Hamiltonian Knight’s tour on a (8x8) chessboard: Algorithm and Implementation

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Abstract:

The Hamiltonian Knight Tour is a very popular problem in graph theory. In chess, the knight can move two squares in the same direction and then one square in the perpendicular direction. The problem statement we have to find a knight tour with the legal move of the knight and we have to visit each square of the chessboard only once. There are two types of knight tours: Closed knight Tour and Open Knight Tour. Warnsdroff has found a very efficient way to find the tour. But this rule cannot find the knight tour all the time. To improve the rule, Roth and Pohl have implemented this rule with some changes. The success rate has increased for these two rules, but these rules also cannot find all successful tours. I have discussed their approach to implementing their rules and the success rate of this rule. Also, I have discussed the Magic Knight Tour which is introduced in 1948 by Beverly. I have also implemented the algorithm and have found their outputs.

Introduction:

In chess, 2 players have a group of items to play around an 8 x 8 grid of squares of the chessboard. Every piece within the chess has its own movement and using the move the piece can be moved in the chessboard. For example, The Queen can be moved in any range of squares in any direction. The bishop will move any range of squares in a diagonal direction, whereas the King is restricted for moving to one step in any direction.

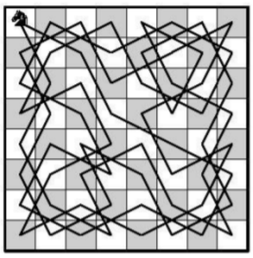
But in the chessboard among all of the moves, the Knight is one of the most interesting pieces to play. The Knight's movement may be represented as a straightforward L form when moving around the board. In terms of the chessboard, the knight may move two squares horizontally and one square vertically or two squares vertically and one square horizontally. If the Knight is in a corner square of the chessboard, then the Knight can move in only two directions. If all the squares of the chessboard are empty and also open to movement, then the Knight can move in eight different squares of the chessboard from its current position. Thus, in an open chessboard that has only one Knight piece, then the Knight can make anywhere from 2 to 8 possible moves on the chessboard.

In this paper, I will explain the Hamiltonian Knight's Tour on a 8x8 chessboard. The problem of the Knight's Tour is as follows: will there exist a path of Knight's movements on the chessboard where each square of the chessboard is visited only once by the Knight? This kind of path is called a Knight's Tour. Knight's Tours will begin on any square of the chessboard since the path should consist of each square of the chessboard. The path is not easily identified as the movement of the Knight in the chessboard is very unique.

These tours can be divided into 2 categories: open tour and closed tour. As shown in the above figure, the example on the left side shows a closed knight tour. In the closed knight tour, a complete loop is made, meaning that the tour begins on one square and moves to every other square before coming back to the starting square. On the right side of the figure is an example of an open knight tour. For this type of knight tour, the knight cannot make a single move from the ending square to get back to the starting square.

History of Knight Tour:

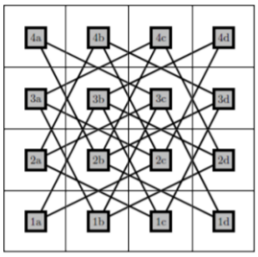
The study of Knight's Tours can be known back to many totally different points in history. One among the first Knight's tours on the two-dimensional 8x8 chessboard were given by Abraham de Moivre within the early eighteenth century. The open tour was created by first creating moves only among the outer 2 rows of the chessboard, moving inward only if necessary. By this methodology, the primary twenty-four moves can be created on the periphery of the chessboard. In 1759, Euler used a special technique to make a closed Knight's Tour on a 8x8 chessboard from a special open knight tour. For this closed knight tour, four squares on the chessboard were incomprehensible. The open knight tour was created using the remaining sixty squares of the chessboard. So as to include the remaining squares into the tour, Euler reordered the portion of the tour. This rearrangement makes it possible to create a move to an incomprehensible square, then go into the remainder of the tour. Through these various steps, Euler took the missing squares and integrated them into the Knight tour, and produced a closed Knight's Tour in the method.

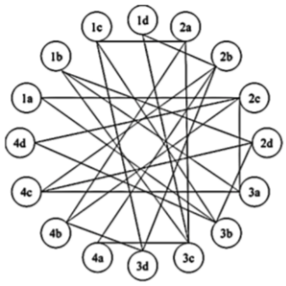


Graph Theory Concepts:

To explain the tour, I have used many ideas from Graph Theory and connected them to the ideas from the Knight's Tour. We know that a graph could be a collection of nodes, or vertices, organized in any means. One vertex is connected to another vertex through a group of edges. We can use the graph to model totally different concepts and ideas, like the movement of the Knight on the chessboard.

Through the Graph Theory ideas, the Knight's Tour will be defined from a special perspective. The Knight's moves and therefore the chessboard will be modelled by a graph, with a group of vertices representing the squares of the chessboard. Edges are placed between 2 vertices if there exists a Knight's move between the corresponding squares of the board. In the below figure, the 4 x 4 chessboard is depicted in each standard and graph form.



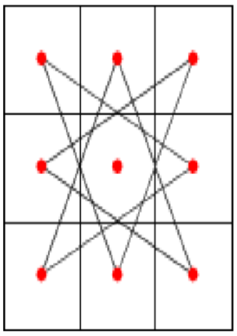


A path in the graph could be a set of movements over edges between connected vertices, wherever no edge is traversed doubly. The path will begin and end in any 2 vertices of the graph. A special form of the path is named as Hamiltonian Path, where every vertex within the graph is enclosed within the path exactly once. In a Hamiltonian Path, the start and end vertex of the graph must not be identical. If the Hamiltonian Path begins and ends on an identical vertex, then it is called a Hamiltonian Cycle. Any graph that contains a Hamiltonian Cycle is known as Hamiltonian Graph.

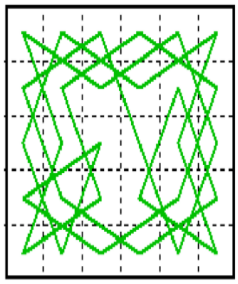
The Knight's Tour on a chessboard is a Hamiltonian Path or Hamiltonian cycle on the graph. By finishing a tour, the Knight is in a position to cover every single square on the chessboard. So, the open Knight's Tour is an example of a Hamiltonian Path, because the starting and ending squares of the Knight's Tour are not identical. Similarly, the closed Knight's Tour is associated with the Hamiltonian Cycle, because the starting and ending square is the same, and therefore the Knight's Tour creates a closed-loop on the chessboard.

Altering the Chess Board:

At the first moment to find the existence of the Hamiltonian knight tour, mathematicians tried to find the tour only in the 8x8 chessboard. But later they have tried to find the knight tour on any type of chessboard. They have stretched and shank the chessboard to find the tour where the row size and column size is not the same. But they have also found that it is not possible to find the tour on every type of chessboard. From this figure, we can say that for some size of the chessboard, the knight tour exists and for some size of the chessboard, it is not possible to find any type of knight tour.



(left) A representation of a 3 x 3 rectangular chess board without a Knight's Tour, as the center square cannot be reached (right) A 5 x 6 chess board with a closed Knight's Tour



Allen Schwenk gave conditions that will be used to find the existence of a closed Knight's tour on an m x n chessboard.

Theorem 1 (Schwenk Theorem): Assume that $1 \leq m \leq n$, where $m, n \in \mathbb{N}$. Then there exists a closed Knight's tour on a m x n chessboard if one among these 3 conditions hold.

1. m and n both are odd numbers,

2. m belongs to {1,2,4},

3. m is equal to 3 and n belongs to {4, 6, 8}

To clarify why m and n cannot each be odd numbers, we will first prove a property of the graphs that represent the chess boards on that a closed Knight tour exists. To prove that a closed tour cannot exist on a chessboard with the odd number of squares, we will show that the graph modelling the Knight's tour on a chessboard be a bipartite graph, or a graph with 2 disjoint sets of vertices known as partite sets, within which edges exist between vertices in opposite partite sets. This may show that there aren't any odd cycles or cycles with an odd range of vertices. Once this can be shown, we will then show that a closed Knight's Tour cannot exist on a chessboard with an odd number of squares.

Proposition: Assume, an m x n chessboard that a closed Knight's Tour exists. Then the graph representing all the moves of the Knight on the chessboard is bipartite. Proof: we will first outline the labelling of the vertices of the graph, especially, we will label a vertex on the graph with the location of the corresponding square on the chessboard. Thus, the vertex of the square of the chessboard in column a and row b is tagged by the ordered pair (a, b), where $1 \leq a \leq n$, and $1 \leq b \leq m$, and (1, 1) represents the squares within the upper left most corner of a m x n board. Note that 2 vertices tagged (a, b) and (c, d) are adjacent if $(a-c, b-d) = (\pm 1, \pm 2)$ or $(a-c, b-d) = (\pm 2, \pm 1)$.

We divide the vertices of the graph into 2 sets X and Y of the chess board's squares by the parity of the total coordinates of the square. That is, $(a, b) \in X$ if $(a + b)$ is even and 8 $(a, b) \in Y$ if $(a + b)$ is odd. Thus, we create 2 disjoint sets. If a Knight makes a move, the total movement is 3 squares (two squares in one direction, followed by one square in a perpendicular direction). In other words, the parity of the square is modified between moves, therefore the Knight will solely enter a square with opposing parity. Thus, the graph is bipartite.

We know that bipartite graphs do have not any odd cycles. Since this graph is bipartite, then the graph of Knight's movements will solely have a cycle with an even number of squares.

If m and n are each odd terms, then there are an odd range of squares within the chessboard, therefore there'll be an uneven distribution of squares between the 2 partite sets. To form a closed Knight's Tour, the ultimate Knight's move would have to be compelled to be created between 2 squares within the same partite set. However, in a bipartite graph, no edges will exist between 2 vertices of a similar partite set. Thus, it's not possible to own a closed Knight's Tour on associate m x n chessboard wherever m and n are odd terms.

Algorithms:

To find the Hamiltonian Knight Tour, we can implement various types of algorithms. One of the most basic methods is Brute Force Algorithm. In this method, we traverse all the squares of the chessboard and check if this particular square is already visited by the knight or not. If the square is unvisited, we will place the knight in that square and this process is continued until all squares are visited by the knight.

Naïve algorithm for knight tour**:**

```
while there are some unvisited squares
{
  start the next tour
  if this Knight tour covers all squares of the chessboard
    print this tour;
}
```

In this algorithm, we have to generate all the tours of the chessboard one by one and then we have to check if this satisfies the given constraints or not. It is a very general problem technique and it is basically a backtracking algorithm. For a regular 8x8 chessboard, there are approximately $4x10^{51}$ possible move sequences. This algorithm does not guarantee the closed Knight tour always.

To implement the backtracking algorithm, I have used the following algorithm:

If there is a successful path

Print the solution

else

go to the next squares to check if there is any path

if do not find any path, then backtrack and find the next path.

if do not find any path, then print no path exist.

I have implemented the backtracking code mentioned in the link.

The output I get using the backtracking method is:

```
The Knight Tour is
  0  59  38  33  30  17   8  63
 37  34  31  60   9  62  29  16
 58   1  36  39  32  27  18   7
 35  48  41  26  61  10  15  28
 42  57   2  49  40  23   6  19
 47  50  45  54  25  20  11  14
 56  43  52   3  22  13  24   5
 51  46  55  44  53   4  21  12
```

Also, I have implemented another backtracking method, where I have blocked some squares and tried to find the Knight Tour using the backtracking method. The code for this tour is mentioned in this link. The output of this code is:

```
The Knight Tour is
  0  37  58  25  30  17   8   X
 57  40  31  36   9  26   X  16
 32   1  38  27  24  29  18   7
 39  56  41   X  35  10  15   X
 42  33   2  23  28   X   6  19
 55  50  53  34  45  20  11  14
 52  43  48   3  22  13  46   5
 49  54  51  44  47   4  21  12
```

In this output, the blocked square is marked with ‘X’ sign. The backtracking algorithm checks every square of the chessboard and tries to find the tour. For this reason, the time complexity is very high in this algorithm. If there are a total of N^2 squares and we can move in 8 possible moves of the knight, then the time complexity of this algorithm is $O(8^{NxN})$. The space complexity is $O(N^2)$ in the algorithm.

The most popular and efficient algorithm to find the Hamiltonian Knight Tour is Warnsdroff Rule. This rule solves this problem in linear time. In 1823, H.C.Warnsdroff introduced this rule. The rule tells us that we can start the knight tour from anywhere of the chessboard and then we have to place the knight in the square that is adjacent, unvisited and also with the minimal degree. This rule can find the closed knight tour in linear time. The rule looks like a very logical rule to implement. But the main logic behind this rule is we cannot deviate from this rule in the last four moves of the knight tour. So, the idea is that if we follow this rule from the beginning of the tour, then we have a high chance to complete the tour. If we start to follow the rule at the end of the tour, then we do not have enough opportunities to complete this tour every time. Also, it is very essential to follow the Warnsdroff Rule if a square has a degree 0, otherwise, we can never visit this square again in the tour. Ganzfired have found that why this rule has successfully found the tour in the small size chessboard. He had proved that the tour cannot deviate from the rule in the last four moves of the tour. To prove that he had assumed the tour is deviated from the rule in the current position of the path and finds the tour. So, let the knight be moved to a square of degree y instead of a square of degree x, where x<y and assume, q denotes the current square, p denotes the square with degree x and r denotes the square with degree y. Here, if x=0, then we cannot visit the square p. So, x=0 is not possible. Then, x must be greater than or equal to 1 and y must be greater than or equal to 2. Assume, there are exactly three squares left on the chessboard. So, we must have x=1 and y=2. Then, r must be adjacent to p as there are only three squares left in the chessboard. So, we get that p, q, r must be adjacent to each other, which is impossible in any chessboard. If there are four squares unvisited and if y=3 and x=1, then also, p, q, r must be mutually adjacent. So, we have found that it is impossible to deviate from the Warnsdroff Rule in the knight tour. Thus, we can say that this rule is not any logical rule and this rule is not specified in the event when there are more than one unvisited, adjacent squares with the minimal degree.

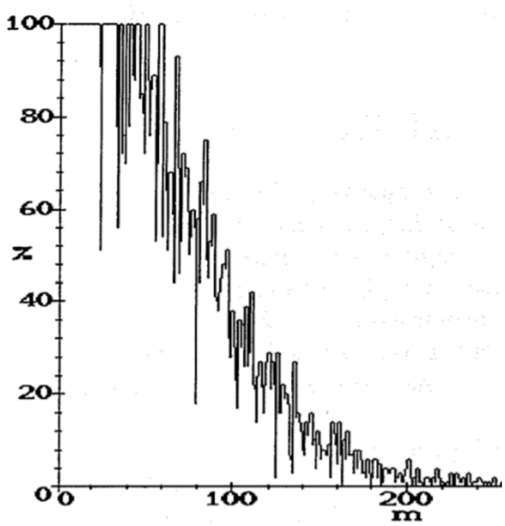
The biggest drawback of the Warnsdroff Rule is that this rule does not provide any method if there are many squares with the minimal degree. Warnsdroff suggests to move to any of them indifferently will result in a tour, in the event that several adjacent squares share the minimal degree. Also, it is assumed that Warnsdroff suggest to start from the corner of the chessboard because the corner square has the smallest degree initially. But if we prefer a random tie break, then there is no guarantee to find the knight tour. In this figure, it can be shown that there is no knight tour but the 8x8 chessboard has followed the Warnsdroff Rule. So, we can say that Warnsdroff Rule cannot always find the knight tour. To prove this, in 1996, Squirrel and Douglas have performed a hundred trials of the Warnsdroff Algorithm for all chessboards of size 5 to 400 and they have found that this rule is successful only for the smaller

1	28	13	46	3	26	39	36
14	43	2	27	48	37	4	25
29	12	47	42	45	40	35	38
–	15	44	49	–	–	24	5
11	30	–	–	41	50	–	34
16	–	18	31	–	–	6	23
19	10	–	–	21	8	33	–
–	17	20	9	32	–	22	7

size chessboard but as the size of the board is increased, the rate of success is dropped very sharply. They have also found that if the board size is less than 50x50, then the success rate is 85% and if the size is less than 100x100, then the success rate is over 50%. But for board size 200x200, the success rate is less than 5%. So, we can say that the success rate of this rule goes to zero rapidly, if we increase the size of the board. Ganzfired had found that there is 98% success rate if the chessboard size is less than 25x25.

To overcome these problems and find a successful rule that can found the knight tour successfully, several attempts are made. At first, Parberry has implemented an algorithm that is combined with Warnsdroff Rule and Euler Rule, which is proposed in 1759. He suggested that we can start any square of the chessboard and then repeatedly goes to another random unvisited neighbour and he has broken the tiebreak using the Warnsdroff Rule. But the success rate of the algorithm decreases exponentially and the time complexity of this algorithm also increases exponentially.

Next Roth has found that the main problem is Warnsdroff's tie-break and he suggested a new method to solve that problem. He proposed we can break the tie in the chessboard by choosing a square that has the largest Euclidian distance from the centre. Roth has claimed that with this technique, the knight tour has failed the first time when the board size is 428x428. But in this rule also, there is no clue if two adjacent, unvisited squares share the same distance from the centre. Next, Pohl has suggested to apply the Wansdroff Rule for the second time, if there is any type of tie-breaking occurs. Pohl has proposed to take the sum of the degrees of all the squares with equal minimal degrees and also choose the square that has the minimal sum among all of the squares.



The graph in the figure is found by Squirrel and here had shown that how the success rate drops with the increase of the size of the chessboard. In the rule to break the tie-break of Warnsdroff Rule, Roth has claimed that his algorithm has a good success rate up to the size of 400x400 and Pohl has claimed that his algorithm was worked fine and get 95% success rate on nearly all the chessboard up to size 100x100.

The Magic Knight Tour

A knight tour is called a full magic knight tour if the sum of all numbers presented in the rows, columns and diagonals are the same. To find the sum, we have to first mark the squares of the chessboard with the moves of the knight. So, in the 8x8 chessboard, there are total 64 marks from 1 to 64. We have to mark these numbers in such a way that we have found the equal sum for all the rows, columns and diagonals. At first, scientists have tried to find the full magic knight tour only for the 8x8 chessboard, but later they have started to find the full magic knight tour for all sizes of the chessboard. If the sum of all squares of the rows and columns are same but the the sum of diagonals is not the same, then this is called a magic knight tour. In 1947, William Beverley has composed the first magic knight tour. In this figure of the chessboard, Beverley has found the magic tour where the sum of all rows and columns is equal to each other which is 260. So, 260 is the magic number for this chessboard. But he cannot find any full knight magic tour. So, the existence of a full knight magic tour on the 8x8 chessboard is a 150-year-old problem. On 5th August, 2003 G. Stertenbrink, J-C. Meyrignac and H. Mackay have proved that it is not possible to find any full magic knight tour on a chessboard of size 8x8. In January 2003 Awani Kumar from India has solved the problem of the diagonally magic tour

1	48	31	50	33	16	63	18
30	51	46	3	62	19	14	35
47	2	49	32	15	34	17	64
52	29	4	45	20	61	36	13
5	44	25	56	9	40	21	60
28	53	8	41	24	57	12	37
43	6	55	26	39	10	59	22
54	27	42	7	58	23	38	11

on 12x12 chessboard. Now it is proved that there are 140 magic knight tours on the 8x8 chessboard.

Warnsdroff Algorithm

In Warnsdroff Rule,

- Any square A is accessible from other square B if the knight can move from square B to square A with the legal knight move and A is an unvisited square.
- The accessibility of square B is the number of squares accessible from A.

Algorithm:

1. First, place the knight in any position on the 8x8 chessboard.
2. Mark the position of the knight on the chessboard.
3. Next, for each move mark the number on the chessboard from 2 to 64.
4. Also, find the minimum accessibility of the knight and place the knight in the next position.
5. At last, we have to return the marked board.

I have implemented the Hamiltonian Knight Tour using the Warnsdroff Rule in this link with CPP. Here I have used srand function so that the algorithm chooses different random starting points every time. This algorithm also checks if the knight tour is closed or open. I have used a function which is used the Warnsdroff Rule to find the next move. To find the next move or to find the minimal degree neighbour, I choose any random neighbour of that current position of the knight. Then I have checked if the next targeted square is visited or not. If the targeted square is unvisited, then find the degree of that square and check if the degree of that targeted square is smaller than the minimum degree or not. If there is a neighbour with the minimal degree and also unvisited, then we get the next move for the knight. Also, I have limit the movement of the knight so that the knight cannot move beyond 64. In a chessboard, the knight can move in the squares {1, 2}, {1, -2}, {2, 1}, {2, -1}, {-1, 2}, {-1, -2}, {-2, 1}, {-2, -1} from the current position. So, to put the knight in that squares I have stored the positions using two arrays, one array is for x-direction move and another array is for y-direction move. I have found the output shown in the below figure using the Warnsdroff Algorithm:

18	3	42	23	20	5	36	33
43	24	19	4	41	34	21	6
2	17	44	61	22	37	32	35
25	62	1	40	47	60	7	38
16	13	54	45	64	39	48	31
53	26	63	14	55	46	59	8
12	15	28	51	10	57	30	49
27	52	11	56	29	50	9	58

From the output, we can say that the starting position is not any corner square of the chessboard. Also, from the endpoint, we can reach the initial square by the legal move of the knight. So, this is a closed knight tour.

Conclusion and Future Issue

The Hamiltonian Knight Tour problem is an NP-Hard problem in graph theory. In this report, I have discussed various types of knight tours and the algorithm to find these knight tours. The backtracking algorithm is a very brute-force algorithm and H.C.Warnsdroff has introduced a new method to find the closed knight tour in a very optimal way. Here, I have discussed the failure of the Warnsdroff Rule to find the closed knight tour on chessboards of different sizes and how the success rate drops with the increase of the size of the chessboard. Also, this rule has failed to break the tie in a randomised way. To improve this rule, Parberry, Roth and Pohl have proposed some rules. But they also have not found any general rule to get the knight tour for all sizes of the chessboard. The scientists are still trying to find any general rule which can be used to find the closed knight tour for all chessboards. We can use the backtracking algorithm to find the tour. But the main problem is there is no guarantee to find the closed knight tour only and the time complexity is very high in the backtracking algorithm. Besides this, the scientists are also trying to find the new structured knight tour with the four knights simultaneously, which can cover all the squares of the chessboard. They are also trying to find the knight tour on n-dimensional chessboards. To find the closed knight tour neural network technique has been introduced. The knight tour currently is used for data hiding in a video by utilizing the least significant method. On an 8x8 chessboard, there are exactly 26,534,728,821,064 closed knight tours possible. Here, it is assumed that the two tours along the same path but in opposite direction travelled by the knight is counted twice as are rotations and reflections. If we count the knight tour in an undirected manner, then the number of the knight tour becomes half of this number.

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