Introduction to Categorical Logic

[DRAFT: October 3, 2019]

Steve Awodey Andrej Bauer

October 3, 2019

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Chapter 1

Algebraic Theories

Algebraic theories are descriptions of structures determined by operations and equations. There are familiar examples from elementary algebra, such as groups, but also many concepts that are not evidently algebraic, such as adjoint functors, can be given algebraic formulations. Thus the scope of algebraic theories is actually much greater than first appears. On the other hand, all such algebraic notions have in common some quite deep and general properties, from the existence of free algebras to Lawvere's duality theory. The most important of these are presented in this chapter. The development also serves as a first example and template for the scheme of "functorial semantics," to be applied to other logical notions in later chapters.

1.1 Syntax and semantics

We begin with a general approach to algebraic structures such as groups, rings, modules, and lattices. These are characterized by axiomatizations which involve only variables, constants, operations, and equations. It is important that the operations are defined everywhere, which excludes two important examples: fields because the inverse of 0 is undefined, and categories because composition is defined only for some pairs of morphisms.

Let us start with the quintessential algebraic theory—the theory of groups. A group can be described as a set G with a binary operation $\cdot : G \times G \to G$, satisfying the two axioms:

$$\forall\, x, y, z \in G \,.\, (x \cdot y) \cdot z = x \cdot (y \cdot z)$$
$$\exists\, e \in G \,.\, \forall\, x \in G \,.\, \exists\, y \in G \,.\, (e \cdot x = x \cdot e = x \wedge x \cdot y = y \cdot x = e)$$

Taking a closer look at the logical form of these axioms, we see that the second one, which expresses the existence of a unit and inverse elements, is somewhat unsatisfactory because it involves nested quantifiers. Not only does this complicate the interpretation, but it is not really necessary, since the unit element and inverse operation in a group are uniquely determined. Thus we can add them to the structure and reformulate as follows. We require the unit to be a distinguished *constant* $e \in G$ and the inverse to be an *operation* ${}^{-1} : G \to G$. We then obtain an equivalent formulation in which all axioms are now *equations*:

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$
$$x \cdot e = x \qquad e \cdot x = x$$
$$x \cdot x^{-1} = e \qquad x^{-1} \cdot x = e$$

Notice that the universal quantifier $\forall x \in G$ is no longer needed in stating the axioms, since we interpret all variables as ranging over all elements of G. Nor do we really need to explicitly mention the particular set G in the specification. Finally, since the constant e can be regarded as a nullary operation, i.e., a function $e : \mathbf{1} \to G$, the specification of the group concept consists solely of operations and equations. This leads us to the general definition of an algebraic theory.

Definition 1.1.1. A *signature* Σ for an algebraic theory consists of a family of sets $\{\Sigma_k\}_{k \in \mathbb{N}}$. The elements of Σ_k are called the *k-ary operations*. In particular, the elements of Σ_0 are the *nullary operations* or *constants*.

The *terms* of a signature Σ are expressions constructed inductively by the following rules:

1. variables x, y, z, ..., are terms,
2. if $t_1, \ldots, t_k$ are terms and $f \in \Sigma_k$ is a k-ary operation then $f(t_1, \ldots, t_k)$ is a term.

Definition 1.1.2 (cf. Definition 1.2.9)**.** An *algebraic theory* $\mathbb{T} = (\Sigma_{\mathbb{T}}, A_{\mathbb{T}})$ is given by a signature $\Sigma_{\mathbb{T}}$ and a set $A_{\mathbb{T}}$ of *axioms*, which are equations between terms (formally, pairs of terms).

Algebraic theories are also called *equational theories*.

Example 1.1.3. The theory of a commutative ring with unit is an algebraic theory. There are two nullary operations (constants) 0 and 1, a unary operation $-$, and two binary operations $+$ and $\cdot$. The equations are:

$$
\begin{aligned}
(x+y)+z &= x+(y+z) & \qquad (x \cdot y) \cdot z &= x \cdot (y \cdot z) \\
x+0 &= x & x \cdot 1 &= x \\
0+x &= x & 1 \cdot x &= x \\
x+(-x) &= 0 & (x+y) \cdot z &= x \cdot z + y \cdot z \\
(-x)+x &= 0 & z \cdot (x+y) &= z \cdot x + z \cdot y \\
x+y &= y+x & x \cdot y &= y \cdot x
\end{aligned}
$$

Example 1.1.4. The “empty” theory with no operations and no equations is the theory of a set.

Example 1.1.5. The theory with one constant and no equations is the theory of a *pointed set*, cf. Example **??**.

Example 1.1.6. Let R be a ring. There is an algebraic theory of left R-modules. It has one constant 0, a unary operation $-$, a binary operation $+$, and for each $a \in R$ a unary operation $\overline{a}$, called *scalar multiplication by* a. The following equations hold:

$$\begin{aligned} (x+y)+z &= x+(y+z)\ , & x+y &= y+x\ , \\ x+0 &= x\ , & 0+x &= x\ , \\ x+(-x) &= 0\ , & (-x)+x &= 0\ . \end{aligned}$$

For every $a, b \in R$ we also have the equations

$$\overline{a}(x+y) = \overline{a}\,x + \overline{a}\,y\ , \qquad \overline{a}(\overline{b}\,x) = \overline{(ab)}\,x\ , \qquad \overline{(a+b)}\,x = \overline{a}\,x + \overline{b}\,x\ .$$

Scalar multiplication by a is usually written as $a \cdot x$ instead of $\overline{a}\,x$. If we replace the ring R by a field $\mathbb{F}$ we obtain the algebraic theory of a vector space over $\mathbb{F}$ (even though the theory of fields is not algebraic!).

Example 1.1.7. In computer science, inductive datatypes are examples of algebraic theories. For example, the datatype of binary trees with leaves labeled by integers might be defined as follows in a programming language:

```
type tree = Leaf of int | Node of tree * tree
```

This corresponds to the algebraic theory with a constant `Leaf` n for each integer n and a binary operation `Node`. There are no equations. Actually, when computer scientists define a datatype like this, they have in mind a particular model of the theory, namely the *free* one.

Example 1.1.8. An obvious non-example is the theory of posets, formulated with a binary relation symbol $x \leq y$ and the usual axioms of reflexivity, transitivity and anti-symmetry, namely:

$$\begin{gathered} x \leq x \\ x \leq y \wedge y \leq z \Rightarrow x \leq z \\ x \leq y \wedge y \leq x \Rightarrow x = x \end{gathered}$$

On the other hand, using an operation of greatest lower bound or "meet" $x \wedge y$, one can make the equational theory of "$\wedge$-semilattices":

$$\begin{gathered} x \wedge x = x \\ x \wedge y = y \wedge x \\ x \wedge (y \wedge z) = (x \wedge y) \wedge z \end{gathered}$$

Then, defining a partial ordering $x \leq y \iff x \wedge y = x$ we arrive at the notion of a "poset with meets", which *is* equational (of course, the same can be done with joins $x \vee y$ as well). We'll have a proof later (in section **??**) that there is no reformulation of the general theory of posets into an equivalent equational one however.

Exercise 1.1.9. Let G be a group. Formulate the notion of a (left) G-set (i.e. a functor $G \to \mathsf{Set}$) as an algebraic theory.

1.1.1 Models of algebraic theories

Let us now consider what a *model* of an algebraic theory is. In classical algebra, a group is given by a set G, an element $e \in G$, a function $m : G \times G \to G$ and a function $i : G \to G$, satisfying the group axioms:

$$
\begin{aligned}
m\,(x, m\,(y, z)) &= m\,(m\,(x, y)\,, z) \\
m\,(x, i\,x) &= m\,(i\,x, x) = e \\
m\,(x, e) &= m\,(e, x) = x
\end{aligned}
$$

This notion can easily be generalized so that we can speak of models of group theory in categories other than Set. This is accomplished simply by translating the equations between certain elements into equations between the operations themselves: thus a group is given by an object $G \in \mathsf{Set}$ and three morphisms

$$e : \mathsf{1} \to G\,, \qquad m : G \times G \to G\,, \qquad i : G \to G\,.$$

Associativity of m is expressed by the commutativity of the following diagram:

$$
\begin{tikzcd}
G \times G \times G \arrow[r, "m \times \pi_2"] \arrow[d, "\pi_0 \times m"'] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
\end{tikzcd}
\tag{1.1}
$$

Similarly, the axioms for the unit and the inverse are expressed by commutativity of the following diagrams:

$$
\begin{tikzcd}
G \times \mathsf{1} \arrow[r, "1_G \times e"] \arrow[dr, "\pi_0"'] & G \times G \arrow[d, "m"] & \mathsf{1} \times G \arrow[l, "e \times 1_G"'] \arrow[dl, "\pi_1"] \\
& G &
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "\langle 1_G, i \rangle"] \arrow[d, "!_G"'] & G \times G \arrow[d, "m"] & G \arrow[l, "\langle i, 1_G \rangle"'] \arrow[d, "!_G"] \\
\mathsf{1} \arrow[r, "e"'] & G & \mathsf{1} \arrow[l, "e"]
\end{tikzcd}
\tag{1.2}
$$

Moreover, this formulation makes sense in any category $\mathcal{C}$ with finite products. So we can define a *group in* $\mathcal{C}$ to consist of an object G equipped with arrows:

$$
\begin{tikzcd}
G \times G \arrow[r, "m"] & G & G \arrow[l, "i"'] \\
& \mathsf{1} \arrow[u, "e"'] &
\end{tikzcd}
$$

such that the above diagrams (1.1) and (1.2) expressing the group equations commute.

There is also an obvious corresponding generalization of a group homomorphism in Set to homomorphisms of groups in $\mathcal{C}$. Namely, an arrow in $\mathcal{C}$ between groups $h : M \to N$ is a homomorphism if it commutes with the interpretations of the basic operations m, i, and e,

$$h \circ m^M = m^N \circ h^2 \qquad h \circ i^M = i^N \circ h \qquad h \circ e^M = e^N$$

as indicated in:

$$\begin{tikzcd} M^2 \arrow[r, "h^2"] \arrow[d, "m^M"'] & N^2 \arrow[d, "m^N"] \\ M \arrow[r, "h"'] & N \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "h"] \arrow[d, "i^M"'] & N \arrow[d, "i^N"] \\ M \arrow[r, "h"'] & N \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "="] \arrow[d, "e^M"'] & 1 \arrow[d, "e^N"] \\ M \arrow[r, "h"'] & N \end{tikzcd}$$

Together with the evident composition and identity arrows inherited from $\mathcal{C}$, this gives a category of groups in $\mathcal{C}$ which we denote:

$$\mathsf{Group}(\mathcal{C})$$

In general, we define an *interpretation* I of a theory $\mathbb{T}$ in a category $\mathcal{C}$ with finite products to consist of an object $I \in \mathcal{C}$ and, for each basic operation f of arity k, a morphism $f^I : I^k \to I$. (More formally, I is the tuple consisting of an underlying set $|I|$ and the interpretations f^I, but we shall write simply I for $|I|$.) In particular, basic constants are interpreted as morphisms $1 \to I$. The interpretation can be extended to all terms as follows: a general term t is always interpreted together with a *context* of variables $x_1, \dots, x_n$, where the variables appearing in t are among the variables appearing in the context. We write

$$x_1, \dots, x_n \mid t \tag{1.3}$$

to indicate that the term t is to be understood in context $x_1, \dots x_n$. The interpretation of a term in context (1.3) is a morphism $t^I : I^n \to I$, determined by the following specification:

1. The interpretation of a variable x_i is the i-th projection $\pi_i : I^n \to I$.

2. A term of the form $f(t_1, \dots, t_k)$ is interpreted as the composite:

$$I^n \xrightarrow{\left({t_1}^I, \dots, {t_k}^I\right)} I^k \xrightarrow{\quad f^I \quad} I$$

   where ${t_i}^I : I^n \to I$ is the interpretation of the subterm t_i, for $i = 1, \dots, k$, and f^I is the interpretation of the basic operation f.

It is clear that the interpretation of a term really depends on the context, and when necessary we shall write $t^I = [\, x_1, \dots, x_n \mid t \,]^I$. For example, the term $f\,x_1$ is interpreted as a morphism $f^I : I \to I$ in context x_1, and as the morphism $f^I \circ \pi_1 : I^2 \to I$ in the context x_1, x_2.

Suppose u and v are terms in context $x_1, \ldots, x_n$. Then we say that the equation $u = v$ is *satisfied* by the interpretation I if u^I and v^I are the same morphism in $\mathcal{C}$. In particular, if $u = v$ is an axiom of the theory, and $x_1, \ldots, x_n$ are all the variables appearing in u and v, we say that I *satisfies the axiom* $u = v$ if $[x_1, \ldots, x_n \mid u]^I$ and $[x_1, \ldots, x_n \mid v]^I$ are the same morphism,

$$\begin{tikzcd} I^n \arrow[r, shift left, "{[x_1, \ldots, x_n \mid u]^I}"] \arrow[r, shift right, "{[x_1, \ldots, x_n \mid v]^I}"'] & I \end{tikzcd} \tag{1.4}$$

which we also write as:

$$I \models u = v \qquad \Longleftrightarrow \qquad u^I = v^I.$$

Of course, we can now define as usual:

Definition 1.1.10 (cf. Definition 1.2.34)**.** A *model* M of an algebraic theory $\mathbb{T}$ in a category $\mathcal{C}$ with finite products is an interpretation I of the theory that satisfies the axioms of $\mathbb{T}$,

$$I \models u = v,$$

for all $(u = v) \in A_{\mathbb{T}}$.

A *homomorphism* of models $h : M \to N$ is an arrow in $\mathcal{C}$ that commutes with the interpretations of the basic operations,

$$h \circ f^M = f^N \circ h^k$$

for all $f \in \Sigma_{\mathbb{T}}$, as indicated in:

$$\begin{tikzcd} M^k \arrow[r, "h^k"] \arrow[d, "f^M"'] & N^k \arrow[d, "f^N"] \\ M \arrow[r, "h"'] & N \end{tikzcd}$$

The category of $\mathbb{T}$-models in $\mathcal{C}$ is written,

$$\mathsf{Mod}(\mathbb{T}, \mathcal{C}).$$

A model of the empty theory $\mathbb{T}_0$ in a category $\mathcal{C}$ with finite products is just an object $A \in \mathcal{C}$, and similarly for homomorphisms, so

$$\mathsf{Mod}(\mathbb{T}_0, \mathcal{C}) = \mathcal{C}.$$

A model of the theory $\mathbb{T}_{\mathsf{Group}}$ of groups in $\mathcal{C}$ is a group in $\mathcal{C}$, in the above sense, and similarly for homomorphisms, so:

$$\mathsf{Mod}(\mathbb{T}_{\mathsf{Group}}, \mathcal{C}) = \mathsf{Group}(\mathcal{C}).$$

In particular, a model in Set is just a group in the usual sense:

$$\mathsf{Mod}(\mathbb{T}_{\mathsf{Group}}, \mathsf{Set}) = \mathsf{Group}(\mathsf{Set}) = \mathsf{Group}.$$

An example of a new kind is provided the following.

Example 1.1.11. A model of the theory of groups in a functor category $\mathsf{Set}^{\mathbb{C}}$ is the same thing as a functor from $\mathbb{C}$ into groups,

$$\mathsf{Group}(\mathsf{Set}^{\mathbb{C}}) \cong \mathsf{Hom}(\mathbb{C}, \mathsf{Group}).$$

Indeed, for each object $C \in \mathbb{C}$ there is an evaluation functor,

$$\mathsf{eval}_C : \mathsf{Set}^{\mathbb{C}} \to \mathsf{Set}$$

with $\mathsf{eval}_C(F) = F(C)$, and evaluation preserves products since these are computed pointwise in the functor category. Moreover, every arrow $h : C \to D$ in $\mathbb{C}$ gives rise to an obvious natural transformation $h : \mathsf{eval}_C \to \mathsf{eval}_D$. Thus for any group G in $\mathsf{Set}^{\mathbb{C}}$, we have groups $\mathsf{eval}_C(G)$ for each $C \in \mathbb{C}$ and group homomorphisms $h_G : C(G) \to D(G)$, comprising a functor $G : \mathbb{C} \to \mathsf{Group}$. Conversely, it is clear that any such functor $H : \mathbb{C} \to \mathsf{Group}$ arises in this way from a group H in $\mathsf{Set}^{\mathbb{C}}$, at least up to isomorphism.

In this way, a group in a category of variable sets can be regarded as a *variable group.*

Exercise 1.1.12. Verify the details of the isomorphism of categories

$$\mathsf{Mod}(\mathbb{T}, \mathsf{Set}^{\mathbb{C}}) \cong \mathsf{Hom}(\mathbb{C}, \mathsf{Mod}(\mathbb{T}, \mathsf{Set}))$$

discussed in example 1.1.11 for arbitrary algebraic theories $\mathbb{T}$.

Exercise 1.1.13. Determine what a group is in the following categories: the category of finite sets $\mathsf{Set}_{\text{fin}}$, the category of topological spaces Top, the category of graphs Graph, and the category of groups Group.

Hint: Only the last case is tricky. Before thinking about it, prove the following lemma [Bor94, Lemma 3.11.6]. Let G be a set provided with two binary operations $\cdot$ and $\star$ and a common unit e, so that $x \cdot e = e \cdot x = x \star e = e \star x = x$. Suppose the two operations commute, i.e., $(x \star y) \cdot (z \star w) = (x \cdot z) \star (y \cdot w)$. Then they coincide, are *commutative* and associative.

1.1.2 Theories as categories

The syntactically presented notion of an algebraic theory, say of groups, is a notational convenience, but as a specification of, say, the mathematical concept of a group it has some defects. We want to find a *presentation-free* notion that captures the group concept without tying it to a specific syntactic presentation. The notion we seek can be given by a category with a certain universal mapping property which determines it uniquely (up to equivalence). This also results in a reformulation of the usual conception of syntax and semantics — so distinctive of conventional logic — bringing it more in line with other fields of modern mathematics.

Let us consider group theory again. The algebraic axiomatization in terms of unit, multiplication and inverse is not the only possible one. For example, an alternative formulation uses the unit e and a binary operation $\odot$, called *double division*, along with a single axiom [McC93]:

$$(x \odot (((x \odot y) \odot z) \odot (y \odot e))) \odot (e \odot e) = z\ .$$

The usual group operations are related to double division as follows:

$$x \odot y = x^{-1} \cdot y^{-1}\,, \qquad x^{-1} = x \odot e\,, \qquad x \cdot y = (x \odot e) \odot (y \odot e)\,.$$

There may be various reasons why we prefer to work with one formulation of group theory rather than another, but this should not be reflected in the general idea of what a group is. We want to avoid particular choices of basic constants, operations, and axioms. This is akin to the situation where an algebra is presented by generators and relations: the algebra itself is regarded as independent of any particular choice of presentation. Similarly, one usually prefers a basis-free theory of vector spaces: it is better to formulate the idea of a vector space without speaking explicitly of vector bases, even though every vector space has one. Without a doubt, vector bases are important, but they really are an auxiliary concept.

As a first step, we could simply take *all* operations built from unit, multiplication, and inverse as basic, and *all* valid equations of group theory as axioms. But we can go a step further and collect all the operations into a category, thus forgetting about which ones were "basic" and which ones "derived", and which equalities were "axioms". We first describe this construction of a category $\mathcal{C}_{\mathbb{T}}$ for a general algebraic theory $\mathbb{T}$, and then determine another characterization of it.

As objects of $\mathcal{C}_{\mathbb{T}}$ we take *contexts*, i.e. sequences of distinct variables,

$$[x_1, \ldots, x_n]\,. \qquad (n \geq 0)$$

Actually, it will be convenient to take equivalence classes under renaming of variables, so that $[x_1, x_3] = [x_2, x_1]$.

A morphism from $[x_1, \ldots, x_m]$ to $[x_1, \ldots, x_n]$ is an n-tuple $(t_1, \ldots, t_n)$, where each t_k is a term in the context, $x_1, \ldots, x_m \mid t_k$. Two such morphisms $(t_1, \ldots, t_n)$ and $(s_1, \ldots, s_n)$ are equal if, and only if, the axioms of the theory imply that $t_k = s_k$ for every $k = 1, \ldots, n$,

$$\mathbb{T} \vdash t_k = s_k$$

Strictly speaking, morphisms are thus (tuples of) *equivalence classes* of terms in context

$$[x_1, \ldots, x_m \mid t_1, \ldots, t_n] : [x_1, \ldots, x_m] \longrightarrow [x_1, \ldots, x_n],$$

where two terms are equivalent when the theory proves them to be equal. Since it is rather cumbersome to work with equivalence classes, we shall work with the terms directly, but keeping in mind that equality between them is equivalence. Note also that the context of the morphism agrees with its domain, so we can omit it from the notation when the domain is clear. The composition of morphisms

$$\begin{aligned}(t_1, \ldots, t_m) &: [x_1, \ldots, x_k] \to [x_1, \ldots, x_m]\\ (s_1, \ldots, s_n) &: [x_1, \ldots, x_m] \to [x_1, \ldots, x_n]\end{aligned}$$

is the morphism $(r_1, \ldots, r_n)$ whose i-th component is obtained by simultaneously substituting in s_i the terms $t_1, \ldots, t_m$ for the variables $x_1, \ldots, x_m$:

$$r_i = s_i[t_1, \ldots, t_m/x_1, \ldots, x_m] \qquad (1 \leq i \leq n)$$

The identity morphism on $[x_1, \ldots, x_n]$ is $(x_1, \ldots, x_n)$. Using the usual rules of deduction for equational logic, it is easy to verify that these specifications are well-defined on equivalence classes and thus make $\mathcal{C}_\mathbb{T}$ a category.

Definition 1.1.14. The category $\mathcal{C}_\mathbb{T}$ just defined is called the *syntactic category* of the theory $\mathbb{T}$.

The syntactic category $\mathcal{C}_\mathbb{T}$ — which may be thought of as the "Lindenbaum-Tarski category" of $\mathbb{T}$ — contains the same "algebraic" information as the theory $\mathbb{T}$ from which it was built, but in a syntax-invariant way. Any two different presentations of $\mathbb{T}$ — like the ones for groups mentioned above — will give rise to essentially the same category $\mathcal{C}_\mathbb{T}$ (i.e. up to isomorphism). In this sense, the category $\mathcal{C}_\mathbb{T}$ is the abstract, algebraic gadget presented by the operations and equations of the theory $\mathbb{T}$, in just the way a group can be presented by generators and relations. But there is another, much more important, sense in which $\mathcal{C}_\mathbb{T}$ represents $\mathbb{T}$, as we next show.

Exercise 1.1.15. Show that the syntactic category $\mathcal{C}_\mathbb{T}$ has all finite products.

1.1.3 Models as functors

Having now represented an algebraic theory $\mathbb{T}$ as a special category $\mathcal{C}_\mathbb{T}$, the syntactic category constructed from $\mathbb{T}$, we next show that $\mathcal{C}_\mathbb{T}$ has the special property that models of $\mathbb{T}$ correspond uniquely to certain functors from $\mathcal{C}_\mathbb{T}$. More precisely, given any FP-category $\mathcal{C}$ there is a natural equivalence,

$$\frac{\mathcal{M} \in \mathsf{Mod}(\mathbb{T}, \mathcal{C})}{\overline{\overline{M : \mathcal{C}_\mathbb{T} \to \mathcal{C}}}}$$

between models $\mathcal{M}$ of $\mathbb{T}$ in $\mathcal{C}$ and FP-functors $M : \mathcal{C}_\mathbb{T} \to \mathcal{C}$. The equivalence is mediated by a "universal model" $\mathcal{U}$ in $\mathcal{C}_\mathbb{T}$, so that every model $\mathcal{M}$ arises as the functorial image $M(\mathcal{U}) \cong \mathcal{M}$ of $\mathcal{U}$ under an essentially unique FP-functor $M : \mathcal{C}_\mathbb{T} \to \mathcal{C}$. The universal model $\mathcal{U}$ is of course the one corresponding to the identity functor $1_{\mathcal{C}_\mathbb{T}} : \mathcal{C}_\mathbb{T} \to \mathcal{C}_\mathbb{T}$ in the above displayed correspondence. The possibility of such universal models (with their attendant property of being logically generic, as described in the next section 1.1.4 below) is a benefit of the generalized notion of a model in a category other than Set. Classical Set-valued models are almost never universal in this way.

To give the details of this correspondence, let $\mathbb{T}$ be an arbitrary algebraic theory and $\mathcal{C}_\mathbb{T}$ the syntactic category constructed from $\mathbb{T}$ as in the foregoing section. It is easy to show that the product in $\mathcal{C}_\mathbb{T}$ of two objects $[x_1, \ldots, x_n]$ and $[x_1, \ldots, x_m]$ is the object $[x_1, \ldots, x_{n+m}]$, and that $\mathcal{C}_{\mathbb{T}}$ has all finite products (including $1 = [-]$, the empty context). Moreover, there is a $\mathbb{T}$-model U in $\mathcal{C}_{\mathbb{T}}$ consisting of the language itself: The underlying object is the context $U = [x_1]$ of length one, and each operation symbol f of, say, arity k is interpreted as itself,

$$f^U = [\, x_1, \ldots, x_k \mid f(x_1, \ldots, x_k)\,] : U^k = [x_1, \ldots, x_k] \longrightarrow [x_1] = U.$$

The axioms are of course all satisfied, since for any terms s, t:

$$U \models s = t \iff s^U = t^U \iff \mathbb{T} \vdash s = t. \tag{1.5}$$

This *syntactic model* U in $\mathcal{C}_{\mathbb{T}}$ is "universal" in the following sense: any model M in any category $\mathcal{C}$ with finite products is the image of U under an essentially unique, finite product preserving functor $\mathcal{C}_{\mathbb{T}} \to \mathcal{C}$. In a certain sense, then, $\mathcal{C}_{\mathbb{T}}$ is the "free finite product category with a model of $\mathbb{T}$". We now proceed to make this more precise.

First, observe that any FP-functor $F : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$ takes the syntactic model U in $\mathcal{C}_{\mathbb{T}}$ to a model FU in $\mathcal{C}$, with interpretations

$$f^{FU} : FU^k \to FU \qquad \text{for each } f \in \Sigma_k.$$

Moreover, any natural transformation $\vartheta : F \to G$ between FP-functors determines a homomorphism of models $h = \vartheta_U : FU \to GU$. In more detail, suppose $f : U \times U \to U$ is a basic operation, then there is a commutative diagram,

$$\begin{tikzcd}
FU \times FU \arrow[r, "h \times h"] \arrow[d, "\cong"] \arrow[dd, bend right=60, "f^{FU}"'] & GU \times GU \arrow[d, "\cong"'] \arrow[dd, bend left=60, "f^{GU}"] \\
F(U \times U) \arrow[r, "\vartheta_{U \times U}"] \arrow[d, "Ff"] & G(U \times U) \arrow[d, "Gf"'] \\
FU \arrow[r, "h = \vartheta_U"'] & GU
\end{tikzcd}$$

where the upper square commutes by preservation of products, and the lower one by naturality. Thus the operation "evaluation at U" determines a functor,

$$\mathsf{eval}_U : \mathsf{Hom}_{\mathrm{FP}}(\mathcal{C}_{\mathbb{T}}, \mathcal{C}) \longrightarrow \mathsf{Mod}(\mathbb{T}, \mathcal{C}) \tag{1.6}$$

from the category of finite product preserving functors $\mathcal{C}_{\mathbb{T}} \to \mathcal{C}$, with natural transformations as arrows, into the category of $\mathbb{T}$-models in $\mathcal{C}$.

Proposition 1.1.16. *The functor* (1.6) *is an equivalence of categories, natural in* $\mathcal{C}$.

Proof. Let M be any model in an FP-category $\mathcal{C}$. Then the assignment $f \mapsto f^M$ given by the interpretation determines a functor $M^\sharp : \mathcal{C}_\mathbb{T} \to \mathcal{C}$, defined on objects by

$$M^\sharp[x_1, \dots, x_k] = M^k$$

and on morphisms by

$$M^\sharp(t_1, \dots, t_n) = \langle {t_1}^M, \dots, {t_n}^M \rangle.$$

In detail, $M^\sharp$ is defined on morphisms

$$[x_1, \dots, x_k \mid t\,] : [x_1, \dots, x_k] \to [x_1, \dots, x_n]$$

in $\mathcal{C}_\mathbb{T}$ by the following rules:

1. The morphism
$$(x_i) : [x_1, \dots, x_k] \to [x_1]$$
is mapped to the i-th projection
$$\pi_i : M^k \to M.$$

2. The morphism
$$(f\,(t_1, \dots, t_m)) : [x_1, \dots, x_k] \to [x_1]$$
is mapped to the composite
$$M^k \xrightarrow{\left(M^\sharp t_1, \dots, M^\sharp t_m\right)} M^m \xrightarrow{\quad M^\sharp f \quad} M$$
where $M^\sharp t_i : M^k \to M$ is the value of $M^\sharp$ on the morphisms $(t_i) : [x_1, \dots, x_k] \to [x_1]$, for $i = 1, \dots, m$, and $M^\sharp f = f^M$ is the interpretation of the basic operation f.

3. The morphism
$$(t_1, \dots, t_n) : [x_1, \dots, x_k] \to [x_1, \dots, x_n]$$
is mapped to the morphism $\langle M^\sharp t_1, \dots, M^\sharp t_n \rangle$ where $M^\sharp t_i$ is the value of $M^\sharp$ on the morphism $(t_i) : [x_1, \dots, x_k] \to [x_1]$, and
$$\langle M^\sharp t_1, \dots, M^\sharp t_n \rangle : M^k \longrightarrow M^n$$
is the evident n-tuple in the FP-category $\mathcal{C}$.

That $M^\sharp : \mathcal{C}_\mathbb{T} \to \mathcal{C}$ really is a functor now follows from the assumption that the interpretation M is a model, which means that all the equations of the theory are satisfied by it, so that the above specification is well-defined on equivalence classes. Observe that the functor $M^\sharp$ is defined in such a way that it obviously preserves finite products, and that there is an isomorphism of models,

$$M^\sharp(U) \cong M.$$

Thus we have shown that the functor "evaluation at U",

$$\mathsf{eval}_U : \mathsf{Hom}_{\mathrm{FP}}(\mathcal{C}_{\mathbb{T}}, \mathcal{C}) \longrightarrow \mathsf{Mod}(\mathbb{T}, \mathcal{C}) \tag{1.7}$$

is essentially surjective on objects, since $\mathsf{eval}_U(M^\sharp) = M^\sharp(U) \cong M$.

We leave the verification that it is full and faithful as an easy exercise.

Exercise 1.1.17. Verify this.

Naturality in $\mathcal{C}$ means the following. Suppose M is a model of $\mathbb{T}$ in any category $\mathcal{C}$ with finite products ("FP-category"). Any finite product-preserving functor ("FP-functor") $F : \mathcal{C} \to \mathcal{D}$ to another FP-category $\mathcal{D}$ then takes M to a model $F(M)$ in $\mathcal{D}$. The interpretation is given by setting $f^{F(M)} = F(f^M)$ for the basic operations f (and composing with the canonical isos coming from preservation of products, $F(M) \times F(M) \cong F(M \times M)$, etc.). Since equations are described by commuting diagrams, F takes a model to a model, and the same is true for homomorphisms. Thus $F : \mathcal{C} \to \mathcal{D}$ induces a functor on $\mathbb{T}$-models,

$$\mathsf{Mod}(\mathbb{T}, F) : \mathsf{Mod}(\mathbb{T}, \mathcal{C}) \longrightarrow \mathsf{Mod}(\mathbb{T}, \mathcal{D}).$$

By naturality of (1.6) we mean that the following square commutes, up to natural isomorphism:

$$\begin{tikzcd}
\mathsf{Hom}_{\mathrm{FP}}(\mathcal{C}_{\mathbb{T}}, \mathcal{C}) \arrow[r, "\mathsf{eval}_U"] \arrow[d, "\mathsf{Hom}_{\mathrm{FP}}(\mathcal{C}_{\mathbb{T}}, F)"'] & \mathsf{Mod}(\mathbb{T}, \mathcal{C}) \arrow[d, "\mathsf{Mod}(\mathbb{T}, F)"] \\
\mathsf{Hom}_{\mathrm{FP}}(\mathcal{C}_{\mathbb{T}}, \mathcal{D}) \arrow[r, "\mathsf{eval}_U"'] & \mathsf{Mod}(\mathbb{T}, \mathcal{C})
\end{tikzcd} \tag{1.8}$$

But this is clear, since for any FP-functor $M : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$ we have:

$$\begin{aligned}
\mathsf{eval}_U \circ \mathsf{Hom}_{\mathrm{FP}}(\mathcal{C}_{\mathbb{T}}, F)(M) &= (\mathsf{Hom}_{\mathrm{FP}}(\mathcal{C}_{\mathbb{T}}, F)(M))(U) \\
&= (F \circ M)(U) \\
&= F(M(U)) \\
&= F(\mathsf{eval}_U(M)) \\
&\cong \mathsf{Mod}(\mathbb{T}, F)(\mathsf{eval}_U(M)) \\
&= \mathsf{Mod}(\mathbb{T}, F) \circ \mathsf{eval}_U(M).
\end{aligned}$$

□

The equivalence of categories

$$\mathsf{Hom}_{\mathrm{FP}}(\mathcal{C}_{\mathbb{T}}, \mathcal{C}) \simeq \mathsf{Mod}(\mathbb{T}, \mathcal{C}) \tag{1.9}$$

actually determines $\mathcal{C}_{\mathbb{T}}$ and the universal model U uniquely, up to equivalence of categories and isomorphism of models. Indeed, to recover U, just put $\mathcal{C}_{\mathbb{T}}$ for $\mathcal{C}$ and the identity functor $1_{\mathcal{C}_{\mathbb{T}}}$ on the left, to get U in $\mathsf{Mod}(\mathbb{T}, \mathcal{C}_{\mathbb{T}})$ on the right! To see that $\mathcal{C}_{\mathbb{T}}$ itself is also determined, observe that (1.9) essentially says that the functor $\mathsf{Mod}(\mathbb{T}, \mathcal{C})$ is representable, with representing object $\mathcal{C}_{\mathbb{T}}$. As usual, this fact can also be formulated in elementary terms as a universal mapping property of $\mathcal{C}_{\mathbb{T}}$, as follows:

Definition 1.1.18. The *classifying category* of an algebraic theory $\mathbb{T}$ is an FP-category $\mathcal{C}_{\mathbb{T}}$ with a distinguished model U, called the *universal model*, such that:

(i) for any model M in any FP-category $\mathcal{C}$, there is an FP-functor

$$M^\sharp : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$$

and an isomorphism of models $M \cong M^\sharp(U)$.

(ii) for any FP-functors $F, G : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$ and model homomorphism $h : F(U) \to G(U)$, there is a unique natural transformation $\vartheta : F \to G$ with

$$\vartheta_U = h.$$

Observe that (i) says that the evaluation functor (1.6) is essentially surjective, and (ii) that it is full and faithful. The category $\mathcal{C}_{\mathbb{T}}$ is clearly determined up to equivalence by this universal mapping property in the usual way. Specifically, if $(\mathcal{C}, U)$ and $(\mathcal{D}, V)$ are both classifying categories for the same theory, then there are classifying functors,

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "V^\sharp"] & \mathcal{D} \arrow[l, bend left, "U^\sharp"] \end{tikzcd}$$

the composites of which are necessarily isomorphic to the respective identity functors, since e.g. $U^\sharp(V^\sharp(U)) \cong U^\sharp(V) \cong U$.

We have now shown not only that every algebraic theory has a classifying category, but also that the syntactic category is essentially determined by that distinguishing property. We record this as the following.

Theorem 1.1.19. *Every algebraic theory $\mathbb{T}$ has the syntactic category $\mathcal{C}_{\mathbb{T}}$ as a classifying category.*

Example 1.1.20. Let us see what the foregoing definitions give us in the case of group theory $\mathbb{G} = \mathbb{T}_{\mathsf{Group}}$. Recall that the category $\mathbb{G}$ consists of contexts $[x_1, \dots, x_n]$ and terms built from variables and the basic group operations. A finite product preserving functor $M : \mathbb{G} \to \mathsf{Set}$ is then determined up to natural isomorphism by its action on the context $[x_1]$ and the terms representing the basic operations. If we set

$$G = M[x_1] \,, \qquad e = M(\cdot \mid e) \,,$$
$$i = M(x_1 \mid {x_1}^{-1}) \,, \qquad m = M(x_1, x_2 \mid x_1 \cdot x_2) \,,$$

then (G, e, i, m) is a just a group with unit e, inverse i and multiplication m. That G satisfies the axioms for groups follows from functoriality of M. Conversely, any group (G, e, i, m) determines a finite product preserving functor $M_G : \mathbb{G} \to \mathsf{Set}$ defined by

$$\begin{aligned} M_G[x_1, \dots, x_n] &= G^n \;, & M_G(\cdot \mid e) \;, \\ M_G(x_1 \mid {x_1}^{-1}) &= i \;, & M_G(x_1, x_2 \mid x_1 \cdot x_2) = m \;. \end{aligned}$$

This shows that $\mathsf{Mod}_{\mathsf{Set}}(\mathbb{G})$ is indeed equivalent to Group, provided both categories have the same notion of morphisms.

Suppose then that (G, e_G, i_G, m_G) and (H, e_H, i_H, m_H) are groups, and let $\phi : M_G \Longrightarrow M_H$ be a natural transformation between the corresponding functors. Then ϕ is already determined by its component at $[x_1]$ because by naturality the following diagram commutes, for $1 \leq k \leq n$:

$$\begin{tikzcd} G^n \arrow[r, "\phi_{[x_1,\dots,x_n]}"] \arrow[d, "G\pi_k = \pi_k"'] & H^n \arrow[d, "H\pi_k = \pi_k"] \\ G \arrow[r, "\phi_{[x_1]}"'] & H \end{tikzcd}$$

If we write $\phi' = \phi_{[x_1]}$ then it follows that $\phi_{[x_1,\dots,x_n]} = \phi' \times \cdots \times \phi'$. Again, by naturality of ϕ we see that the following diagram commutes:

$$\begin{tikzcd} G \times G \arrow[r, "\phi' \times \phi'"] \arrow[d, "m_G"'] & H \times H \arrow[d, "m_h"] \\ G \arrow[r, "\phi'"'] & H \end{tikzcd}$$

Similar commutative squares show that ϕ' preserves the unit and commutes with the inverse operation, therefore $\phi' : G \to H$ is indeed a group homomorphism. Conversely, a group homomorphism $\psi' : G \to H$ determines a natural transformation $\psi : G \Longrightarrow H$ whose component at $[x_1, \dots, x_n]$ is the n-fold product $\psi' \times \cdots \psi' : G^n \to H^n$. This demonstrates that

$$\mathsf{Mod}_{\mathsf{Set}}(\mathbb{G}) \simeq \mathsf{Group} \;.$$

Example 1.1.21. Recall from 1.1.11 that a group G in the functor category $\mathsf{Set}^{\mathbb{C}}$ is essentially the same thing as a functor $G : \mathbb{C} \to \mathsf{Group}$. From the point of view of algebras as functors, this amounts to the observation that product-preserving functors $\mathbb{G} \to \mathsf{Hom}(\mathbb{C}, \mathsf{Set})$ correspond (by exponential transposition) to functors $\mathbb{C} \to \mathsf{Hom}_{\mathsf{FP}}(\mathbb{G}, \mathsf{Set})$, where the latter Hom-set consists just of product-preserving functors. Indeed, the correspondence extends to natural transformations to give the previously observed equivalance of categories,

$$\mathsf{Group}(\mathsf{Set}^{\mathbb{C}}) \simeq (\mathsf{Group}(\mathsf{Set}))^{\mathbb{C}} \simeq \mathsf{Group}^{\mathbb{C}}.$$

1.1.4 Completeness

Consider an algebraic theory $\mathbb{T}$ and an equation $s = t$ between terms of the theory. If the equation can be proved from the axioms of the theory, then every model of the theory satisfies the equation; this is just the *soundness* of the equational calculus with respect to models in categories. The converse statement is:

$$\text{“Every model of } \mathbb{T} \text{ satisfies } s = t\text{.”} \Rightarrow \text{“}\mathbb{T} \text{ proves equation } s = t\text{.”}\ .$$

This property is called *completeness*, and (together with soundness) it says that the calculus of equations suffices for proving all (and only) the ones that hold in the semantics. This holds in an especially strong sense for categorical semantics, as shown by the following.

Theorem 1.1.22 (Strong completeness)**.** *Suppose $\mathbb{T}$ is an algebraic theory.*

1. *For any equation $s = t$: every model M of $\mathbb{T}$ in every FP-category $\mathbb{C}$ satisfies $s = t$, i.e. $M \models s = t$, if and only if $\mathbb{T} \vdash s = t$.*

2. *Then there exists an FP-category $\mathbb{C}$ and a model $U \in \mathsf{Mod}_{\mathbb{C}}(\mathbb{T})$ with the property that, for every equation $s = t$ between terms of the theory $\mathbb{T}$,*

$$U \models s = t \iff \mathbb{T} \vdash s = t.$$

   That is, satisfaction by U is equivalent to provability in $\mathbb{T}$.

We will say that the equational calculus of algebraic theories is strongly complete *with respect to general categorical semantics.*

Proof. The second statement follows from the syntactic construction of the classifying category 1.1.19 as follows: Let $\mathcal{C} = \mathcal{C}_{\mathbb{T}}$ be the classifying category and U the universal model. If $\mathbb{T} \vdash s = t$, then by the syntactic construction of $\mathcal{C}_{\mathbb{T}}$ we have $s^U = t^U$. Conversely, if $U \models s = t$, then $s^U = t^U$. But by the syntactic construction of $\mathcal{C}_{\mathbb{T}}$, it then must be the case that $\mathbb{T} \vdash s = t$.

For the first statement, any model M in an FP-category $\mathcal{C}$ has a classifying functor $M^\sharp : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$, which preserves the interpretations of s and t in the sense that (up to canonical isomorphism):

$$M^\sharp(s^U) = s^{M^\sharp(U)} = s^M$$

and similarly for t. Thus from $s^U = t^U$ we can infer $s^M = t^M$, i.e. $M \models s = t$. Thus $M \models s = t$ for every model M if and only if this holds in the universal model U, which is equivalent to provability by part (2). □

Definition 1.1.23. A single model with the property mentioned in the theorem, of satisfying all and only those equations that are provable from the theory, shall be said to be *logically generic.*

Thus, by the foregoing, the universal model is logically generic. Classically, it is seldom the case that there exists a single, logically generic model; instead, for classical completeness, we consider the range of all models in Set. Completeness with respect to such a restricted range of models is of course a stronger statement than completeness with respect to all models in all categories. Toward the classical result, we first consider completeness with respect to "variable models" in Set, i.e. models in presheaf categories $\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$.

Proposition 1.1.24. *Let $\mathbb{T}$ be an algebraic theory. The Yoneda embedding*

$$\mathsf{y} : \mathcal{C}_{\mathbb{T}} \to \widehat{\mathcal{C}_{\mathbb{T}}}$$

is a generic model for $\mathbb{T}$.

Proof. The Yoneda embedding $\mathsf{y} : \mathcal{C}_{\mathbb{T}} \to \widehat{\mathcal{C}_{\mathbb{T}}}$ preserves limits, and in particular finite products, hence it corresponds to a model $U' = \mathsf{y}(U)$ of $\mathbb{T}$ in $\widehat{\mathcal{C}_{\mathbb{T}}}$. Simply because y is a functor, U' satisfies all equations that hold in U, but because it is faithful, U' does not validate any equations that do not already hold in U. Since U is logically generic, so is U'. □

Example 1.1.25. We consider group theory one last time. As a presheaf on the theory of groups, the universal group satisfies every equation that is satisfied by all groups, and no others. Let us describe it explicitly as a variable set. Recall that the theory of groups is the category $\mathbb{G}$ whose objects are contexts $[x_1, \ldots, x_n]$, $n \in \mathbb{N}$. The carrier U of the universal group is the presheaf represented by the context with one variable,

$$U = \mathsf{y}[x_1] = \mathbb{G}(-, [x_1]) \ .$$

This is a set parametrized by the objects of $\mathbb{G}$. For every $n \in \mathbb{N}$, we get the set $U_n = \mathbb{G}([x_1, \ldots, x_n], [x_1])$ that consists of all terms built from n variables, modulo equations of group theory; but this is precisely the free group on n generators! Unit, inverse, and multiplication on U are defined at each stage U_n as the corresponding operations on the free group on n generators (the reader should verify this in detail).

To summarize, as a presheaf, the universal group is the free group on n-generators, where $n \in \mathbb{N}$ is a parameter.

Finally, we consider completeness with respect to Set-valued models $M : \mathcal{C}_{\mathbb{T}} \to \mathsf{Set}$, which of course correspond to classical models. We need the following:

Lemma 1.1.26. *For any small category $\mathcal{C}$, there is a jointly faithful set of FP-functors $E_i : \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}} \to \mathsf{Set}, i \in I$. That is, for any maps $f, g : A \to B$ in $\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$, if $E_i(f) = E_i(g)$ for all $i \in I$, then $f = g$.*

Proof. Consider the evaluation functors $\mathrm{ev}_C : \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}} \to \mathsf{Set}$ for all $C \in \mathcal{C}$. These are clearly jointly faithful, and they preserve all limits and colimits, since these are constructed pointwise in presheaves. □

Proposition 1.1.27. *Suppose $\mathbb{T}$ is an algebraic theory. For every equation $s = t$ between terms of the theory $\mathbb{T}$,*

$$M \models s = t \quad \textit{for all models } M \textit{ in } \mathsf{Set} \iff \mathbb{T} \vdash s = t.$$

Thus the equational calculus of algebraic theories is complete with respect to Set*-valued semantics.*

Proof. Combine the foregoing lemma with the fact, from Proposition 1.1.24, that the Yondea embedding is a generic model. □

Exercise 1.1.28. The universal group U is a functor $\mathbb{G}^{\mathsf{op}} \to \mathsf{Set}$. In the last example we described the object part of U. What is the action of U on morphisms? Also describe the group structure on U explicitly.

Exercise 1.1.29. Let s be a term of group theory with variables $x_1, \ldots, x_n$. On one hand we can think of s as an element of the free group U_n, and on the other we can consider the interpretation of s in the universal group U, namely a natural transformation $Us : U^n \Longrightarrow U$. Suppose t is another term of group theory with variables $x_1, \ldots, x_n$. Show that $Us = Ut$ if, and only if, $s = t$ in the free group U_n.

1.1.5 Functorial semantics

Let us now summarize our treatment of algebraic theories so far. We have reformulated the traditional *logical* notions in terms of "algebraic" or *categorical* ones. The traditional approach to logic may be described as involving four different parts:

Type theory
: There is an underlying type theory, which is a calculus of types and terms. For algebraic theories the calculus of types is trivial, since there is only one type which is not even explicitly mentioned. The terms are built from variables and basic operations.

Logic
: A variety of different kinds of logic can be considered. Algebraic theories have a very simple kind of logic that only involves equations between terms and equational reasoning.

Theory
: A theory is given by basic types, basic terms, and axioms. The types and the terms are expressed in the type theory of the system, and the axioms are be expressed in the logic of the system.

Interpretations and Models
: The type theory and logic of a logical system can be interpreted in any category of the appropriate kind. For algebraic theories we considered categories with finite products. The interpretation is *denotational*, in the sense that the types and terms

of the theory are assigned to objects and morphisms (which they "denote") by induction on the structure of types, terms, and logical formulas. An interpretation of a theory is a *model* if it satisfies all the axioms of the theory, where in the present case the notion of satisfaction just means that the arrows interpreting the terms occurring in the equations are actually equal.

The alternative approach developed here — called *functorial semantics* — may be summarized as follows:

Theories are categories

From a theory we can construct a category which expresses essentially the same information as the theory but is syntax-invariant, in the sense that it does not depend on a particular presentation by (basic) operations and axioms. The structure of the category reflects the underlying type theory and logic. For example, single sorted algebraic theories give rise to categories with finite products.

Models are functors

A model is a (structure-preserving) functor from a (category representing a) theory to a category with appropriate structure to interpret the logic. The requirement that all axioms of the theory must be satisfied by a model translates to the requirement that the model is a functor and that it preserves the structure of the theory. For models of algebraic theories we only required that they preserve finite products, whereas functoriality ensures that all valid equations of the theory are preserved, thus satisfying the axioms.

Homomorphisms are natural transformations

We obtain a notion of homomorphisms between models for free: since models are functors, homomorphisms are natural transformations between them. Homomorphisms between models of algebraic theories turned out to be the usual notion of morphisms that preserved the algebraic structure.

Universal model

By admitting models in categories other than Set, functorial semantics allows the possibility of *universal models*: a model U in the classifying category $\mathcal{C}_{\mathbb{T}}$, such that every model anywhere is a functorial image of U by an essentially unique, logic-preserving functor. Such a universal model is then "logically generic", in the sense that it has all and only those logical properties had by all models, since such properties are preserved by the functors in question.

Logical completeness

The construction of the classifying category from the logical syntax of the theory shows the *soundness and completeness* of the theory with respect to general categorical semantics. Completeness with respect to a special class of models (e.g. Set-valued ones) results from an embedding theorem for the classifying category.

1.2 Lawvere duality

The scheme of functorial semantics that we have developed also applies to a wide range of logics other than algebraic theories, and we shall consider some of these in later chapters. A further aspect of functorial semantics is not nearly as transparent in the general case as it is in that of algebraic theories, however; namely, a deep and fascinating duality relating syntax and semantics. We devote the rest of this chapter to its investigation.

1.2.1 Logical duality

There is a remarkable and far-reaching duality in logic of the form:

$$\text{Syntax} \quad \simeq \quad \text{Semantics}^{\mathsf{op}}$$

It was first presented by F.W. Lawvere in his thesis, and developed in some early papers [Law63a, Law63b, Law65], but it has hardly even been noticed by conventional logicians—probably because its recognition requires the tools of category theory.

We can see this duality quite clearly in the case of algebraic theories. Let $\mathcal{C}_{\mathbb{T}}$ be the classifying category for an equational theory $\mathbb{T}$, like the theory of groups, constructed syntactically as in section 1.1.2 above. So the objects of $\mathcal{C}_{\mathbb{T}}$ are contexts of variables $[x_1 \ldots, x_n]$, up to renaming, and the arrows are terms in context $[x_1 \ldots, x_n \mid t]$, up to $\mathbb{T}$-provable equality. We will see that $\mathcal{C}_{\mathbb{T}}$ is actually dual to a certain subcategory $\mathbb{M}$ of classical models of $\mathbb{T}$ (in Set). Specifically, there is a full subcategory, $\mathbb{M} \hookrightarrow \mathsf{Mod}(\mathbb{T})$ and an equivalence of categories,

$$\mathcal{C}_{\mathbb{T}} \simeq \mathbb{M}^{\mathsf{op}},$$

making the *syntactic* category $\mathcal{C}_{\mathbb{T}}$ dual to a subcategory of the *semantic* category $\mathsf{Mod}(\mathbb{T})$. Thus, in particular, there is an invariant representation of the syntax of the theory $\mathbb{T}$ "hidden" inside the category of models of $\mathbb{T}$. Indeed, it is quite easy to specify $\mathbb{M}$: it is the *full* subcategory on the finitely generated free models of $\mathbb{T}$,

$$\mathsf{Mod}_{\mathrm{fg}}(\mathbb{T})_0 = \left\{ F(n) \;\middle|\; F(n) \text{ free } \mathbb{T}\text{-model}, n \in \mathbb{N} \right\}.$$

Theorem 1.2.1. *Let $\mathbb{T}$ be an algebraic theory, and let*

$$\mathbb{M} = \mathsf{Mod}_{\mathrm{fg}}(\mathbb{T}) \hookrightarrow \mathsf{Mod}(\mathbb{T})$$

be the full subcategory of finitely generated, free models of $\mathbb{T}$. Then $\mathbb{M}^{\mathsf{op}}$ classifies $\mathbb{T}$ models. That is to say, for any FP-category $\mathcal{C}$, there is an equivalence of categories,

$$\mathsf{Hom}_{\mathrm{FP}}(\mathbb{M}^{\mathsf{op}}, \mathcal{C}) \simeq \mathsf{Mod}(\mathbb{T}, \mathcal{C}), \tag{1.10}$$

which is natural in $\mathcal{C}$.

Before giving the somewhat lengthy proof, let us observe that the claimed duality follows almost directly. Namely, there is an equivalence,

$$\mathcal{C}_{\mathbb{T}} \simeq \mathsf{Mod}_{\mathrm{fg}}(\mathbb{T})^{\mathsf{op}} \tag{1.11}$$

between the syntactic category $\mathcal{C}_{\mathbb{T}}$ and the opposite of the category $\mathsf{Mod}_{\mathrm{fg}}(\mathbb{T})$ of finitely generated, free models. This is because both objects $\mathcal{C}_{\mathbb{T}}$ and $\mathsf{Mod}_{\mathrm{fg}}(\mathbb{T})^{\mathsf{op}}$ represent the same functor, $\mathsf{Mod}(\mathbb{T}, \mathcal{C})$.

of theorem 1.2.1. First, observe that $\mathbb{M}^{\mathsf{op}}$ has all finite products, since $\mathbb{M}$ has all finite coproducts. Indeed, we have

$$F(n) + F(m) \cong F(n+m),$$
$$0 \cong F(0),$$

since the left adjoint F preserves all colimits.

To determine the universal $\mathbb{T}$-algebra in M^{op}, let,

$$U = F(1),$$

so that every object is a power of U in M^{op},

$$F(n) \cong U^n.$$

Next, we interpret the signature. For each basic operation f of $\mathbb{T}$, with arity n, there is an element of $F(n)$ built from f and the n generators $x_1, \dots, x_n$, namely

$$f(x_1, \dots, x_n) \in F(n).$$

E.g. in the theory of groups, there is the element $x \cdot y$ in the free group on the two generators x, y. By freeness of $F(1)$, each element $t \in F(n)$ determines a unique homomorphism $\bar{t} : F(1) \to F(n)$ in $\mathbb{M}$ taking the generator in $F(1)$ to t. Thus there is a homomorphism

$$\overline{f(x_1, \dots, x_n)} : F(1) \to F(n) \qquad \text{in } \mathbb{M}$$

associated to $f(x_1, \dots, x_n)$ in $F(n)$. This map is the interpretation of f,

$$f^U : U^n \to U \qquad \text{in } \mathbb{M}^{\mathsf{op}}.$$

Similarly, if $x_1 \dots, x_n \mid t$ is any term in context, then the interpretation

$$[x_1 \dots, x_n \mid t] : U^n \to U$$

is just the unique homomorphism corresponding to the element $t \in F(n)$ (proof by induction!).

It now follows that for every equation of $\mathbb{T}$,

$$s = t$$

we have $U \models s = t$. Indeed,

$$[x_1 \ldots, x_n \mid s] = [x_1 \ldots, x_n \mid t] : U^n \to U$$

if $\mathbb{T} \vdash s = t$, since clearly these terms must agree in the free algebra $F(n)$. For instance, $x \cdot y = y \cdot x$ for the two generators x, y of the free *abelian* group $F(2)$, but not in the free (non-abelian) group.

Thus we indeed have a model U of $\mathbb{T}$ in $\mathbb{M}^{\mathsf{op}}$, made from the free algebras. We show that this model has the required universal property, in three steps:

Step 1. Let A be any $\mathbb{T}$-algebra in Set. Then there is a product-preserving functor,

$$A^\sharp : \mathbb{M}^{\mathsf{op}} \to \mathsf{Set}$$

with $A^\sharp(U) \cong A$ (as $\mathbb{T}$-models), namely:

$$A^\sharp(-) = \mathsf{Hom}_{\mathsf{Mod}(\mathbb{T})}(-, A),$$

where we of course restrict the representable functor $\mathsf{Hom}_{\mathsf{Mod}(\mathbb{T})}(-, A) : \mathsf{Mod}(\mathbb{T})^{\mathsf{op}} \to \mathsf{Set}$ along the (full) inclusion

$$\mathbb{M} = \mathsf{Mod}_{\mathrm{fg}}(\mathbb{T}) \hookrightarrow \mathsf{Mod}(\mathbb{T})$$

of the finitely generated, free algebras. The functor

$$A^\sharp : \mathbb{M}^{\mathsf{op}} \to \mathsf{Set}$$

clearly preserves products: for each object $U^n \in \mathbb{M}^{\mathsf{op}}$, we have

$$A^\sharp(U^n) = \mathsf{Hom}_{\mathsf{Mod}(\mathbb{T})}(F(n), A) \cong= \mathsf{Hom}_{\mathsf{Set}}(n, V(A)) \cong |A|^n.$$

And in particular $A^\sharp(U) \cong |A|$.

Finally, let us show that for any basic operation f, we have $A^\sharp(f^U) = f^A$, up to isomorphism. Indeed, given any algebra A and operation $f^A : |A|^n \to |A|$, we have a commutative diagram,

$$\begin{tikzcd}
|A|^n \arrow[r, "f^A"] \arrow[d, "\cong"'] & |A| \arrow[d, "\cong"] \\
\mathsf{Hom}(F(n), A) \arrow[r, "f^*"'] & \mathsf{Hom}(F(1), A)
\end{tikzcd} \tag{1.12}$$

where f^* is precomposition with the homomorphism

$$F(n) \xleftarrow{\overline{f(x_1, \ldots, x_n)}} F(1)$$

To see that (1.12) commutes, take any $(a_1, \dots, a_n) \in |A|^n$ with associated homomorphism $\overline{(a_1, \dots, a_n)} : F(n) \to A$ and precompose with $\overline{f(x_1, \dots, x_n)}$ to get a map $F(1) \to A$ picking out the element

$$\begin{aligned}
\overline{(a_1, \dots, a_n)} \circ \overline{f(x_1, \dots, x_n)}(x) &= \overline{(a_1, \dots, a_n)}(f(x_1, \dots, x_n)) \\
&= \overline{(a_1, \dots, a_n)} \circ f^{F(n)}(x_1, \dots, x_n) \\
&= f^A \circ \overline{(a_1, \dots, a_n)}(x_1, \dots, x_n) \\
&= f^A(a_1, \dots, a_n)
\end{aligned}$$

where x is the generator of $F(1)$, and using the fact that $\overline{(a_1, \dots, a_n)}$ is a homomorphism and therefore commutes with the respective interpretations of f.

But now note that

$$F(n) \xleftarrow{\overline{f(x_1, \dots, x_n)}} F(1)$$

in $\mathbb{M}$ is

$$U^n \xrightarrow[f^U]{} U$$

in $\mathbb{M}^{\mathsf{op}}$, and that $\mathsf{Hom}(F(n), A) = A^\sharp(U^n)$ and $f^* = A^\sharp(f^U)$. Thus (1.12) shows that indeed $A^\sharp(f^U) = f^A$, up to isomorphism.

Thus, as algebras, $A^\sharp(U) \cong A$, as required.

We leave it to the reader to verify that any homomorphism $h : F(U) \to G(U)$ of $\mathbb{T}$-algebras $F(U), G(U)$ arising from FP-functors $F, G : \mathbb{M}^{\mathsf{op}} \to \mathsf{Set}$ is of the form $h = \vartheta_U$ for a unique natural transformation $\vartheta : F \to G$.

Exercise 1.2.2. Show this.

Step 2. Let $\mathbb{C}$ be any (locally small) category, and $\mathcal{A}$ a $\mathbb{T}$-algebra in $\mathsf{Set}^{\mathbb{C}}$. Since

$$\mathsf{Mod}(\mathbb{T}, \mathsf{Set}^{\mathbb{C}}) \cong \mathsf{Mod}(\mathbb{T})^{\mathbb{C}},$$

each $\mathcal{A}(C)$ is a $\mathbb{T}$-algebra (in Set), which by Step 1 has a classifying functor,

$$\mathcal{A}(C)^\sharp : \mathbb{M}^{\mathsf{op}} \to \mathsf{Set}.$$

Together, these determine a single functor $\mathcal{A}^\sharp : \mathbb{M}^{\mathsf{op}} \to \mathsf{Set}^{\mathbb{C}}$, defined on $U \in \mathbb{M}^{\mathsf{op}}$ by:

$$(\mathcal{A}^\sharp(U))(C) \cong \mathcal{A}(C) = \mathcal{A}(C)^\sharp(U),$$

and on U^n by

$$(\mathcal{A}^\sharp(U^n))(C) \cong \mathcal{A}(C)^n = \mathcal{A}(C)^\sharp(U^n).$$

The functor $\mathcal{A}^\sharp(U) : \mathbb{C} \to \mathsf{Set}$ acts on an arrow $g : C \to D$ in $\mathbb{C}$ as indicated in the diagram:

$$\begin{tikzcd}
\mathcal{A}^\sharp(U)(C) \arrow[r, "\mathcal{A}^\sharp(U)(g)"] \arrow[d, "\cong"'] & \mathcal{A}^\sharp(U)(C) \arrow[d, "\cong"] \\
\mathcal{A}(C) \arrow[r, "\mathcal{A}(g)"'] & \mathcal{A}(D).
\end{tikzcd} \tag{1.13}$$

The case of U^n is precisely analogous. Finally, the action of $\mathcal{A} : \mathbb{M}^{\mathsf{op}} \to \mathsf{Set}^{\mathbb{C}}$ on arrows $U^n \to U^m$ in $\mathbb{M}^{\mathsf{op}}$ is similarly determined pointwise, i.e. by the components

$$(\mathcal{A}^\sharp(U^n))(C) \cong \mathcal{A}(C)^\sharp(U^n) \to \mathcal{A}(C)^\sharp(U^m) = (\mathcal{A}^\sharp(U^m))(C),$$

for all $C \in \mathbb{C}$.

Step 3. For the general case, let $\mathcal{C}$ be any (locally small) FP-category, and A a $\mathbb{T}$-algebra in $\mathcal{C}$. Use the Yoneda embedding

$$\mathsf{y} : \mathcal{C} \hookrightarrow \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$$

to send A to an algebra $\mathcal{A} = \mathsf{y}(A)$ in $\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$ (since y preserves finite products). Now apply Step 2 to get a classifying functor,

$$\mathcal{A}^\sharp : \mathbb{M}^{\mathsf{op}} \to \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}.$$

We claim that $\mathcal{A}^\sharp$ factors through the Yoneda embedding,

$$\begin{tikzcd}
 & \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}} \\
\mathbb{M}^{\mathsf{op}} \arrow[ur, "\mathcal{A}^\sharp"] \arrow[r, dotted, "A^\sharp"'] & \mathcal{C}. \arrow[u, hook, "\mathsf{y}"']
\end{tikzcd}$$

Indeed, we know that the objects of $\mathbb{M}^{\mathsf{op}}$ all have the form U^n, so their images

$$\mathcal{A}^\sharp(U^n) \cong \mathcal{A}^n \cong \mathsf{y}(A)^n \cong \mathsf{y}(A^n)$$

are all representable. Since y is full and faithful, the claim is established, and the resulting functor $A^\sharp : \mathbb{M}^{\mathsf{op}} \to \mathcal{C}$ preserves finite products because $\mathcal{A}^\sharp$ does so, and y creates them. Clearly,

$$A^\sharp(U) \cong A.$$

Naturality of the equivalence

$$\mathsf{Hom}_{\mathrm{FP}}(\mathbb{M}^{\mathsf{op}}, \mathcal{C}) \simeq \mathsf{Mod}(\mathbb{T}, \mathcal{C}),$$

in $\mathcal{C}$ is essentially automatic, using the fact that it is induced by evaluating an FP functor $F : \mathbb{M}^{\mathsf{op}} \to \mathcal{C}$ at the universal model U in $\mathbb{M}^{\mathsf{op}}$. $\square$

As already mentioned, since the classifying category is uniquely determined, up to equivalence, by its universal property, combining the foregoing theorem with the syntactic construction of $\mathcal{C}_{\mathbb{T}}$ given in theorem 1.1.19 yields the following:

Corollary 1.2.3 (Logical duality for algebraic theories)**.** *For any algebraic theory* $\mathbb{T}$*, there is an equivalence,*

$$\mathcal{C}_{\mathbb{T}} \simeq \mathsf{Mod}_{\mathrm{fg}}(\mathbb{T})^{\mathsf{op}} \tag{1.14}$$

between the syntactic category $\mathcal{C}_{\mathbb{T}}$ *and the opposite of the category* $\mathsf{Mod}_{\mathrm{fg}}(\mathbb{T})$ *of finitely generated, free models.*

Thus the syntactic construction of the classifying category $\mathcal{C}_{\mathbb{T}}$, on the one hand, and the semantic construction of it as $\mathsf{Mod}_{\mathrm{fg}}(\mathbb{T})^{\mathsf{op}}$, taken together imply that there is an invariant representation of the syntax of $\mathbb{T}$ just sitting there, as it were, in the opposite of the semantics $\mathsf{Mod}(\mathbb{T})$. In section 1.2.4 below, we shall consider how to actually *recover* this category $\mathcal{C}_{\mathbb{T}}$ from the semantics $\mathsf{Mod}(\mathbb{T})$, by identifying the subcategory $\mathsf{Mod}_{\mathrm{fg}}(\mathbb{T})$ intrinsically.

Before doing so, however, let us examine the equivalence (1.14) explicitly in a very special case: the “empty” theory $\mathbb{T}_0$ with no basic operations or equations. A model of this theory in Set is just a set X, equipped with no operations, and satisfying no further conditions (and similarly in any other FP category). Thus $\mathbb{T}_0$ is the pure theory of equality on an object.

All $\mathbb{T}_0$-algebras are free, and the finitely generated ones are the finite sets, so

$$\mathsf{Mod}_{\mathrm{fg}}(\mathbb{T}_0) = \mathsf{Set}_{\mathrm{fin}},$$

is the category of finite sets (to be more specific, let us take one n-element set $[n]$ for each $n \in \mathbb{N}$). Our theorem 1.2.1 tells us that, for any FP category $\mathcal{C}$, there is an equivalence

$$\mathsf{Hom}_{\mathrm{FP}}(\mathsf{Set}_{\mathrm{fin}}^{\mathsf{op}}, \mathcal{C}) \simeq \mathsf{Mod}(\mathbb{T}_0, \mathcal{C}) \simeq \mathcal{C}.$$

This simply says that $\mathsf{Set}_{\mathrm{fin}}^{\mathsf{op}}$ is the free FP category on one object. Equivalently, $\mathsf{Set}_{\mathrm{fin}}$ is the free finite *coproduct* category on one object. This is indeed the case, as can easily be seen directly (the objects are $0, 1, 1+1, 1+1+1, \dots$).

The duality of corollary 1.2.3 now tells us that the dual to the category of finite sets is the *syntactic category* of the theory of equality $\mathbb{T}_0$. The terms of this theory are simply tuples of variables $(x_1, \dots, x_n)$, and the valid equations are those that are true of them as terms, like $(x_2, x_5) = (x_2, x_5)$. Our corollary tells us that this is the category of finite sets, if we read the contexts $[x_1, \dots, x_n]$ as coproducts $1 + \cdots + 1$ and the tuples $(x_1, \dots, x_n)$ as *cotuples* $[x_1, \dots, x_n] : 1 + \cdots + 1 \to 1$, etc.

Example 1.2.4. For a less trivial example, consider the theory $\mathbb{T}_{\mathsf{Ab}}$ of abelian groups. Duality tells us that the syntactic category $\mathcal{C}_{\mathbb{T}_{\mathsf{Ab}}}$ is dual to the category of finitely generated, free abelian groups $\mathsf{Ab}_{\mathrm{fg}}$,

$$\mathcal{C}_{\mathbb{T}_{\mathsf{Ab}}} \simeq \mathsf{Ab}_{\mathrm{fg}}^{\mathsf{op}}.$$

This gives us a representation of the syntax of (abelian) group theory in the category of abelian groups, which is summarized as follows:

- the basic types of variables $[-] = 1,\ [x_1] = U,\ [x_1, x_2] = U \times U, \dots$ are represented by the groups $\{0\},\ \mathbb{Z},\ \mathbb{Z} + \mathbb{Z} = \mathbb{Z}^2, \dots,$
- the group unit $0 : 1 \to U$ is the (unique) homomorphism $0 : \mathbb{Z} \to \{0\}$,
- the inverse $i : U \to U$ is the (unique) homomorphism $- : \mathbb{Z} \to \mathbb{Z}$ taking 1 to -1,
- the group operation $U \times U \to U$ is the (unique) homomorphism $+ : \mathbb{Z} \to \mathbb{Z} + \mathbb{Z}$ taking 1 to $\langle 1, 1\rangle = \langle 1, 0\rangle + \langle 0, 1\rangle$ (using $\mathbb{Z} + \mathbb{Z} \cong \mathbb{Z} \times \mathbb{Z}$),
- the laws of abelian groups (and no further ones!) hold under this interpretation, because the group structure on any abelian group A is induced by precomposing with these "co-operations", as indicated in the following diagram for the sum $a + b$ of elements $a, b \in A$.

$$\begin{tikzcd}
\mathbb{Z} + \mathbb{Z} \arrow[r, "{(a,b)}"] & A \\
\mathbb{Z} \arrow[u, "+"] \arrow[ur, "a+b"'] &
\end{tikzcd}$$

Example 1.2.5. The category of *affine schemes* is by definition the dual of the category of commutative rings with unit,

$$\mathsf{Scheme}_{\mathrm{aff}} = \mathsf{Ring}^{\mathsf{op}}$$

There is therefore a ring object in affine schemes – called the *affine line* – based on the finitely generated free algebra $F(1) = \mathbb{Z}[x]$, the ring of polynomials in one variable x with integer coefficients. The "co-operations" of $+$ and $\cdot$ are given in rings by the homomorphisms $\mathbb{Z}[x] \to \mathbb{Z}[x, y]$ taking the generator x to the elements $x + y$ and $x \cdot y$.

Exercise 1.2.6. Prove directly that $\mathsf{Set}_{\mathrm{fin}}$ is the free finite coproduct category on one object.

Exercise 1.2.7. Show that for any algebraic theory $\mathbb{T}$, the forgetful functor $V : \mathsf{Mod}(\mathbb{T}) \to \mathsf{Set}$ is an algebra in the functor category $\mathsf{Set}^{\mathsf{Mod}(\mathbb{T})}$. In more detail, each n-ary operation f determines a natural transformation $f^V : V^n \to V$, since the homomorphisms in $\mathsf{Mod}(\mathbb{T})$ commute with the various operations interpreting f. Indeed, given any algebra A we have the underlying set $V(A) = |A|$ and an operation $f^A : |A|^n \to |A|$, and for every homomorphism $h : A \to B$ to another algebra B, there is a commutative square,

$$\begin{tikzcd}
{|A|^n} \arrow[r, "{|h|^n}"] \arrow[d, "f^A"'] & {|B|^n} \arrow[d, "f^B"] \\
{|A|} \arrow[r, "{|h|}"'] & {|B|.}
\end{tikzcd} \tag{1.15}$$

So we can set $(f^V)_A = f^A$ to get a natural transformation $f^V : V^n \to V$. Now check that this really is an algebra in $\mathsf{Set}^{\mathsf{Mod}(\mathbb{T})}$.

Exercise 1.2.8. * Show that the algebra described in the previous exercise is represented by the universal algebra $U = F(1)$ in $\mathbb{M}^{\mathsf{op}}$, in the sense of the (covariant) Yoneda embedding,

$$\mathsf{y} : \mathsf{Mod}(\mathbb{T})^{\mathsf{op}} \longrightarrow \mathsf{Set}^{\mathsf{Mod}(\mathbb{T})}.$$

1.2.2 Lawvere algebraic theories

Nothing in the foregoing account of duality for algebraic theories really depended on the primarily syntactic nature of such theories, i.e. their specification in terms of operations and equations. We can thus immediately generalize it to "abstract" algebraic theories, which can be regarded as providing a *presentation-free* notion of an algebraic theory.

Definition 1.2.9 (cf. Definition 1.1.2)**.** A *Lawvere algebraic theory* $\mathbb{A}$ is a small category with finite products whose objects form a sequence $A^0, A^1, A^2, \dots$ such that $A^m \times A^n = A^{m+n}$ for all $m, n \in \mathbb{N}$. In particular, $\mathbf{1} = A^0$ is the terminal object and every object is a product of finitely many copies of $A = A^1$.

A *model* of a Lawvere algebraic theory $\mathbb{A}$ in any category $\mathcal{C}$ with finite products is a finite-product-preserving functor $M : \mathbb{A} \to \mathcal{C}$, and a *homomorphism of models* is a natural transformation $\vartheta : M \to M'$.

We could just as well have taken the natural numbers $0, 1, 2, \dots$ themselves as the objects of a Lawvere algebraic theory $\mathbb{A}$, but the notation A^n is more suggestive. A Lawvere algebraic theory $\mathbb{A}$ in the sense of the above definition determines an algebraic theory in the sense of Definition 1.1.2 as follows. As basic operations with arity k we take all of the morphisms $A^k \to A$:

$$\Sigma(\mathbb{A})_k = \mathsf{Hom}_{\mathbb{A}}(A^k, A) \tag{1.16}$$

There is a canonical interpretation in $\mathbb{A}$ of terms built from variables and morphisms $A^k \to A$, namely each morphism is interpreted by itself, and variables are interpreted as product projects, as usual. An equation $u = v$ is taken as an axiom of the theory $\mathbb{A}$ if the canonical interpretations of u and v coincide. Of course, the conventional logical notions of model 1.1.10 and homomorphism of models then also correspond to the new, functorial ones in an obvious way.

This new, abstract view of algebraic theories immediately suggest some interesting examples.

Example 1.2.10. The algebraic theory $\mathcal{C}^\infty$ of smooth maps is the category whose objects are n-dimensional Euclidean spaces $\mathbf{1}$, $\mathbb{R}$, $\mathbb{R}^2$, ..., and whose morphisms are $\mathcal{C}^\infty$-maps between them. Recall that a $\mathcal{C}^\infty$-map $f : \mathbb{R}^n \to \mathbb{R}$ is a function which has all higher partial derivatives, and that a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is a $\mathcal{C}^\infty$-map when its compositions $\pi_k \circ f : \mathbb{R}^n \to \mathbb{R}$ with projections $\pi_k : \mathbb{R}^m \to \mathbb{R}$ are $\mathcal{C}^\infty$-maps.

A model of this theory in Set is a finite product preserving functor $A : \mathcal{C}^\infty \to \mathsf{Set}$. Up to natural isomorphism it can be described as follows. A $\mathcal{C}^\infty$-model is given by a set A and for every smooth map $f : \mathbb{R}^n \to \mathbb{R}$ a function $Af : A^n \to A$ such that if $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^m \to \mathbb{R}$, $i = 1, \ldots, n$, are smooth maps then, for all $a_1, \ldots, a_m \in A$,

$$Af\big((Ag_1)\langle a_1, \ldots, a_m\rangle, \ldots, (Ag_n)\langle a_1, \ldots, a_m\rangle\big) = A(f \circ \langle g_1, \ldots, g_n\rangle)\langle a_1, \ldots, a_m\rangle \; .$$

In particular, since multiplication and addition are smooth maps, A is a commutative ring with unit. Such structures are known as *$\mathcal{C}^\infty$-rings*. Therefore, the models in Set (cf. [MR91])

Example 1.2.11. Recall that a *(total) recursive function* $f : \mathbb{N}^m \to \mathbb{N}^n$ is one that can be computed by a Turing machine. This means that there exists a Turing machine which on input $\langle a_1, \ldots, a_m\rangle$ outputs the value of $f\langle a_1, \ldots, a_m\rangle$. The algebraic theory Rec of recursive functions is the category whose objects are finite powers of the natural numbers $\mathsf{1}$, $\mathbb{N}$, $\mathbb{N}^2$, …, and whose morphisms are recursive functions between them. The models of this theory in a category $\mathcal{C}$ with finite products give a theory of computability in $\mathcal{C}$.

Indeed, let us consider the category of its set-theoretic models $\mathbb{R} = \mathsf{Mod}_{\mathsf{Set}}(\mathsf{Rec})$. First, there is the "identity" model $I \in \mathbb{R}$, defined by $I\mathbb{N}^k = \mathbb{N}^k$ and $If = f$. Given any model $S \in \mathbb{R}$, its object part is determined by $S_1 = S\mathbb{N}$ since $S\mathbb{N}^k = S_1^k$. For every $n \in \mathbb{N}$ there is a morphism $\mathsf{1} \to \mathbb{N}$ in Rec defined by $\star \mapsto n$. Thus we have for each $n \in \mathbb{N}$ an element $s_n = S(\star \mapsto n) : \mathsf{1} \to S_1$. This defines a function $s : \mathbb{N} \to S_1$ which in turn determines a natural transformation $\sigma : I \Longrightarrow S$ whose component at $\mathbb{N}^k$ is $s \times \cdots \times s : \mathbb{N}^k \to S_1^k$.

Example 1.2.12. In a category $\mathcal{C}$ with finite products every object $A \in \mathcal{C}$ determines a full subcategory consisting of the finite powers $\mathsf{1}, A, A^2, A^3, \ldots$ and all morphisms between them. This is the *total theory* $\mathbb{T}(A)$ of the object A in $\mathcal{C}$.

Free algebras

In order to extend the duality theory of the foregoing section to the abstract case, we will require the notion of a *free model* of an abstract algebraic theory. Of course, we already have the conventional notion of free models determined in terms of the associated conventional algebraic theory given by (1.16). But we can also determine free models directly in terms of the abstract theory, which then of course applies to the conventional case as well.

Let $\mathbb{A}$ be a Lawvere algebraic theory, with objects $1, A, A^2, \ldots$. We have the category of models,

$$\mathsf{Mod}(\mathbb{A}) = \mathsf{Hom}_{\mathsf{FP}}(\mathbb{A}, \mathsf{Set}).$$

Let us first define the *forgetful functor* by,

$$U : \mathsf{Mod}(\mathbb{A}) \to \mathsf{Set} \tag{1.17}$$

$$(M : \mathbb{A} \to \mathsf{Set}) \mapsto M(A). \tag{1.18}$$

We shall also write
$$|M| = U(M) = M(A).$$

Now for the (finitary) free functor $F_{\text{fin}} : \mathsf{Set}_{\text{fin}} \to \mathsf{Mod}(\mathbb{A})$, we set:

$$\begin{aligned} F(0) &= \mathsf{Hom}_{\mathbb{A}}(1, -) \\ F(1) &= \mathsf{Hom}_{\mathbb{A}}(A, -) \\ &\vdots \\ F(n) &= \mathsf{Hom}_{\mathbb{A}}(A^n, -). \end{aligned}$$

Note that this is a composite of the two (contravariant) functors,

$$\mathsf{Set}_{\text{fin}} \to \mathbb{A}^{\mathsf{op}}, \quad \mathbb{A}^{\mathsf{op}} \to \mathsf{Mod}(\mathbb{A}),$$

given by $n \mapsto A^n$ and $X \mapsto \mathsf{Hom}_{\mathbb{A}}(X, -)$, and is therefore functorial. Note also that the representables $\mathsf{Hom}_{\mathbb{A}}(A^n, -)$ do indeed preserve finite products, and are therefore in the full subcategory $\mathsf{Mod}(\mathbb{A}) \hookrightarrow \mathsf{Set}^{\mathbb{A}}$.

For adjointness we need to check that:

$$\mathsf{Hom}_{\mathsf{Mod}(\mathbb{A})}(F(n), M) \cong \mathsf{Hom}_{\mathsf{Set}}(n, |M|) \tag{1.19}$$

(naturally in both arguments, of course). The right-hand side is plainly just

$$\mathsf{Hom}_{\mathsf{Set}}(n, |M|) \cong |M|^n.$$

For the left-hand side we have:

$$\begin{aligned} \mathsf{Hom}_{\mathsf{Mod}(\mathbb{A})}(F(n), M) &= \mathsf{Hom}_{\mathsf{Mod}(\mathbb{A})}(\mathsf{Hom}_{\mathbb{A}}(A^n, -), M) \\ &= \mathsf{Hom}_{\mathsf{Set}^{\mathbb{A}}}(\mathsf{Hom}_{\mathbb{A}}(A^n, -), M) \\ &\cong M(A^n) && \text{(by Yoneda)} \\ &\cong M(A)^n && (M \text{ is FP}) \\ &= |M|^n && (1.17). \end{aligned}$$

The full definition of the free functor

$$F : \mathsf{Set} \to \mathsf{Mod}(\mathbb{A})$$

is then given by writing an arbitrary set X as a (filtered) colimit of its finite subsets $X_i \subseteq X$, and then setting $F(X) = \operatorname{colim}_i F(X_i)$ in the category $\mathsf{Set}^{\mathbb{A}}$. Since filtered colimits commute with finite products, these colimits taken in $\mathsf{Set}^{\mathbb{A}}$ and will remain in $\mathsf{Mod}(\mathbb{A})$.

Theorem 1.2.13. *For any set X with free algebra $F(X)$ as just defined, there is a natural isomorphism,*
$$\mathsf{Hom}_{\mathsf{Mod}(\mathbb{A})}(F(X), M) \cong \mathsf{Hom}_{\mathsf{Set}}(X, |M|)\,. \tag{1.20}$$

Proof. The proof is an easy exercise. □

By definition, the finitely generated free models $F(n)$ are just the representables $\mathsf{Hom}_{\mathbb{A}}(A^n, -)$; therefore as the "semantic dual" $\mathsf{Mod}_{\mathrm{fg}}(\mathbb{A}) \hookrightarrow \mathsf{Mod}(\mathbb{A})$ of the theory $\mathbb{A}$, in the sense of corollary 1.2.3, we simply have the full subcategory of $\mathsf{Hom}_{\mathsf{FP}}(\mathbb{A}, \mathsf{Set})$ on the image of the Yoneda embedding,

$$\begin{tikzcd}
\mathsf{Mod}_{\mathrm{fg}}(\mathbb{A}) \arrow[r, hook] & \mathsf{Mod}(\mathbb{A}) = \mathsf{Hom}_{\mathrm{FP}}(\mathbb{A}, \mathsf{Set}) \arrow[r, hook] & \mathsf{Set}^{\mathbb{A}} \\
\mathbb{A}^{\mathsf{op}} \arrow[u, "\cong"] \arrow[rr, "="'] & & \mathbb{A}^{\mathsf{op}}. \arrow[u, hook, "\mathsf{y}"']
\end{tikzcd}$$

In the abstract case, then, the logical duality

$$\mathbb{A} \simeq \mathsf{Mod}_{\mathrm{fg}}(\mathbb{A})^{\mathsf{op}}$$

comes down to the fact that the (contravariant) Yoneda embedding

$$\mathbb{A}^{\mathsf{op}} \hookrightarrow \mathsf{Set}^{\mathbb{A}}$$

presents $\mathbb{A}$ as (the dual of) a full subcategory of (product-preserving!) functors. Thus we have shown:

Theorem 1.2.14. *For any Lawvere algebraic theory $\mathbb{A}$, there is an equivalence,*

$$\mathbb{A} \simeq \mathsf{Mod}_{\mathrm{fg}}(\mathbb{A})^{\mathsf{op}}$$

between $\mathbb{A}$ and the full subcategory of finitely generated free models.

Exercise 1.2.15. Prove theorem 1.2.13.

1.2.3 Algebraic categories

Given an *arbitrary* category $\mathcal{A}$, we may ask, when is $\mathcal{A}$ the category of models for some algebraic theory? Such categories are often called *varieties*, at least in universal algebra, and there are well-known "recognition theorems" such the famous "HSP-theorem" of Birkhoff, which says that a collection of algebas for some fixed signature are exactly those satisfyig a set of equations if the collection is closed under Products, Subalgebras, and Homomorphic images (i.e. quotient algebras). Toward the goal of "recognizing" a *category* of algebras (without even being given the signature), let us define:

Definition 1.2.16. An *algebraic category* $\mathcal{A}$ is a (locally small) category equivalent to one of the form

$$\mathsf{Hom}_{\mathsf{FP}}(\mathbb{A}, \mathsf{Set}) \hookrightarrow \mathsf{Set}^{\mathbb{A}}$$

where $\mathbb{A}$ is any small finite product category and $\mathsf{Hom}_{\mathsf{FP}}(\mathbb{A}, \mathsf{Set})$ is the full subcategory of finite product preserving, set-valued functors and natural transformations. If $\mathbb{A}$ is a Lawvere algebraic theory (i.e. the objects are generated under finite products by a single object), then we will say that $\mathcal{A}$ is a *Lawvere algebraic category.*

If $\mathcal{A}$ is the category of models of a Lawvere algebraic theory $\mathbb{A}$, with generating object A, then in particular there will be a forgetful functor

$$U : \mathcal{A} \to \mathsf{Set},$$

determined by evaluation at A, and we know moreover that U preserves all limits, and one can show without too much difficulty that it also preserves all filtered colimits (cf. exercise 1.2.21). We require only one further condition to "recognize" $\mathcal{A}$ as algebraic, namely creation of "U-absolute coequalizers".

Definition 1.2.17. In any category $\mathcal{C}$, a coequalizer $c : Y \to Z$ of maps $a, b : X \rightrightarrows Y$ is *absolute* if, for every category $\mathcal{D}$ and functor $F : \mathcal{C} \to \mathcal{D}$, the image $Fc : FY \to FZ$ is a coequalizer of the images $Fa, Fb : FX \rightrightarrows FY$. A functor $F : \mathcal{C} \to \mathcal{D}$ *creates F-absolute coequalizers* if for every parallel pair of maps $a, b : X \rightrightarrows Y$ in $\mathcal{C}$ and absolute coequalizer $q : FY \to Q$ for $Fa, Fb : FX \rightrightarrows FY$ in $\mathcal{D}$, there is a unique object Z and map $c : Y \to Z$ in $\mathcal{C}$ with $FZ = Q$ and $Fc = q$, which, moreover, is a coequalizer in $\mathcal{C}$.

$$\begin{tikzcd}
\mathcal{C} \arrow[d, "F"'] & X \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & Y \arrow[r, dotted, "c"] & Z \\
\mathcal{D} & FX \arrow[r, shift left, "Fa"] \arrow[r, shift right, "Fb"'] & FY \arrow[r, "q"'] & Q
\end{tikzcd} \tag{1.21}$$

Theorem 1.2.18. *Given a category $\mathcal{A}$, equipped with a functor $U : \mathcal{A} \to \mathsf{Set}$, the following conditions are equivalent.*

1. *$\mathcal{A}$ is a Lawvere algebraic category; i.e. there is a Lawvere algebraic theory $\mathbb{A}$, and an equivalence,*
$$\mathcal{A} \simeq \mathsf{Hom}_{\mathsf{FP}}(\mathbb{A}, \mathsf{Set}) \hookrightarrow \mathsf{Set}^{\mathbb{A}}$$
between $\mathcal{A}$ and the full subcategory of finite product preserving functors on $\mathbb{A}$, associating $U : \mathcal{A} \to \mathsf{Set}$ to the evaluation at the generating object of $\mathbb{A}$.

2. *$U : \mathcal{A} \to \mathsf{Set}$ has a left adjoint $F : \mathsf{Set} \to \mathcal{A}$, preserves all (small) filtered colimits, and creates U-absolute coequalizers.*

3. *$\mathcal{A}$ is monadic over Set (via $U : \mathcal{A} \to \mathsf{Set}$),*
$$\mathcal{A} \simeq \mathsf{Set}^T$$
for a finitary monad $T : \mathsf{Set} \to \mathsf{Set}$.

Proof. (1⇒2) Suppose first that $\mathcal{A}$ is (Lawvere) algebraic, so

$$\mathcal{A} \simeq \mathsf{Mod}(\mathbb{A}) = \mathsf{Hom}_{\mathsf{FP}}(\mathbb{A}, \mathsf{Set}) \hookrightarrow \mathsf{Set}^{\mathbb{A}}$$

for an algebraic theory $\mathbb{A}$. First, observe that U preserves all limits: since these are computed pointwise in $\mathsf{Set}^{\mathbb{A}}$ and U is evaluation at the generating object of $\mathbb{A}$, it suffices to show that the limit of a diagram of FP functors, calculated in $\mathsf{Set}^{\mathbb{A}}$, is again an FP functor. But this is true because limits commute with finite products. The same argument works for filtered colimits in place of limits, since these also commute with finite products. Thus U preserves both limits and filtered colimits.

We also know by theorem 1.2.13 that we can construct a free algebra $F(X)$ for any set X, so (by the adjoint functor theorem), U has a left adjoint $F : \mathsf{Set} \to \mathcal{A}$ (we could also have used the fact that U has a left adjoint to infer that it preserves limits).

For creation of U-absolute coequalizers, suppose we have maps $f, g : A \rightrightarrows B$ in $\mathcal{A}$ and an absolute coequalizer $c : UB \to C$ for $Uf, Ug : UA \rightrightarrows UB$ in Set; we want to put an algebra structure on C making c a homomorphism $c : B \to C$ in $\mathcal{A}$ and a coequalizer of f and g.

$$\begin{tikzcd} UA^n \arrow[r, shift left, "Uf^n"] \arrow[r, shift right, "Ug^n"'] \arrow[d, "\sigma^A"'] & UB^n \arrow[r, "c^n"] \arrow[d, "\sigma^B"] & C^n \arrow[d, dotted, "\sigma^C"] \\ UA \arrow[r, shift left, "Uf"] \arrow[r, shift right, "Ug"'] & UB \arrow[r, "c"'] & C \end{tikzcd} \tag{1.22}$$

For each function symbol $\sigma \in \Sigma$ we have commutative squares as on the left in the above diagram, because f and g are homomorphisms. It follows by a simple diagram chase that $c \circ \sigma^B$ coequalizes the pair $Uf^n, Ug^n : UA^n \rightrightarrows UB^n$. Since $c : UB \to C$ is absolute, it is preserved by the functor $(-)^n$, and therefore $c^n : UB^n \to C^n$ is a coequalizer of Uf^n, Ug^n. There is therefore a unique map $\sigma^C : C^n \to C$ as indicated, making the right hand square commute. Doing this for each $\sigma \in \Sigma$ gives an interpretation of Σ on C. This is seen to be an algebra structure because the maps c^n are surjections. Thus $c : B \to C$ is a homomorphism, which is easily seen to be a coequalizer in $\mathcal{A}$.

(2⇒3) Taking the standard monad (T, η, μ) on Set with underlying functor $T = U \circ F$, we want to show that the canonical comparison map

$$\mathcal{A} \to \mathsf{Set}^T$$

to the category of T-algebras is an isomorphism. This follows from the condition that U creates absolute coequalizers by Beck's theorem; see [Lan71, VI.7]. Moreover, T preserves filtered colimits (i.e. is "finitary") because each of F and U do so.

(3⇒1) Let (T, η, μ) be a finitary monad on Set and $U : \mathsf{Set}^T \to \mathsf{Set}$ the forgetful functor from the category of T-algebras. We want an algebraic theory $\mathbb{A}$ and an equivalence

$$\mathsf{Set}^T \simeq \mathsf{Mod}(\mathbb{A})$$

over U and evaluation at the generator of $\mathbb{A}$, where recall $\mathsf{Mod}(\mathbb{A}) = \mathsf{Hom}_{\mathsf{FP}}(\mathbb{A}, \mathsf{Set})$. Let

$$\mathbb{A} = \mathsf{FGF}(\mathsf{Set}^T)^{\mathsf{op}} \tag{1.23}$$

be the dual of the full subcategory of finitely generated free T-algebras. The objects of $\mathbb{A}$ are of the form $T_0, T_1, T_2, ...$ where $T_n = T(n)$, equipped with the multiplication $\mu_n : T^2(n) \to T(n)$ as algebra structure map. Since, as free algebras, $T(n+m) \cong T(n) + T(m)$ we indeed have $T_n \times T_m \cong T_{n+m}$ as objects of $\mathbb{A}$, and T_1 as the generating object.
By the first two steps of this proof, we know that the algebraic category $\mathsf{Mod}(\mathbb{A})$ is also (finitary) monadic,

$$\mathsf{Mod}(\mathbb{A}) \simeq \mathsf{Set}^M,$$

with monad $M = U_M \circ F_M$, where $F_M \dashv U_M$ is the free-forgetful adjunction for $\mathsf{Mod}(\mathbb{A}) = \mathsf{Hom}_{\mathsf{FP}}(\mathbb{A}, \mathsf{Set})$, and $U_M \cong \mathsf{eval}_{T_1}$. Thus it will suffice to show that $M \cong T$, as monads, in order to conclude that

$$\mathsf{Mod}(\mathbb{A}) \simeq \mathsf{Set}^M \simeq \mathsf{Set}^T.$$

Moreover, since both M and T are finitary, it suffices to show that their respective restrictions to the dense subcategory $\mathsf{Set}_{\mathsf{fin}} \hookrightarrow \mathsf{Set}$ are isomorphic. By (1.19), we know that the finite free functor $F_M(n)$ has the form

$$F_M(n) = \mathsf{Hom}_{\mathbb{A}}(T_n, -) = \mathsf{Hom}_{\mathsf{FGF}(\mathsf{Set}^T)}(-, \langle T(n), \mu_n \rangle)$$

thus using the fact that $U_M \cong \mathsf{eval}_{T_1}$ we see that

$$\begin{aligned} M(n) = U_M(F_M(n)) &= U_M\big(\mathsf{Hom}_{\mathsf{FGF}(\mathsf{Set}^T)}(-, \langle T(n), \mu_n \rangle)\big) \\ &\cong \mathsf{Hom}_{\mathsf{FGF}(\mathsf{Set}^T)}(\langle T(1), \mu_1 \rangle, \langle T(n), \mu_n \rangle) \\ &\cong \mathsf{Hom}_{\mathsf{Set}}(1, T(n)) \cong T(n). \end{aligned}$$

□

Remark 1.2.19. Another "recognition theorem" that can be found in [Bor94] is the following:

Theorem (Borceux II.3.9)**.** *Given a category $\mathcal{A}$, equipped with a functor $U : \mathcal{A} \to \mathsf{Set}$, the following conditions are equivalent.*

1. *$\mathcal{A}$ is equivalent to the category of models of some algebraic theory $\mathbb{T}$,*

$$\mathcal{A} \simeq \mathsf{Mod}(\mathbb{T})$$

   with $U : \mathcal{A} \to \mathsf{Set}$ the corresponding forgetful functor.

2. *$\mathcal{A}$ has coequalizers and kernel pairs, and $U : \mathcal{A} \to \mathsf{Set}$ has a left adjoint $F : \mathsf{Set} \to \mathcal{A}$, preserves all (small) filtered colimits and regular epimorphisms, and reflects isomorphisms.*

Condition (1) is of course equivalent to condition (1) or our theorem 1.2.18, by theorem 1.2.1.

Exercise 1.2.20. A *split coequalizer* for maps $f, g : A \rightrightarrows B$ is a map $e : B \to C$ together with s and t as indicated below,

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "e"] \arrow[l, bend left, "t"] & C \arrow[l, bend left, "s"] \end{tikzcd} \tag{1.24}$$

satisfying the equations

$$ef = eg, \quad ft = 1_B, \quad gt = se, \quad es = 1_C.$$

Show that a split coequalizer is an absolute coequalizer.

Exercise 1.2.21. A filtered colimit of algebras can be described directly as follows: First consider the case of sets. Let the index category $\mathbb{J}$ be filtered and $D : \mathbb{J} \to \mathsf{Set}$ a diagram. The colimiting set $\mathrm{colim}_j\, D_j$ can be described as the quotient of the coproduct $(\coprod_j Dj)/\sim$, where the equivalence relation $\sim$ is defined by:

$$(d_i \in D_i) \sim (d_j \in D_j) \;\Leftrightarrow\; t_{ik}(d_i) = t_{jk}(d_j) \text{ for some } t_{ik} : i \to k \text{ and } t_{jk} : j \to k \text{ in } \mathbb{J}.$$

1. Show that this is an equivalence relation using the filteredness of $\mathbb{J}$.
2. Now assume that the D_j all have an algebra structure and that all the transition maps $t_{ik} : D_i \to D_k$ are homomorphisms. Show that the colimit set $D_\infty = \mathrm{colim}_j\, D_j$ is also an algebra of the same kind by defining each of the operations $\sigma_\infty : D_\infty \times \ldots \times D_\infty \to D_\infty$ on equivalence classes as
$$\sigma_\infty\langle [d_i], \ldots, [d'_j] \rangle = [\sigma_k \langle t_{ik}(d_i), \ldots, t_{jk}(d'_j) \rangle]$$
for suitable k. Show that this is well-defined, and that D_∞, so equipped, also satisfies the equations satisfied by the D_j.

Example 1.2.22. A field is a ring in which every non-zero element has a multiplicative inverse. The theory of fields is (apparently) not algebraic, because the axiom

$$x \neq 0 \;\Rightarrow\; \exists y (x \cdot y = 1)$$

is not simply an equation, but in principle there could be an equivalent algebraic formulation of the theory which would somehow circumvent this problem. We can show that this is not the case by proving that the category Field of fields and field homomorphisms is not algebraic.

First observe that a category of models $\mathsf{Mod}(\mathbb{A})$ always has a terminal object because Set has a terminal object 1, and the constant functor $\Delta_1 : \mathbb{A} \to \mathsf{Set}$ which maps everything to 1 is a model. The functor Δ_1 is the terminal object in $\mathsf{Mod}(\mathbb{A})$ because it is the terminal functor in the functor category $\mathsf{Set}^{\mathbb{A}}$. Now in order to see that Field is not algebraic it suffices to show that there is no terminal field.

Exercise 1.2.23. Show that the category Field does not have a terminal object. (Hint: suppose that T is the terminal field and use the unique homomorphism $\mathbb{Z}_2 \to T$ to see that $1 + 1 = 0$ in T, then reason similarly using the unique homomorphism $\mathbb{Z}_3 \to T$.)

1.2.4 Definability and duality

A *syntactic translation* of one algebraic theory into another is an assignment of types to types and terms to terms, and thus can be represented as a finite product preserving functor,

$$T : \mathbb{A} \to \mathbb{B}$$

between the associated Lawvere algebraic theories. Every such translation induces a "definable" functor on the semantics:

$$T^*(M) = M \circ T.$$

$$\mathsf{Mod}(\mathbb{A}) \xleftarrow{T^*} \mathsf{Mod}(\mathbb{B}) \tag{1.25}$$

$$\begin{tikzcd} \mathbb{A} \arrow[r, "T"] \arrow[dr, "T^*M"'] & \mathbb{B} \arrow[d, "M"] \\ & \mathsf{Set} \end{tikzcd}$$

For instance, let $\mathbb{F} = (\mathsf{Set}_{\mathrm{fin}})^{\mathsf{op}}$, the "empty" theory of an object, so that $\mathsf{Mod}(\mathbb{F}) = \mathsf{Set}$. Then the object underlying the universal model $U \in \mathbb{A}$ has a classifying functor

$$U : \mathbb{F} \to \mathbb{A}$$

which induces the forgetful functor by precomposition,

$$\mathsf{Set} = \mathsf{Mod}(\mathbb{F}) \xleftarrow{U^*} \mathsf{Mod}(\mathbb{A})$$

$$\begin{tikzcd} \mathbb{F} \arrow[r, "U"] \arrow[dr, "U^*M"'] & \mathbb{A} \arrow[d, "M"] \\ & \mathsf{Set} \end{tikzcd}$$

More generally, every translation corresponds to a "model in syntax":

$$\frac{T : \mathbb{A} \to \mathbb{B}}{\hat{T} \in \mathsf{Mod}(\mathbb{A}, \mathbb{B})}$$

by the universal property of $\mathbb{A}$. For instance, since every ring R has an underlying group $\mathsf{Grp}(R)$, the universal ring $U_{\mathbb{R}}$ in $\mathbb{R}$ has one $\mathsf{Grp}(U_{\mathbb{R}})$, which is classified by a unique functor from the theory of groups,

$$\mathsf{Grp}(U_{\mathbb{R}}) : \mathbb{G} \to \mathbb{R}.$$

This translation induces a functor on the corresponding categories of models,

$$\mathsf{Grp}(U_{\mathbb{R}})^* : \mathsf{Group} \leftarrow \mathsf{Ring}, \tag{1.26}$$

which of course is just the forgetful functor $\mathsf{Grp} : \mathsf{Ring} \to \mathsf{Group}$.

Now let us ask, which functors $f : \mathsf{Mod}(\mathbb{B}) \to \mathsf{Mod}(\mathbb{A})$ between algebraic categories are of the form $f = T^*$ for a translation $T : \mathbb{A} \to \mathbb{B}$ of theories? Let us call these *algebraic functors*. We consider first the case where T takes the generator A_1 of $\mathbb{A}$ to the generator B_1 of $\mathbb{B}$,

$$T(A_1) \cong B_1.$$

Then T^* commutes with the forgetful functors, which, recall, are evaluation at the generators, $U_{\mathbb{A}}(M) = M(A_1)$, etc.:

$$\begin{tikzcd}
\mathsf{Mod}(\mathbb{B}) \arrow[rr, "T^*"] \arrow[dr, "U_{\mathbb{B}}"'] & & \mathsf{Mod}(\mathbb{A}) \arrow[dl, "U_{\mathbb{A}}"] \\
& \mathsf{Set} &
\end{tikzcd}$$

because:

$$(U_{\mathbb{A}} \circ T^*)(M) = U_{\mathbb{A}}(M \circ T) = (M \circ T)(A_1) \cong M(T(A_1)) \cong M(B_1) = U_{\mathbb{B}}(M).$$

In fact, this condition is already sufficient!

Proposition 1.2.24. *Given any functor* $f : \mathsf{Mod}(\mathbb{B}) \to \mathsf{Mod}(\mathbb{A})$ *with*

$$U_{\mathbb{B}} \cong U_{\mathbb{A}} \circ f,$$

there is a unique (up to iso) translation $T : \mathbb{A} \to \mathbb{B}$ *such that*

$$f \cong T^*.$$

Proof. (Cf. [Bor94, 3.9.2]) Consider the diagram:

$$\begin{tikzcd}
\mathsf{Mod}(\mathbb{B}) \arrow[rr, "f"] \arrow[dr, shift left, "U_{\mathbb{B}}"] & & \mathsf{Mod}(\mathbb{A}) \arrow[dl, shift right, "U_{\mathbb{A}}"'] \\
& \mathsf{Set} \arrow[ul, shift left, "F_{\mathbb{B}}"] \arrow[ur, shift right, "F_{\mathbb{A}}"'] &
\end{tikzcd}$$

where in each pair $F \dashv U$. We seek an FP-functor $T : \mathbb{A} \to \mathbb{B}$; thus, using the explicit description of $\mathbb{A}$ as the subcategory of finitely generated free models $\mathsf{Mod}_{\mathrm{fg}}(\mathbb{A})^{\mathsf{op}}$ (and the same for $\mathbb{B}$), it suffices to give a functor T as indicated in:

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "T"] \arrow[d, "\cong"'] & \mathbb{B} \arrow[d, "\cong"'] \\
\mathsf{Mod}_{\mathrm{fg}}(\mathbb{A})^{\mathsf{op}} \arrow[r, dotted] & \mathsf{Mod}_{\mathrm{fg}}(\mathbb{B})^{\mathsf{op}}
\end{tikzcd}$$

So we need a functor T with:

$$T(F_{\mathbb{A}}(n)) \cong F_{\mathbb{B}}(n).$$

Consider the following construction: beginning with the unit $\eta_n : n \to U_{\mathbb{B}} F_{\mathbb{B}}(n)$, we have a natural bijection

$$\frac{\eta_n : n \to U_{\mathbb{B}} F_{\mathbb{B}}(n) = U_{\mathbb{A}} f U_{\mathbb{B}}}{h_n : F_{\mathbb{A}}(n) \to f F_{\mathbb{B}}(n)}$$

The pair $(F_{\mathbb{A}}(n), h_n)$ has the following universal property: for any $M \in \mathsf{Mod}(\mathbb{B})$ and $k : F_{\mathbb{A}}(n) \to fM$, there is a unique "lift" $\bar{k} : F_{\mathbb{B}}(n) \to M$ with $k = f(\bar{k}) \circ h_n$, as indicated in the following.

$$\begin{tikzcd}
\mathsf{Mod}(\mathbb{B}) \arrow[d, "f"'] & F_{\mathbb{B}}(n) \arrow[r, "\bar{k}"] & M \\
\mathsf{Mod}(\mathbb{A}) & f F_{\mathbb{B}}(n) \arrow[r, "f\bar{k}"] & fM \\
& F_{\mathbb{A}}(n) \arrow[u, "h_n"] \arrow[ur, "k"'] &
\end{tikzcd}$$

Indeed, we have natural bijections:

$$\frac{\dfrac{k : F_{\mathbb{A}}(n) \longrightarrow fM}{k' : n \longrightarrow U_{\mathbb{A}} fM \cong U_{\mathbb{B}} M}}{\bar{k} : F_{\mathbb{B}}(n) \longrightarrow M} \tag{1.27}$$

Now, to define T, take any $\alpha : F_{\mathbb{A}}(n) \to F_{\mathbb{A}}(m)$ and use

$$\begin{tikzcd}
& f F_{\mathbb{B}}(m) \\
F_{\mathbb{A}}(n) \arrow[ur, "h_m \alpha"] \arrow[r, "\alpha"'] & F_{\mathbb{A}}(m) \arrow[u, "h_m"']
\end{tikzcd}$$

to get the lift:

$$T(\alpha) := \overline{h_m \alpha} : F_{\mathbb{B}}(n) \longrightarrow F_{\mathbb{B}}(m).$$

This assignment is easily seen to be functorial. Finally, to show that it is induced by precomposing with $T : \mathbb{A} \to \mathbb{B}$, we need the following to commute up to natural isomorphism.

$$\begin{tikzcd}
\mathsf{Mod}(\mathbb{B}) \arrow[r, "f"] \arrow[d, "\cong"'] & \mathsf{Mod}(\mathbb{A}) \arrow[d, "\cong"'] \\
\mathsf{FP}(\mathbb{B}, \mathsf{Set}) \arrow[r, "T^*"'] & \mathsf{FP}(\mathbb{A}, \mathsf{Set})
\end{tikzcd}$$

Identifying $\mathbb{B} = \mathsf{Mod}_{\mathrm{fg}}(\mathbb{B})^{\mathsf{op}}$ and $\mathbb{A} = \mathsf{Mod}_{\mathrm{fg}}(\mathbb{A})^{\mathsf{op}}$, going aroung the Southwest, we have

$$M \mapsto \mathsf{Hom}_{\mathsf{Mod}(\mathbb{B})}(-, M) \mapsto \mathsf{Hom}_{\mathsf{Mod}(\mathbb{A})}(T(-), M).$$

Going around the Northeast gives

$$M \mapsto fM \mapsto \mathsf{Hom}(-, fM).$$

But by (1.27), we have a natural bijection:

$$\frac{F_{\mathbb{A}}(n) \to fM}{T(F_{\mathbb{A}}(n)) = F_{\mathbb{B}}(n) \to M}$$

□

Corollary 1.2.25. *For a functor $f : \mathsf{Mod}(\mathbb{B}) \to \mathsf{Mod}(\mathbb{A})$ between algebraic categories, the following are equivalent.*

1. *f is algebraic, $f = T^*$, for some FP functor $T : \mathbb{A} \to \mathbb{B}$ which, moreover, preserves the generator, $T(A_1) \cong B_1$*
2. *f commutes with the forgetful functors, $U_{\mathbb{A}} \circ f \cong U_(B)$*

□

Exercise 1.2.26. Show that for any algebraic theory $\mathbb{A}$, the full inclusion $\mathsf{Mod}(\mathbb{A}) \hookrightarrow \mathsf{Set}^{\mathbb{A}}$ has a left adjoint. (Hint: use the Adjoint Functor Theorem.)

Exercise 1.2.27. Assuming the result of the previous exercise, show that the precomposition functor $T^* : \mathsf{Mod}(\mathbb{B}) \longrightarrow \mathsf{Mod}(\mathbb{A})$ induced by any translation $T : \mathbb{A} \to \mathbb{B}$ (not necessarily preserving the generator) always has a left adjoint $T_! : \mathsf{Mod}(\mathbb{A}) \longrightarrow \mathsf{Mod}(\mathbb{B})$.

Exercise 1.2.28. Assuming the results of the previous two exercises, show that a functor $f : \mathsf{Mod}(\mathbb{B}) \longrightarrow \mathsf{Mod}(\mathbb{A})$ is induced by a translation $T : \mathbb{A} \to \mathbb{B}$ as $f = T^*$ iff the following conditions hold:

1. f has a left adjoint $f_! : \mathsf{Mod}(\mathbb{A}) \longrightarrow \mathsf{Mod}(\mathbb{B})$.
2. $f_!(U_{\mathbb{A}}) \cong (U_{\mathbb{B}})^n$ for some $0 \leq n$, where as usual we identify $\mathbb{A} = \mathsf{Mod}_{\mathrm{fgf}}(\mathbb{A})^{\mathsf{op}}$ and similarly for $\mathbb{B}$.

Remark 1.2.29. The general question, when is a "semantic functor"

$$f : \mathsf{Mod}(\mathbb{B}) \longrightarrow \mathsf{Mod}(\mathbb{A})$$

between (Lawvere) algebraic categories induced by a "syntactic translation" $T : \mathbb{A} \to \mathbb{B}$ of the Lawvere algebraic theories (not necessarily preserving the generating objects) can also be answered in the abstract, providing a definition of an *algebraic functor*: it is one that preserves all limits, filtered colimits, and regular epimorphisms. However, this characterization requires a slight modification in the notion of algebraic theory to include closure under retracts — so-called *Cauchy completeness*. In that setting, there is a neat duality theory between the "syntax category" of theories and translations and the "semantics category" of algebraic categories and algebraic functors, which is developed in detail in [ALR03].