Cosmology Notes

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Proofs not shown may not be trivial but are left as exercises to the reader anyways

LaTeX compiled by Fahim Rajit Hossain and Arnab

1 Preliminary Concepts

The equations in cosmology are all derived from General Relativity. But the universe is symmetric. It has translational symmetry (homogeneity), rotational symmetry (isotropy), time symmetry and so on. If you are familiar with Noether's theorem, you'll know that each symmetry leads to a conservation law, viz the conservation of linear momentum, angular momentum, energy etc. The beauty of this symmetry is that we can now derive the correct equations describing an expanding universe using Newtonian physics.

1.1 Hubble–Lemaître Law

The math in cosmology begins with the **Hubble–Lemaître Law**. To understand Hubble's Law we must first understand redshift. Redshift is a phenomenon where electromagnetic radiation from an object undergoes an increase in wavelength. There are three main causes of redshifts:

1. Objects move apart (or closer together) in space. This is the Doppler Effect.
2. Space itself is expanding, causing objects to become separated without changing their positions in space. This is known as cosmological redshift.
3. Gravitational redshift, observed due to strong gravitational fields, which distort spacetime and exert a force on light.

For all galaxies, we find a "redshift" z which is proportional to the distance D light has traveled from the galaxy, i.e. $z \propto D$. The constant of proportionality is related to the Hubble constant H_0 which we discuss in greater detail later on. If λ_0 is the laboratory wavelength measurement of a single absorption or emission line and λ_{obs} is the observed wavelength of the redshifted line then $\Delta\lambda = \lambda_{\text{obs}} - \lambda_0$ is the difference between the two, then redshift is defined as–

$$z \equiv \frac{\Delta\lambda}{\lambda_0} = \frac{H_0}{c} D \qquad [\text{“Hubble law” for redshifts}]$$

This is often incorrectly interpreted as a Doppler effect where[1],

$$z = \frac{v}{c} = \frac{\Delta\lambda}{\lambda_0}$$

Nonetheless, this idea lends itself to the more familiar version of the Hubble law:

$$v = H_0 D \qquad [\text{“Hubble law” for velocities}]$$

The Hubble constant is now known from 2018 Planck results to be $67.4 \pm 0.5\,\text{kms}^{-1}/\text{Mpc}$ (assuming the base- ΛCDM cosmology).

[1] The relation $z = \frac{v}{c}$ in cosmology can only be used for very small redshifts, $z \ll 1$.

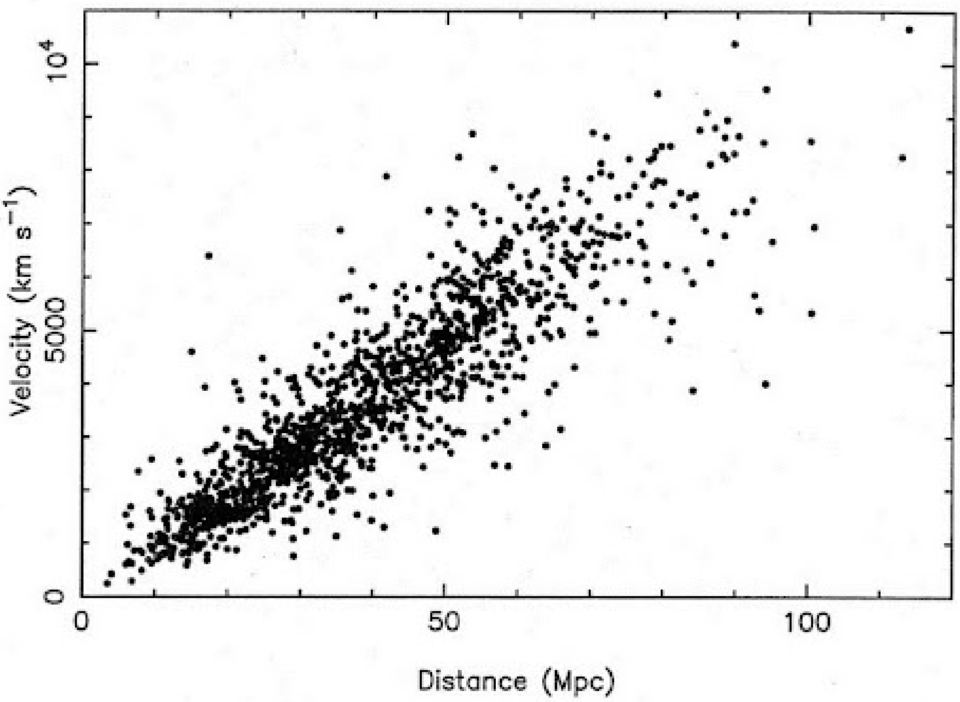


Figure A plot of velocity versus estimated distance for a set of 1355 galaxies. A straight-line relation implies Hubble's law. The considerable scatter is due to observational uncertainties and random galaxy motions, but the best-fit line accurately gives Hubble's law.

Problem 1 (folklore): The observed redshift of a QSO is $z = 0.20$, estimate its distance.

It is an observed fact that wavelengths are redshifted. Historically, people were convinced that this was evidence of a velocity-induced doppler effect. However, this immediately leads to further questions about why the galaxies are moving away from each other.

***Q* 1.1**

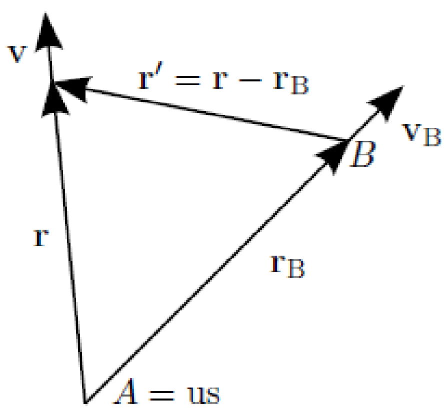
Does the Hubble law define a privileged observer, i.e. us?

Answer: No! Consider the view with respect to different observers.
Note: Hubble's law is a vector law.

We observe: $v = H_0 r$ and $v_B = H_0 r_B$

Observer B: $v' = v - v_B = H_0 (r - r_B) = H_0 r'$

This means that all observers see the same law!



***Q* 1.2**

What is the proper interpretation of the Hubble law?

Answer: The old (wrong) idea was that galaxies are in flight with respect to fixed (absolute) space. This resembles some kind of cosmic super-explosion, hence the name Big Bang. Everything would fly away from a special location, but WHY should there be such a special point in space? The better (correct) idea is that space itself is expanding! The effect is much like raisins in an expanding fruit cake.

***Q* 1.3**

But observation tells us some galaxies don't obey Hubble's law. Andromeda is even moving toward us!

Answer: Hubble's law is only observable for galaxies far enough away. And by that I mean at least a few Megaparsecs, that's what Mpc^{-1} in the units mean. All galaxies have some sort of random motion due to their interaction with other galaxies. This is called peculiar motion/velocity. Peculiar velocity can be a few hundred km/s. So a galaxy has to be pretty far away before Hubble's velocity dominates.

Problem 2 (Andrew Liddle): Supposing that a typical galaxy peculiar velocity is 600 km/s, how far away would a galaxy have to be before it could be used to determine the Hubble constant to ten percent accuracy?
Hint: The Hubble velocity vector only points away from us, but the peculiar velocity vector can point in any direction.

***Q* 1.4**

So, space is expanding. Wouldn't that mean space at a small scale, including intermolecular space, is also expanding? Isn't that just scaling the universe where literally everything gets bigger and effectively nullifies the increasing distance between galaxies?

Answer: Well, space does expand at molecular levels, but that doesn't mean that everything will get bigger. No, it's not because the expansion at molecular levels is negligible. It's because the intermolecular forces are really strong at this scale and they keep the actual intermolecular distances fixed. Specifically, electromagnetic forces between atoms and gravitational forces in our solar system or within galaxies completely nullify the effect of the expansion. You'd need galaxies at huge distances for the expansion of space to overcome that threshold and slowly move things apart.

Think of it like your entire neighborhood is stretching and the houses are getting further apart, but your cat is tied to you by a leash and the distance between you and your cat stays the same. That leash is the electromagnetic force between particles or gravity between solar systems, galaxies.

A better explanation is that our current model of the universe assumes a homogeneous distribution of mass. Every result calculated has been based on that assumption. But that's only true at large scales. At small scales, the universe is far from homogeneous where all sorts of random motion dominate. Hence our model and calculated results do not apply.

***Q* 1.5**

How do we describe expanding space?

Answer: We introduce a scale factor $a(t)$ which compares distances at different times. Then physical distances r and comoving distances x are related through a as follows:

$$r = a(t) \cdot x$$

Where,

$$\begin{aligned} a &= \text{cosmic scale factor} \\ x &= \text{comoving coordinates} \\ r &= \text{physical (or proper) coordinates} \end{aligned}$$

Take a balloon and draw a grid on it. The type of grid and how you draw it doesn't matter. Draw some galaxies on it too. When you blow up the balloon, you can note two things. The grid spacing between the galaxies doesn't change, these are the comoving coordinates. You can view these coordinates as being carried along with the expansion. The next is the physical distance between the galaxies (measured along the surface of the balloon if you want to get technical) that increases with the expansion. This is the physical distance that we are actually used to. The scale factor is literally just a scaling factor to go from comoving to physical coordinates

x stays the same during expansion so the entire effect of the expanding space is encoded in $a(t)$. Furthermore, only r has physical meaning as comoving distances are not directly observable.

The main difference between comoving distance and physical distance is that the comoving coordinates are just imaginary coordinates that move along with spacetime, thus the name comoving. While physical distances are actually, according to the name, physical distances that don't expand with spacetime. Thus objects moving along with space have a constant comoving distance but face increasing physical distance.

For an expanding universe the scale factor is an increasing function with time, i.e. $a(t_1) < a(t_2)$, so we often normalize the scale factor. The parametrization is chosen such that the present-day value of the scale factor is one:

$$a_0 \equiv a(t_o) \equiv 1$$

where, $t_0 \equiv$ today

Hence, the scale factor is just another way of measuring cosmic time.

***Q* 1.6**

An ant starts walking from one end of a 10 cm long rubber rope at 1 cm/s. The rope is being stretched uniformly at 1 cm/s. Will the ant ever reach the other end?

1.2 Relation between scale factor and other quantities

Now we have a completely different (and correct) interpretation of the redshift - light travels through expanding space - so the physical wavelength changes according to $\lambda \propto a$.

$$\text{At emission:} \qquad \lambda_{\text{em}}(\lambda_0) \propto a(t_{\text{em}})$$
$$\text{Observed today:} \qquad \lambda_{\text{obs}}(\lambda) \propto a(t_0)$$

Therefore, the redshift is,

$$z = \frac{\Delta\lambda}{\lambda_0} = \frac{\lambda - \lambda_0}{\lambda_0} = \frac{\lambda}{\lambda_0} - 1 = \frac{a_0}{a(t_{\text{em}})} - 1,$$

providing the cosmological definition of redshift once we substitute $a(t_{\text{em}}) \to a(t) \to a$ and $a_0 \to 1$.

$$1 + z = \frac{1}{a}$$

From Wien's law we have $T \propto \dfrac{1}{\lambda}$ and thus $T \propto \dfrac{1}{a}$ providing us the relation between temperature and scale factor.

It is useful to remember that the average temperature of the Cosmic Microwave Background (CMB) or equivalently the universe is currently $T_0 = 2.73$ K.

Problem 3 (IOAA 2011): Given that the cosmic background radiation has the spectrum of a black body throughout the evolution of the Universe, give the temperature of the background radiation when $z \approx 10$.

Because we can measure them, we define two distances in the universe: d_{ang}, known as the angular diameter distance, and d_L, known as the luminosity distance.

$$d_{ang} = \frac{d}{\theta} = r.a$$

$$d_L = \sqrt{\frac{L}{4\pi F}} = \frac{r}{a}$$

These two distances are related by the Etherington Reciprocity theorem as

$$d_{ang}(1+z) = \frac{d_L}{(1+z)} = r$$

Problem 4 (IOAA 2008): Consider a type Ia supernova in a distant galaxy that has a luminosity of $5.8 \times 10^9 L_\odot$ at its maximum light. Suppose you observe this supernova using your telescope and find that its brightness is 1.6×10^{-7} times the brightness of Vega. The redshift of its host galaxy is known to be $z = 0.05$. Calculate the distance of this galaxy (in pc). Given $m_\odot = -26.72$.

1.3 The Hubble Parameter

The term Hubble constant is a bit misleading. Although certainly it is constant in space, it is not constant in time. We now generalize this, the idea is to describe expansion with a Hubble law that is valid for all times, not just today. Formally, the Hubble relation is $v = H(t)r$ but velocity is fundamentally defined by $v = \dot{r} = \frac{d}{dt}a(t)\,x = \dot{a}x$ so we obtain,

$$H(t) \equiv \frac{\dot{a}}{a} \qquad \text{(Hubble parameter)}$$

$$H_0 \equiv H(t_0) \equiv \frac{\dot{a}_0}{a_0} \quad \text{(Hubble constant)}$$

It is best to use the phrase 'Hubble parameter' for the quantity H, as a function of time; reserving 'Hubble constant', H_0 for its present value.

Problem 5 (folklore): What are the dimensions of the Hubble parameter? One can define a characteristic timescale for the expansion of the Universe (i.e. Hubble time t_H) using the Hubble parameter. Calculate the present-day Hubble time t_{H_0}.

Answer: The dimensions of Hubble parameter is just $[T^{-1}]$. We can estimate the time elapsed since the Big Bang by the following simple argument. If we assume a constant expansion rate, so that $H(t) = H_0$ for all t, then it follows that

$$t_H = H_0^{-1}$$

Therefore, if $H_0 = 70\text{kms}^{-1}/\text{Mpc}$ the Hubble time t_H and Hubble radius R_H, a characteristic scale for the size of the universe, are roughly

$$t_H \approx 10 \text{ Gyr} \qquad R_H \approx c\,t_H \approx 10^{26} \text{ m} = \text{a few Gpc}$$

Problem 6 (GeCAA 2020): A large mirror is placed at a distance S_0 from the Solar System. From Earth, a laser beam is directed towards it. After a time T, the radiation returns and is detected, allowing the determination of the constant H_0.

a) Find an expression for H_0 as a function of S_0, c (speed of light) and T . Consider that the separation S between the Solar System and the mirror increases only due to the expansion of the universe according to the law $S = S_0 e^{H_0 t}$. You may use $e^x \approx 1 + x$ for $x \ll 1$.

b) Imagine that such a mirror is located in the vicinity of the star Vega (parallax, $p = 0.125''$). Estimate the total duration of this H_0 measurement experiment.

Author's note: For part (b) the mirror is too close for the expansion of the universe to have any effect, and you'll see that ignoring the expansion will give the same result.

Problem 7 (4th Singapore Astronomy Olympiad): Calculate the apparent magnitude of a quasar with redshift $z = 7.081$ and a luminosity of $6.3 \times 10^{13} L_\odot$. **Hint**: The comoving distance can be calculated from $d_c(z) = d_H\chi(z)$ where $\chi(\text{z})$ is a dimensionless factor that is equal to 2.0 for this quasar, d_H is the Hubble radius

2 The cosmic equation of motion

The dynamics of the universe can be mapped into the motion of a point particle. We can use that to derive a cosmic equation of motion a.k.a the Friedmann equation that describes the expansion of the universe. We'll try to derive a simplified form of it using only Newtonian mechanics.

Consider a sphere of radius R and uniform matter density ρ. Find an equation of motion, in terms of a , ρ and other constants, using the laws of Energy conservation.

We assume the universe to be an expanding fluid and we look at a particle in the universe and see its Kinetic Energy,

$$\frac{1}{2}mv^2 = \frac{1}{2}m\dot{a}^2$$

The particle also has Potential Energy, which can be modeled similar to the energy due to a spherical body,

$$-\frac{GMm}{a}$$

The total energy will be equal to some constant, which itself will be proportional to m,

$$\frac{1}{2}m\dot{a}^2 - \frac{GMm}{a} = k'm$$
$$\frac{1}{2}\dot{a}^2 - \frac{G \cdot 4\pi a^3 \rho}{3a} = k'$$

Rearranging we obtain our 'simplified Friedmann equation'.

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2}}$$

Note: The k in the equation represents the geometric curvature of the universe. The state of cosmology for most of the 20th century was one where the value of k was quite uncertain. Fortunately, the theorist's perfect model of a universe with $k = 0$ turns out to correspond to ours. This is what we call a flat universe.

Being flat means that if any three points in the universe are connected by straight lines, the triangle will have the same properties as a triangle drawn on a flat piece of paper. I won't go into the details on the curvature, but here's a table that summarizes the consequences of the three possible values of k.

Curvature	Geometry	Angle of triangle	Circumference of circle	Type of Universe
$k > 0$	Spherical	$> 180°$	$c < 2\pi r$	Closed
$k = 0$	Flat	$180°$	$c = 2\pi r$	Flat
$k < 0$	Hyperbolic	$< 180°$	$c > 2\pi r$	Open

3 Evolution of the Components

3.1 Baryonic Matter

By baryonic matter, we mean normal non-relativistic matter made of baryons. Suppose $\rho(t)$ is the matter density of the universe at any time t and ρ_0 is that density today. We can find a relation between $\rho(t)$, ρ_0 and a. Consider a cubical region of the expanding space. Conservation of mass dictates that the mass M within a given comoving volume is constant in time. Comparing the mass at two different times (t and t_0),

$$M(t) = a^3 x^3 \, \rho(t)$$
$$M(t_0) = a_0^3 x^3 \, \rho(t_0) = x^3 \rho_0,$$

provides the evolution of the cosmic matter density

$$\rho(t) = \rho_0 \, a^{-3} = \rho_0 \, (1+z)^3$$

Problem 8 (Carroll & Ostlie): Assuming that the present density of baryonic matter is $\rho_m = 4.17 \times 10^{-28}$ kg/m^3 what was the density of baryonic matter at the time of the Big Bang nucleosynthesis (when $T \sim 10^{10}$ K)?

3.2 Radiation

Radiation is a confusing term in cosmology as it refers to all relativistic materials. This includes both photons and the almost massless neutrinos. In case of radiation, a useful term is number density, n, which is simply the number of particles in a given volume. If the mean energy of particle (including mass-energy) is E, then the number density is related to the radiation mass density by

$$\rho_{\text{rad}} c^2 = n \times E$$

Note that the units of ρ_{rad} are kg/m^3.

In a thermodynamic equilibrium, like the current state of the universe, the total number of particles must remain constant. Another point to note is that photons lose their energy as the universe expands and their wavelength is stretched, so their energy is $E_{\text{rad}} \propto 1/a$.

How does radiation mass density ρ_{rad} relate to the scale factor?

Since the number of particles remain constant, the only thing changing the number density is the expansion of the universe, $n_{\text{rad}} \propto \dfrac{1}{a^3}$,

$$\rho_{\text{rad}} \propto n_{\text{rad}} \times E_{\text{rad}} \propto \frac{1}{a^3} \times \frac{1}{a} \propto \frac{1}{a^4}$$

Thus we get

$$\rho_{\text{rad}} = \rho_{\text{rad}0} (1+z)^4$$

3.3 Dark Matter

Dark matter is basically matter we can't see because it does not interact except gravitationally. Most of the matter we can see in this universe is basically stars, but there's a huge chunk of matter we can't see. They're not planets or space rocks cause they're not massive enough and there's not much of them. We don't know what they are but we do know that they kinda behave like matter and most importantly, they evolve like matter.

3.4 Dark Energy

Dark energy is the energy of empty space. Unlike matter and radiation, dark energy does not evolve i.e. its density remains constant throughout the expansion of the universe. A proposed form of the Dark Energy is the cosmological constant Λ, which is the energy of space or vacuum energy.

Problem 9 (folklore): Let us define the critical density ρ_c as the matter density required to explain the expansion of a flat universe without any radiation or dark energy. Find an expression of the critical density, in terms H and G. Calculate the present critical density ρ_{c0}.

Answer: From Friedmann Equation, the critical density is defined in the way of

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_c$$

thus,

$$\rho_c = \frac{3H^2}{8\pi G}$$

The modern day value of the critical density is thus 1.06×10^{-26} kg/m^3.

Problem 10 (IOAA 2008): The CMB is at redshift $z_{CMB} = 1100$. The current densities of the Dark Energy, Dark Matter, and Normal Matter components of the Universe as a whole are, $\rho_{\mathrm{DE}} = 7.1 \times 10^{-30}$ g/cm^3, $\rho_{\mathrm{DM}} = 2.4 \times 10^{-30}$ g/cm^3 and $\rho_{\mathrm{NM}} = 0.5 \times 10^{-30}$ g/cm^3, respectively.

What is the ratio between the density of Dark Matter to the density of Dark Energy at the time CMB was emitted?

Problem 11 (IOAA 2009): Assume the mass of neutrinos is $m_\nu = 10^{-5}\, m_e$. Calculate the number density of neutrinos needed to compensate for the dark matter of the universe. Assume the universe is flat and 25% of its mass is dark matter.

4 Einstein de-Sitter Universe

The Einstein de-Sitter Universe is one where the geometric curvature and the cosmological constant are set to zero, leaving only a *flat* matter dominated universe. Density becomes the critical density. Thus the e.o.m becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 a^{-3}$$

We make an educated guess and relate the age of an EdS universe to the scale factor using the power law[2]:

$$a(t) = A\,t^{\alpha} \Rightarrow \alpha^2\,2t^{-2} = \frac{H_0^2}{A^3}\,t^{-3\alpha} \Rightarrow \alpha = \frac{2}{3}A = \left[\frac{3}{2}\,H_0\right]^{2/3}$$

Remember that here t is the time since the Big-Bang. This leads to the solution for an EdS universe:

$$a(t) = \left[\frac{3}{2}\,H_0\right]^{2/3}\,t^{2/3}$$

or simply,

$$a(t) \propto t^{2/3}$$

Problem 12 (folklore): Calculate the present age of a flat EdS Universe.

Answer: In this model the exact age of the universe is $t_{\mathrm{H,EdS}} = \dfrac{2}{3H_0} \sim 9$ Gyr, which is younger than the age of the oldest globular clusters (11.2 billion years). Hence, we cannot live in an EdS universe without suffering from a ‘cosmic age crisis’. However, this model is still useful to make assumptions about the matter dominated era. Furthermore, because when we view distance regions of the universe we are viewing them as they were in the past it turns out that all the parts of the universe we can see beyond a redshift of about $z = 2$ are still well described by an Einstein de Sitter model.

It is useful to remember that the age of the universe according to current models is 13.8×10^9 years.

Problem 13 (IOAA 2011): Based on the spectrum of a galaxy with redshift $z = 6.03$ it was determined that the age of the stars in the galaxy is from 560 to 600 million years. At what z did the epoch of star formation occur in this galaxy?

Problem 14 (IOAA 2012): Compute how long it will take for the universe to cool down by 0.1K.

Problem 15 (folklore): Show that for a radiation dominated universe the relation between scale factor and time since the Big Bang is $a \propto t^{1/2}$.

[2]Mathematically the power law is, $Y = AX^{\alpha}$ where Y and X are two related variables.

5 Density Parameter

It is often useful to express densities as a fraction of the critical density. As a refresher, critical density is the average density of matter required for a universe, with zero curvature and zero dark energy, to just halt its expansion, but only after an infinite time. Thus we define the density parameter as,

$$\Omega = \frac{\rho}{\rho_c}$$

Problem 16 (IOAA 2018):
The complete Friedmann Equation is usually written as follows:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left(\rho_m + \rho_r\right) + \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2}$$

The Friedmann Equation can be rewritten using the dimensionless density parameters simply as, $\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k = 1$

Use this information to find expressions for Ω_Λ and Ω_k, in terms of H, c, Λ, k, a.
E.A.: Find an expression for Ω_m & Ω_r as well.

Note: In practice we consider $c = 1$ & drop the 'c' in the Friedmann equations.

Answer: This problem is purely mathematical.

$$\frac{8\pi G}{3}\left(\rho_m + \rho_r\right) + \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2} = \left(\frac{\dot{a}}{a}\right)^2$$

$$\frac{8\pi G}{3}\rho_m + \frac{8\pi G}{3}\rho_r + \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2} = H^2$$

$$\frac{8\pi G}{3H^2}\rho_m + \frac{8\pi G}{3H^2}\rho_r + \frac{\Lambda c^2}{3H^2} - \frac{kc^2}{a^2H^2} = 1 \tag{1}$$

$$\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k = 1 \tag{2}$$

Comparing 1 and 2,

$$\Omega_m = \frac{8\pi G}{3H^2}\rho_m \qquad \Omega_r = \frac{8\pi G}{3H^2}\rho_r$$

$$\Omega_\Lambda = \frac{\Lambda c^2}{3H^2} \qquad \Omega_k = -\frac{kc^2}{a^2H^2}$$

Problem 17 (folklore): Manipulate the Friedmann Equation to obtain the form

$$\dot{a} = H_0\sqrt{\Omega_{r0}a^{-2} + \Omega_{m0}a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0}a^2}$$

Differentiate this to obtain $\ddot{a} = -\dfrac{1}{2}H_0^2\left(2\Omega_{r0}a^{-3} + \Omega_{m0}a^{-2} - 2\Omega_{\Lambda 0}a\right)$

From this, estimate the scale factor, $a(t)$ when our Universe switched from decelerating to accelerating expansion.

6 The Early Universe

The scale factor of the universe depends on the matter and energy density in a way that depends on which component is dominant. That is, which component has a higher density. The other components can be considered negligible for now. A transition in the dominance from one component to another will cause a transition of how the scale factor evolves. Now to find the epoch (z) of this transition. While doing this we will assume the transition is fairly instantaneous compared to the timescale of the universe. At the moment of transition, the densities of the components in consideration will be equal.

During the matter-radiation equality,

$$\Omega_r = \Omega_m$$
$$\Omega_{r0}(1+z)^4 = \Omega_{m0}(1+z)^3$$
$$z = \frac{\Omega_{m0}}{\Omega_{r0}} - 1$$

At the matter-dark energy equality,

$$\Omega_\Lambda = \Omega_m$$
$$\Omega_{\Lambda 0} = \Omega_{m0}(1+z)^3$$
$$z = \left(\frac{\Omega_{\Lambda 0}}{\Omega_{m0}}\right)^{1/3} - 1$$

We use the following values of the density parameters to find the epochs

$$\Omega_{r0} = 9.1 \times 10^{-5}; \qquad \Omega_{m0} = 0.31; \qquad \Omega_{\Lambda 0} = 0.69$$

Thus at matter-radiation equality, $z \approx 3400$,
At matter-dark energy equality, $z \approx 0.3$.

Note: These values were taken from the 4th Singapore Astronomy Olympiad (SAO).

Values of the parameters will be generally given to you in the problems or in the constants table and you are required to follow that. Nonetheless, the facts that

$$\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k = 1 \text{ and } \Omega_r, \Omega_k \ll 1$$

thus,

$$\Omega_m + \Omega_\Lambda = 1$$

will always hold.

Problem 18 (4th SAO): Estimate the age of the Universe at the radiation-matter equality, given that the matter-dark energy equality occurred 4.05 billion years ago.
Hint: In a radiation-, matter- and dark-energy-dominated Universe, $a \propto t^{1/2}$, $a \propto t^{2/3}$ and $a \propto \exp\left(t\sqrt{\frac{\Lambda}{3}}\right)$ respectively.

Problem 19 (folklore): Show that at the time of radiation-dark energy equality, the evolution of the scale factor didn't change.

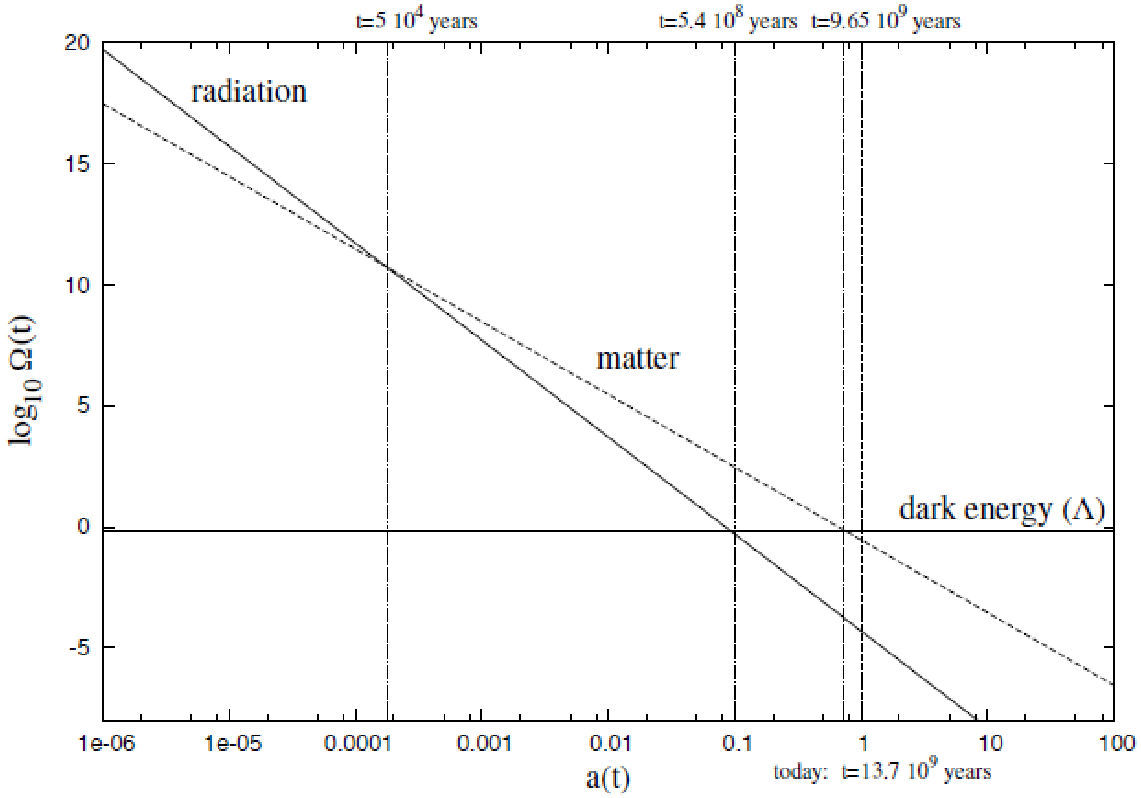


Fig: Three epochs in the evolution of the universe. $t = 5 \times 10^4$ years is the matter-radiation equality, $t = 9.65 \times 10^9$ years is the matter-dark energy equality, $t = 13.7 \times 10^9$ years is the present time

Problem 20 (folklore): Consider a flat universe with only matter and dark energy. For this universe, write Ω_m as a function of Ω_{m_0} and z.

Answer: We have $\Omega_m + \Omega_\Lambda = 1$ and $\Omega_{m_0} + \Omega_{\Lambda_0} = 1$

Now,

$$\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c} = \frac{\rho_{\Lambda 0}}{\rho_{c0}} \cdot \frac{\rho_{c0}}{\rho_c} = \Omega_{\Lambda_0}\frac{H_0^2}{H^2} = (1-\Omega_{m_0})\frac{H_0^2}{H^2}$$

And,

$$\Omega_m = \frac{\rho_m}{\rho_c} = \frac{\rho_{m0}(1+z)^3}{\rho_{c0}} \cdot \frac{\rho_{c0}}{\rho_c} = \Omega_{m_0}(1+z)^3\frac{H_0^2}{H^2}$$

Comparing equations we get,

$$\Omega_\Lambda = (1-\Omega_{m_0})\frac{\Omega_m}{\Omega_{m_0}(1+z)^3}$$

Plugging this into the Friedmann Equation,

$$\Omega_m = \frac{\Omega_{m_0}(1+z)^3}{\Omega_{m_0}(1+z)^3 + 1 - \Omega_{m_0}}$$

6.1 Radiation Density

Remember that radiation is a combination of photons and neutrinos. So we'll use the subscripts - r for radiation, γ for photons, ν for neutrinos.

We all understand radiation energy density, it's intuitive. But radiation mass density can be puzzling. And that is probably because we see mass as an intrinsic property. But in fact mass arises when energy is confined in some volume. Thus mass is not a fundamental property, it's an emergent property. Think of yourself, almost all of your mass comes from protons and neutrons, 99% of which are made of massless gluons. You have mass just because they are confined in a very small space.

So what's radiation mass density? It's the mass that emerges when you confine radiation energy in a unit volume.

The relation between radiation mass density ρ_r, and radiation energy density ϵ_r, is -

$$\epsilon_r = \rho_r c^2$$

The calculation of neutrino density in the universe is difficult. So instead, we'll only calculate the photon radiation density and use the information $\rho_\nu = 0.68\rho_\gamma$ to find the neutrino radiation density.

By integrating Planck's law and from the Stefan-Boltzmann law, we get the following relation,
Photon radiation energy density of black body,

$$\epsilon_\gamma = \frac{\pi^2}{15\hbar^2 c^2}(k_B T)^4 = \frac{4\sigma T^4}{c}$$

where, σ is is the Stefan-Boltzmann constant.

So the photon radiation density parameter,

$$\Omega_\gamma = \frac{\rho_\gamma}{\rho_c} = \frac{\epsilon_\gamma}{\rho_c\, c^2} = \frac{1}{\rho_c\, c^2}\,\frac{4\sigma T^4}{c}$$

We plug in the current value of the critical density as 1.06×10^{-26} kg/m^3. And the remnant of the radiation density is in fact the CMB, and its current temperature is $T = 2.73$ K.

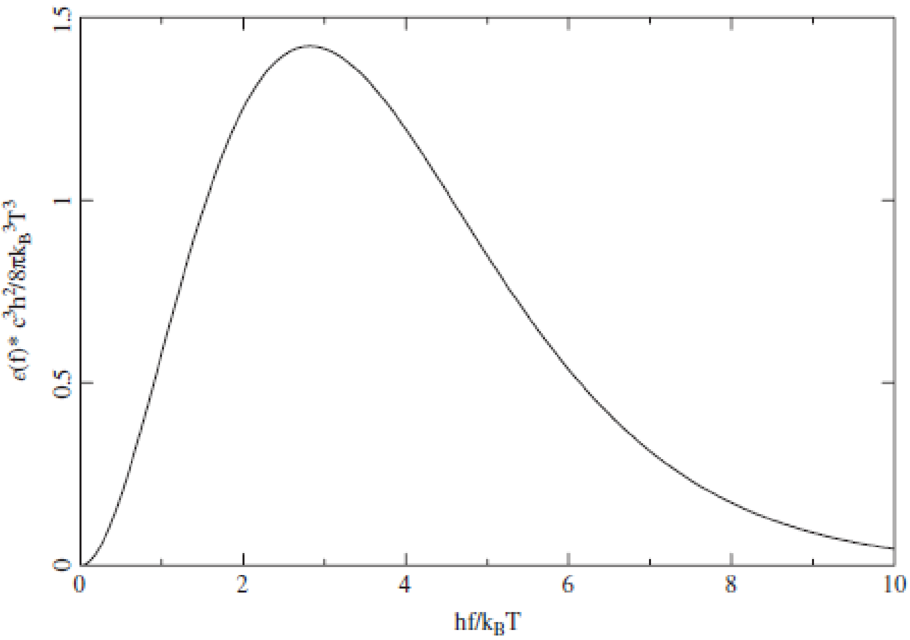
$$\Omega_{\gamma,0} = 1.98 \times 10^{-5}; \qquad \Omega_{\nu,0} = 1.34 \times 10^{-5}; \qquad \Omega_{r,0} = 3.32 \times 10^{-5}.$$

6.2 Temperature unit conversion

At times you'll see characteristic temperatures expressed in eV rather than K. This serves two purposes - one is to allows easier comparison of very high temperatures. The other is that we can use this to compare formation energies of various species.

eV is basically the unit of energy where 1 eV $= 1.6 \times 10^{-19}$ J. $k_B T$ is a characteristic energy of a distribution of photons. Not to be confused with hf, which is the energy of a single photon. Also not to be confused with $1/2 k_B T$, which is the energy per degree of freedom of gas particles.

To go from energy to temperature units, convert eV to Joules and divide that by k_B.



If we look at the energy density distribution of a black-body spectrum, we will see that the total energy in the radiation is dominated by photons with energies of order $k_B T$. Indeed, the mean energy of a photon in this distribution is $3k_B T$.

Problem 21 (IOAA 2018): The neutrinos decoupled from the primordial soup when the temperature of the universe was around 1 MeV. At this time, the radiation density in the universe was much more than all other components. Estimate the time ($t = \frac{1}{2H}$) when neutrinos decoupled.

Problem 22 (Concept review:): Assuming a matter dominated universe where $H_0 = 70$km/s/Mpc and $T_0 = 2.73$ K, for a redshift of $z = 9$, find:

$T(\text{in K and eV}), \lambda, a, t, \rho_c, H, \dot{a}, \ddot{a}$

Now do this again for a radiation dominated and dark energy dominated universe.

7 Fluid Equation

We've seen how the densities evolve with the scale factor. Now we'll be deriving an equation to formulate these in a more formal manner. We start with the first law of thermodynamics.

$$dE = dQ - pdV$$

Now there are two assumptions. First, we can consider that the universe is expanding adiabatically. Or we could assume reversible expansion and thus $dS = 0$. Both of these conclude to $dQ = 0$.

Second, we consider the galaxies to have negligible potential energy because they are really far away from each other and they have negligible kinetic energy because their velocities with respect to the comoving distances are much less than the speed of light. So the only type of internal energy they have is due to $E = mc^2$.

Applying this to a unit comoving radius,

$$E = \frac{4\pi}{3} a^3 \rho c^2$$

The change of energy in a time dt,

$$\frac{dE}{dt} = 4\pi a^2 \rho c^2 \dot{a} + \frac{4\pi}{3} a^3 \dot{\rho} c^2$$

While the change in volume is,

$$\frac{dV}{dt} = 4\pi a^2 \dot{a}$$

Putting these into the first law and rearranging gives,

$$\boxed{\dot{\rho} + 3\,\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0}$$

which is called the fluid equation in cosmology.

7.1 Pressure

It is stressed that there are no pressure forces in a homogeneous universe, because density and pressure are the same everywhere. A pressure gradient is required to supply a force. Instead of thinking of pressure as a force per unit area ($P = F/A$), it is better to think of it as an energy per unit volume i.e. $P = E/V$. So pressure does not contribute a force helping the expansion along; its effect is through the work done as the universe expands.

It is a usual assumption in cosmology that there is a unique pressure associated with each component of density. That is, the fluid equation applies to matter, radiation and dark energy separately; and there is a unique relation between pressure and density for each component such that p is a function of ρ. This relation is called the **equation of state**.

Problem 23 (IOAA 2018): The Fluid Equation $\rho + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0$ is valid for matter, radiation and dark energy. Radiation contains photons and massless neutrinos, and they both travel at the speed of light. The pressure exerted by these particles is 1/3 of their energy density. Show that the density of radiation $\rho_r \propto (1+z)^4$. You may note that if $\frac{\dot{\rho}}{\rho} = n\frac{\dot{a}}{a}$ then $\rho \propto a^n$.

E.A.: Show that for matter, $\rho_m \propto (1+z)^3$ if matter applies no pressure.

Answer:

$$\begin{aligned}&\text{Energy, } E = mc^2\\ &\text{Energy density, } U = \rho c^2\\ &\text{Radiation pressure, } p = \frac{\rho_r c^2}{3}\end{aligned}$$

Plugging this in the fluid equation we get,

$$\frac{\dot{\rho}}{\rho} = -4\frac{\dot{a}}{a}$$

Thus, $\rho_r \propto a^{-4} \Rightarrow \rho_r \propto (1+z)^4$,

For matter putting $p = 0$ one can similarly show that,

$$\frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a}$$

Thus, $\rho_m \propto a^{-3} \Rightarrow \rho_m \propto (1+z)^3$,

Problem 24 (IOAA 2018): We know that the value of the cosmological constant Λ doesn't evolve. Its equation of state has a form $p = w\rho_\Lambda c^2$, where w is an integer. Find the value of w.

Problem 25 (Andrew Liddle): We examined solutions for the expansion when the Universe contained either matter ($p = 0$) or radiation ($p = \rho c^2/3$). Suppose we have a more general equation of state, $p = (\gamma - 1)\,\rho c^2$, where γ is a constant in the range $0 < \gamma < 2$. Find solutions for $\rho(a)$, $a(t)$ and hence $\rho(t)$ for universes containing such matter. Assume $k = 0$ in the Friedmann equation.

Component	EoS Coefficient w	Scaling with Time
Radiation	1/3	$t^{1/2}$
Matter	0	$t^{2/3}$
Dark Energy	-1	$\exp\left(t\sqrt{\frac{\Lambda}{3}}\right)$

8 Acceleration Equation [optional]

The acceleration equation can be derived from the first Friedmann Equation and the Fluid Equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_m + \rho_r) + \frac{\Lambda}{3} - \frac{k}{a^2}$$

$$\dot{\rho} + 3\,\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0$$

to get:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{k}{a^2} + \frac{\Lambda}{3}$$

9 Olympiad level Problems

Problem 26 (PoTW): Suppose Bombom is living in a universe that is filled by radiation only. This universe is expanding with Hubble constant H_0, and its temperature at present is T_0.

a. Find the condition for H_0 in terms of T_0 so that the universe has: flat curvature ($\kappa = 0$), positive curvature ($\kappa = 1$), and negative curvature ($\kappa = -1$).

b. Find the radius of the curvature at the present (R_0).

c. Find the age of the universe (t_0) in terms of radiation density parameter (Ω_0) and H_0.

d. Using $H_0 = 70$ km/s/Mpc and $T_0 = 2.73$ K, calculate R_0, t_0, and also the temperature of the universe at $t = t_0/2$

10 Sources and Acknowledgments

1. An Introduction to Modern Cosmology by Andrew Liddle
2. Lecture Notes by Prof. Volker Bromm
3. Wikipedia
4. IOAA Problems
5. SAO Problems
6. Cosmology Slides by SAO
7. USAAAO Cosmology Handout by Faraz Ahmed
8. An Introduction to Modern Astrophysics - Carroll & Ostlie