0.1 Convex Quadratic Programming Problem

A general convex quadratic programming problem is in the following two forms. Given, $Q \succ 0$, is a real symmetric matrix of order $n \times n$, $c, x \in R^n, b \in R^m$, A is real matrix of order $m \times n$.

$$(QP_1) \qquad \min \frac{1}{2} x^T Q x + c^T x$$
$$subject\ \ to Ax = b$$

$$(QP_2) \qquad \min \frac{1}{2} x^T Q x + c^T x$$
$$subject\ \ to Ax \leq b, x \geq 0$$

Both the quadratic programming problems are convex programming problems since (i) $Q \succ 0 \Rightarrow x^T Q x\ > 0$ and $Ax \leq, = b$ is a linear system.

0.1.1 Solution of (QP_1)

Since (QP_1) is a convex programming problem so KKT conditions are both necessary and sufficient conditions for the existence of solution.
The lagrange function for (QP_1) is

$$L(x, \mu) = f(x) + \mu^T h(x) = \frac{1}{2} x^T Q x + c^T x + \mu^T (Ax - b)$$

KKT optimality conditions are

$$\nabla_x L(x, \mu) = 0$$
$$Ax = b, \mu \in R^m$$

Here $Q \succ 0$ so Q^{-1} exists and $Q = Q^T$ since Q is symmetric.

$$\begin{aligned}
&\nabla_x L(x, \mu) = 0 \\
&\equiv Qx + c + A^T\mu = 0 \\
&\equiv x = -Q^{-1}(c + A^T\mu)
\end{aligned}$$

What is μ?

$$\begin{aligned}
Ax = b \equiv &- AQ^{-1}(c + A^T\mu) = b \\
&\equiv AQ^{-1}A^T\mu = -(b + AQ^{-1}c)
\end{aligned}$$

For any nonzero vector $x \in R^n$, $x^TAQ^{-1}A^Tx = (A^Tx)^TQ^{-1}(A^Tx) = z^TQ^{-1}z > 0$ since $Q \succ 0$. Hence inverse of $AQ^{-1}A^T$ exists.
Therefore $\mu = -(AQ^{-1}A^T)^{-1}(b + AQ^{-1}c)$.

Example 0.1.1. *Using the above process find the solution of the following quadratic programming problems:*

1. $\min 3x_1^2 + x_2^2 + x_1x_2 + 2x_1 - x_2,\ s.to\ \ x_1 + 2x_2 = 4$
2. $\min 3x_1^2 + x_2^2 + 2x_3^2 + x_1x_2 + x_1x_3 + 2x_1 - x_2 + x_3,\ s.to\ \ x_1 + 2x_2 + x_3 = 4, 3x_1 - x_2 + 3x_3 = 5$

0.1.2 Solution of (QP_2)

$$\begin{aligned}
(QP_2) \qquad\qquad &\min \frac{1}{2}x^TQx + c^Tx \\
&subject\ \ to Ax \leq b, x \geq 0
\end{aligned}$$

This is also a convex programming problem. Hence KKT optimality conditions are both necessary and sufficient for the existence of solution. Denote

$$L(x,\mu) = f(x) + \lambda^T g(x) = \frac{1}{2}x^T Q x + c^T x + \lambda^T (Ax - b)$$

KKT optimality conditions are

$$\begin{aligned} \nabla_x L(x,\lambda) &= 0 \\ Ax &\leq b, \\ (\lambda(Ax-b))_j = 0, \lambda_i &\geq 0 x_j \geq 0 \end{aligned}$$

$\nabla_x L(x,\lambda) = 0$ is same as $Qx + c + A^T\lambda = 0$. This is a system of linear equations in x and λ

$Ax \leq b$ is same as $Ax + s = b$, where $s \in R^m$, $s_j \geq 0$ are slack variables.

$(\lambda(Ax-b))_j = 0$ is same as $\lambda_i s_i = 0$ for $i = 1, 2, ..., m$

Hence KKT optimality conditions are

$$\begin{aligned} Qx + c + A^T\lambda &= 0 \\ Ax + s &= b \\ \lambda_i s_i &= 0 \\ x_j, \lambda_i &\geq 0 \end{aligned}$$

This system can be solved by solving the following linear programming problem with artificial variables $z_j \geq 0, j = 1, 2, ..., n$, and complementary slackness condition $\lambda_i s_i = 0 \; \forall i$.

$$\begin{aligned} \min \quad & z_1 + z_2 + ... + z_n \\ subject\ to \quad & QxA^T\lambda + z = -c \\ & Ax + s = b \\ & x_j, \lambda_i, z_j \geq 0 \end{aligned}$$

Complementary slackness condition $\lambda_i s_i = 0 \ \forall i$ is also known as restricted basis entry rule. The above linear programming problem can be solved by simplex method taking care the restricted basis entry rule. This method is also known as WOLFES MODIFIED SIMPLEX METHOD. This modified simplex method is discussed in one example in the class. For details please see Quadratic Programming chapter of Engineering Optimization by S.S.Rao.

Exercise: Solve the following quadratic programming problems

1. $\min \ \ 2x_1^2 + 2x_2^2 + 2x_1x_2 - 4x_1 - 6x_2$ subject to $x_1 + 2x_2 \leq 2$, $x_1, x_2 \geq 0$

2. $\min \ \ x_1^2 - x_2 - 2x_1$ subject to $2x_1 + 3x_2 \leq 6$, $2x_1 + x_2 \leq 4$, $x_1, x_2 \geq 0$

3. Convert the quadratic programming problem $\min \ \ x_1^2 - x_2 - 2x_1$ subject to $2x_1 + 3x_2 \leq 6$, $2x_1 + x_2 = 4$, $x_1, x_2 \geq 0$ to a modified linear programming problem using KKT Optimality condition.

0.2 Quadratic Programming Application in Portfolio Analysis:Markowitz Mean Variance Model

- Suppose X amount has to be invested in the assets $A_1, A_2, ..., A_n$ at the time t_0.

- X_i is the amount of investment in A_i

- $P : (A_1, A_2, ..., A_n)$ is known as a portfolio.

- $\sum_{i=1}^{n} X_i = X$.
  X_i may be $+ve$ or $-ve$. If $X_i < 0$ then it is called short selling.

- $w_i = \frac{X_i}{X}$ = Proportion of total investment in i^{th} asset. $\sum_{i=1}^{n} w_i = 1$ or $e^T w = 1$ in vector form.

- r_i = Rate of return of the asset A_i, which is a random variable=$\frac{r_{i,j} - r_{i,j-1}}{r_{i,j-1}}$, r_{ij}=return of i^{th} asset calculated at T_j time $j = 1, 2, ..., n$).

- $\mu_i = E(r_i)$=Mean/Expected/Average return of i^{th} asset for time T=$\frac{1}{n-1} \sum_{j=1}^{n} \frac{r_{i,j} - r_{i,j-1}}{r_{i,j-1}}$
  (For large data set you may use time series to calculate this.)

- $R =$ Rate of return of the portfolio P, which is also a random variable$=\sum_{i=1}^{n} w_i r_i$.
- Expected return of the portfolio $P = \mu_P = E(\sum_{i=1}^{n} w_i r_i) = \sum_{i=1}^{n} w_i E(r_i) = \sum_{i=1}^{n} w_i \mu_i = w^T \mu$.

0.3 Risk of the portfolio

There are several types of risk functions used in portfolio analysis.

0.4 Variance risk

Variance of portfolio return is $\sigma^2 = E[(R - E(R))^2]$, which is known as variance risk.
Denote the variance of return of i^{th} asset $=\sigma_i^2 = E[(r_i - E(r_i))^2]$.
Covariance of the return of asset i and asset j $=\sigma_{ij} = E[(r_i - E(r_i))(r_j - E(r_j))]$.

$$\begin{aligned}
\sigma^2 =& E[(R - E(R))^2] \\
=& E\Big(\sum_{i=1}^{n} w_i r_i - \sum_{i=1}^{n} w_i \mu_i\Big)^2 \\
=& E\Big(\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j (r_i - \mu_i)(r_j - \mu_j)\Big) \\
=& \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j E[(r_i - \mu_i)(r_j - \mu_j)] \\
=& \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} = w^T \Omega w,
\end{aligned}$$

where $w = (w_1, w_2, ..., w_n)^T$, $\Omega = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & . & . & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & . & . & \sigma_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ \sigma_{n1} & \sigma_{n2} & . & . & \sigma_n^2 \end{pmatrix}$

Example-1 :

1. Q1(a)Construct Markowitz portfolio optimization models for three assets with following data which represents deviation form mean return. This should be understood in percentage.

	A	B	C
T_0	10	15	10
T_1	15	20	15
T_2	13	18	11
T_3	15	15	14

Ans: Denote the return table as

A=

10	15	10
15	20	15
13	18	11
15	15	14
$E(r_1) = 13.25$	$E(r_2) = 17$	$E(r_3) = 12.5$

Denote the deviation matrix as $D = (r_J - E(r_j))$.

$$D = \begin{bmatrix} -3.25 & -2 & -2.5 \\ 1.75 & 3 & 2.5 \\ -0.25 & 1 & -1.5 \\ 1.75 & -2 & 1.5 \end{bmatrix}$$

Then covariance matrix is $\Omega = 1/4D^T D$,

$$\Omega = \begin{bmatrix} 4.1875 & 2 & 3.875 \\ 2 & 4.5 & 2 \\ 3.875 & 2 & 4.25 \end{bmatrix}$$

$$QP1: \quad \min \quad w^T \Omega w \;\; s.to \;\; e^T w = 1,$$

which is

$$\min \;\; 4.1875w_1^2 + 4.5w_2^2 + 4.25w_3^2 + 4w_1w_2 + 7.75w_1w_3 + 4w_2w_3$$

subject to

$$w_1 + w_2 + w_3 = 1$$

0.4.1 Model I

$$QP1 \;\; \min w^T \Omega w \;\; subject \;\; to \;\; e^T w = 1$$

$QP1$ is a convex programming problem since the variance matrix $\Omega \succ 0$. Hence this can be solved using KKT optimality conditions. Consider the lagrange function with primal vector w and dual variable λ as

$$L(w, \lambda) = w^T \Omega w + \lambda(1 - e^T w)$$

KKT optimality conditions for the existence of solution are $\nabla_w L(w, \lambda) = 0, e^T w = 1$. Since $\Omega \succ 0$ so Ω^{-1} exists. Hence
$\nabla_w L(w, \lambda) = 0$
$\equiv 2\Omega w - \lambda e = 0$
$\equiv 2\Omega^{-1}\Omega w = \Omega^{-1}\lambda e$
$\equiv \;\; w = \frac{\lambda}{2}\Omega^{-1}e$
Next, $e^T w = 1 \equiv e^T(\frac{\lambda}{2}\Omega^{-1}e) = 1$. Hence $\lambda = \frac{2}{e^T\Omega^{-1}e}$, since $\Omega \succ 0$.
Substituting this value in w, we have the optimal portfolio as $w^* = \frac{\Omega^{-1}e}{e^T\Omega^{-1}e}$.

Consider Q1(a). Here optimal portfolio is $w^* = (w_1^*, w_2^*, w_3^*)^T$, where

$$w^* = \frac{\Omega^{-1}e}{e^T\Omega^{-1}e} = \frac{1}{0.32}\begin{vmatrix} 1.5488 & -0.0768 & -1.376 \\ -0.0768 & 0.2848 & -0.064 \\ -1.376 & -0.064 & 1.52 \end{vmatrix} . \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix} = \begin{vmatrix} 0.3 \\ 0.45 \\ 0.25 \end{vmatrix}$$

0.4.2 Model II

For given R,

$$(QP2) : \min w^T \Omega w \text{ subject to } \mu^T w = R, e^T w = 1$$

This is also a convex programming problem. Hence KKT conditions are both necessary and sufficient for the existence of solution. With dual variables λ_1 and λ_2, consider the lagrange function as $L(w, \lambda_1, \lambda_2) = w^T \Omega w + \lambda_1 (R - \mu^T w) + \lambda_2 (1 - e^T w)$. Then KKT conditions are $\nabla_w L(w, \lambda_1, \lambda_2) = 0, \mu^T w = R, e^T w = 1$. $\nabla_w L(w, \lambda_1, \lambda_2) = 0$

$\equiv 2\Omega w - \lambda_1 \mu - \lambda e = 0$

$\equiv w = \frac{1}{2}\Omega^{-1}(\lambda_1 \mu + \lambda_2 e)$

$\mu^T w = R \Rightarrow \lambda_1 \mu^T \Omega^{-1} \mu + \lambda_2 \mu^T \Omega^{-1} e = 2R$

$e^T w = 1 \Rightarrow \lambda_1 e^T \Omega^{-1} \mu + \lambda_2 e^T \Omega^{-1} e] = 2$

After solving above two equations,

$$\lambda_1 = \frac{det \begin{pmatrix} 2R & \mu^T\Omega^{-1}e \\ 2 & e^T\Omega^{-1}e \end{pmatrix}}{det \begin{pmatrix} \mu^T\Omega^{-1}\mu & \mu^T\Omega^{-1}e \\ e^T\Omega^{-1}\mu & e^T\Omega^{-1}e \end{pmatrix}} \quad \lambda_2 = \frac{det \begin{pmatrix} \mu^T\Omega^{-1}\mu & 2R \\ e^T\Omega^{-1}\mu & 2 \end{pmatrix}}{det \begin{pmatrix} \mu^T\Omega^{-1}\mu & \mu^T\Omega^{-1}e \\ e^T\Omega^{-1}\mu & e^T\Omega^{-1}e \end{pmatrix}}$$

Hence the optimal portfolio is,

$$w^* = \frac{det \begin{pmatrix} 2R & \mu^T\Omega^{-1}e \\ 2 & e^T\Omega^{-1}e \end{pmatrix}}{det \begin{pmatrix} \mu^T\Omega^{-1}\mu & \mu^T\Omega^{-1}e \\ e^T\Omega^{-1}\mu & e^T\Omega^{-1}e \end{pmatrix}} \Omega^{-1}\mu + \frac{det \begin{pmatrix} \mu^T\Omega^{-1}\mu & 2R \\ e^T\Omega^{-1}\mu & 2 \end{pmatrix}}{det \begin{pmatrix} \mu^T\Omega^{-1}\mu & \mu^T\Omega^{-1}e \\ e^T\Omega^{-1}\mu & e^T\Omega^{-1}e \end{pmatrix}} \Omega^{-1}e$$

0.4.3 Model III

$$(QP3) \quad \min w^T \Omega w \text{ subject to } \mu^T w \geq R, e^T w = 1$$

This problem is also a convex quadratic programming problem, which can be solved using KKT optimality conditions. Covert the problem to LPP and solve by Wolfe's method taught in pre-requisite classes.

$L(w, \lambda_1, \lambda_2) = w^T \Omega w + \lambda_1(\mu^T w - R) + \lambda_2(1 - e^T w)$

KKT Conditions are :

Normalized condition: $\nabla_w L(w, \lambda_1, \lambda_2) = 0 \equiv 2\Omega w + \lambda_1 \mu - \lambda_2 e = 0$

Feasibility conditions: $\mu^T w \geq R, e^T w = 1$

Complementary Slackness condition: $\lambda_1(\mu^T w - R) = 0$

Dual restrictions: $\lambda_1 \geq 0$

This can be converted to following LPP with restricted entry rule.

$\min$

$2\Omega w + \lambda_1 \mu - \lambda_2 e + z = 0$

$\mu^T w - s + a_1 = R$

$e^T w + a_2 = 1$

$\lambda_1, z_j, s, a_1, a_2 \geq 0$ with respect to the restricted basis entry rule $\lambda_1.s = 0$, $z = (z_1, z_2, ..., z_n)^T$, z_j, a_1, a_2 are artificial variables. s is surplus variable.

This problem can be solve by LPP technique keeping in mind the restricted basis entry rule.

0.5 Assignment

1. Derive the solution of $\min \quad 1/2.x^T Q x + c^T x \quad s.to \;\; Ax = b$ in closed form. Q is a positive definite matrix of order $n \times n$, A is a real matrix of order $m \times n$, $rank(A) = m, m \leq n, x, c \in R^n, b \in R^m$.(Discussed in class)

2. Convert the following quadratic programming problems to a modified linear programming problem, with restricted basis entry rule.(Discussed in class)

$$(a) \min \quad 1/2x^T Q x + c^T x \quad s.to \;\; Ax \leq b, x \geq 0$$

$$(b) \min \quad 1/2x^T Q x + c^T x \quad s.to \;\; Ax \leq b, Dx = p$$

3. solve the following problem using Wolfe's method

$$\min \quad 3x^2 + 4y^2 - 2x - 5y \;\; s.to \;\; 2x + 3y \leq 6, x + 4y = 5, x, y \geq 0$$

4. Consider following data of rate of return(r_j) in percentage of three assets A, B,C.

	A	B	C
T_0	10	15	10
T_1	15	20	15
T_2	13	18	11
T_3	15	15	14

   (a) With this data set find the optimal portfolio of Markowitz models (Theory discussed in class)
   I: $\min w^T \Omega w \ \ s.to \ \ e^T w = 1$
   II :$\min w^T \Omega w \ \ s.to \ \ e^T w = 1, \mu^T w = 0.3$
   III: $\min w^T \Omega w \ \ s.to \ \ e^T w = 1, \mu^T w \geq 0.3$

5. Collect 20 assets data(closing/opening/max/min, any one from BSE or NSE), either month wise or day wise, in excel, for some time period of your choice. Total time period should be large. Asset names should start with any one letter from first three letters of your name. Calculate return and variance risk. You may use time series or general mathematical calculations discussed in the class. Save this file as yourrollno. You will use this file for future assignments as well as term project. Throughout this course this data will be used.

   - Develop code in Python or R to solve Markowitz model I , find optimal portfolio.
   - Develop code in Python or R to solve Markowitz model II , find optimal portfolio.
   - Develop code in Python or R to solve Markowitz model III , find optimal portfolio.

   Call the saved data for Deviation matrix D–Find Ω matrix— use in-built codes for solving models I,II,III. Save your code as MK1ROLLNO, MK2ROLLNO,MK3ROLLNO. In a folder with name MKROLLNO, save your complete data set and python or R code. Send me for evaluation.