17
Mathematical Programming Applications in Financial Mathematics

17.1 Introduction

Modern finance is a flavor of recent times. With exceptional contributions coming in from economists Kiyoshi Itō in 40's and 50's, Harry M. Markowitz in 50's, Fischer Black, Myron Scholes and Robert C. Merton in late 60's and early 70's, and many more, the subject of financial mathematics has witnessed an explosive growth. Financial mathematics focusses on applying mathematical or numerical techniques on the problems arising in financial economics. The necessity to understand the theoretical and computational aspects of the subject for the success of any organization makes it more appealing. Now a days the subject commands high level of attention among the researchers and students across the globe.

The aim of the chapter is to take a closer look at a particular problem of finance called *portfolio selection* from optimization viewpoint. The theory of optimal selection of portfolio was developed by H. M. Markowitz, a US economist, in 50's. He shared the Nobel prize in Economics with M. M. Miller and W. F. Sharpe in 1990 for his pioneer work in portfolio theory. Here, we present a brief description of the models and relate them to quadratic programming problems (QPP) through mean-variance analysis.

17.2 Technical Terminologies

Let us first get familiar with some terminologies often used in the chapter.

Definition 17.2.1 (Asset). *An asset is a valuable economic entity from which the future economic benefits are expected to flow to the owner of the asset. It is controlled by the owner who legally acquired it as a result of past transactions or other events.*

Assets can be classified in many categories. Here we list only few of them. Physical or tangible assets are fixed assets like real estate, machinery, furniture etc. These assets are also known as capital assets in accounting. Intangible assets are defined as those non-monetary assets that cannot be physically measured like patents, goodwill, competitive knowledge etc. Liquid or financial assets include cash, bonds, shares, mutual funds, currency etc., gold and other precious metals are assets that are both tangible and liquid.

Throughout the chapter, an asset always means the financial asset only.

Definition 17.2.2 (Return). *The return on an asset is an indicator of gain/loss in the investment of an asset in the financial market. It is determined by the following formula*

$$\text{return} = \frac{\text{amount received} - \text{amount invested}}{\text{amount invested}}.$$

Suppose that the current price of an asset is $A(0)$ and after $\mathbf{T}$ time period the asset is sold off at an amount equals $A(\mathbf{T})$. Then the return on an asset for $\mathbf{T}$ time period is given by

$$r = \frac{A(\mathrm{T}) - A(0)}{A(0)}.$$

The positive value of return on an asset signifies gain while the negative return signifies loss, zero return means neither gain nor loss from the investment.

Remark 17.2.1 *It is important to mention here that definition 17.2.2 is given in percentage and hence return on an asset is actually to be understood as a rate of return on an asset. However, we shall continue to call it as return on an asset for consistency with the financial market glossary.*

Definition 17.2.3 (Risk). *The risk is often defined as the degree of uncertainty of return on an asset. It signifies the possibility of loss in the investment.*

The risk can either be zero, implying that the asset is risk-free, or positive, implying the asset is risky. If the asset is risk-free then the future value of the asset is known with certainty otherwise the future value of the risky asset is uncertain. The financial asset can thus be classified as risk-free asset like bond or fixed deposit and risky asset like share or mutual fund.

There are two kinds of risks associated with a risky asset viz., systematic risk and unsystematic risk.

(i) The systematic risk is common to all business. It is concerned with the institutional structure of the banking system, money and capital markets, credit and fiscal policies and economic policies which govern the market economy.

(ii) The unsystematic risk, also called the diversifiable risk or residual risk, is unique to a company such as, workers strike, the outcome of unfavorable litigation, sudden discovery of deficiencies in a product of a company, a natural catastrophe etc. Risks of this nature can be eliminated through diversification by investing the original amount of money in more than one asset.

Definition 17.2.4 (Short Selling). *It refers to a situation where an investor actually does not own an asset but he establishes a market position by selling an asset in anticipation that the price of that asset will fall. We say that the investor has taken a short position. Mathematically, this situation can be explained by taking the number of assets owned by the investor as negative.*

(i) A short position in risk-free asset simply means borrowing cash from the market at some interest rate or borrowing rate, repaying that loan along with the interest at later stage and close the short position.

(ii) In case of risky asset, a short position is realized by short selling. Here, an investor borrows an asset, sells it at some price, say S_0. The short position is closed when the investor buys back the asset at, say S_1, and returns it to the owner. The difference $S_0 - S_1$ is the gain/loss of the investor.

Remark 17.2.2 *Short selling is generally considered to be very risky. If the price S_1 of an asset in the short position of an investor shoots up than the investor stands to loose enormously. Many perceive short selling to be a cause of market crashes and hence short selling is prohibited in certain markets but of course it is not completely forbidden. However, the very notion that, a short seller can cause a permanent fall in the share prices, itself is debateable for any security which is short sold is to be bought back and hence there is no permanent supply of the shares in the market by the short sellers. Many economists strongly believes that banning short selling does more harm than good to the market as in that situation the market prices are controlled by the alleged manipulators and irrational investors. Till some time ago, in India, short-selling was only available to retail investors (an individual who purchases small amounts of securities for himself/herself, also called small investor) and not allowed to the institutional investors (entity with large amounts to invest, such as investment companies, mutual funds, brokerages, insurance companies, pension funds, investment banks and endowment funds). But very recently guidelines have been issued by securities and exchange board of India (SEBI) that enable institutional investors to sell stocks without owning them under certain rules.*

Remark 17.2.3 *If the number of assets of a particular kind owned by the investor is positive then the investor is said to have taken a long position.*

In order to build a mathematical model of the real world financial scenario we make certain assumptions.

1. The prices of all assets at any time are strictly positive.
2. The return r on an asset is a random variable taking finitely many values.

3. An investor can own a fraction of an asset. This assumption is known as *divisibility.*
4. An asset can be bought or sold on demand in any quantity at the market price. This assumption is known as *liquidity.*
5. There are no commissions/transaction costs.

We are now in a position to move towards our main aim, i.e. *portfolio optimization.* We first define a portfolio and then consider *two-asset* and *multi-asset* portfolio theories in the subsequent sections.

Definition 17.2.5 (Portfolio). *A portfolio is a collection of two or more assets, say, $a_1, \ldots, a_n$, represented by an ordered n-tuple $\Theta = (x_1, \ldots, x_n)$, where $x_i \in \mathbf{R}$ is the number of units of the asset a_i $(i = 1, \ldots, n)$ owned by the investor.*

We consider only a single period model, that is, in between the initial time taken as $t = 0$ and the final transaction time taken as $t = \mathrm{T}$, no transaction ever takes place.

Let $V_i(0)$ and $V_i(\mathrm{T})$ be the values of the i-th asset at $t = 0$ and $t = \mathrm{T}$, respectively. Let $V(0)$ and $V(\mathrm{T})$ denote the values of the portfolio $\Theta = (x_1, \ldots, x_n)$ at $t = 0$ and $t = \mathrm{T}$, respectively. Then, we have

$$V(0) = \sum_{i=1}^{n} x_i V_i(0),$$

$$V(\mathrm{T}) = \sum_{i=1}^{n} x_i V_i(\mathrm{T}).$$

Definition 17.2.6 (Asset Weights). *The weight w_i of the asset a_i is the percentage of the value of the asset in the portfolio at $t = 0$, i.e.*

$$w_i = \frac{x_i V_i(0)}{\sum_{i=1}^{n} x_i V_i(0)} \qquad (i = 1, \ldots, n).$$

It can be observed that $w_1 + \cdots + w_n = 1$.

Remark 17.2.4 *In a portfolio, if $w_i < 0$, for some i, it indicates that the investor has taken a short position on the i-th asset a_i.*

Let r_i be the return (definition 17.2.2) on the i-th asset. Then

$$r_i = \frac{V_i(\mathrm{T}) - V_i(0)}{V_i(0)} \qquad (i = 1, \ldots, n).$$

Suppose the portfolio comprises of two assets a_1 and a_2 with initial prices $V_1(0)$ =Rs 25 and $V_2(0)$ =Rs 50. We purchase 20($= x_1$) units of a_1 and 15($= x_2$) units of a_2. Then the initial worth of portfolio is $V(0) = x_1V_1(0) + x_2V_2(0) =$ Rs 1250. The proportion of allocation of Rs 1250 between two assets is $w_1 = \frac{(25)(20)}{1250} = 40\%$ and $w_2 = \frac{(50)(15)}{1250} = 60\%$. After one year, suppose the assets return values are $V_1(1) =$Rs 30 and $V_2(1) =$Rs 45. Then the portfolio worth would be $V(1) = (20)(30) + (15)(45) =$Rs 1275. The return on a_1 is $r_1 = \frac{30-25}{25} = 20\%$, and $r_2 = \frac{45-50}{50} = -25\%$.

Due to our assumption, the return value of any asset is a random variable and thus we can talk about the expected value μ_i of the return r_i, i.e.

$$\mu_i = E(r_i) \qquad (i = 1, \ldots, n).$$

The risk associated with the asset is given by the variance of the return, i.e.

$$\sigma_i^2 = var(r_i) \qquad (i = 1, \ldots, n).$$

Example 17.2.1 *Suppose there are two investment opportunities O_1 and O_2 giving the following returns in two different market scenarios*

scenario	*probability of scenario*	*return O_1*	*return O_2*
ω_1	0.25	8%	12%
ω_2	0.75	11%	9%

Which of the two is a more risky opportunity?

Solution Here

$$\mu_1 = E(O_1) \quad = (0.25)(0.08) + (0.75)(0.11) = 0.1025,$$
$$\mu_2 = E(O_2) \quad = (0.25)(0.12) + (0.75)(0.09) = 0.0975.$$

$$\sigma_1^2 = var(O_1) = (0.25)(0.08-0.1025)^2 + (0.75)(0.11-0.1025)^2 = 0.00016875,$$
$$\sigma_2^2 = var(O_2) = (0.25)(0.12-0.0975)^2 + (0.75)(0.09-0.0975)^2 = 0.00016875.$$

Thus both opportunities are equally risky although the expected return on the first opportunity (10.25%) is higher than that on the second opportunity (9.75%).

17.3 Two Asset Portfolio Optimization

Consider a portfolio with two assets, say, a_1, a_2 with weights w_1, w_2, returns r_1, r_2 and standard deviations σ_1, σ_2, respectively. Then the portfolio expected return μ and portfolio variance σ are respectively given by

$$\mu = E(w_1r_1 + w_2r_2) = w_1\mu_1 + w_2\mu_2, \tag{17.1}$$
$$\sigma^2 = var(w_1r_1 + w_2r_2) = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2\rho w_1w_2\sigma_1\sigma_2. \tag{17.2}$$

Here ρ is the coefficient of correlation between r_1 and r_2, and the value of ρ lies in $[-1, 1]$. We pause here to analyze the effect of ρ on the risk involved in a portfolio.

What we shall be observing is that the value of ρ provides a measure of the extent of diversification of portfolio possible to reduce risk. The more negative the value of ρ, the greater are the benefits of the portfolio diversification.

As w_1 and w_2 are weights representing the proportions of total investment in two assets a_1 and a_2, respectively, we have $w_1 + w_2 = 1$. Moreover, in case of short selling, the weights can be negative. Subsequently, we write $w_1 = 1 - s$, and so, $w_2 = s$, $s \in \mathbf{R}$. Now, it follows from relations (17.1) and (17.2) that

$$\mu = (1-s)\mu_1 + s\mu_2, \tag{17.3}$$
$$\sigma^2 = (1-s)^2\sigma_1^2 + s^2\sigma_2^2 + 2\rho(1-s)s\sigma_1\sigma_2. \tag{17.4}$$

Relation (17.4) can be simplified as

$$\sigma^2 = (\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)s^2 - 2\sigma_1(\sigma_1 - \rho\sigma_2)s + \sigma_1^2. \tag{17.5}$$

Without loss of generality we assume that $0 < \sigma_1 \leq \sigma_2$. We discuss the following two independent cases.

$$(i) \quad \rho = \pm 1, \qquad\qquad (ii) \quad -1 < \rho < 1.$$

Case (i). $\rho = \pm 1$. From relations (17.3) and (17.5), the portfolio expected return and variance are respectively given by

$$\mu = (1-s)\mu_1 + s\mu_2,$$
$$\sigma = |(1-s)\sigma_1 \pm s\sigma_2|.$$

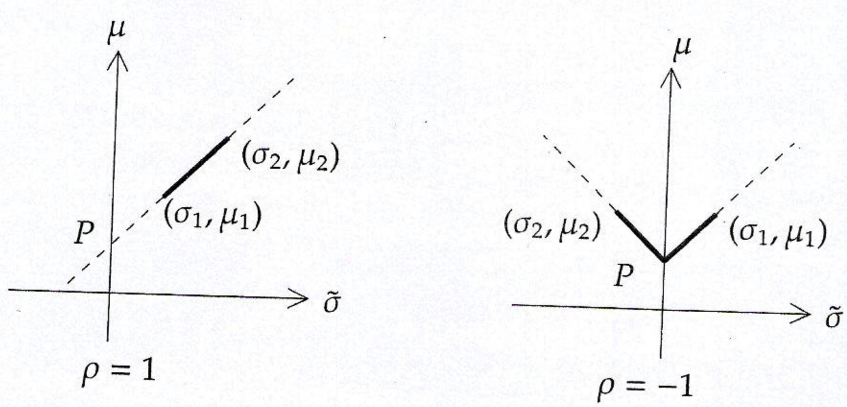

Fig. 17.1.

For $s \in [0,1]$ both weights are non-negative and thus the portfolio has no short positions. If $s > 1$ then $w_1 < 0$, it means that asset a_1 is held short, while if $s < 0$ then $w_2 < 0$ and it indicates that asset a_2 is held short. It may be noted that an investor can not take short position on both the assets simultaneously.

For $\mu = (1-s)\mu_1 + s\mu_2$, $\tilde{\sigma} = (1-s)\sigma_1 + s\sigma_2$, we plot the $(\tilde{\sigma}, \mu)$-graph, illustrated in the first graph of Fig 17.1. The second graph in Fig 17.1 corresponds to the case when $\mu = (1-s)\mu_1 + s\mu_2$, $\tilde{\sigma} = (1-s)\sigma_1 - s\sigma_2$. The bold parts in the graphs correspond to $s \in [0,1]$.

Subsequently, we plot the standard deviation-mean diagram of the portfolio, i.e. the (σ, μ)-graph, for $\rho = \pm 1$. Observe that $\sigma = |\tilde{\sigma}|$. The two graphs are depicted in Fig 17.2.

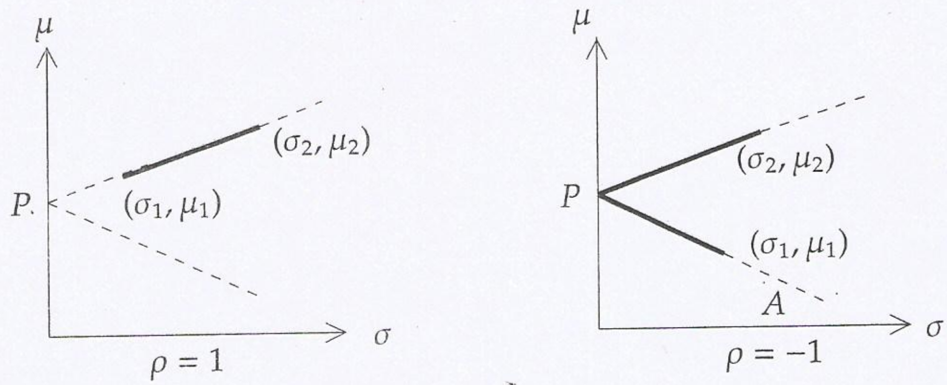

Fig. 17.2.

From Fig 17.2 we have the following observations to share.

Remark 17.3.1 *(i) When the returns of the two assets are perfectly positively correlated, i.e. $\rho = 1$, the higher expected return of the portfolio comes along with the higher risk. Furthermore, the risk of the portfolio can be completely eliminated by taking a short position on asset a_2 (point P in the first figure of Fig 17.2).*
(ii) When the returns of the two assets are perfectly negatively correlated, i.e. $\rho = -1$, the expected portfolio return increases with the gradual decrease in the overall risk of the portfolio (point A to point P in the second figure of Fig 17.2) till the risk is completely eliminated (point P). After that the higher return comes with higher risk as weight of riskier asset increases.

Below we provide mathematical justification of the above observations.
When $\rho = 1$ and $\sigma_1 = \sigma_2$, then $\sigma_{\min} = \sigma_1$.
When $\rho = 1$ and $\sigma_1 < \sigma_2$, then we have

$$\sigma^2 = (1-s)^2\sigma_1^2 + s^2\sigma_2^2 + 2(1-s)s\sigma_1\sigma_2.$$

Our aim is to minimize σ, or equivalently σ^2. Now

$$\begin{aligned} \frac{d\sigma^2}{ds} &= s(\sigma_1 - \sigma_2)^2 - \sigma_1(\sigma_1 - \sigma_2), \\ \frac{d^2\sigma^2}{ds^2} &= (\sigma_1 - \sigma_2)^2 > 0. \end{aligned} \tag{17.6}$$

Thus, to minimize the risk σ, we must choose the weight s such that $\dfrac{d\sigma^2}{ds} = 0$. Thereby (17.6) yield

$$
\begin{aligned}
s &= \frac{\sigma_1}{\sigma_1 - \sigma_2} < 0, \\
\mu_{\min} &= \frac{\sigma_1\mu_2 - \sigma_2\mu_1}{\sigma_1 - \sigma_2}.
\end{aligned}
$$

Since $s < 0$ (i.e. $w_2 < 0$), an investor can eliminate risk in the portfolio by taking a short position with asset a_2.

When $\rho = -1$ and $\sigma_1 < \sigma_2$, then we have

$$\sigma^2 = (1-s)^2\sigma_1^2 + s^2\sigma_2^2 - 2(1-s)s\sigma_1\sigma_2.$$

Thus, we have

$$
\begin{aligned}
s &= \frac{\sigma_1}{\sigma_1 + \sigma_2} > 0, \\
1 - s &= \frac{\sigma_2}{\sigma_1 + \sigma_2} > 0, \\
\mu_{\min} &= \frac{\sigma_1\mu_2 + \sigma_2\mu_1}{\sigma_1 + \sigma_2}.
\end{aligned}
$$

Since $s > 0$ and $1 - s > 0$ hence the investor can eliminate the risk in the portfolio without restoring to short selling.

Case (ii). We now consider the second case when $-1 < \rho < 1$.

Recall relations (17.3) and (17.5), we have

$$
\begin{aligned}
&\mu = (1-s)\mu_1 + s\mu_2, \\
&\sigma^2 = (\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)s^2 - 2\sigma_1(\sigma_1 - \rho\sigma_2)s + \sigma_1^2.
\end{aligned}
\tag{17.7}
$$

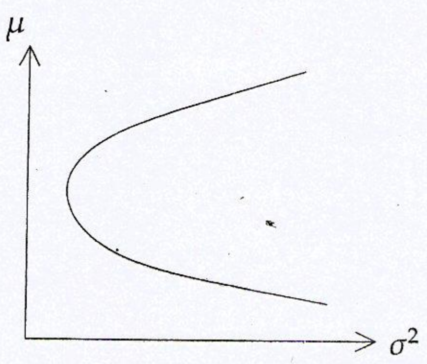

Fig. 17.3.

(17.7) is a quadratic equation in s representing a parabola depicted in Fig 17.3. We wish to minimize σ^2. Now,

$$\frac{d\sigma^2}{ds} = 0 \Rightarrow s = \frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}.$$

And

$$\frac{d^2\sigma^2}{ds^2} = 2((\sigma_1 - \rho\sigma_2)^2 + \sigma_2^2(1-\rho^2)) > 0.$$

Consequently, the minimum value of σ^2 is given by

$$\sigma^2_{\min} = \frac{\sigma_1^2\sigma_2^2(1-\rho^2)}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2},$$

Moreover, the minimum expected portfolio return equals

$$\mu_{\min} = (\mu_2 - \mu_1)s_{\min} + \mu_1.$$

Remark 17.3.2 *It is important to take note of the following points.*

(i) The condition $-1 \le \rho \le \dfrac{\sigma_1}{\sigma_2}$ is equivalent to $0 < s_{\min} < 1$. Thus the minimum risk can be achieved without short selling. Also,

$$\sigma^2_{\min} = 0 \Leftrightarrow \rho = -1.$$

(ii) $\rho = \dfrac{\sigma_1}{\sigma_2} \Leftrightarrow s_{\min} = 0 \Leftrightarrow \sigma^2_{\min} = \sigma_1^2.$

(iii) The condition $\dfrac{\sigma_1}{\sigma_2} < \rho \le 1$ is equivalent to $s_{\min} < 0$. In this case the investor has taken a short position on asset a_2 in order to minimize the portfolio risk. Further,

$$\sigma^2_{\min} = 0 \Leftrightarrow \rho = 1.$$

The entire theory of this section is summarized in Fig 17.4.

The risk-return relation of two assets for various values of ρ provides us with a triangle ΔAPB. The points A and B signify undiversified portfolios. Since $-1 \le \rho \le 1$, ΔAPB specifies the limit of diversification. The risk-return relation for all values of ρ except ± 1 lie within this triangle.

Let us verify the two-assets portfolio theory by considering the following example and plotting the corresponding (σ, μ)-graph.

Let A and B be two assets with expected returns 12% and 16% and standard deviation 16% and 20%, respectively. Let x_A and x_B denote the number of units of asset A and asset B, respectively, in a portfolio.

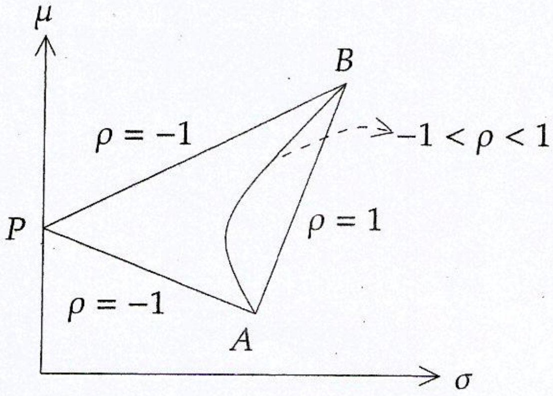

Fig. 17.4.

x_A	x_B	μ	σ				
			$\rho = 1$	$\rho = 0.5$	$\rho = 0$	$\rho = -0.5$	$\rho = -1$
100	0	12.00	16.00	16.00	16.00	16.00	16.00
90	10	12.40	16.40	16.50	14.54	13.51	12.40
80	20	12.80	16.80	15.20	13.41	11.34	8.80
70	30	13.20	17.20	15.12	12.71	9.71	5.20
60	40	13.60	17.60	15.26	12.50	8.91	1.60
50	50	14.00	18.00	15.62	12.81	9.17	2.00
40	60	14.40	18.40	16.18	13.60	10.40	5.60
30	70	14.80	18.80	16.92	14.80	12.32	9.20
20	80	15.20	19.20	17.82	16.32	14.66	12.80
10	90	15.60	19.60	18.85	18.07	17.26	16.40
0	100	16.00	20.00	20.00	20.00	20.00	20.00

Remark 17.3.3 *From the above table we can easily observe the following.*

(i) The two assets can be combined in such a way that the portfolio risk is less than the individual risks. For instance, when the assets are taken in the ratio 80 : 20 *and* $\rho = 0.5$ *then the risk in the portfolio is* 15.20% *whereas if the entire investment is put in asset A only or in asset B only then the risks involved are* 16% *and* 20%*, respectively. While if* $\rho = -0.5$ *then* 60 : 40 *ratio can bring down the risk to* 8.91%. *It signifies that in many circumstances diversification of the portfolio is advisable for reducing the risk. The underline principle is thus that 'do not put all the eggs in one basket'.*

(ii) For a given weight combination the risk reduces as ρ *moves from 1 to -1, i.e. the more uncorrelated the assets are the better is the risk-return relation.*

(iii) When $\rho < 1$ *then certain combinations of the assets are better than the others. For*

example, for $\rho = 0$ *the 60:40 combination with* $\mu = 13.60\%$ *and* $\sigma = 12.50\%$ *is better than the 70:30 combination with* $\mu = 13.20\%$ *and* $\sigma = 12.71\%$*. For* $\rho = 0.5$, $70 : 30$ *combination yields return* 13.20% *and risk* 15.12% *which is better than the* $80 : 20$ *ratio that gives lower return* 12.80% *with higher risk* 15.20%.

From the above discussion we can thus conclude that by choosing appropriate ratio of investment between two assets, the risk can be reduced considerably.

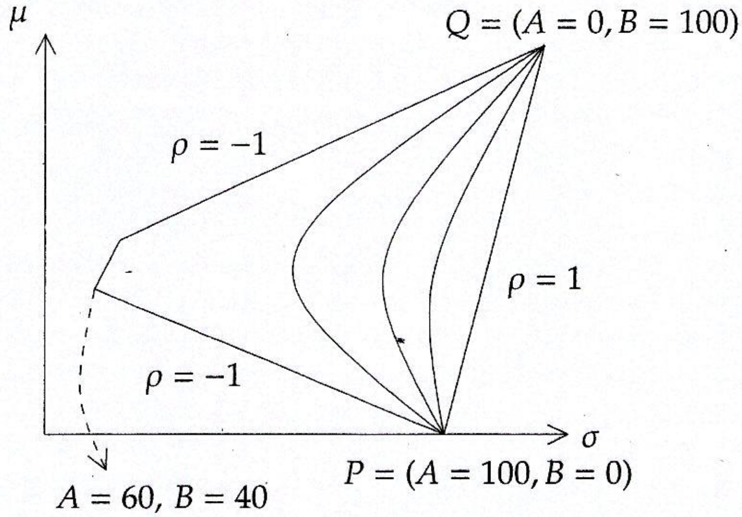

Fig. 17.5.

In Fig 17.5, as the weights of the assets A and B vary from 100 to 0 and 0 to 100, respectively, the curve between risk and return moves from extreme right (south-east point P) to extreme left and turns to move from extreme left to extreme right (north east point Q) corresponding to the various values of ρ between -1 and 1.

17.4 Multi Asset Portfolio Optimization

In this section we extend the two-asset portfolio theory to n - asset portfolio theory.

The weights of the various assets $a_1, \ldots, a_n$ in the portfolio are written in the vector form $w = col(w_1, \ldots, w_n)$. Let $e^T = (1, \ldots, 1) \in \mathbf{R}^n$. Then $w_1 + \cdots + w_n = 1$ can be expressed as $e^T w = 1$. Let $m^T = (\mu_1, \ldots, \mu_n)$ be the expected return vector of the portfolio, where, $\mu_i = E(r_i)$ $(i = 1, \ldots, n)$, and $C = [c_{ij}]$ denotes the $n \times n$ variance-covariance matrix with entries $c_{ij} = \mathrm{cov}(r_i, r_j)$ $(i, j = 1, \ldots, n)$. Note that $c_{ii} = \sigma_i^2$ $(i = 1, \ldots, n)$. Obviously C is a symmetric matrix. Also, C is a positive definite matrix and thus it is invertible.

Now the expected return μ of the portfolio is given by

$$\mu = E\left(\sum_{i=1}^{n} w_i r_i\right) = \sum_{i=1}^{n} w_i \mu_i = m^T w,$$

and the risk σ^2 of the portfolio is

$$\sigma^2 = var\left(\sum_{i=1}^{n} w_i r_i\right) = \sum_{i,j=1}^{n} c_{ij} w_i w_j = w^T C w. \qquad (17.8)$$

During a portfolio selection, every investor is faced with a choice of either minimizing a risk with respect to certain value of return or maximizing a return with respect to certain value of risk.

Now, from (17.8), we observe that the portfolio risk σ depends on three factors, viz.,

(i) risk of each individual asset;

(ii) coefficient of correlation between assets returns;

(iii) weights of the assets.

Out of these contributing factors, the only factor that an investor can control is the weights of the assets. Our main aim is to examine the optimal choice of these weights.

Consider the n-dimensional hyperplane $e^T w = 1$ in which the weight vector w resides. Let f be the mapping that takes each weight vector in the weight hyperplane to the corresponding portfolio point in the (σ, μ)-graph. We try and find the image of any straight line in the weight hyperplane $e^T w = 1$ under the mapping f.

The parametric equation of any line in the weight hyperplane is of the form

$$\begin{aligned} l(\xi) &= (s_1\xi + b_1, \ldots, s_n\xi + b_n) \\ &= \xi s + b, \qquad -\infty < \xi < \infty \end{aligned}$$

where $s = (s_1, \ldots, s_n)$ and $b = (b_1, \ldots, b_n)$. Let w be any point on this line. Then

$$\begin{aligned} \mu &= m^T w \\ &= m^T(\xi s + b) \\ &= \xi(m^T s) + (m^T b). \end{aligned}$$

Let $\alpha = (m^T s)^{-1}$, $\beta = -(m^T b)(m^T s)^{-1}$. Then, $\xi = \mu\alpha + \beta$. Moreover,

$$\begin{aligned} \sigma^2 &= w^T C w \\ &= (\xi s + b)^T C(\xi s + b) \\ &= (s^T C s)\xi^2 + (s^T C b + b^T C s)\xi + b^T C b \\ &\equiv \gamma\xi^2 + \delta\xi + \eta. \end{aligned}$$

Substituting the value of ξ we get

$$\sigma^2 = \gamma(\alpha\mu + \beta)^2 + \delta(\alpha\mu + \beta) + \eta. \qquad (17.9)$$

As ξ varies from $-\infty$ to ∞, the ordered pair (σ^2, μ) traces out a parabola given by (17.9) which lies in (σ, μ)- graph with axis parallel to σ-axis and sides open on the right.

We are actually interested in (σ, μ)-graph. Taking the square root of σ^2, the resulting curve is

$$\sigma = \sqrt{\gamma(\alpha\mu + \beta)^2 + \delta(\alpha\mu + \beta) + \eta}. \tag{17.10}$$

This curve is called a *Markowitz curve.* Thus, each line in the weight hyperplane is mapped onto a Markowitz curve. This phenomena is depicted in Fig 17.6.

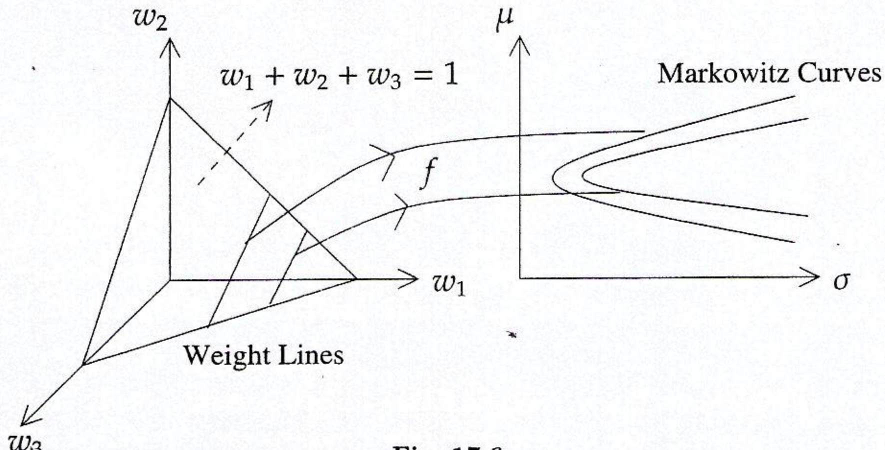

Fig. 17.6.

Remark 17.4.1 *Here it is important to note that the Markowitz curve (17.10) is not a parabola. In fact the main difference between the parabola (17.9) and the Markowitz curve (17.10) in (σ, μ)-graph is that a tangent can be drawn to the parabola (17.9) from any point on the μ-axis, whereas the Markowitz curve behaves almost as a straight line as $\mu \to \infty$, thereby, it is not possible to draw a tangent to the Markowitz curve as $\mu \to \infty$. This difference may not sound significant right now but it plays a vital role when the portfolio consists of one risk-free asset. We shall be addressing to this type of portfolio in the next section. For the current discussion we have assumed that all the assets in the portfolio are risky.*

As already observed that among all the factors affecting the risk σ^2 of the portfolio, the only factor that can be controlled by an investor is the weight vector w. In other words it is same as saying that an investor can decides upon appropriate weight vector to minimize the overall risk in the investment.

Theorem 17.4.1 *A portfolio with minimum risk has weights given by*

$$w = \frac{C^{-1}e}{e^T C^{-1} e}.$$

Proof. The problem is to

$$\text{Min} \qquad \sigma^2 = w^T C w$$
$$\text{subject to}$$
$$e^T w = 1. \tag{17.11}$$

Using the Lagrange multiplier $\lambda \in \mathbf{R}$, we minimize the Lagrangian

$$L(w, \lambda) = w^T C w + \lambda(1 - e^T w). \tag{17.12}$$

Note that λ is unrestricted in sign because the constraint in the risk minimization problem is an equation $e^T w = 1$. Now, differentiating (17.12) with respect to w, we obtain

$$2w^T C - \lambda e^T = 0 \implies w = \frac{\lambda}{2} C^{-1} e.$$

Using (17.11), we get

$$e^T \left(\frac{\lambda}{2} C^{-1} e \right) = 1 \implies \frac{\lambda}{2} = \frac{1}{e^T C^{-1} e}.$$

Thus the requisite result follows. □

Definition 17.4.1 (Markowitz Efficient Frontier). *The set of points which provides minimum risk for each expected return value μ is a Markowitz curve called the Markowitz efficient frontier.*

Definition 17.4.2 (Minimum Risk Weight Line). *Corresponding to the Markowitz efficient frontier in (σ, μ)- graph, the line in the weight hyperplane $e^T w = 1$ is called the minimum risk weight line.*

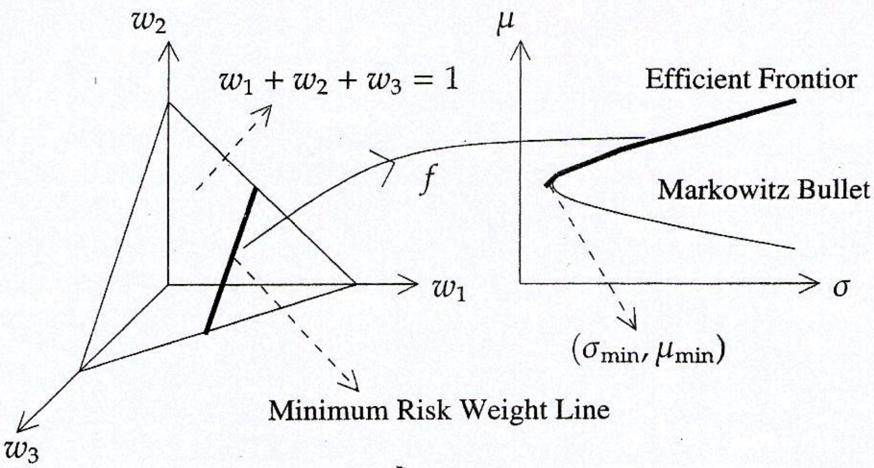

Fig. 17.7.

The Markowitz efficient frontier and the minimum risk weight line are depicted in Fig 17.7.

Example 17.4.1 *Suppose there are two assets a_1 and a_2 with $\mu_1 = 0.4$, $\mu_2 = 0.8$, $\sigma_1^2 = 2 = \sigma_2^2$, $\sigma_{12} = 1$. Obtain the minimum variance point and sketch the entire efficient frontier.*

Solution Here, we can note that

$$\rho = \frac{\sigma_{12}}{\sigma_1\sigma_2} = \frac{1}{2}; \qquad \sigma^2_{\text{min}} = \frac{\sigma_1^2\sigma_2^2(1-\rho^2)}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2} = \frac{3}{2};$$

$$s_{\text{min}} = \frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2} = \frac{1}{2}; \qquad \mu_{\text{min}} = (\mu_2-\mu_1)s_{\text{min}} + \mu_1 = \frac{3}{5}.$$

Thus $(\sigma_{\text{min}}, \mu_{min}) = (1.2247, 0.6)$. The corresponding Markowitz curve and efficient frontier are shown in bold in Fig 17.8.

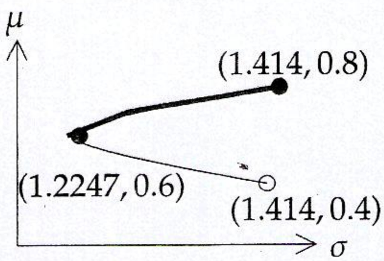

Fig. 17.8.

In many cases, it is more likely that an investor provides a fixed value of the expected return, say μ, that he desired to achieve. He has to decide the right investment strategy to obtain the return μ with the minimum risk. We look at this scenario in the result to follow.

Theorem 17.4.2 *For a given expected return μ, the portfolio with minimum risk has weights given by*

$$w = \frac{det\begin{pmatrix} \mu & m^TC^{-1}e \\ 1 & e^TC^{-1}e \end{pmatrix} C^{-1}m + det\begin{pmatrix} m^TC^{-1}m & \mu \\ e^TC^{-1}m & 1 \end{pmatrix} C^{-1}e}{det\begin{pmatrix} m^TC^{-1}m & m^TC^{-1}e \\ e^TC^{-1}m & e^TC^{-1}e \end{pmatrix}}. \tag{17.13}$$

Proof. We wish to solve the following quadratic programming problem

$$\text{Min} \quad \sigma^2 = w^TCw$$

subject to

$$\begin{aligned} m^Tw &= \mu \\ e^Tw &= 1. \end{aligned} \tag{17.14}$$

This is a convex quadratic programming problem with unrestricted variable vector w. Using the Lagrange multipliers $\alpha, \beta \in \mathbf{R}$, we minimize the Lagrangian

$$L(w, \alpha, \beta) = w^T C w + \alpha(\mu - m^T w) + \beta(1 - e^T w)$$

to obtain

$$2w^T C - \alpha m^T - \beta e^T = 0 \implies w = \frac{1}{2} C^{-1}(\alpha m + \beta e). \tag{17.15}$$

Substituting the value of w in (17.14), we get

$$\begin{aligned} (m^T C^{-1} m)\alpha + (m^T C^{-1} e)\beta &= 2\mu \\ (e^T C^{-1} m)\alpha + (e^T C^{-1} e)\beta &= 2. \end{aligned}$$

Solving the above two equations for α and β, we obtain

$$\alpha = \frac{det\begin{pmatrix} \mu & m^T C^{-1} e \\ 1 & e^T C^{-1} m \end{pmatrix}}{det\begin{pmatrix} m^T C^{-1} m & m^T C^{-1} e \\ e^T C^{-1} m & e^T C^{-1} e \end{pmatrix}}, \quad \beta = \frac{det\begin{pmatrix} m^T C^{-1} m & \mu \\ e^T C^{-1} m & 1 \end{pmatrix}}{det\begin{pmatrix} m^T C^{-1} m & m^T C^{-1} e \\ e^T C^{-1} m & e^T C^{-1} e \end{pmatrix}}$$

Substituting these values in the expression (17.15) for w , we get the required expression (17.13).

□

Definition 17.4.3 (Markowitz Bullet). *Since the efficient frontier contains all the points of minimum risk for the preassigned value of return, therefore all the feasible points of problem (17.14) lie on or to the right of this frontier. Due to its shape, this region is called the Markowitz Bullet.*

It is worthwhile to pause here to make an important observation about the minimum-variance set for a fixed return. Recall from (17.14) and (17.15) that, for a given value of return μ, the points of minimum variance must satisfy the following system of $(n+2)$ linear equations in $(n+2)$ unknowns $w \in \mathbf{R}^n$, $\alpha \in \mathbf{R}$, $\beta \in \mathbf{R}$.

$$\begin{aligned} 2w^T C - \alpha m^T - \beta e^T &= 0 \\ m^T w &= \mu \\ e^T w &= 1. \end{aligned} \tag{17.16}$$

Suppose we solve the system (17.16) for two distinct values of expected return μ, say $\bar{\mu}^1$ and $\bar{\mu}^2$. Let the two solutions be $(w^1 = (w_1^1, \ldots, w_n^1)^T, \alpha^1, \beta^1)^T$ and $(w^2 = (w_1^2, \ldots, w_n^2)^T, \alpha^2, \beta^2)^T$, respectively. Then it is simple to verify that the combination portfolio, $\lambda(w^1, \alpha^1, \beta^1)^T + (1-\lambda)(w^2, \alpha^2, \beta^2)^T$, $\lambda \in \mathbf{R}$, is also a solution of the system (17.16) corresponding to the return $\lambda\bar{\mu}^1 + (1-\lambda)\bar{\mu}^2$. Therefore, in order to solve (17.16) for every value of μ, one is only required to solve it for two distinct values of μ and then form the combination of the two solutions. Thus, the knowledge of two distinct portfolios yielding the minimum variances is sufficient to generate the entire minimum variance set. This result is significant from investor point of view. Also, it demonstrates a good application of KKT optimality conditions. The result is known as the *two fund theorem.*

Theorem 17.4.3 (Two Fund Theorem). *Two efficient portfolios can be established so that any other efficient portfolio can be duplicated, in terms of mean and variance, as a linear combination of these two assets. In other words, it says that, an investor seeking an efficient portfolio need to invest only in the combination of these two assets.*

The most convenient way to get two solutions of (17.16) is to assign two distinct values to α and β, and then work out the solutions. The most convenient choices are $\alpha = 1,\ \beta = 0$ and $\alpha = 0,\ \beta = 1$. The above discussion is illustrated through the following example.

Example 17.4.2 *Consider three risky assets with the covariance matrix and expected returns as follows.*

variance - covariance matrix(C)			return(M)
2	*1*	*0*	*0.4*
1	*2*	*1*	*0.8*
0	*1*	*2*	*0.8*

Find two portfolios yielding the minimum variance. Also, determine the expected returns from these two portfolios. Using the two fund theorem, construct the portfolio giving the return of 33.4% *with minimum risk.*

Solution Taking $\alpha = 0,\ \beta = 1$ in (17.16), we need to solve: $\sum_{j=1}^{3} \sigma_{ij} v_j^1 = 1$, $(i = 1, 2, 3)$, resulting in the following system of linear equations

$$\begin{aligned} 2v_1^1 + v_2^1 \quad &= 1 \\ v_1^1 + 2v_2^1 + v_3^1 \quad &= 1 \\ v_2^1 + 2v_3^1 \quad &= 1. \end{aligned}$$

The solution is $V^1 = (0.5,\ 0,\ 0.5)^T$. We next take $\alpha = 1,\ \beta = 0$ in (17.16), to solve $\sum_{j=1}^{3} \sigma_{ij} v_j^2 = \mu_i$, $(i = 1, 2, 3)$, i.e.

$$\begin{aligned} 2v_1^2 + v_2^2 \quad &= 0.4 \\ v_1^2 + 2v_2^2 + v_3^2 \quad &= 0.8 \\ v_2^2 + 2v_3^2 \quad &= 0.4. \end{aligned}$$

The solution of the above system is $V^2 = (0.1,\ 0.2,\ 0.3)^T$.

Note that $\sum_{j=1}^{3} v_j^1 = 1$, thus we take $w^1 = V^1 = (1/2, 0, 1/2)$. Normalizing V^2, we get, $w^2 = (1/6, 1/3, 1/2)^T$ (so that, $\sum_{j=1}^{3} w_j^2 = 1$). The corresponding returns from the two portfolios with weights w^1 and w^2 are $\bar{\mu}^1 = m^T w^1 = 0.6$ and $\bar{\mu}^2 = m^T w^2 = 0.733$, respectively.

Next, an investor desired a return of $\mu = 0.334$ at minimum risk. It is simple to check that for $\lambda = 3$, $\lambda\bar{\mu}^1 + (1-\lambda)\bar{\mu}^2 = 0.334$. Thus, by the two fund theorem the requisite portfolio is given by $w = \lambda w^1 + (1-\lambda)w^2 = (7/6, -2/3, 1/2)$. Observe that the second asset has a short position in this portfolio. The variance corresponding to this portfolio is

$$w^T C w = \begin{pmatrix} 7/6 & -2/3 & 1/2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 7/6 \\ -2/3 \\ 1/2 \end{pmatrix} = 2/9.$$

17.5 Capital Asset Pricing Model (CAPM)

So far we have assumed that all assets in the portfolio are risky assets. So it is natural to query as to what would be the scenario if one risk-free asset is included in the portfolio? In this section we make an attempt to study this aspect of portfolio selection.

Consider a portfolio with n risky assets, $a_1, \ldots, a_n$ with weights $w_1, \ldots, w_n$ and one risk-free asset a_{rf} with weight w_{rf}. Then

$$w_{\text{risky}} + w_{\text{rf}} = \sum_{i=1}^{n} w_i + w_{\text{rf}} = 1 \tag{17.17}$$

$$\Rightarrow \quad w_{\text{risky}} = \sum_{i=1}^{n} w_i \leq 1.$$

Also, the expected return and the variance associated with this portfolio are respectively given by

$$\mu = \sum_{i=1}^{n} w_i\mu_i + w_{\text{rf}}\mu_{\text{rf}} = \mu_{\text{risky}} + w_{\text{rf}}\mu_{\text{rf}},$$

$$\sigma^2 = var\left(\sum_{i=1}^{n} w_i\mu_i + w_{\text{rf}}\mu_{\text{rf}}\right) = var\left(\sum_{i=1}^{n} w_i\mu_i\right) = \sigma^2_{\text{risky}}.$$

If we remove the risk-free asset from the portfolio and readjust the weights of the risky assets so that their sum remain 1, the resultant portfolio so obtained is referred to as the *derived risky portfolio*. We use μ_{der} and σ^2_{der} tc denote the derived risky portfolio expected return and risk, respectively. Then,

$$
\begin{aligned}
\mu &= \sum_{i=1}^{n} w_i\mu_i + w_{\text{rf}}\mu_{\text{rf}} \\
&= w_{\text{risky}}\left(\sum_{i=1}^{n} \frac{w_i}{w_{\text{risky}}}\mu_i\right) + w_{\text{rf}}\mu_{\text{rf}} \\
&= w_{\text{risky}}\mu_{\text{der}} + w_{\text{rf}}\mu_{\text{rf}}.
\end{aligned}
\tag{17.18}
$$

$$
\begin{aligned}
\sigma^2 &= var\left(\sum_{i=1}^{n} w_i\mu_i\right) \\
&= w_{\text{risky}}^2 var\left(\sum_{i=1}^{n} \frac{w_i}{w_{\text{risky}}}\mu_i\right) \\
&= w_{\text{risky}}^2\ \sigma_{\text{der}}^2.
\end{aligned}
\tag{17.19}
$$

Using (17.16) and (17.19) in (17.18), we obtain

$$
\mu = \mu_{\text{risky}} + \frac{\mu_{\text{der}} - \mu_{\text{risky}}}{\sigma_{der}}\sigma\,. \tag{17.20}
$$

(17.20) is an equation of a line joining $(0, \mu_{\text{risky}})$ and $(\sigma_{\text{der}}, \mu_{\text{der}})$ in the (σ, μ)-graph.

Now, for a given risk σ, suppose we choose various weight combinations of risk-free asset and risky assets satisfying (17.17), we generate different lines represented by (17.20) in (σ, μ)-graph. Obviously, among all such lines, the line that produces the points with highest expected return for a given risk is tangent to the upper portion of the Markowitz bullet. This is illustrated in Fig 17.9.

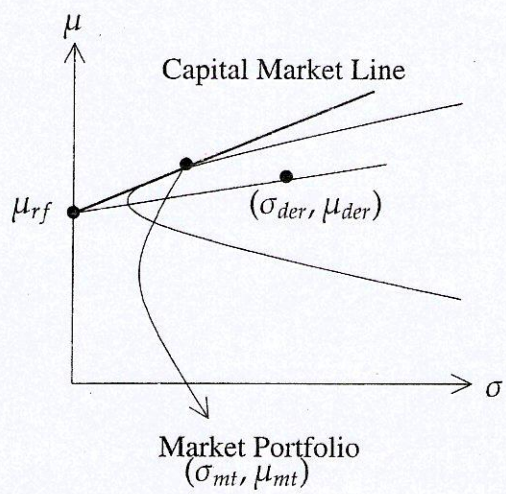

Fig. 17.9.

Definition 17.5.1 (Capital Market Line). *Among all the lines (17.20) for various weight combinations of risk-free asset and risky assets, the line giving the highest return for a given risk is called the capital market line.*

The basic idea of the capital asset pricing model (CAPM) is that an investor can improve the risk-expected return balance by investing partially in a portfolio of risky assets and partially in a risk-free asset. All investors will end up with portfolios along the capital market line as all *efficient portfolios* lie along this line while any other combination of risk-free asset and risky assets, except those which are efficient, lies below the capital market line. It is thus important to observe that all investors will hold combinations of only two assets, viz. a market portfolio and a risk-free asset. This fund scenario is summarized in the following theorem.

Theorem 17.5.1 (One Fund Theorem). *There exists a single portfolio, M, of risky assets such that any efficient portfolio can be constructed as a linear combination of the tangent portfolio and the risk-free asset.*

Unlike with the two-fund theorem where any two efficient portfolios are sufficient, in this case, the tangent portfolio is a specific portfolio.

Definition 17.5.2 (Market Potfolio). *The point on the Markowitz bullet where the capital market line is tangential is said to represent the market portfolio.*

Importance of Market Portfolio

(i) The market portfolio must contains all risky assets, for if some asset is not in it then it will wither and die.

(ii) Since the market portfolio contains all risky assets, it is a completely diversified portfolio with no unsystematic risk.

Theorem 17.5.2 *For any expected risk-free return μ_{rf}, the weights of the capital market portfolio is given by*

$$w = \frac{C^{-1}(m - \mu_{\text{rf}}\, e)}{e^T C^{-1}(m - \mu_{\text{rf}}\, e)}.$$

Proof. From Fig 17.9, we observe that for any point (σ, μ) in the Markowitz bullet, the slope of the line joining $(0, \mu_{\text{rf}})$ to (σ, μ) is

$$s = \frac{\mu - \mu_{\text{rf}}}{\sigma} = \frac{\sum_{i=1}^{n} \mu_i w_i - \mu_{\text{rf}}}{\sum_{i,j=1}^{n} c_{ij} w_i w_j}.$$

For the line joining $(0, \mu_{\text{rf}})$ to (σ, μ) to be a tangent line to the Markowitz bullet, we need to solve the following programming problem

$$\begin{aligned} \text{Max} \quad & \frac{m^T w - \mu_{\mathrm{rf}}}{(w^T C w)^{1/2}} \\ \text{subject to} \quad & \\ & e^T w = 1. \end{aligned} \tag{17.21}$$

The Lagrange function $L : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ is described as

$$L(w, \lambda) = \frac{m^T w - \mu_{\mathrm{rf}}}{(w^T C w)^{1/2}} + \lambda(1 - e^T w).$$

Now, solving (17.21) is same as minimizing $L(w, \lambda)$. So, $\nabla_w L(w, \lambda) = 0$, giving,

$$\frac{1}{w^T C w}\Big((w^T C w)^{1/2} m - (m^T w - \mu_{\mathrm{rf}})(w^T C w)^{-1/2} C w\Big) = \lambda e.$$

The above expression can be rewritten as

$$\sigma m - (\mu - \mu_{\mathrm{rf}})\frac{Cw}{\sigma} = \lambda \sigma^2 e.$$

Multiplying by σ, we obtain

$$\sigma^2 m - (\mu - \mu_{\mathrm{rf}}) C w = \lambda \sigma^3 e, \tag{17.22}$$

which in turn yields

$$\sigma^2 w^T m - (\mu - \mu_{\mathrm{rf}}) w^T C w = \lambda \sigma^3 w^T e,$$

Since $e^T w = 1$, $\mu = w^T m$, and $\sigma = w^T C w$, we get

$$\lambda = \frac{\mu_{\mathrm{rf}}}{\sigma}. \tag{17.23}$$

The requisite value of weight vector w now follows from (17.22) and (17.23). □

Example 17.5.1 *Suppose a portfolio comprises of one risk-free asset with return* 0.5%, *and three mutually independent risky assets with expected returns* 1%, 2%, 3% *and variances* 1%, 1%, 1%, *respectively. Determine the equation of the capital market line.*

Solution The given information gives, $m^T = (\mu_1, \mu_2, \mu_3) = (1, 2, 3)$, $\mu_{\mathrm{rf}} = 0.5$, $C = [\sigma_{ij}] = I_{3\times 3}$, $e^T = (1, 1, 1)$. Therefore, the weight vector of the market portfolio is given by

$$w_{\mathrm{mt}} = \frac{C^{-1}(m - \mu_{\mathrm{rf}} e)}{e^T C^{-1}(m - \mu_{\mathrm{rf}} e)} = \begin{pmatrix} 1/9 \\ 1/3 \\ 5/9 \end{pmatrix}.$$

Consequently, the expected return and variance of the market portfolio are

$$\mu_{mt} = m^T w_{mt} = \frac{22}{9}\%, \quad \sigma_{mt} = ((w_{mt})^T C w_{mt})^{1/2} = \frac{\sqrt{35}}{9}\%.$$

Thus, the equation of the capital market line is

$$\begin{aligned} \mu &= \mu_{rf} + \frac{\mu_{mt} - \mu_{rf}}{\sigma_{mt}}\sigma \\ &= \frac{1}{2} + \frac{\sqrt{35}}{2}\sigma. \end{aligned}$$

Remark 17.5.1 *Suppose the market portfolio (σ_{mt}, μ_{mt}) is known. Then, from (17.20), the equation of the capital market line is given by*

$$\mu = \mu_{rf} + \frac{\mu_{mt} - \mu_{rf}}{\sigma_{mt}}\sigma.$$

If the investor is willing to take a positive risk σ, he can earn an additional return $\left(\frac{\mu_{mt} - \mu_{rf}}{\sigma_{mt}}\sigma\right)$ over and above the risk-free return μ_{rf} to compensate the risk taken by him.

In practice there are certain assets which are listed in the stock called *index stocks.* These limited assets are significant ones that can capture the pulse of the whole market. The most regularly quoted market indices are broad-base indices comprising of the stocks of large companies listed on a nations' largest stock exchanges, such as the American Dow Jones Industrial Average and S&P 500 Index, the British FTSE 100, the French CAC 40, the Japanese Nikkei 225. The Bombay Stock Exchange is the largest in India, with over 6000 stocks listed and it accounts for over two thirds of the total trading volume in the country. The index stocks finally help us to compute the market portfolio (σ_{mt}, μ_{mt}). The knowledge of the market portfolio yields the equation of capital market line, see Remark 17.5.1. Now suppose an investor P is willing to take risk σ_P. Then for this risk, the expected return μ_P is maximum if the point (σ_P, μ_P) lies on the capital market line. Thus,

$$\mu_P = \mu_{rf} + \frac{\mu_{mt} - \mu_{rf}}{\sigma_{mt}}\sigma_P.$$

If we let $w_P = \dfrac{\sigma_P}{\sigma_{mt}}$ then

$$\mu_P = w_P \mu_{mt} + (1 - w_P)\mu_{rf}.$$

Thus the expected return on an efficient portfolio can be thought of as
(Expected return) = (Price of time) + (Price of risk) $\times$ (Amount of risk).

Remark 17.5.2 *The above relation suggests that if an investor is willing to take a risk σ_P, then he should invest $w_P = \dfrac{\sigma_P}{\sigma_{mt}}$ proportion of investment in index fund and $(1 - w_P)$ proportion of investment in the risk-free investment schemes.*

We now aim to examine how an individual asset behaves with respect to the market portfolio. For this, we attempt to build a relationship between the expected return along with the risk of an individual asset with the market portfolio.

Theorem 17.5.3 *Suppose the market portfolio is* $(\sigma_{\mathrm{mt}}, \mu_{\mathrm{mt}})$. *The expected return of an asset* a_i *is given by*

$$\mu_i = \mu_{\mathrm{rf}} + \beta_i\,(\mu_{\mathrm{mt}} - \mu_{\mathrm{rf}}), \quad \text{where} \quad \beta_{\mathrm{i}} = \frac{\mathrm{cov}\,(\mu_{\mathrm{i}}, \mu_{\mathrm{mt}})}{\sigma_{\mathrm{mt}}^2}. \tag{17.24}$$

Proof. Suppose an investor portfolio comprises of asset a_i with weight w and the market portfolio M with weight $1-w$. Then the expected return and risk of the investor portfolio are respectively given by

$$\begin{aligned} \mu &= w\mu_i + (1-w)\mu_{\mathrm{mt}}, \\ \sigma^2 &= w^2\sigma_i^2 + (1-w)^2\sigma_{\mathrm{mt}}^2 + 2\rho w(1-w)\sigma_i\sigma_{\mathrm{mt}}, \end{aligned} \tag{17.25}$$

where ρ is the coefficient of correlation between the asset a_i and the market portfolio M.

As w varies, these values trace out a curve in the (σ, μ)-graph. It can be observed from Fig 17.10 that as w passes through zero, the capital market line becomes tangent to the curve at M. This tangency condition can be translated into the condition that the slope of the curve is equal to the slope of the capital market line at M (corresponding to $w = 0$).

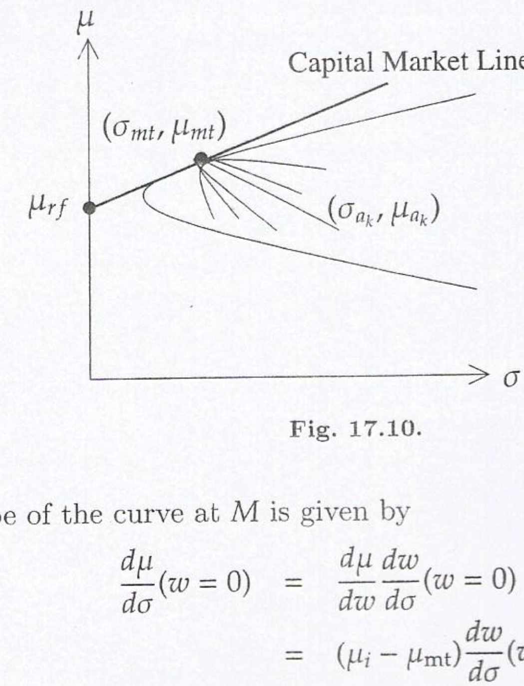

Fig. 17.10.

Now the slope of the curve at M is given by

$$\begin{aligned} \frac{d\mu}{d\sigma}(w=0) &= \frac{d\mu}{dw}\frac{dw}{d\sigma}(w=0) \\ &= (\mu_i - \mu_{\mathrm{mt}})\frac{dw}{d\sigma}(w=0). \end{aligned}$$

Differentiating (17.25) with respect to w and computing its value at $w = 0$, we get

$$\begin{aligned}\frac{d\sigma}{dw}(w=0) &= \frac{w\sigma_i^2 - (1-w)\sigma_{\text{mt}}^2 + \rho\sigma_i\sigma_{\text{mt}}(1-2w)}{\sigma}(w=0)\\ &= \frac{\sigma_{\text{imt}} - \sigma_{\text{mt}}^2}{\sigma_{\text{mt}}}, \qquad \sigma_{\text{imt}} = \rho\sigma_i\sigma_{\text{mt}}.\end{aligned}$$

Consequently,

$$\frac{d\mu}{d\sigma}(w=0) = \frac{(\mu_i - \mu_{\text{mt}})\sigma_{\text{mt}}}{\sigma_{\text{imt}} - \sigma_{\text{mt}}^2}. \tag{17.26}$$

As discussed above, the slope of the curve needs to be equal to the slope of the capital market line at M, thereby yielding that

$$\frac{\mu_{\text{mt}} - \mu_{\text{rf}}}{\sigma_{\text{mt}}} = \frac{d\mu}{d\sigma}(w=0).$$

The above relation along with (17.26), on simplification, yields

$$\begin{aligned}\mu_i &= \mu_{\text{rf}} + \frac{\mu_{\text{mt}} - \mu_{\text{rf}}}{\sigma_{\text{mt}}^2}\sigma_{\text{imt}}\\ &= \mu_{\text{rf}} + \beta_i(\mu_{\text{mt}} - \mu_{\text{rf}}).\end{aligned}$$

□

Remark 17.5.3 *Here, $\beta_i = \dfrac{\sigma_{\text{imt}}}{\sigma_{\text{mt}}^2}$ is called the Beta of an asset. Note that, for the market portfolio, $\beta_{mt} = 1$. Beta is generally calculated for individual assets using regression analysis. As can be observed, beta measures an asset volatility or risk in relation to the rest of the market. It is thus appropriately referred to as financial elasticity or correlated relative volatility, and it is all what is required to be known about the asset's risk characteristics in CAPM formula. In other words, an investor ready to bear some systematic risk gets rewarded for it. For instance, if $\beta_i = 2$, it indicates that the i^{th} asset return is expected to increase (decrease) by 2% when the market increases (decreases) by 1%. Equivalently, if the market return fluctuates over a specific range of values, the asset returns will fluctuate over a larger range of values. Thus, the market risk is magnified in the asset risk.*

Definition 17.5.3 (Beta of the Portfolio). *The overall Beta β of the portfolio is the weighted average of the Betas of the individual assets in the portfolio, with the weights being those that define the portfolio, i.e. $\beta = \sum_{i=1}^{n} w_i\beta_i$.*

Definition 17.5.4 (Security Market Line). *A linear equation*

$$\mu = \mu_{\text{rf}} + \beta(\mu_{\text{mt}} - \mu_{\text{rf}}), \quad \text{where} \quad \beta = \frac{\text{cov}(\mu, \mu_{\text{mt}})}{\sigma_{\text{mt}}^2},$$

that describes the expected return for all assets in the market is called the security market line. All portfolio investments lie along this line in a beta-return space. The line is pictorially depicted in Fig 17.11 in bold.

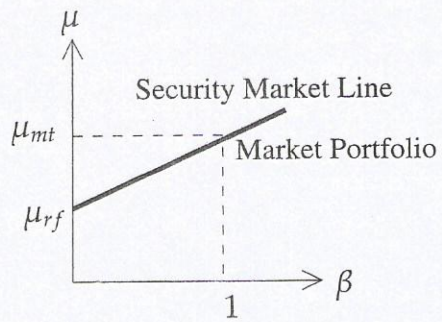

Fig. 17.11.

17.6 Summary and Additional Notes

- In this chapter we analyzed the advantages of diversifying the total investment among several assets optimally so as to get a 'decent' return with minimum risk. The theory described here is mainly based on the work of Markowitz[113].
- The requisite terminologies were introduced in Section 2, followed by a simple case of two assets portfolio optimization in Section 3, to get the feel of the subject. The ideas of Section 3 lead us to extend the analysis to the multi asset scenario in Section 4.
- Section 5 continues with the multi asset portfolio optimization with the difference that one risk-free asset is included in the portfolio. This inclusion results in the Capital Asset Pricing model (CAPM) and a new concept of market portfolio. It was shown that the market portfolio can guide the investor to determine the advantage of taking more risk with his investment.
- The contents of the chapter are kept very simple with the intention to familiarize the readers with the basics behind portfolio optimization. However things can be very complex in financial circuits. One needs to realize that any study related to financial problems requires extra skills and high level of understanding. Several interesting books covering various aspects of portfolio theory can be referred to, like, Bartholomew-Biggs [10], Capinski and Zastwawniak [31], Cornuejols and Tütüncü [39], Elton et al. [53], Luenberger [107], Roman [132], Ross [133], to name a few.
- A prominent area where portfolio optimization has gained momentum in recent years is the asset-liability management of the institutional investors, like, insurance com-

panies, pension funds or mutual funds. The institutional investors make huge investments in the markets and simultaneously repay the maturity amounts to the other investors who had invested with them. For this reasons, they need to constantly rebalance their portfolio after every time frame, which is generally very small. An institutional investor will get some inflow of money at t instance of time as the return from various investments that had been made in the market earlier and which had subsequently matured at t time, and also the institutional investor needs to pay the maturity amount to all those investors who had invested with him and whose funds have matured at the end of $t-1$ time. The remaining amount is reinvested in the market. The time scale involved in such asset-liability problems has been captured by using stochastic linear programming models (see, [88] for stochastic LPP). Number of research papers can be found on asset-liability management, like, [146, 157, 171] and references therein.

- There are several commercial packages, for instance, CPLEX, LINGO, MATLAB, SAS that provide lot of inbuilt functions for Portfolio Analysis. The major disappointment with all the commercial packages is that they can best generate only the approximate piecewise linear representation of the efficient frontier in portfolio optimization. With large number of assets involved, say 600-800, the performance of these software in computing the efficient frontier deteriorates. The MPQ (multi-parametric quadratic programming), programmed in Java and available in public domain, performs exceptionally well on large-scale applications in a reasonable time and yields the exact efficient frontier. For more on MPQ, we refer to Steuer et al. [149].
- The effect of introduction of transaction costs and/or different lending and borrowing rates in portfolio optimization theory has also been analyzed in literature. Some of the books mentioned above contain subject matter on this issue.
- Several problems of mathematical finance have been modeled as optimization problems in literature. Among them, one of the most widely studied problem is the pricing of the derivative securities, and in particular, financial options. The fundamental theorem of asset pricing showing the existence of risk neutral probability has been nicely proved using LPP duality in [39]. Besides the references listed above, one can also look for the books Ammann [3], Brigo and Mercurio [28] and David [45] for more applications of optimization problems in credit-risk models, interest rate models, volatility estimation and other financial problems.
- 'Optimal trading strategies' is another area of finance where optimization plays a key role. The mean-variance theory of Markowitz and the capital asset pricing model of Sharp tell us a great deal about what we should *hold* in our portfolio. But it is equally important to know how to *acquire* them. and this leads to the optimization of the execution cost and timing risk for obtaining an optimal trading strategy. An appropriate text on this topic is Kissell and Glantz [97]. Some important contributions in this direction are due to Almgren and Chriss [2] who introduced the concepts of

efficient trading frontiers and *capital trade line.* Bertsimas and Lo [21] studied the problem of optimal control of the execution cost by applying the stochastic dynamic programming. Recently the technique of reinforcement learning has also been used in the area of optimal trading strategies.

- Mathematical programming has also been applied to the problem of *credit scoring* where the decision models are developed that aid lenders in granting the consumer credit. These techniques assess the risk in lending to a particular consumer. An appropriate source for this topic is the text by Thomas et al. [158]

17.7 Exercises

17.1 *Suppose there are three financial market scenarios $\Omega = \{w_1, w_2, w_3\}$ with different probabilities of occurrence. Consider the following table showing the returns on two different stocks in these three scenarios*

scenario	prob	return k_1%	return k_2%
w_1	0.2	−10	−30
w_2	0.5	0	20
w_3	0.3	20	15

(a) What is the expected returns on the stocks?
(b) Suppose 60% *of the available fund is invested in stock 1 and the remaining is invested in stock 2, then what is the expected return of the portfolio?*
(c) Compute the weights if the expected return on a portfolio is 20%.

17.2 *Consider the following data*

scenario	prob	return k_1%	return k_2%
w_1	0.4	−10	20
w_2	0.2	0	20
w_3	0.4	20	10

Suppose a portfolio comprises of 40% *of total investment in stock 1 and* 60% *in stock 2. Compare the risk of the portfolio with the risks of its individual components. What will be the risk situation if a portfolio is designed with investment of* 80% *in stock 1 and the remaining in stock 2.*

17.3 *Prove that if short sales are not allowed then the risk of the portfolio can not exceed the greater of the risks of the individual components of the portfolio.*

17.4 *Show that if short sale is allowed in stock 1 to* 50% *and all the other data being the same as in exercise* **17.2** *the conclusion of exercise* **17.3** *fails.*

17.5 *Suppose the portfolios are constructed using three securities a_1, a_2, a_3 with expected returns, $\mu_1 = 20\%$, $\mu_2 = 13\%$, $\mu_3 = 17\%$, standard deviations of returns, $\sigma_1 = 25\%$, $\sigma_2 = 28\%$, $\sigma_3 = 20\%$, and the correlation between returns, $\rho_{12} = 0.3$, $\rho_{31} = 0.15$. Among all the attainable portfolios, find the one with minimum variance. What are the weights of the three securities in this portfolio? Also compute the expected return and standard deviation of this portfolio.*

17.6 *Among all attainable portfolios with expected return 20% constructed using the data provided in exercise* **17.5**, *find the portfolio with minimum variance. Compute the weights of individual assets in this portfolio.*

17.7 *Consider the following data*

	μ	σ
asset 1	10%	5%
asset 2	8%	2%

For each correlation coefficient $\rho = -1, -0.5, 0, 0.5, 1$, what is the combination of the two assets that yields the minimum standard deviation and what is the minimum value of the standard deviation?

17.8 *Compute the minimum risk portfolio for the following rate return (%) data:*

	Jan	Feb	Mar	Apr	May	June
asset 1	12	10	5	7	15	12
asset 2	7	12	10	10	12	15

Also compute the expected return for the optimal portfolio.

17.9 *Consider three risky assets with the covariance matrix and expected returns (all data in %) as follows.*

variance - covariance matrix(C)			return(M)
10	4	0	5
4	12	6	6
0	6	10	1

Find two portfolios yielding the minimum variance. Also, determine the expected returns from these two portfolios. Using the two fund theorem, construct the portfolio giving the return of 2.8% with minimum risk.

17.10 *Suppose an investor is interested in constructing a portfolio with one risk-free asset a_1, with risk-free return 6%, and three risky assets a_2, a_3, a_4 with expected returns 10%, 12%, 18%, respectively. Given the covariance matrix of the three assets (data in %) as*

$$C = \begin{pmatrix} 4 & 20 & 40 \\ 20 & 10 & 70 \\ 40 & 70 & 14 \end{pmatrix},$$

what is the optimum portfolio for the investor? What is the expected return of this portfolio?

17.11 *Consider the data of two risky assets a_1, a_2 with $\mu_1 = 12.5\%$, $\mu_2 = 10.5\%$, $\sigma_1 = 14.9\%$, $\sigma_2 = 14\%$, $\rho = 0.33$.*

(a) *Is it advisable to diversify the investment? If so then what composition of the assets will minimize the risk?*
(b) *What is the minimum value of the risk?*
(c) *If the risk-free rate of return is 5% then derive the equation of the capital market line?*

17.12 *Given the following information about the one risk-free asset and three risky assets, find the expected return and standard deviation of the market portfolio. Also determine the equation of the capital market line.*

$$\mu_{rf} = 5\%,\ \mu_1 = 14\%,\ \mu_2 = 8\%,\ \mu_3 = 20\%;$$
$$\sigma_1 = 6\%,\ \sigma_2 = 3\%,\ \sigma_3 = 15\%;\ \sigma_{12} = 0.5,\ \sigma_{13} = 0.2,\ \sigma_{23} = 0.4.$$

17.13 *Assume that the following assets are correctly priced according to the security market line. Derive the security market line. What is the expected return on an asset with $\beta = 2$?*

$$\mu_1 = 6\%,\ \beta_1 = 0.5;\quad \mu_2 = 12\%,\ \beta_2 = 1.5.$$

17.14 *If the following two assets are correctly priced according to the security market line, what is the return of the market portfolio? What is the risk-free return?*

$$\mu_1 = 9.5\%,\ \beta_1 = 0.8;\quad \mu_2 = 13.5\%,\ \beta_2 = 1.3.$$