Trečias laboratorinis darbas

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1 Analitinis sprendimas

Turime antros eilės tiesinę nehomogeninę lygtį su pradinėmis sąlygomis $x(0) = 0, x'(0) = 1$.

$$x'' + 9x = 4\sin(t) - \cos(4t) \tag{1}$$

Pirmiausia spręsime homogeninį atvejį norint gauti bendrąjį sprendinį:

$$x'' + 9x = 0 \tag{2}$$

Lygtis (2) turi pastovius koeficientus, todėl galime formuoti charakteringąjį polinomą

$$\lambda^2 + 9 = 0 \implies \lambda = \pm 3i$$

Šiuo atveju bendrasis sprendinys $x_B(t)$ turės formą

$$x_B(t) = C_1 \sin(3t) + C_2 \cos(3t) \tag{3}$$

Nehomogeninę lygtį (1) spręsime konstantų variavimo metodu, todėl:

$$x(t) = C_1(t)\sin(3t) + C_2(t)\cos(3t) \tag{4}$$

Pagal variavimo metodą, darome prielaidą, kad

$$C_1' \sin(3t) + C_2' \cos(3t) = 0 \tag{5}$$

Prieš įstatant varijuotą bendrąjį sprendinį (4) apskaičiuosime x'' ir $x'' + 9x$:

$$\begin{aligned}
x'' &= (C_1' \sin(3t) + 3C_1 \cos(3t) + C_2' \cos(3t) - 3C_2 \sin(3t))' \\
&= (\underbrace{C_1' \sin(3t) + C_2' \cos(3t)}_{0} + 3C_1 \cos(3t) - 3C_2 \sin(3t))' \\
&= (3C_1 \cos(3t) - 3C_2 \sin(3t))' \\
&= 3C_1' \cos(3t) - 9C_1 \sin(3t) - 3C_2' \sin(3t) - 9C_2 \cos(3t)
\end{aligned}$$

tada

$$\begin{aligned}
x'' + 9x &= 3C_1' \cos(3t) - 9C_1 \sin(3t) - 3C_2' \sin(3t) - 9C_2 \cos(3t) + 9C_1 \sin(3t) + 9C_2 \cos(3t) \\
&= 3C_1' \cos(3t) - 3C_2' \sin(3t)
\end{aligned}$$

Taigi, dabar sprendžiame lygtį

$$3C_1' \cos(3t) - 3C_2' \sin(3t) = 4\sin(t) - \cos(4t) \tag{6}$$

Iš prielaidos (5) galime išsireikšti $C_1' = -C_2' \frac{\cos(3t)}{\sin(3t)}$ ir įsistatyti į (6)

$$\begin{aligned}
-3C_2' \frac{\cos(3t)}{\sin(3t)} \cos(3t) - 3C_2' \sin(3t) &= 4\sin(t) - \cos(4t) \\
-3C_2' \left(\frac{\cos^2(3t)}{\sin(3t)} + \frac{\sin^2(3t)}{\sin(3t)} \right) &= 4\sin(t) - \cos(4t) \\
-\frac{3}{\sin(3t)} C_2' &= 4\sin(t) - \cos(4t) \\
C_2' = \frac{1}{3}\cos(4t)\sin(3t) &- \frac{4}{3}\sin(t)\sin(3t) \qquad (7) \\
C_2 = \frac{1}{3}\int \cos(4t)\sin(3t)dt &- \frac{4}{3}\int \sin(t)\sin(3t)dt \qquad (8)
\end{aligned}$$

Šiuos integralus išspręsime atskirai. Pradėsime nuo antro; naudodamiesi tapatybe $\sin(\alpha)\sin(\beta) = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$ galime suprastinti integralą:

$$\begin{aligned}
\int \sin(3t)\sin(t)dt \\
\frac{1}{2}\int (\cos(2t) - \cos(4t))dt \\
\frac{1}{4}\int \cos(2t)d(2t) - \frac{1}{8}\int \cos(4t)d(4t) \\
\frac{1}{4}\sin(2t) - \frac{1}{8}\sin(4t) \\
\frac{1}{2}\sin(t)\cos(t) - \frac{1}{4}\sin(2t)\cos(2t) \\
\frac{1}{2}\sin(t)\cos(t) - \frac{1}{2}\sin(t)\cos(t)(\cos^2(t) - \sin^2(t)) \\
\frac{1}{2}\sin(t)\cos(t)(1 - \cos^2(t) + \sin^2(t)) \\
\frac{1}{2}\sin(t)\cos(t)(\sin^2(t) + \cos^2(t) - \cos^2(t) + \sin^2(t)) \\
\sin^3(t)\cos(t)
\end{aligned}$$

Naudodami tapatybę $\sin(\alpha)\cos(\beta) = \frac{1}{2}(\sin(\alpha - \beta) + \sin(\alpha + \beta))$ galime išspręsti pirmąjį integralą:

$$\begin{aligned}
\int \sin(3t)\cos(4t)dt \\
\frac{1}{2}\int (\sin(-t) + \sin(7t))dt \\
-\frac{1}{2}\int \sin(t)dt + \frac{1}{14}\int \sin(7t)d(7t) \\
-\frac{1}{2}\int \sin(t)dt + \frac{1}{14}\int \sin(7t)d(7t) \\
\frac{1}{2}\cos(t) - \frac{1}{14}\cos(7t)
\end{aligned}$$

Galiausiai įsistatome gautas išraiškas atgal į (8) ir gauname $C_2(t)$:

$$C_2 = \frac{1}{3}\left(\frac{1}{2}\cos(t) - \frac{1}{14}\cos(7t)\right) - \frac{4}{3}\sin^3(t)\cos(t) + \tilde{C}_2$$
$$C_2(t) = \frac{1}{6}\cos(t) - \frac{1}{42}\cos(7t) - \frac{4}{3}\sin^3(t)\cos(t) + \tilde{C}_2$$

Norint rasti $C_1(t)$ galim įstatyti (7) į jau minėtą formulę $C_1' = -C_2' \frac{\cos(3t)}{\sin(3t)}$:

$$C_1' = -\frac{\cos(3t)}{\sin(3t)}\left(\frac{1}{3}\cos(4t)\sin(3t) - \frac{4}{3}\sin(t)\sin(3t)\right)$$
$$C_1' = \frac{4}{3}\sin(t)\cos(3t) - \frac{1}{3}\cos(4t)\cos(3t) \tag{9}$$
$$C_1 = \frac{4}{3}\int \sin(t)\cos(3t)dt - \frac{1}{3}\int \cos(4t)\cos(3t)dt \tag{10}$$

Šiuos integralus taip pat išspręsime atskirai. Šį kart pradėsime nuo pirmojo. Čia dar kartą galime pritaikyti tapatybę $\sin(\alpha)\cos(\beta) = \frac{1}{2}(\sin(\alpha-\beta) + \sin(\alpha+\beta))$

$$\int \sin(t)\cos(3t)dt$$
$$\frac{1}{2}\int (\sin(-2t) + \sin(4t))dt$$
$$-\frac{1}{4}\int \sin(2t)d(2t) + \frac{1}{8}\int \sin(4t)d(4t)$$
$$\frac{1}{4}\cos(2t) - \frac{1}{8}\cos(4t)$$

Antrajam integralui išspręsti galime panaudoti dar vieną tapatybę:

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}(\cos(\alpha-\beta) + \cos(\alpha+\beta))$$

tada

$$\int \cos(4t)\cos(3t)dt = \frac{1}{2}\int (\cos(t) + \cos(7t))dt = \frac{1}{2}\sin(t) + \frac{1}{14}\sin(7t)$$

Viską įstatę atgal į (8) gauname, kad

$$C_1 = \frac{4}{3}\left(\frac{1}{4}\cos(2t) - \frac{1}{8}\cos(4t)\right) - \frac{1}{3}\left(\frac{1}{2}\sin(t) + \frac{1}{14}\sin(7t)\right) + \tilde{C}_1$$
$$C_1(t) = \frac{1}{3}\cos(2t) - \frac{1}{6}\cos(4t) - \frac{1}{6}\sin(t) - \frac{1}{42}\sin(7t) + \tilde{C}_1$$

Galiausiai nehomogeninės lygties (1) analitinis sprendinys yra

$$x(t) = \left(\frac{\cos(2t)}{3} - \frac{\cos(4t)}{6} - \frac{\sin(t)}{6} - \frac{\sin(7t)}{42} + \tilde{C}_1\right)\sin(3t) \tag{11}$$
$$+\left(\frac{\cos(t)}{6} - \frac{\cos(7t)}{42} - \frac{4\sin^3(t)\cos(t)}{3} + \tilde{C}_2\right)\cos(3t) \tag{12}$$

Liko rasti Koši sprendinį:

$$
\begin{aligned}
x(0) &= 0 \\
\left(\frac{1}{3} - \frac{1}{6} - \frac{0}{6} - \frac{0}{42} + \tilde{C_1}\right) 0 + \left(\frac{1}{6} - \frac{1}{42} - \frac{0}{3} + \tilde{C_2}\right) 1 &= 0 \\
\frac{1}{6} - \frac{1}{42} + \tilde{C_2} &= 0 \\
\tilde{C_2} &= -\frac{1}{7}
\end{aligned}
$$

$$
\begin{aligned}
x'(0) &= 1 \\
3\tilde{C_1}\cos(0) + \frac{3}{7}\sin(0) &= 1 \\
\tilde{C_1} &= \frac{1}{3}
\end{aligned}
$$

Taigi, Koši sprendinys yra:

$$
\begin{aligned}
x(t) = &\left(\frac{\cos(2t)}{3} - \frac{\cos(4t)}{6} - \frac{\sin(t)}{6} - \frac{\sin(7t)}{42} + \frac{1}{3}\right)\sin(3t) \\
&+ \left(\frac{\cos(t)}{6} - \frac{\cos(7t)}{42} - \frac{4\sin^3(t)\cos(t)}{3} - \frac{1}{7}\right)\cos(3t)
\end{aligned}
$$

2 Palyginimas su programos sprendiniu

Programa rasti lygties (1) sprendiniui buvo rašyta su python programavimo kalba bei sympy paketu. Kodas randantis Koši sprendinį:

```
from sympy import *
t = symbols('t')
x = Function('x')(t)
equation = Eq(x.diff(t, 2) + 9*x, 4 * sin(t) - cos(4 * t))
general_solution = dsolve(equation, x).rhs
conditions = [
    Eq(general_solution.subs(t, 0), 0),
    Eq(general_solution.diff(t).subs(t, 0), 1)
]
C_values = solve(conditions, symbols("C1, C2"))
general_solution.subs(C_values)
```

Programos rastas sprendinys:

$$x(t) = \frac{\sin(t)}{2} + \frac{\sin(3t)}{6} - \frac{\cos(3t)}{7} + \frac{\cos(4t)}{7}$$