Problem 4

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1 Problem formulation

$$\frac{\partial u}{\partial t} = \alpha \Delta u$$

2 Boundary and initial conditions

$$\begin{aligned}
u(x, y, 0) &= 0, \quad x \in [0, L], \quad y \in [0, L] \\
u(0, y, t) &= 100, \quad y \in [0, L], \quad t \in [0, T] \\
u(x, L, t) &= 0, \quad x \in [0, L], \quad t \in [0, T] \\
\frac{\partial u}{\partial x} &= 0\Big|_{x=L}, \quad y \in [0, L], \quad t \in [0, T] \\
\frac{\partial u}{\partial y} &= 0\Big|_{y=0}, \quad x \in [0, L], \quad t \in [0, T]
\end{aligned} \tag{1}$$

Where L is the side length of the plate and T is the duration of the simulation.

3 Numerical models

We will use explicit finite difference method to get a solution for the heat function

$$\begin{aligned}
\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} &= \alpha \left(\frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{\Delta x^2} + \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{\Delta y^2} \right) \\
u_{i,j}^{n+1} &= u_{i,j}^{n} - \alpha \Delta t \left(\frac{2u_{i,j}^{n}}{\Delta x^2} + \frac{2u_{i,j}^{n}}{\Delta y^2} \right) + \alpha \Delta t \left(\frac{u_{i+1,j}^{n} + u_{i-1,j}^{n}}{\Delta x^2} + \frac{u_{i,j+1}^{n} + u_{i,j-1}^{n}}{\Delta y^2} \right) \\
u_{i,j}^{n+1} &= \left(1 - 2\alpha \Delta t \left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} \right) \right) u_{i,j}^{n} + \alpha \Delta t \left(\frac{u_{i+1,j}^{n} + u_{i-1,j}^{n}}{\Delta x^2} + \frac{u_{i,j+1}^{n} + u_{i,j-1}^{n}}{\Delta y^2} \right)
\end{aligned} \tag{2}$$

To ensure numerical stability we need to make sure the coefficient in front of $u_{i,j}^{n}$ is always positive so

$$\begin{aligned}
1 - 2\alpha \Delta t \left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} \right) &\geqslant 0 \\
\frac{1}{2} &\geqslant \alpha \Delta t \left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} \right) \\
\Delta t &\leqslant \frac{1}{2\alpha} \left(\frac{\Delta x^2 \Delta y^2}{\Delta x^2 + \Delta y^2} \right)
\end{aligned} \tag{3}$$