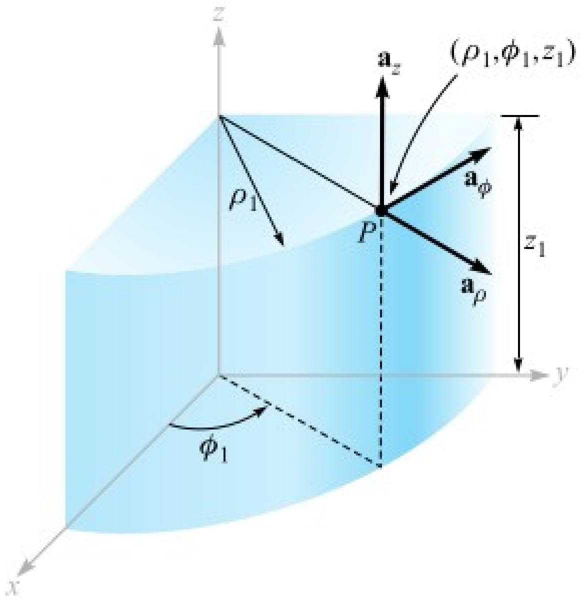
All the above **3** surfaces are **perpendicular to each other**.



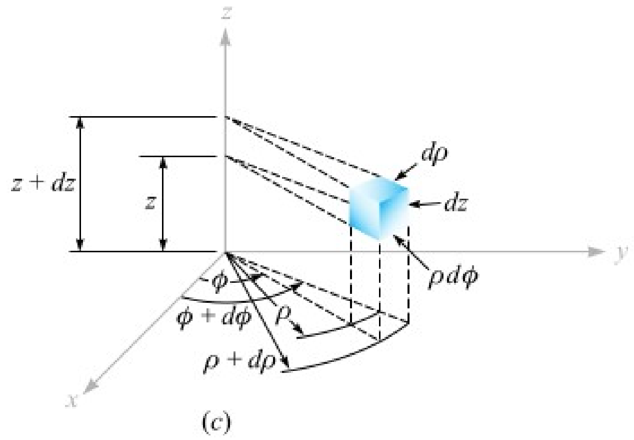
(*a*)

Intersection of 3 Mutually perpendicular planes in Cylindrical Coordinates

Three unit vectors must also be defined, but we may no longer direct them along the "coordinate axes," for such axes exist only in rectangular coordinates. Instead, we take a broader view of the unit vectors in rectangular coordinates and realize that they are directed toward increasing coordinate values and are perpendicular to the surface on which that coordinate value is constant (i.e., the unit vector $\mathbf{a}_x$ is normal to the plane $x =$ constant and points toward larger values of x). In a corresponding way we may now define three unit vectors in cylindrical coordinates, $\mathbf{a}_\rho$, $\mathbf{a}_\phi$, and $\mathbf{a}_z$.



Three Mutually perpendicular unit vectors of the Cylindrical Coordinate System



*A **Small Differential Volume** element in case of Cylindrical Coordinate System*

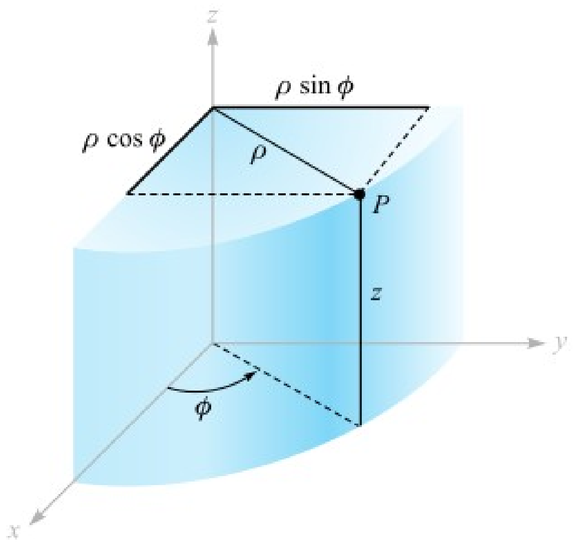
In rectangular coordinates, the unit vectors are not functions of the coordinates. Two of the unit vectors in cylindrical coordinates, $\mathbf{a}_\rho$ and $\mathbf{a}_\phi$, however, *do* vary with the coordinate ϕ, as their directions change. In integration or differentiation with respect to ϕ, then, $\mathbf{a}_\rho$ and $\mathbf{a}_\phi$ must not be treated as constants.

Surface Areas of the above given Small Volume Element

$$\rho\partial\rho\partial\varphi$$

$$\partial\rho\partial z$$

$$\rho\partial\varphi\partial z$$



Relationship between rectangular coordinates and circular cylindrical coordinates.

Expressing rectangular coordinates in terms of cartesian coordinates

$$x = \rho\cos\phi$$

$$y = \rho\sin\phi$$

$$z = z$$

Expressing Cylindrical coordinates in terms of rectangular coordinates

$$\rho = \sqrt{x^2 + y^2} \quad (\rho \geq 0)$$

$$\phi = \tan^{-1}\frac{y}{x}$$

$$z = z$$

Using the above relations we can easily convert a **Scalar Function** given in terms of one coordinate system into another and vice versa.

But converting a **Vector Function** from one coordinate system to another is essentially a **two step process**.

This is because we may be given a vector with different vector components such as:

$$\mathbf{A} = A_x \mathbf{a}_x + A_y \mathbf{a}_y + A_z \mathbf{a}_z$$

where each component is in general a function of x,y, and z.
and we need to convert this into an entirely different set of vector components:

$$\mathbf{A} = A_\rho \mathbf{a}_\rho + A_\phi \mathbf{a}_\phi + A_z \mathbf{a}_z$$

similarly, here the individual components are functions of ρ, φ, z in general.

From the Dot Product discussion we extracted that the component of any general vector along the direction of another 2nd vector vector is the dot product of the 1st vector with a unit vector in the direction of the 2nd vector.

i.e.

$$A_\rho = \mathbf{A} \cdot \mathbf{a}_\rho \qquad \text{and} \qquad A_\phi = \mathbf{A} \cdot \mathbf{a}_\phi$$

$$A_\rho = (A_x \mathbf{a}_x + A_y \mathbf{a}_y + A_z \mathbf{a}_z) \cdot \mathbf{a}_\rho = A_x \mathbf{a}_x \cdot \mathbf{a}_\rho + A_y \mathbf{a}_y \cdot \mathbf{a}_\rho$$

$$A_\phi = (A_x \mathbf{a}_x + A_y \mathbf{a}_y + A_z \mathbf{a}_z) \cdot \mathbf{a}_\phi = A_x \mathbf{a}_x \cdot \mathbf{a}_\phi + A_y \mathbf{a}_y \cdot \mathbf{a}_\phi$$

$$A_z = (A_x \mathbf{a}_x + A_y \mathbf{a}_y + A_z \mathbf{a}_z) \cdot \mathbf{a}_z = A_z \mathbf{a}_z \cdot \mathbf{a}_z = A_z$$

$$\mathbf{a}_z.\mathbf{a}_{rho} = \mathbf{a}_z.\mathbf{a}_{phi} = 0$$

as both $\mathbf{a}_{rho}$ and $\mathbf{a}_{phi}$ are unit vectors lying on the z = const plane and $\mathbf{a}_z$ is a unit vector perpendicular to the z = const plane.

To carry out the above given component transformations we need to know the following dot products:

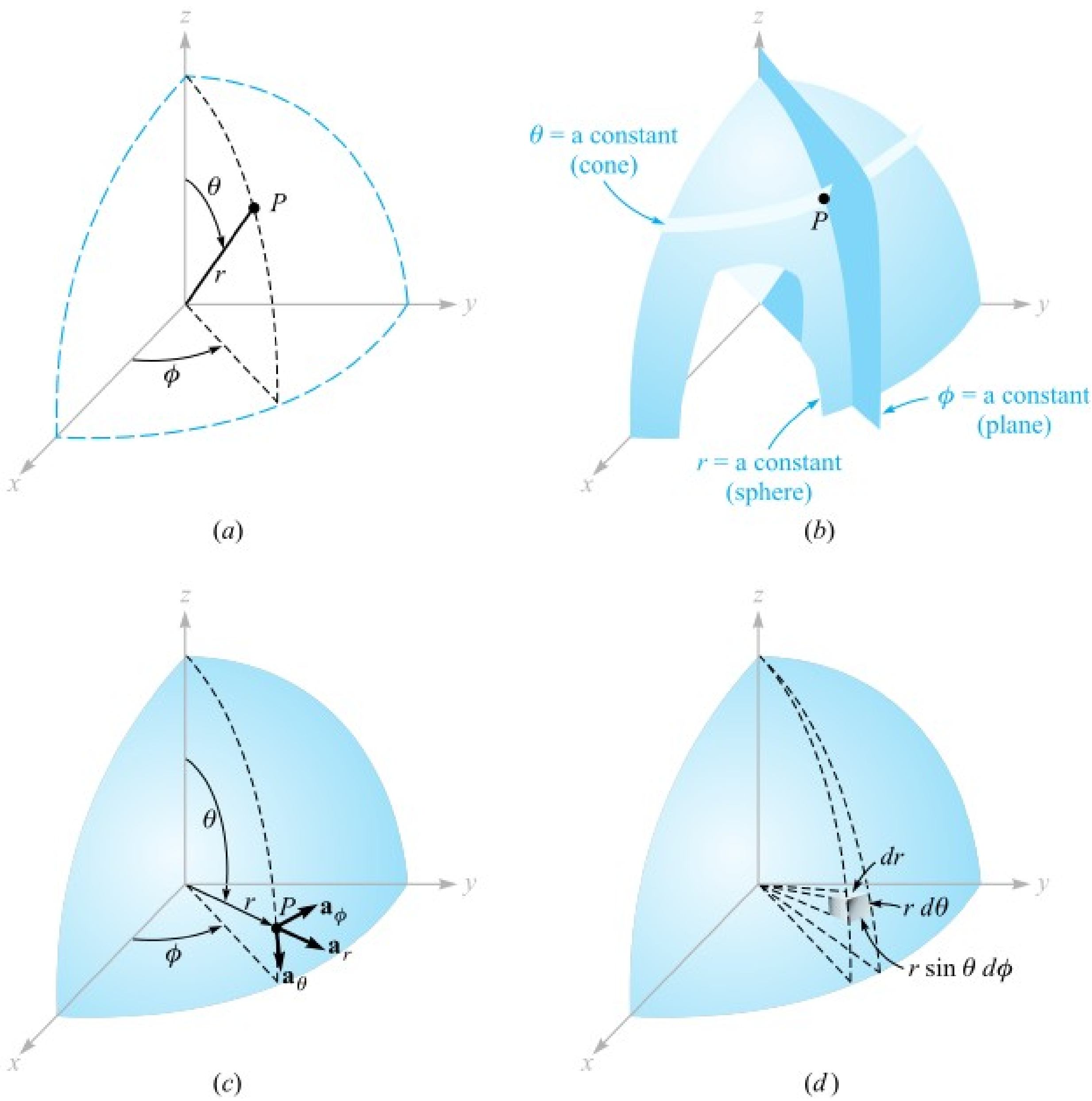
$$\mathbf{a}_x \cdot \mathbf{a}_\rho \qquad \mathbf{a}_y \cdot \mathbf{a}_\rho \qquad \mathbf{a}_x \cdot \mathbf{a}_\phi \qquad \mathbf{a}_y \cdot \mathbf{a}_\phi$$

The values of the 4 dot products is summarized in the table below:

	$\mathbf{a}_\rho$	$\mathbf{a}_\phi$	$\mathbf{a}_z$
$\mathbf{a}_x \cdot$	$\cos\phi$	$-\sin\phi$	0
$\mathbf{a}_y \cdot$	$\sin\phi$	$\cos\phi$	0
$\mathbf{a}_z \cdot$	0	0	1

Circular Coordinates

The surface $r = const$ is sphere.
The surface $\theta = const$ is a cone. (where θ is the angle between the z-axis and the line joining the origin to the point in question)
The conical surface and the spherical surface are **perpendicular to each other everywhere**, where the **circle has a radius** of of $r.\sin\theta$.



a) The parameters r, θ, φ in a spherical coordinate system.

b) The three mutually perpendicular surfaces.

c) The 3 unit vectors perpendicular to each of the 3 mutually perpendicular surfaces.

d) A **Small Differential Volume** element in the spherical coordinate system.

The third coordinate ϕ is also an angle and is exactly the same as the angle ϕ of cylindrical coordinates. It is the angle between the x axis and the projection in the $z = 0$ plane of the line drawn from the origin to the point. It corresponds to the angle of longitude, but the angle ϕ increases to the "east." The surface ϕ = constant is a plane passing through the $\theta = 0$ line (or the z axis).

A right handed spherical coordinate system is defined as one which obeys:

$$\mathbf{a}_r \times \mathbf{a}_\theta = \mathbf{a}_\phi$$

The surface areas of the small differential volume element:

$$\begin{gathered} r \partial r \partial \theta \\ r \sin \theta \partial r \partial \theta \\ r^2 \sin \theta \partial \theta \partial \varphi \end{gathered}$$

The transformations of scalar functions from the spherical coordinate system to rectangular coordinate system can accomplished by the following substitutions:

Rectangular → Spherical

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned}$$

Spherical → Rectangular

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} && (r \geq 0) \\ \theta &= \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}} && (0^\circ \leq \theta \leq 180^\circ) \\ \phi &= \tan^{-1} \frac{y}{x} \end{aligned}$$

Again in this case, the individual unit vectors vary with the coordinate points at which they are drawn and hence cannot be taken as constant as in the case of rectangular coordinate system.

Coulomb's Law and Electric Field Intensity

Objectives:

1. Coulomb's Electrostatic Force Law.
2. Generalize the above law using the Field Theory.
3. Use in evaluation of forces among static point charges.
4. Determination of the Electric Field produced by various charge distributions.
   (Initially, we will consider only E-fields in *vacuum* or *free space*.)

Coulomb's Law

Coulomb's Law states that the force between two very small objects separated in vacuum or free space by a distance which is much greater than their size is directly proportional to the charge on each of them and inversely proportional to the square of the distance separating them.

$$F = k\frac{Q_1 Q_2}{R^2}$$

$$\boxed{F = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2}}$$

Where,

$Q_1, Q_2 \rightarrow$ *Charges on each of the small objects.*

$R \rightarrow$ *The distance of separation between the two objects.*

$k \rightarrow$ *Constant of proportionality.*

$$k = \frac{1}{4\pi\epsilon_0}$$

Where,

$\epsilon_0 \rightarrow$ *Permittivity of free space.* Units : Farads/metre.

$$\boxed{\epsilon_0 = 8.854 \times 10^{-12} \doteq \frac{1}{36\pi}10^{-9} \text{ F/m}}$$

According to Coulomb's Law:

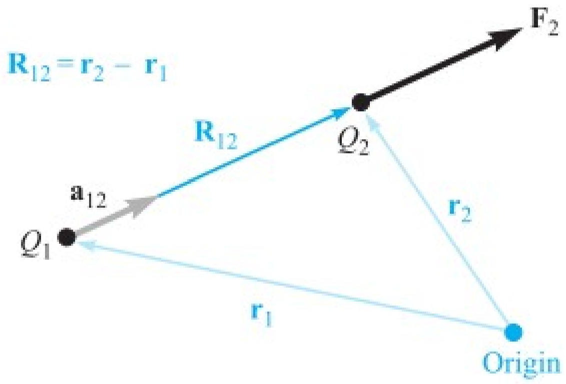
Units of $\epsilon_0 \rightarrow$ *Permittivity of free space.* $\rightarrow$ $C^2/N \cdot m^2$ F/m

F $\rightarrow$ $C^2/N \cdot m$

To formulate the vector version of the coulomb's law we need to the following:

1. The Coulomb's force always acts along the straight line joining the two charges.
2. Each of the charge experiences a force of equal magnitude and in opposite direction that is, the coulomb force is mutual in nature.

3. Like charges repel and unlike charges attract each other.



If charges Q_1, Q_2 are of like sign the force F_2 is directed along the direction of R_{12} .

$$\mathbf{R}_{12} = \mathbf{r}_2 - \mathbf{r}_1$$

$$\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi \epsilon_0 R_{12}^2} \mathbf{a}_{12}$$

where $\mathbf{a}_{12}$ = a unit vector in the direction of R_{12}, or

$$\mathbf{a}_{12} = \frac{\mathbf{R}_{12}}{|\mathbf{R}_{12}|} = \frac{\mathbf{R}_{12}}{R_{12}} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|}$$

Therefore F_1, F_2 are related as follows:

$$\boxed{\mathbf{F}_1 = -\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi \epsilon_0 R_{12}^2} \mathbf{a}_{21} = -\frac{Q_1 Q_2}{4\pi \epsilon_0 R_{12}^2} \mathbf{a}_{12}}$$

IMP:

Coulomb's law is linear, for if we multiply Q_1 by a factor n, the force on Q_2 is also multiplied by the same factor n. It is also true that the force on a charge in the presence of several other charges is the sum of the forces on that charge due to each of the other charges acting alone.

Electric Field

The Force Field associated with any charge in its neighbourhood is referred to as its *Electric Field*.

The Force experienced by any test charge (Q_t) in this neighbourhood is given by the coulomb's law:

Let the charge due to which the electric field is considered is Q_1 .

$$\mathbf{F}_t = \frac{Q_1 Q_t}{4\pi \epsilon_0 R_{1t}^2} \mathbf{a}_{1t}$$

Then, if this force F_t is expressed as the force per unit test charge Q_t, then this gives us the Electric Field (E_1) due to the charge Q_1 .

Therefore, E_1 is defined as:

$$\boxed{\mathbf{E} = \frac{\mathbf{F}_t}{Q_t}}$$

In which E_1 is a vector, which is the electric field intensity evaluated at the test charge location due to the charge Q_1 .

Units of Electric Field Intensity:

1. Newton/Coulomb (N/C)
2. Volts/metre (V/m)

Since, Volts → N. m/C or, J/C

Therefore, the Electric Field due to a Single Point charge **Q :**

$$\boxed{\mathbf{E} = \frac{Q}{4\pi \epsilon_0 R^2} \mathbf{a}_R}$$

Where, R → Magnitude of the R vector.
$\mathbf{a}_R$ → Unit Vector along the direction of the **R** vector.

And **R** vector is directed for the charge Q to the point at which we want to find the electric field intensity.

When a source charge Q is assumed to be lying in the centre of our spherical coordinate system then the unit vector $\mathbf{a}_R$ simply reduces to the radial unit vector $\mathbf{a}_r$. And hence R is r in this case.

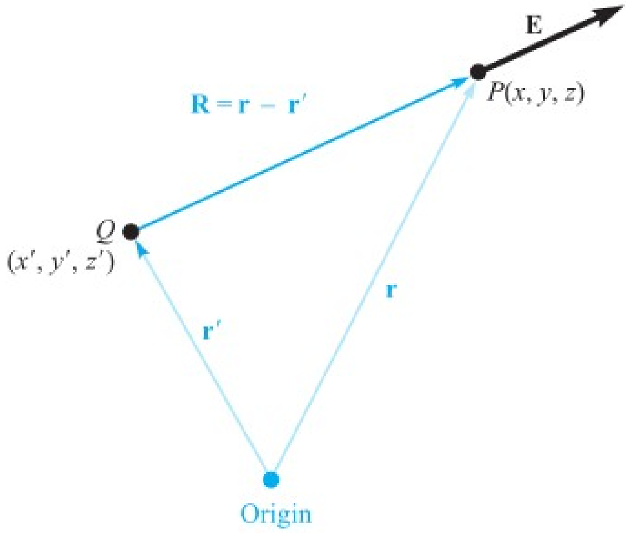
$$\mathbf{E} = \frac{Q_1}{4\pi \epsilon_0 r^2} \mathbf{a}_r$$

However, if we assume the charge not be at origin, there is no more the spherical symmetry, we can take advantage of, and hence we can just apply the rectangular coordinate system as follows:

coordinates. For a charge Q located at the source point $\mathbf{r}' = x'\mathbf{a}_x + y'\mathbf{a}_y + z'\mathbf{a}_z$, as illustrated in Figure 2.2, we find the field at a general field point $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ by expressing $\mathbf{R}$ as $\mathbf{r} - \mathbf{r}'$, and then

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} = \frac{Q(\mathbf{r}-\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^3}$$

$$= \frac{Q[(x-x')\mathbf{a}_x + (y-y')\mathbf{a}_y + (z-z')\mathbf{a}_z]}{4\pi\epsilon_0[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}$$



A vector function is symbolized as $\mathbf{E}(\mathbf{r})$.
Where, ‘bold’ E → A Vector Function.
‘bold’ r → Input is also a vector.

Because th ... ntensity arising from
two point charg ... ces on Q_t caused by
Q_1 and Q_2 acti

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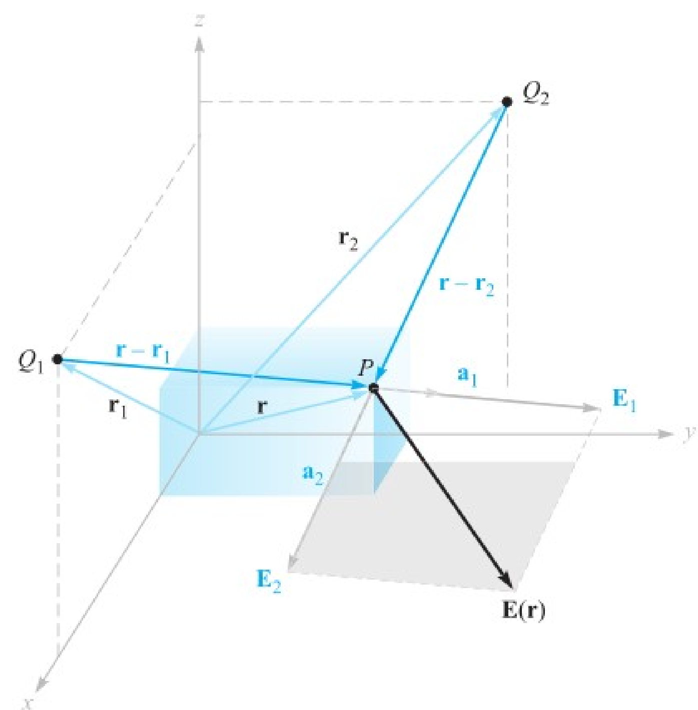
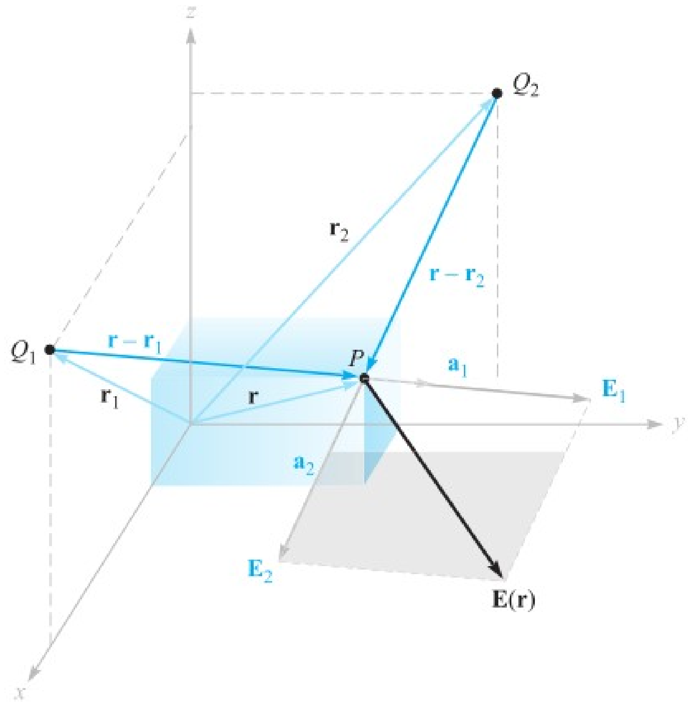
where $\mathbf{a}_1$ and $\mathbf{a}_2$ ... $(\mathbf{r} - \mathbf{r}_2)$, respectively.



The vectors **r, r₁, r₂, r-r₁, r-r₂** are shown below:



In general if we find the E-field due to n charges at a point whose position is denoted by **r.** Then, we have:

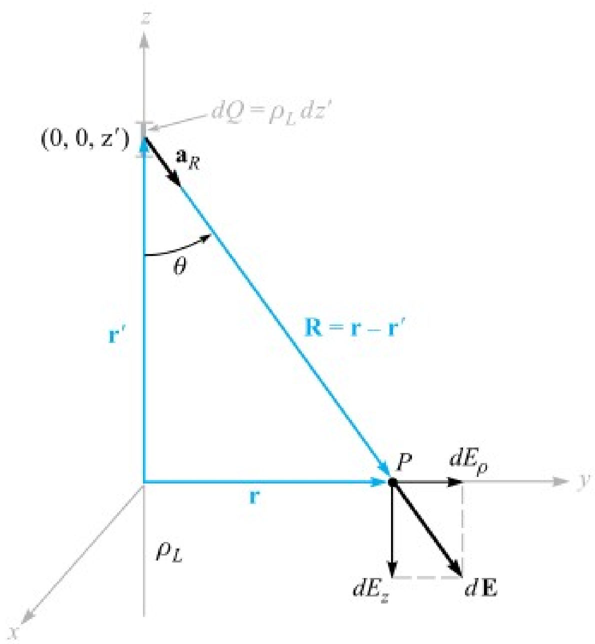
$$\mathbf{E}(\mathbf{r}) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_m|^2}\mathbf{a}_m$$

Field due to a Continuous Volume Charge Distribution

Field Due to an Infinite Line Charge

In this section we consider a special case of the volume charge distribution i.e. a long thin cylindrical charge distribution stretching from ∞ to +∞ along the z-axis.

If we let the radius of this thin cylindrical conductor tend to 0, then we effectively have a line charge distribution with a uniform Linear Charge Density of ρ_L C/m (Coulomb/metre).



Symmetry consideration should resolve 2 specific factors:

1> Which components of the electric are *present* and which are *not present.*
2> With which coordinates the E-field *varies* and with which it *does not vary.*

$$d\mathbf{E} = \frac{\rho_L dz'(\mathbf{r} - \mathbf{r}')}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|^3}$$

According to the above diagram:

$$\mathbf{r} = y\mathbf{a}_y = \rho\mathbf{a}_\rho$$
$$\mathbf{r}' = z'\mathbf{a}_z$$

$$\mathbf{r} - \mathbf{r}' = \rho \mathbf{a}_\rho - z' \mathbf{a}_z$$

Therefore,

$$d\mathbf{E} = \frac{\rho_L dz'(\rho \mathbf{a}_\rho - z' \mathbf{a}_z)}{4\pi\epsilon_0(\rho^2 + z'^2)^{3/2}}$$

Since only the E_ρ component is present:

$$dE_\rho = \frac{\rho_L \rho dz'}{4\pi\epsilon_0(\rho^2 + z'^2)^{3/2}}$$

$$E_\rho = \int_{-\infty}^{\infty} \frac{\rho_L \rho dz'}{4\pi\epsilon_0(\rho^2 + z'^2)^{3/2}}$$

By taking the help of the trigonometric substitution $z' = \rho.\cot\theta$ the above integral is evaluated as follows:

$$E_\rho = \frac{\rho_L}{4\pi\epsilon_0}\rho \left(\frac{1}{\rho^2} \frac{z'}{\sqrt{\rho^2 + z'^2}} \right)_{-\infty}^{\infty}$$

or,

$$E_\rho = \frac{\rho_L}{2\pi\epsilon_0 \rho}$$

Finally,

$$\boxed{\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0 \rho} \mathbf{a}_\rho}$$

Field due an Infinite Sheet of Charge