Cosmology II

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Abstract

Proofs not shown may or may not be trivial but are left as exercises anyways

Table 1: **Useful Constants**

Rest Energy of Neutron	=	939.57 MeV
Rest Energy of Proton	=	938.27 MeV
Rest Energy of Electron	=	0.511 MeV
Typical Nuclear Binding Energy	$\approx$	1 MeV
Ionization Energy of Hydrogen Atom	=	13.6 eV
$n \to \bar{\nu_e} + p + e^-$		
$n + \nu_e \rightleftharpoons p + e^-$		
$n + e^+ \rightleftharpoons p + \bar{\nu_e}$		

0.1 Hubble–Lemaître Law

The math in cosmology begins with the **Hubble–Lemaître Law**. To understand Hubble's Law we must first understand redshift. Redshift is a phenomenon where electromagnetic radiation from an object undergoes an increase in wavelength. Specifically, we shall be dealing with cosmological redshift. Space itself is expanding, causing objects to become separated without changing their positions in space. This is known as cosmological redshift.

For all galaxies, we find a "redshift" z which is proportional to the distance D light has traveled from the galaxy, i.e. $z \propto D$. The constant of proportionality is related to the Hubble constant H_0. If λ_0 is the laboratory wavelength measurement of a single absorption or emission line and λ_{obs} is the observed wavelength of the redshifted line then $\Delta\lambda = \lambda_{\text{obs}} - \lambda_0$ is the difference between the two, then redshift is defined as–

$$z \equiv \frac{\Delta\lambda}{\lambda_0} = \frac{H_0}{c} D \qquad \text{[“Hubble law” for redshifts]}$$

This is often incorrectly interpreted as a Doppler effect where[1],

$$z = \frac{v}{c} = \frac{\Delta\lambda}{\lambda_0}$$

Nonetheless, this idea lends itself to the more familiar version of the Hubble law:

$$v = H_0 D \qquad \text{[“Hubble law” for velocities]}$$

How do we describe expanding space?

We introduce a scale factor $a(t)$ which compares distances at different times. Then physical distances r and comoving distances x are related through a as follows:

$$r = a(t) \cdot x$$

Where,

$$\begin{aligned} a &= \text{cosmic scale factor} \\ x &= \text{comoving coordinates} \\ r &= \text{physical (or proper) coordinates} \end{aligned}$$

Take a balloon and draw a grid on it. Draw some galaxies on it too. When you blow up the balloon, you can note two things. The grid spacing between the galaxies doesn't change, these are the comoving coordinates. You can view these coordinates as being carried along with the expansion. The next is the physical distance between the galaxies (measured along the surface of the balloon if you want to get technical) that increases with the expansion. This is the physical distance that we are actually used to. The scale factor is literally just a scaling factor to go from comoving to physical coordinates. Thus objects moving along with space have a constant comoving distance but face increasing physical distance.

[1] The relation $z = \frac{v}{c}$ in cosmology can only be used for very small redshifts, $z \ll 1$.

We often normalize the scale factor. The parametrization is chosen such that the present-day value of the scale factor is one:

$$a_0 \equiv a(t_o) \equiv 1$$

where, $t_0 \equiv$ today

Hence, the scale factor is just another way of measuring cosmic time.

0.2 Relation between scale factor and other quantities

Now we have an interpretation of the redshift - light travels through expanding space - so the physical wavelength changes according to $\lambda \propto a$.

$$\begin{aligned} &\text{At emission:} && \lambda_{\text{em}}(\lambda_0) \propto a(t_{\text{em}}) \\ &\text{Observed today:} && \lambda_{\text{obs}}(\lambda) \propto a(t_0) \end{aligned}$$

Therefore, the redshift is,

$$z = \frac{\Delta\lambda}{\lambda_0} = \frac{\lambda - \lambda_0}{\lambda_0} = \frac{\lambda}{\lambda_0} - 1 = \frac{a_0}{a(t_{\text{em}})} - 1,$$

providing the cosmological definition of redshift once we substitute $a(t_{\text{em}}) \to a(t) \to a$ and $a_0 \to 1$.

$$1 + z = \frac{1}{a}$$

From Wien's law we have $T \propto \frac{1}{\lambda}$ and thus $T \propto \frac{1}{a}$ providing us the relation between temperature and scale factor. It is useful to remember that the average temperature of the Cosmic Microwave Background (CMB) or equivalently the universe is currently $T_0 = 2.73$ K.

0.3 Critical Density and Density Parameter

Let us define the critical density ρ_c as the average matter density required for a universe, with zero curvature and zero dark energy, to just halt its expansion, but only after an infinite time. Let us find an expression of the critical density, in terms H and G. Consider the universe to be a sphere made of fluid, of radius a and uniform matter density ρ. Using the laws of Energy conservation on a particle of mass m in the sphere:

$$\frac{1}{2}mv^2 - \frac{GMm}{a} = 0$$
$$\frac{1}{2}v^2 - \frac{G \cdot 4\pi a^3 \rho}{3a} = 0$$

$$\left(\frac{v}{a}\right)^2 = \frac{8\pi G\rho}{3}$$

$$\rho_c = \frac{3H^2}{8\pi G}$$

It is often useful to express densities as a fraction of the critical density. Thus we define the density parameter as,

$$\Omega = \frac{\rho}{\rho_c}$$

1 Origin of the CMB

A hydrogen atom has an ionization energy of 13.6 eV. As long as the Universe is hot enough, photons will easily have this energy and are able to keep the hydrogen fully ionized. The Universe at that time was therefore a sea of free nuclei and electrons, and because photons interact strongly with free electrons (via Thomson scattering), the mean free path of any photon was short (approximately $1/n_e\sigma_e$ where n_e is the electron number density and σ_e the Thomson scattering cross-section). So we picture a sea of frequently-colliding particles, forming an ionized plasma. This situation is actually not very exotic; if you calculate the density of material at that time you will find it's very low– considerably less than water– and it's very easy to heat a gas up until it becomes a plasma. As the Universe expanded and cooled, the photons of light lost energy and became less and less able to ionize any atoms that form. The situation is exactly that of the photoelectric effect, where long-wavelength photons, however numerous, are unable to knock electrons out of metal atoms. Eventually all the electrons found their way into the ground state and the photons were no longer able to interact at all. Over a short interval of time, the Universe suddenly switched from being opaque to being completely transparent. The photons were then able to travel unimpeded for the entire remainder of the Universe's evolution. This process is known as decoupling.

2 CMB Origins - Decoupling

In this problem, we will explore the origins of the Cosmic Microwave Background (CMB). First we need some prerequisites. The total radiation energy density of a blackbody at temperature T is given by,

$$\epsilon_{rad} = \alpha T^4$$

where $\alpha = 7.565 \times 10^{-16}\mathrm{Jm^{-3}K^{-4}}$. You also know that the typical energy of a photon of light in a thermal distribution is

$$E = 3k_BT$$

a. At the current temperature of the universe, what is the number density of photons?

From Planck results, it is also known that the current energy density of baryonic matter is $\epsilon_{bar} = 3.9 \times 10^{-11}\mathrm{Jm^{-3}}$. And the rest energies of a neutron and proton are 939.57 MeV and 938.27 MeV respectively.

b. What is the current number density of baryons?

c. Find the photon to baryon ratio, η.

Now, we're in a position to estimate the epoch of the origin of the CMB. One can naively estimate that the first H atoms formed when the mean energy of the photons was equal to the first ionization energy of Hydrogen, $Q = 13.6\,\mathrm{eV}$

d. Using this assumption, what was the temperature of the universe when the formation of H atoms was first possible?

However, from the solution you found in (c), you should realize that there is a long tail of photons with energies much higher than 13.6 eV. These prevent the formation of H atoms, and we have to consider that. The fraction of photons with energy exceeding I is given approximately by

$$\eta = \exp(-I/k_B T)$$

e. What is the revised temperature now? This is approximately the temperature when 'decoupling' occurs.

f. How does the value of the baryon-to-photon ratio, η, affect the recombination temperature in the early universe. Express the fractional ionization X as a function of temperature. First assuming $\eta = 4 \times 10^{-10}$, then assuming $\eta = 8 \times 10^{-10}$, if we define T_{rec} as the temperature at which $X = 1/2$

g. Assuming a baryon-to-photon ratio $\eta = 6.1 \times 10^{-10}$, at what temperature T will there be one ionizing photon, with $hf > Q = 13.6$ eV, per baryon? Is the temperature you calculate greater than or less than $T_{rec} = 3760$ K?

3 What if the CMB had a different origin story?

We know the origin story of the CMB, photons didn't have enough energy to ionize Hydrogen atoms and thus 'decoupled' from electrons. But what if the Hydrogen stayed ionized? Photons would still decouple from the electrons at some point, but by a different mechanism. To understand that we need some primary concepts.

The likelihood of interactions between two particles is given by a quantity called the cross-section. The cross-section for Thomson scattering (photon-electron interaction) is $\sigma_e = 6.65 \times 10^{-29}\text{m}^2$. This means, if you have a photon within a gas of electrons, they will act like particles with a cross-sectional area of σ_e

a. If the number density of such gas of electrons is n_e, what is the mean free path, λ of the photon? [1]

b. Given the current electron number density is $n_{e,0} = 0.25\text{m}^{-3}$, what is the scattering rate of the photon, Γ at any scale factor a? [2]

When the photon scattering rate drops below the cosmic expansion rate aka the Hubble parameter, H, then the electrons are being diluted by expansion more rapidly than the photons can interact with them. The photons then decouple from the electrons and the universe becomes transparent. We assume the decoupling occurred during the matter dominated era.

c. Find an expression for H in the matter dominated era. Take $\Omega_{m,0} = 0.31$ and $H_0 = 2.2 \times 10^{-18}\text{s}^{-1}$.

d. At what redshift, z and temperature, T do photons decouple from electrons?

4 IOAA-23 T11: 'X-ray emission from galaxy clusters'

Clusters of galaxies are strong X-ray sources. It has been established that the emission mechanism is thermal bremsstrahlung (free-free radiation) from a hot hydrogen and helium plasma inside the cluster. The luminosity L_X (in Watts) of each component of the plasma is described by the formula:

$$L_X = 6 \times 10^{-41} N_e N_X T^{\frac{1}{2}} V Z_X^2 \tag{1}$$

where the symbols represent:

- X - Hydrogen (H) or Helium (He),
- N_e - number density of electrons [m^{-3}],
- N_X - number density of ions X [m^{-3}],
- Z_X - atomic number of ion X,
- T - temperature of the plasma [K],
- V - volume occupied by the plasma [m^3].

(a) Determine the total mass (in solar masses) of the plasma which emits the X-rays, assuming that:

  - the plasma is fully ionized with 1 helium ion for every 10 hydrogen ions;
  - $L_{\text{total}} = 1.0 \times 10^{37}$ W,
  - $T = 80 \times 10^6$ K,
  - the plasma is uniformly distributed in a sphere of radius $R = 500$ kpc,
  - self-absorption is negligible.

  (16 points)

  The photons of the cosmic microwave background (CMB) interact with plasma in a process known as inverse Compton scattering. The CMB normally has a thermal blackbody spectrum at a temperature of 2.73 K. However, interaction with the plasma leads to distortion of the CMB spectrum (known as the Sunyaev–Zeldovich effect).

(b) Estimate the mean free path of CMB photons in the plasma, i.e. the average distance travelled by a photon before interacting with an electron. Express it in Mpc. The effective cross section for photon–electron interactions is $\sigma = 6.65 \times 10^{-29}$ m^2. (5 points)

(c) Estimate the typical energy of CMB photons. (3 points)

(d) The energy of CMB photons can be increased by a factor of up to $(1+\beta)/(1-\beta)$ due to the inverse Compton scattering, where $v = \beta c$ is the velocity of electrons. Estimate the energy of scattered CMB photons. (6 points)

5 Big Bang Nucleosynthesis

In the very early universe, everything is in thermodynamic equilibrium and particles are freely created, destroyed, and converted between each other due to the high temperature. In one such process, the reaction converting between neutrons and protons happens at a very high rate. In thermal equilibrium, the relative number density of particle species is given approximately by the Boltzmann factor:

$$n_i \propto m_i^{3/2} \exp\left[-\frac{m_i c^2}{k_B T}\right]$$

a. Find an expression for the ratio of number densities of neutron to proton?

Thus, the temperature dictates the ratio of neutrons to protons when they are in thermal equilibrium. This thermal equilibrium is maintained through the interactions between baryons and neutrinos. At one point in time, the neutrinos decouple from the neutrons and protons, and the ratio of neutrons to protons is "frozen". This happens at a temperature where $k_B T \approx 0.8 MeV$ known as the freeze-out temperature.

b. What was the ratio of number densities of neutrons to protons at this temperature?

After the freezing out, ^{4}He and other lighter elements started forming. This process is called the Big Bang Nucleosynthesis. However, $0.8 MeV$ is still close to the binding energy of a nucleus, so the temperature had to fall just a little more to allow all neutrons to fuse. You can consider this lower energy to be reached 200s later (roughly 3 minutes). Also, free neutrons are unstable and they'll decay into protons with a half life of about 614s. By the time all the neutrons combined with a proton, some of them decayed.

c. Thus find the new number density of neutron to proton after the BBN.

d. If $f = n_n/n_p$, find Y_p, the fraction of baryonic mass in the form of ^{4}He, in terms of f. Assume all neutrons go into forming ^{4}He.

e. The total luminosity of the stars in our galaxy is L = 3×10^{10} $L_\odot$. Suppose that the luminosity of our galaxy has been constant for the past 10 Gyr. How much energy has our galaxy emitted in the form of starlight during that time?

f. Most stars are powered by the fusion of H into ^{4}He, with the release of 28.4 MeV for every helium nucleus formed. How many helium nuclei have been created within stars in our galaxy over the course of the past 10 Gyr, assuming that the fusion of H into ^{4}He is the only significant energy source?

g. If the baryonic mass of our galaxy is $M \approx 10^{11} M_\odot$, by what amount has the helium fraction Y of our galaxy been increased over its primordial value $Y_p = 0.24$?