CS6301: Optimization in Machine Learning
Lecture 8: Accelerated Gradient Descent and Practical Aspects of Gradient Descent

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https://sites.google.com/view/cs-6301-optml/home

February 10th, 2020

Project and Assignment

- Project Deadline 1: Finalize on your Project Topics and partners: **February 15th 2020**.
- Projects can be done in Groups with 1-3 students per group
- You need to upload the following:
  - A Project Proposal File with a) Team members, b) Introduction and Motivation of the Project, and c) Expected Outcomes
  - A 5-7 slide summary of this for each group. You will have around 5 mins to present this on Monday (and possibly Wednesday) next week

Outline

- Summary of Results for Gradient Descent: Continuous, Smooth and Strong Convex
- Accelerated Gradient Descent and Lower Bounds
- Practical Implementational Aspects

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- Case III: Smooth: Any black box procedure have an error of at least $\frac{3L}{32}\frac{R^2}{(T+1)^2}$ (GD: $\frac{LR^2}{2T}$)

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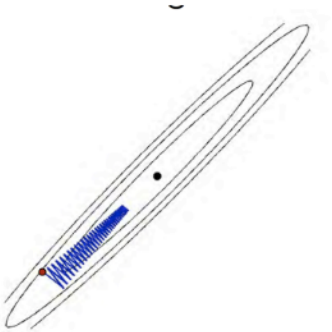
- Case III: Smooth: Any black box procedure have an error of at least $\frac{3L}{32}\frac{R^2}{(T+1)^2}$ (GD: $\frac{LR^2}{2T}$)
- Case IV: Smooth + Strongly Convex: Define $\kappa = \frac{L}{\mu}$. Then Any black box procedure will have an error of at least $\frac{\mu}{2}(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1})^{2(T-1)}$ (GD: $\frac{LR^2}{2}(1-\frac{\mu}{L})^T = \frac{L}{2}(\frac{\kappa-1}{\kappa})^T$)

Why can GD be slow?

- GD has suboptimal rates for smooth and smooth + strongly convex case.
- GD relies just on local gradient information
- Can we add some momentum from the progress made so far to push it faster towards the optimal?



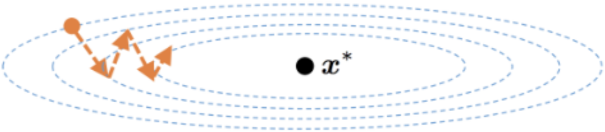
Attempt 1: Heavy Ball Momentum

- Recall standard gradient descent is $x_{t+1} = x_t - \gamma_t \nabla f(x_t)$
- Idea of Momentum: Add inertia to the Ball:

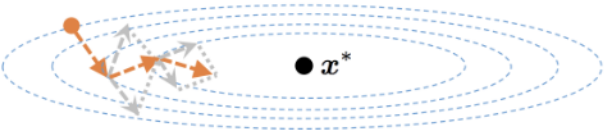
$$x_{t+1} = x_t - \gamma_t \nabla f(x_t) + \beta_t (x_t - x_{t-1})$$

- Heavy Ball result: For smooth + strongly convex functions, the heavy ball algorithm converges in $\frac{R^2}{2}(1 - \sqrt{\frac{1}{\kappa}})^T = \frac{L}{2}(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}})^T$ instead of $\frac{R^2}{2}(1 - \frac{1}{\kappa})^T = \frac{L}{2}(\frac{\kappa-1}{\kappa})^T$ (GD convergence) iterations.
- Heavy Ball momentum not optimal for the Smooth case (though it is optimal for the strongly convex + smooth class).

GD vs Momentum



gradient descent



heavy-ball method

Nesterov's Accelerated Gradient Descent

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- Define $\lambda_0 = 0, \lambda_t = \frac{1+\sqrt{1+4\lambda_{t-1}^2}}{2}$ and $\beta_t = \frac{1-\lambda_t}{\lambda_{t+1}}$. Note $\beta_t \leq 0$

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- Matches the lower bound upto contant factors!

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- Difference between Theory and Practice and the need to connect the two!
- Next, we will implement some of these algorithms for various ML Loss Functions!

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[python]

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[python]

- ‘funObj‘ is the

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def gd(funObj,w,maxEvals,alpha,X,y,lam, verbosity):
    [f,g] = funObj(w,X,y,lam)
    funEvals = 1
    funVals = []
    while(1):
        [f,g] = funObj(w,X,y,lam)
        optCond = LA.norm(g, np.inf)
        if (verbosity > 0):
            print(funEvals,alpha,f,optCond)
        w = w - alpha*g
        funEvals = funEvals+1
        if ((optCond < 1e-2) and (funEvals > maxEvals)
            break
        funVals.append(f)
    return funVals
```

Gradient Descent in Practice: Basic Version

- Run this by invoking:

```
funV = gd(LogisticLoss,w,200,1e-1,X,y,1,1,10)
```

- Try running this with different values of learning rates: $\alpha = 1e-1, 1e-3, 1e-5, \ldots$
- How do we find the optimal learning rate every time?
- Can there be better strategies to adapt the learning rates?
- Next, we shall see a few line search based strategies.

Armijo Backtracking Line-Search V1

- We don't want to tune α every time
- This is the idea behind line search
- Simple Line search strategy:
  - Start with a large value of α
  - Divide α by 1/2 if it doesn't satisfy Armijo's condition:

$$f(w - \alpha g) \leq f(w) - \gamma\alpha ||g||^2$$

  - Basically find α such that there is a reduction in function value by atleast $\gamma\alpha ||g||^2$
  - Idea: Choose α and γ such that this happens.

Armijo Backtracking Line-Search V2

- Danger with the simple backtracking is that α may quickly become very small quickly
- Easy fix: Reset α every time!
- Issue with this: Too many function evaluations lost in repeated backtracking!

Armijo Backtracking Line-Search V3

- Just halving the step size ignores the information collected during line search!
- Reduce the number of backtracks using a polynomial interpolation!
- Minimize a quadratic passing through $f(w), f'(w)$ and $f(w - \alpha g)$
- Choose α using a polynomial interpolation as follows:

$$\alpha = \frac{\alpha^2 g^T g}{2(fcurr + \alpha g^T g - f)}$$

- Here *fcurr* is the function evaluation with the current value of α and f is the function value before starting backtracking!

Armijo Backtracking Line-Search V4

- Final Issue to fix is better initialization of α.
- Initializing $\alpha = 1$ is too large in practice
- Wasted backtracks because of this.
- Use a hueristic like $\alpha = 1/||g||$
- On subsequent iterations again use a polynomial interpolation:

$$\alpha = \min(1, 2(f_{old} - f)/g^T g)$$

- A lot of this is tried empirically and based on empirical knowledge..

Finally: Accelerated Gradient Descent

- Algorithm:
  - Define $\lambda_0 = 0, \lambda_t = \frac{1+\sqrt{1+4\lambda_{t-1}^2}}{2}$ and $\gamma_t = \frac{1-\lambda_t}{\lambda_{t+1}}$.
  - Note $\gamma_t \leq 0$
  - Initialize $x_1 = y_1$ as an arbitrary point
  - Step 1: $y_{t+1} = x_t - \alpha \nabla f(x_t)$ (like normal GD)
  - Step 2: $x_{t+1} = (1-\gamma_t)y_{t+1} + \gamma_t y_t = y_{t+1} - \gamma_t(y_{t+1} - y_t)$ (slide a little bit further than y_{t+1} towards the previous point y_t!)
- In practice, we club this with Armijo line search for tuning the learning rate α.