

Problem 5

Arnas Vaicekauskas

September 15, 2024

1 Problem formulation

$$\frac{\partial u}{\partial t} = \alpha \Delta u$$

2 Boundary and initial conditions

$$\begin{aligned} A &= [0, 0.5L] \times [0, 0.5L] \cup [0.5L, L] \times [0.5L, L] \\ u(x, y, 0) &= 100, \quad (x, y) \in A \\ u(x, y, 0) &= 0, \quad (x, y) \notin A \\ \frac{\partial u}{\partial x} \Big|_{x=0} &= \frac{\partial u}{\partial x} \Big|_{x=L} = 0, \quad y \in [0, L], \quad t \in [0, T] \\ \frac{\partial u}{\partial y} \Big|_{y=0} &= \frac{\partial u}{\partial y} \Big|_{y=L} = 0, \quad x \in [0, L], \quad t \in [0, T] \end{aligned} \tag{1}$$

Where L is the side length of the plate and T is the duration of the simulation.

3 Numerical models

We will use explicit finite difference method to get a solution for the heat function

$$\begin{aligned} \frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} &= \alpha \left(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{\Delta x^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{\Delta y^2} \right) \\ u_{i,j}^{n+1} &= u_{i,j}^n - \alpha \Delta t \left(\frac{2u_{i,j}^n}{\Delta x^2} + \frac{2u_{i,j}^n}{\Delta y^2} \right) + \alpha \Delta t \left(\frac{u_{i+1,j}^n + u_{i-1,j}^n}{\Delta x^2} + \frac{u_{i,j+1}^n + u_{i,j-1}^n}{\Delta y^2} \right) \\ u_{i,j}^{n+1} &= \left(1 - 2\alpha \Delta t \left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} \right) \right) u_{i,j}^n + \alpha \Delta t \left(\frac{u_{i+1,j}^n + u_{i-1,j}^n}{\Delta x^2} + \frac{u_{i,j+1}^n + u_{i,j-1}^n}{\Delta y^2} \right) \end{aligned} \tag{2}$$

To ensure numerical stability we need to make sure the coefficient in front of $u_{i,j}^n$ is always positive so

$$\begin{aligned} 1 - 2\alpha \Delta t \left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} \right) &\geq 0 \\ \frac{1}{2} &\geq \alpha \Delta t \left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} \right) \\ \Delta t &\leq \frac{1}{2\alpha} \left(\frac{\Delta x^2 \Delta y^2}{\Delta x^2 + \Delta y^2} \right) \end{aligned} \tag{3}$$