Introduction to Linear System Control

& Kalman Filter

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Outline

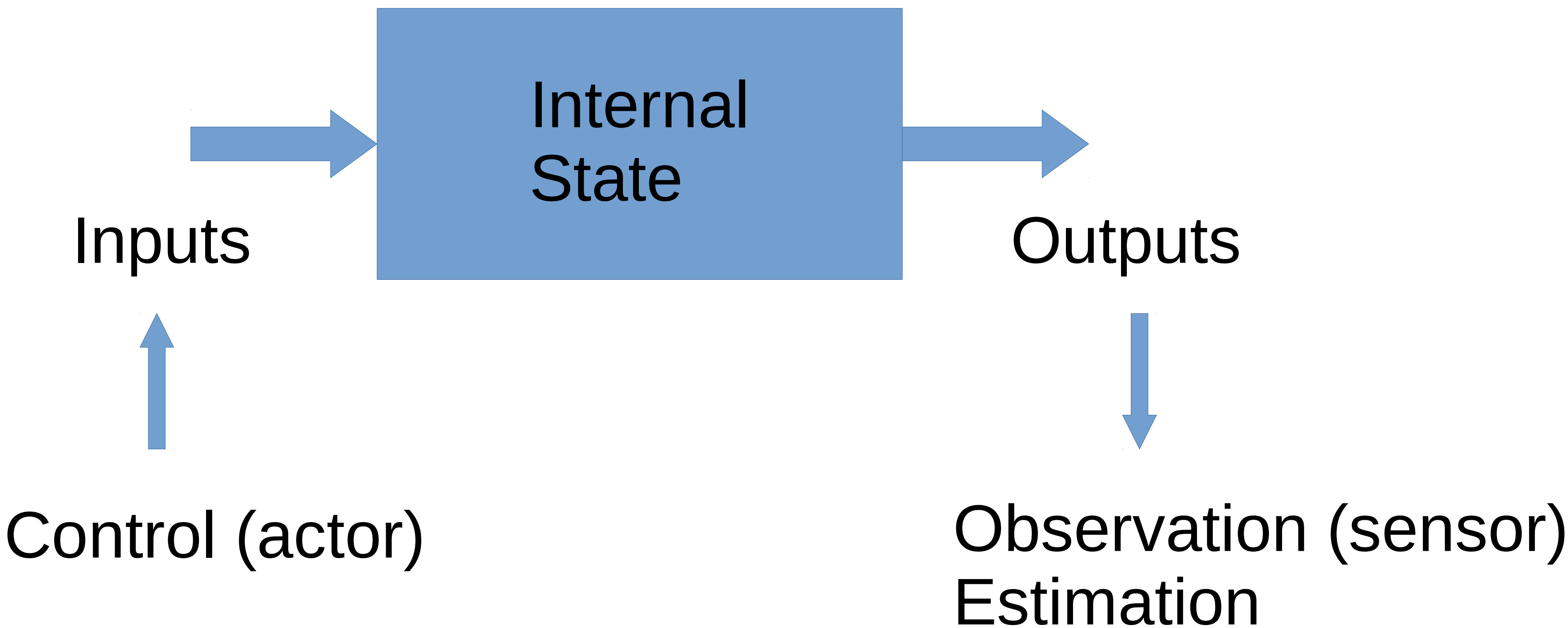
- System Model Equation

- Input = Theory of Control
  (Riccati Equation – Kalman Gain)

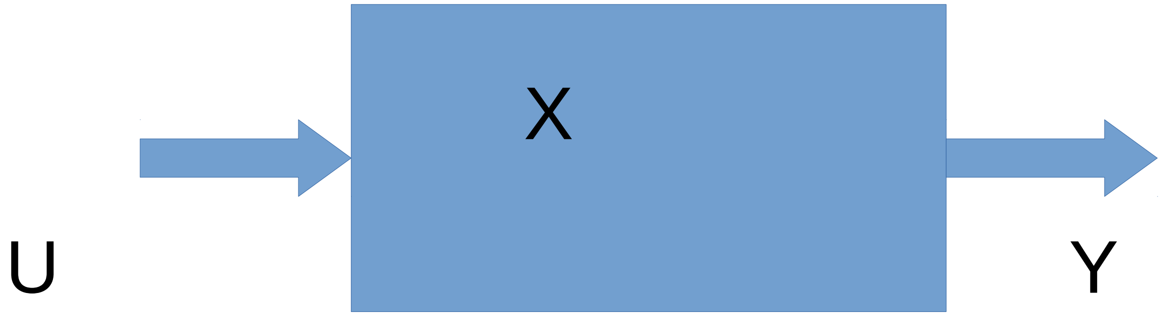
- Output = Theory of Estimation

  (Kalman Filter)

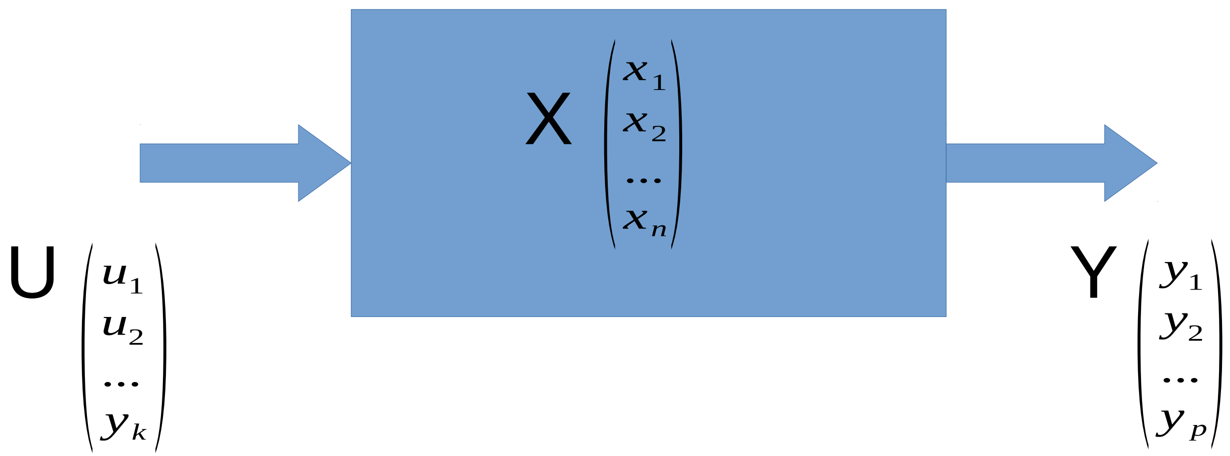
System … Black Box



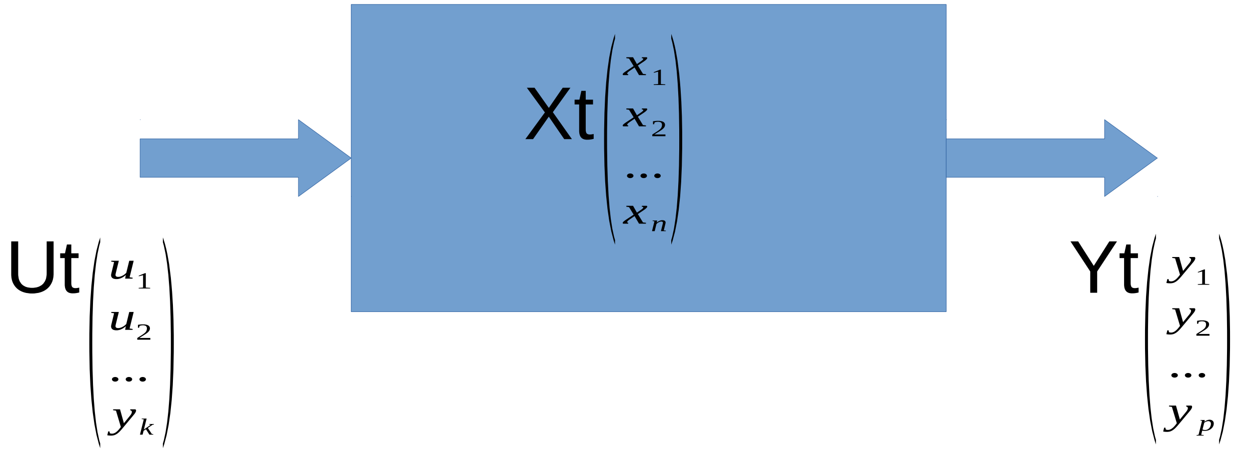
Notation



Dimension / Freedom Degrees



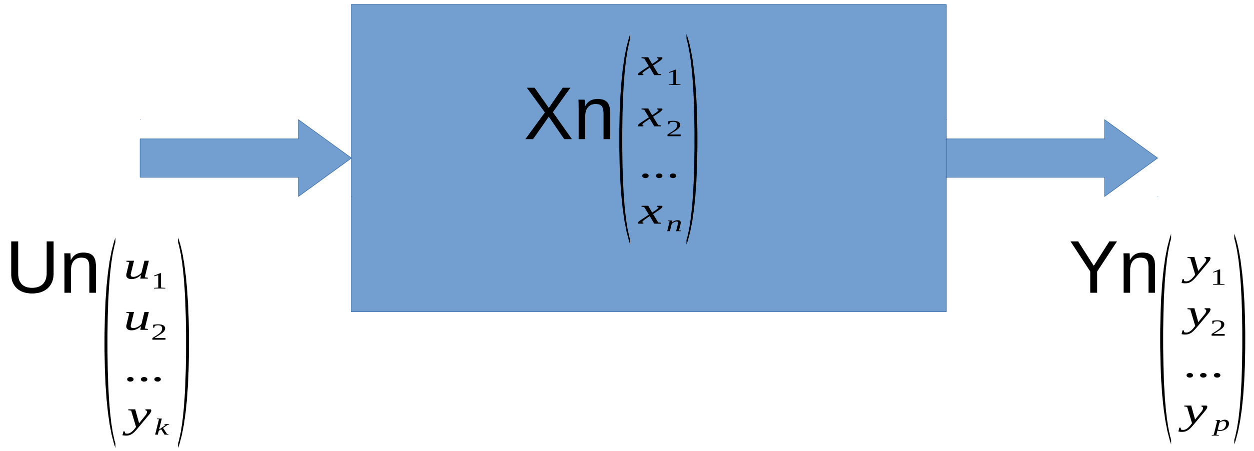
State Model Equation Continuous case



$$\dot{X}_t = \frac{\partial X_t}{\partial t} = f\left(X_t, U_t, W_t\right)$$

with Wt $\begin{pmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{pmatrix}$: white noise (brownian)

State Model Equation
Discrete case



$$X_{n+1} = f\left(X_n, U_n, W_n\right)$$

with Wn $\begin{pmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{pmatrix}$: white noise (normal / gaussian)

Time Discretisation

$$t_{n+1} = t_n + dt$$

$$\ldots t_n = n \, . \, dt + t_0$$

$$X_n = X\left(t_n\right)$$

$$\ldots \dot{X}_t \, . \, dx = X\left(t_{n+1}\right) - X\left(t_n\right) = X_{n+1} - X_n$$

Note on CPU Timers : for fixed delay => use Real Time linux or hardware …
Typical loop timer: 10ms

State Model Differential Equation Order N => Order1 - Dimension N

$$\frac{\partial^n X_t}{\partial^n t} = f\left(\frac{\partial^{n-1} X_t}{\partial^{n-1} t}, \frac{\partial^{n-2} X_t}{\partial^{n-2} t}, \ldots, X_t\right)$$

let

$$X_t = \begin{pmatrix} X_t \\ \frac{\partial X_t}{\partial t} \\ \frac{\partial^2 X_t}{\partial^2 t} \\ \vdots \\ \frac{\partial^{n-1} X_t}{\partial^{n-1} t} \end{pmatrix}$$

then

$$\dot{X}_t = \begin{pmatrix} \frac{\partial X_t}{\partial t} \\ \frac{\partial^2 X_t}{\partial^2 t} \\ \vdots \\ \frac{\partial^n X_t}{\partial^n t} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & \\ 0 & 0 & 1 & \cdots & \\ \vdots & & & \ddots & 0 \\ 0 & & & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} X_t \\ \frac{\partial X_t}{\partial t} \\ \vdots \\ \frac{\partial^{n-1} X_t}{\partial^{n-1} t} \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ f(\ldots, X_t) \end{pmatrix}$$

Which can be written as $X_t = A.X_t + B.U_t$

where A=matrix(n,k) B=matrix(1,k) Ut=vector(k)

Example Order 2
Newtown Mecanic

$$m.\vec{a}=\sum \vec{F}$$

Write with x,y coord:

$$\begin{pmatrix} m\ddot{x}=f_x \\ m\ddot{y}=f_y \end{pmatrix}$$

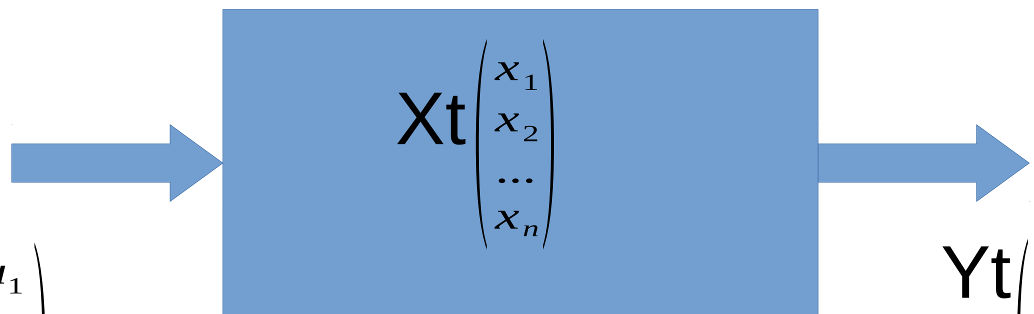
using:

$$X_t=\begin{pmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{pmatrix} \quad A=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B=\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad U=\begin{pmatrix} f_x/m \\ f_y/m \end{pmatrix}$$

Finally:

$$\dot{X}_t=A.X_t+B.U_t$$

Linear State Model Equation
Continuous case



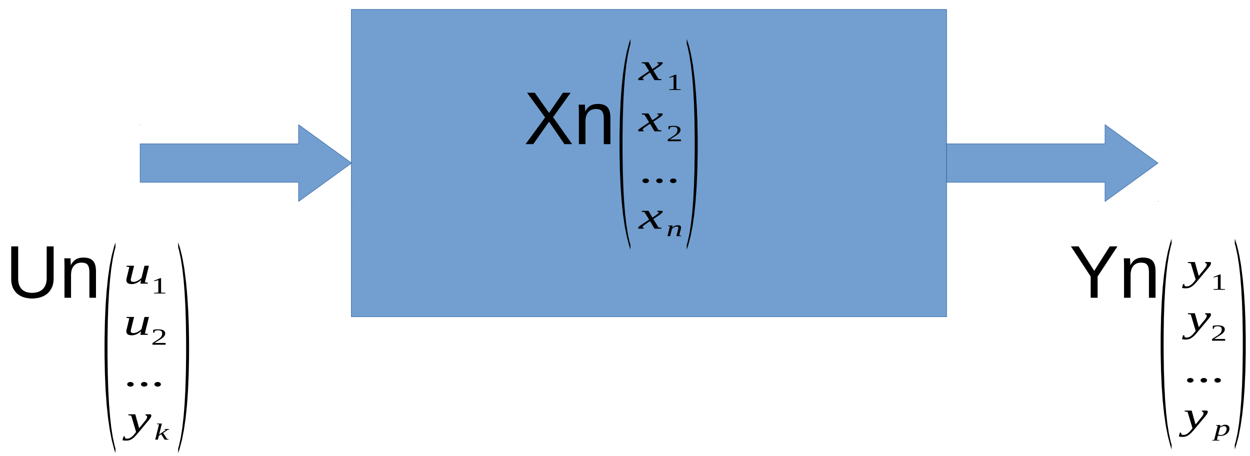
Ut $\begin{pmatrix} u_1 \\ u_2 \\ \dots \\ y_k \end{pmatrix}$

Yt $\begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_p \end{pmatrix}$

$$\dot{X}_t = A.X_t + B.U_t + D W_t$$

with Wt $\begin{pmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{pmatrix}$: white noise (brownian)

Linear State Model Equation
Discrete case



$$X_{n+1} = A.X_n + B.U_n + D.W_n$$

With Wn $\begin{pmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{pmatrix}$: white noise (normal / gaussian)

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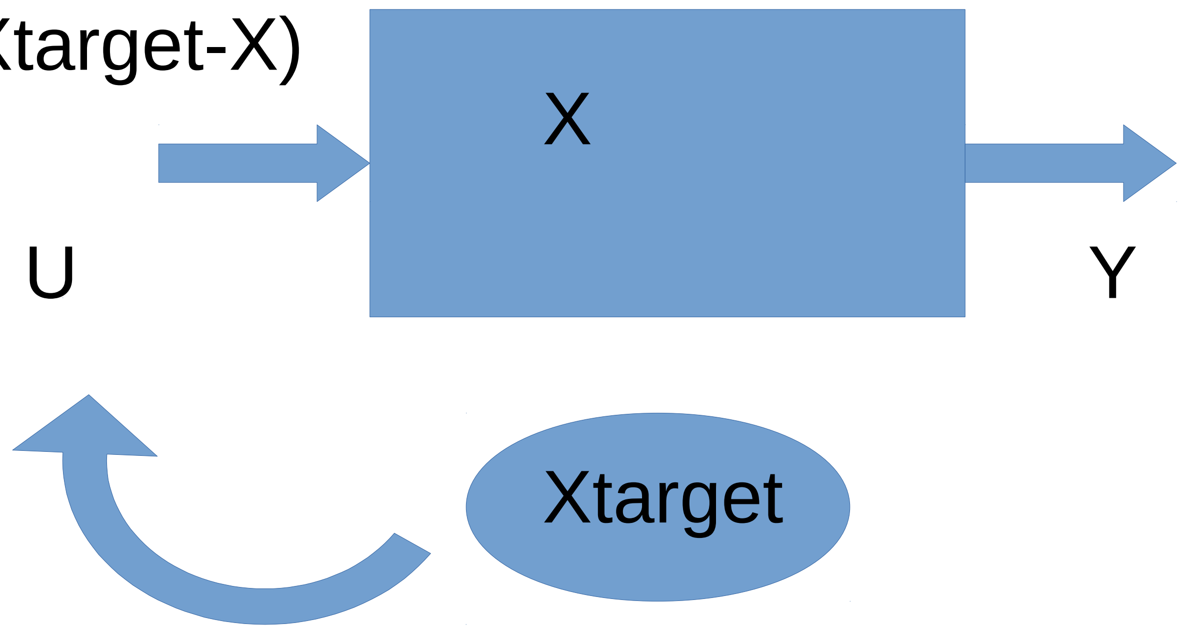
  (Kalman Filter)

Feedback Loop : Follow Trajectory

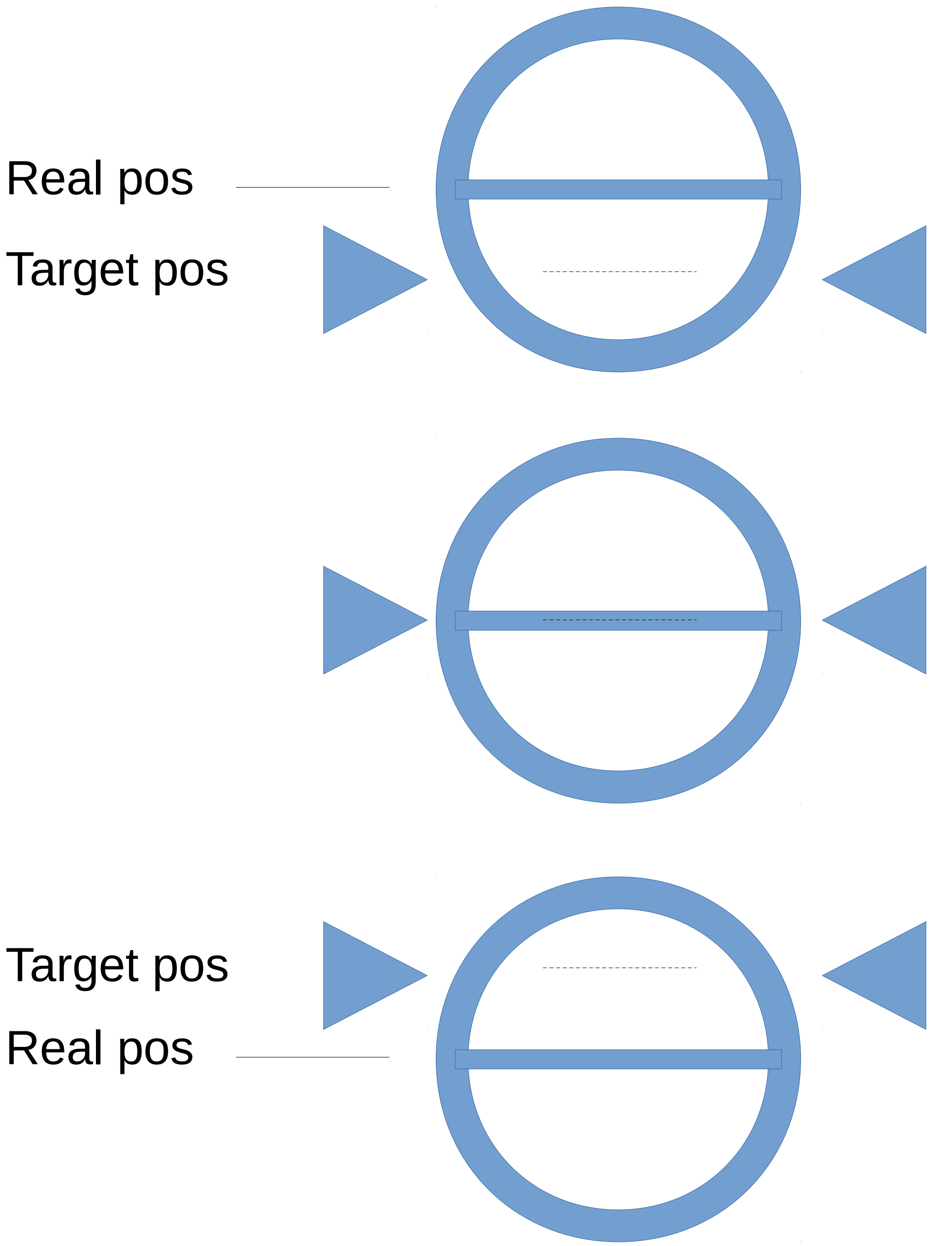
$U = -K(Xtarget-X)$



Xtarget-X = target error to correct

K = feedback gain (= kalman control gain)

Example



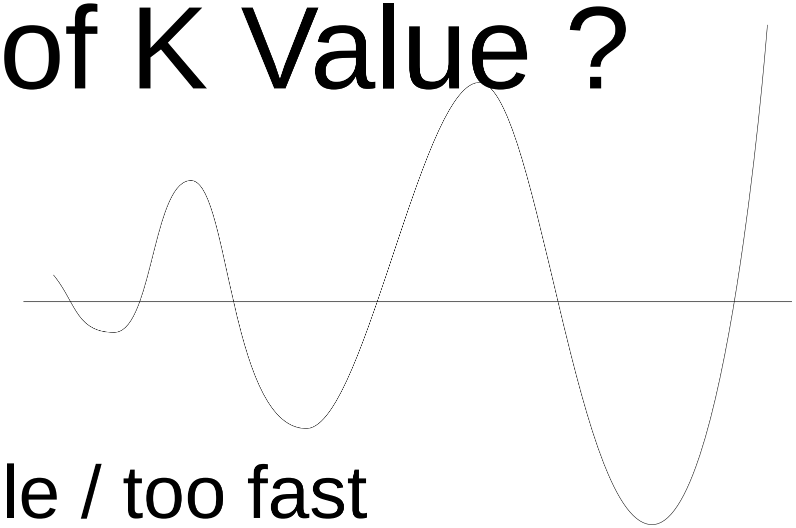
Too HIGH => GO Down

OK => don't move

Too LOW => GO Up

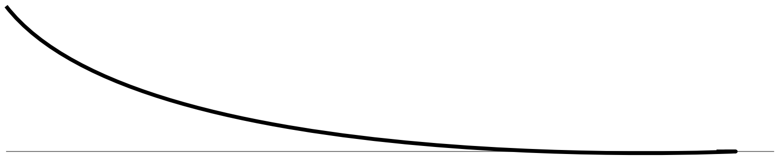
Choice of K Value ?

K Too BIG
(K > Kmax)

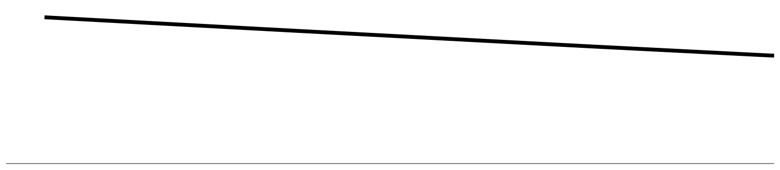


=> Divergent / unstable / too fast

K Optimal?
(Kmin < K < Kmax)



K Too SMALL
(0 < K < Kmin)



=> Not convergent / not reactive / too slow

Response Time Shift

Optimal in Theory :

$$U_t = -K\,(X_{target} - X_t)$$

In practise ...

$$U_{t+delay_t} = -K\,(X_{target} - X_t)$$

If delayt too big:
    response time of computation > typical time of system
Then instability, vibration..
(example: you are drunk … reaction > 100ms .. don't drive)

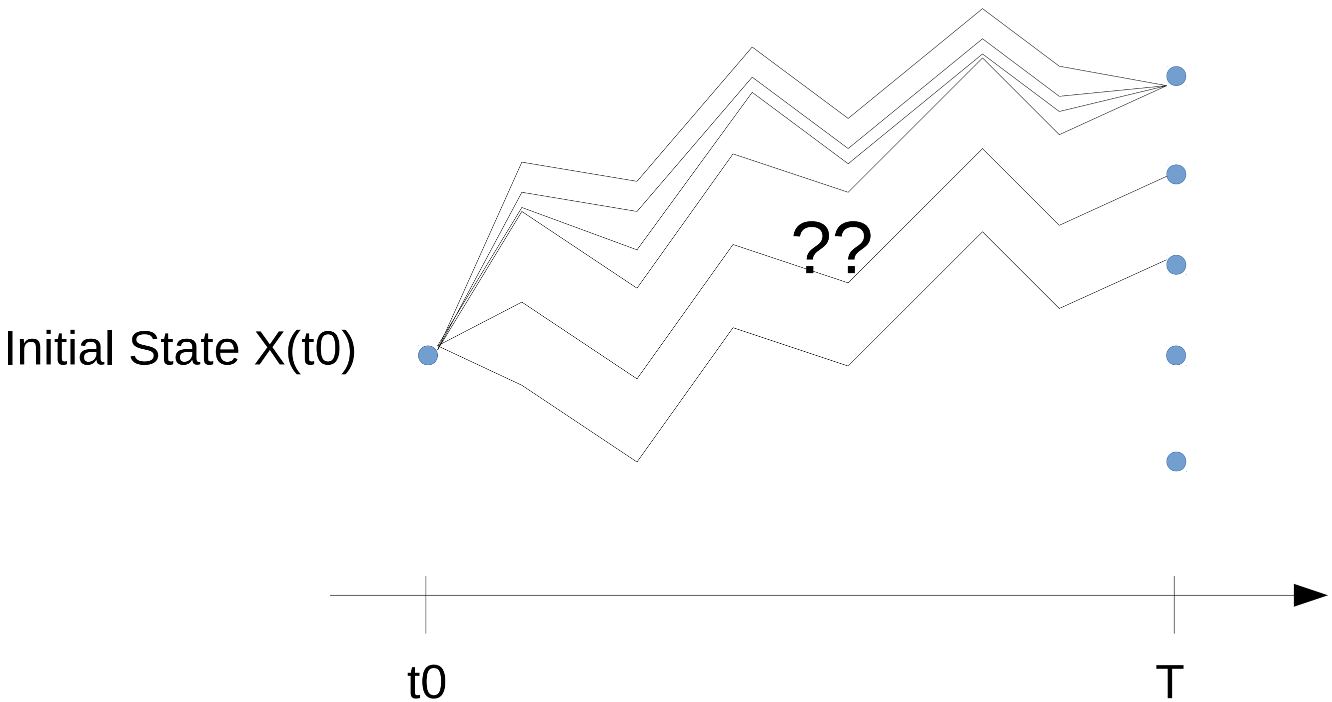
Achieve smallest delayt
Maybe split K1, K2...Kn with
        K1= in hardward=nanos / K2: 10ms / K3: 100ms ...

Computation for Optimal K ?

$$V(trajectory) = \int_{t_0}^{T} \underbrace{{}^t(X_t - Xtarget_t)\, Q\, (X_t - Xtarget_t)}_{\text{Term for cumulated error of trajectory following}} + \underbrace{{}^t U_t\, R\, U_t}_{\text{Term for cumulated energy of control}}\, dt$$

Choose 2 symmetric matrices Q=(n,n) & R=(k,k)

Compute for All Trajectories??

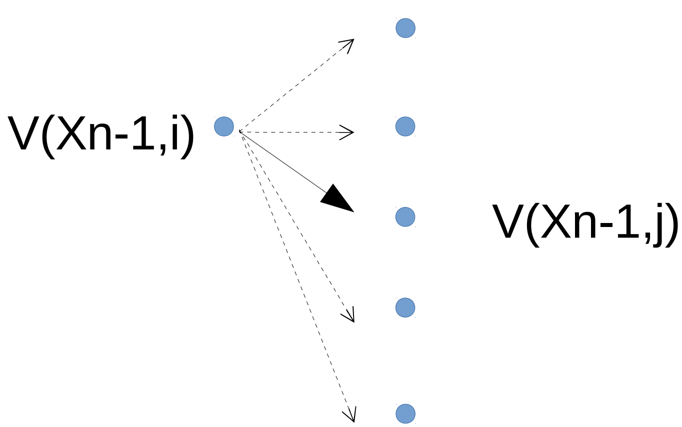


Dynamic Programming 1/3

Principle:
If X0 → X1 → X2 → …. Xn is the optimal trajectory from X0 to any Xn
Then Xi → Xi+1 → … Xn is the optimal (sub) trajectory from Xi to any Xn

Compute => Xn-1 → Xn the optimal last step for Xn-1 to any Xn

Computation for last Step N:
Foreach K, compute
V(Xn-1,i) = arg min j cost(Xn-1,i → Xn,j) + V(Xn,j)

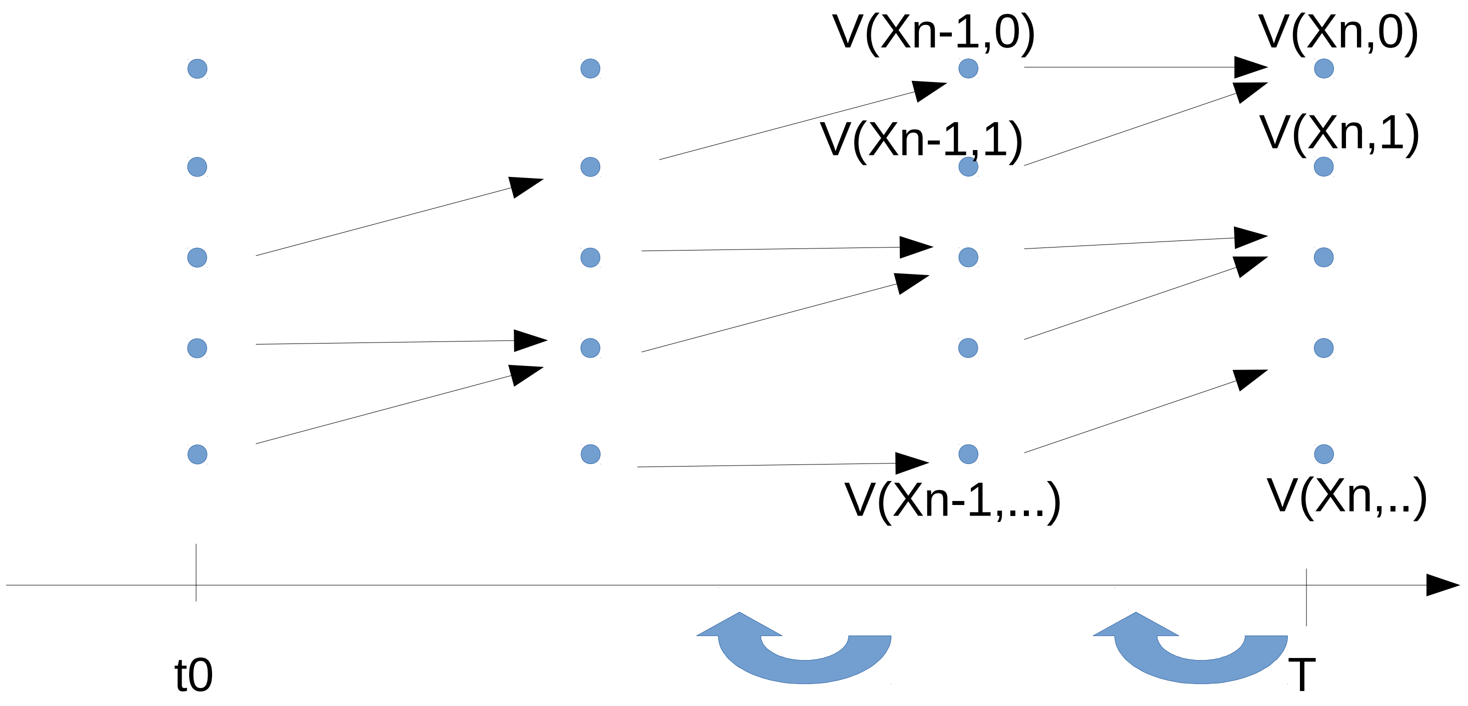


Algorithm similar to
Bellman-Kalaba “shortest path”
To any destinations

Dynamic Programming 2/3

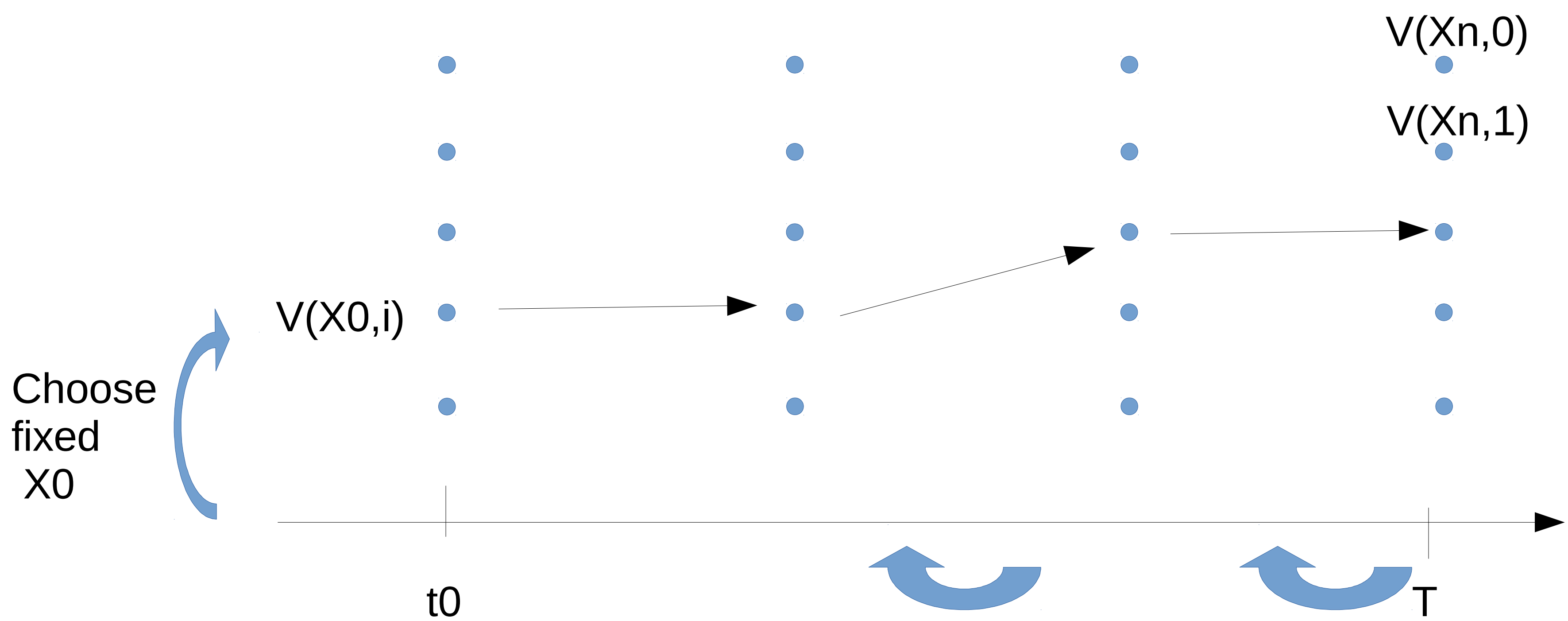
Recurse … to get optimal Trajectories from any points:

- initialize V(Xn,i)
- recurse n → n-1 :   compute V(Xn-1,i) … remember direction from Xn-1,i → Xn



Dynamic Programming 3/3

Pick up optimal trajectory from t0
(remembering each step directions Xn-1,i → Xn)



Application Computation of Optimal Kalman Gain

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Algebraic Riccati equation

From Wikipedia, the free encyclopedia

An **algebraic Riccati equation** is a type of nonlinear equation that arises in the context of infinite-horizon optimal control problems in continuous time or discrete time.

A typical algebraic Riccati equation is similar to one of the following:

the continuous time algebraic Riccati equation (CARE):

$$A^T X + XA - XBR^{-1}B^T X + Q = 0$$

or the discrete time algebraic Riccati equation (DARE):

$$X = A^T XA - (A^T XB)(R + B^T XB)^{-1}(B^T XA) + Q.$$

X is the unknown n by n symmetric matrix and A, B, Q, R are known real coefficient matrices.

It is called Riccati Equation (but nothing to do with Italian Mathematician)
This is from Kalman !

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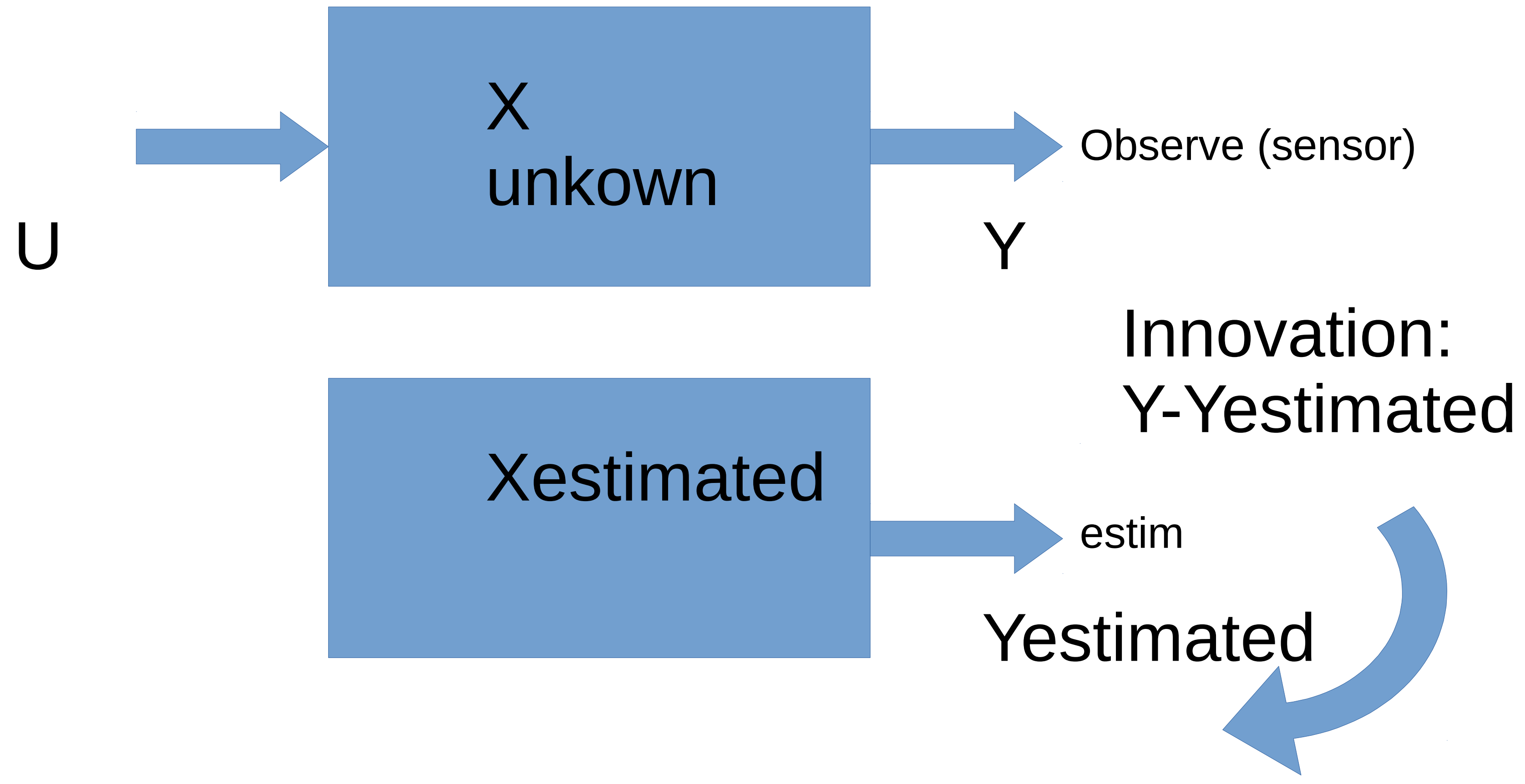
Estimation Feedback Loop from Observations



Dimensions / Observation vs Degrees of Freedom

$$\text{Xn}\begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} \qquad \text{Yn}\begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_p \end{pmatrix}$$

In general $n > p$: not every variables are observables

You mesure only a projection

Example 1: $\text{Xn}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\text{Yn}\begin{pmatrix} x \\ y \end{pmatrix}$ In 3D word ...only seing 2D images

Example 2: $\text{Xn}\begin{pmatrix} x \\ \dot{x} \\ \ddot{x} \end{pmatrix}$ $\text{Yn}\begin{pmatrix} \ddot{x} \end{pmatrix}$ On your cell-phone, you only have an accelerometer

Redundant Sensors Measures => merge for Accuracy

Same “measure” several times by different sensors
=> get different conflicting values for same variable !

=> use ponderations between sensors accuracy / speeds

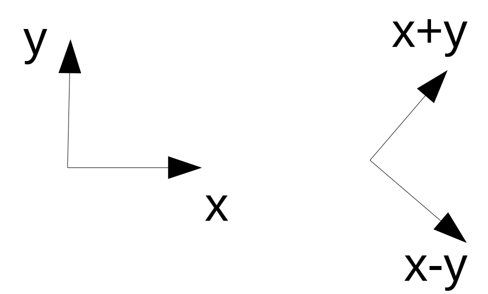
Example 1: Xn $\begin{pmatrix} x \\ y \end{pmatrix}$ Yn$\begin{pmatrix} x \\ x+1 \\ 2x \\ 3x-1 \end{pmatrix}$

Example: Take several Pictures
(with different expositions)
=> merge pictures to remove noise

Deduce State from Linear Combination of Sensors ?

Example 1: Xn $\begin{pmatrix} x \\ y \end{pmatrix}$ Yn $\begin{pmatrix} x+y \\ x-y \end{pmatrix}$

Measure in different basis
=> change basis (inverse basis matrix)



Example 2: Xn $\begin{pmatrix} x \\ y \end{pmatrix}$ Yn $\begin{pmatrix} dist_{satellite\,1} \\ dist_{satellite\,2} \\ dist_{satellite\,3} \\ dist_{satellite\,4} \end{pmatrix}$

GPS :
Compute latitude/longitude position
from distances to Satellites...
in your cell-phone also

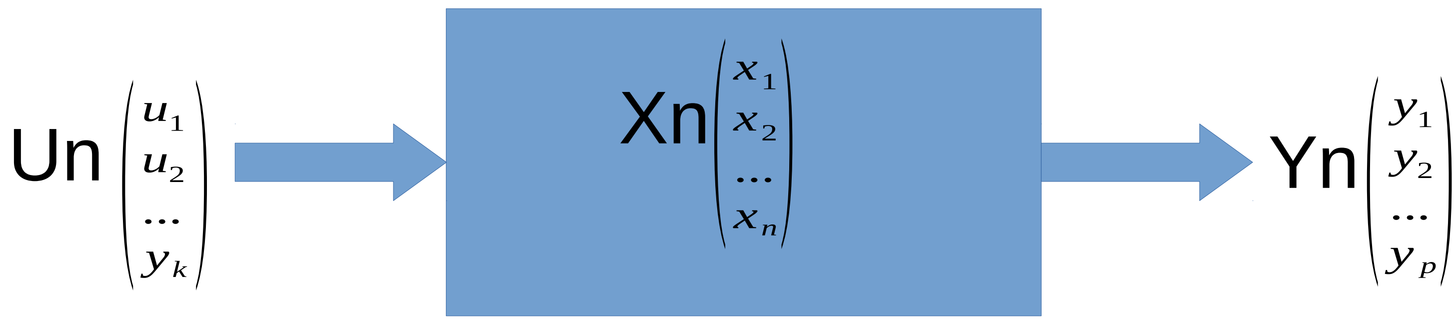
Deduce State From Current Sensor + N Past Sensor Measures

Mathematical Theorem on Linear Observable System …

if ($AC, A^2C, A^3C, ..A^nC$) has dimension N
=> then system is observable

Notice Transposition:
if ($AB, A^2B, A^3B, ..A^nB$) has dimension N
=> then system is controlable

Observation Model Equation
Continuous / Discrete case



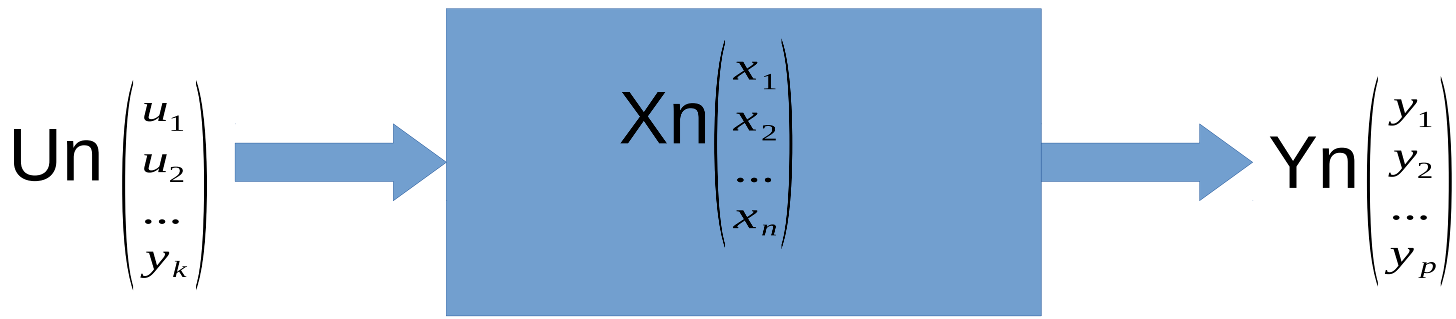
Discrete case : $$Y_n = f\left(X_n, U_n, W_n\right)$$

with Wn $\begin{pmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{pmatrix}$: white noise (normal / gaussian)

Continuous case : $$Y_t = f\left(X_t, U_t, W_t\right)$$

with Wt $\begin{pmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{pmatrix}$: white noise (brownian)

Linear Observation Model Equation Continuous / Discrete case



C=matrix(p,n), F=matrix(p,k) in general=0, E=matrix(p,p)

Discrete case :

$$Y_n = C\,X_n \left(+F\,U_n\right) + E\,W_n$$

with Wn $\begin{pmatrix} w_1 \\ w_2 \\ \dots \\ w_k \end{pmatrix}$: white noise (normal / gaussian)

Continuous case :

$$Y_t = C\,X_t \left(+F\,U_t\right) + E\,W_t$$

with Wt $\begin{pmatrix} w_1 \\ w_2 \\ \dots \\ w_k \end{pmatrix}$: white noise (brownian)

Sensor (so Measures) have many Weaknesses !

White Noise : random additive noise each time you measure

Other weaknesses :

Model Error … what you measure is not really what you think, your equation are wrong

Biased Noise : the average measure is shifted compared to real
you need to re-calibrate your sensor

Discrete Sampling Time : your eletronic sensor is not very fast ...example every 10ms

Sampling Not Strictly Regular : every 10ms? but sometime 9ms, sometime 11ms

Integer Rounding (=Quantification) : your Analog To Digital converter is only precise
to 10 bits for instance (= 1024 values)

Time Delay : you get results 5ms late after they are really measured

Missing/Irregular Measures Arrival : sometime you don't have measure,
or measures are not regular at all (example: satellites)

Handling Irregular Measures

Example of handling “Missing/Irregular Measures Arrival”

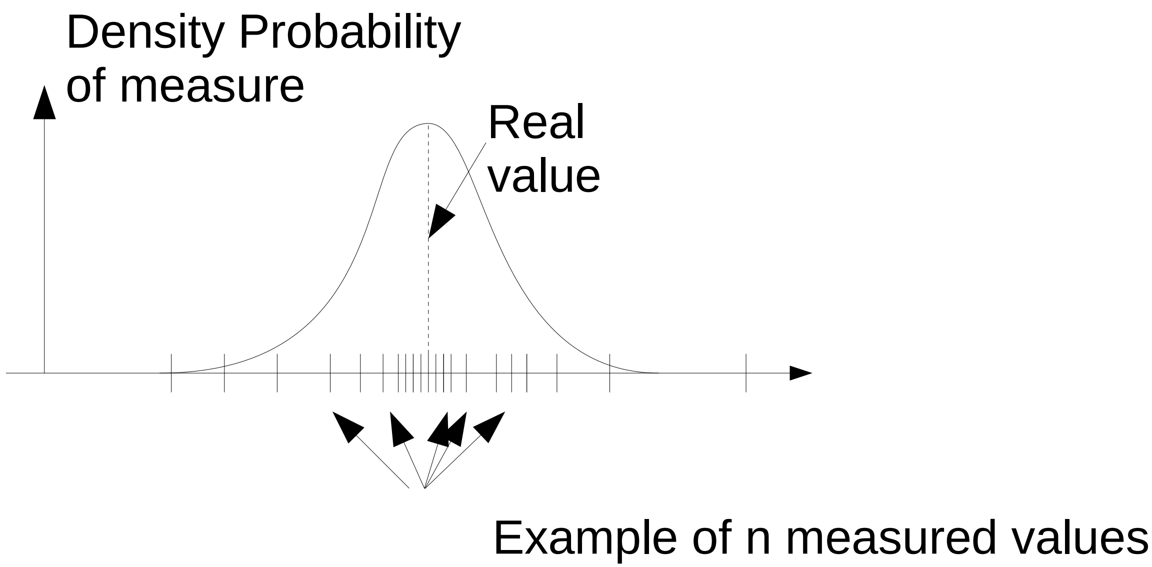
When receiving measures :

measure set1 => then use equation1 $$Y_n = C_1 X_n + E_1 W_n$$

measure set2 => then use equation2 $$Y_n = C_2 X_n + E_2 W_n$$

measure set3 => then use equation3 ...

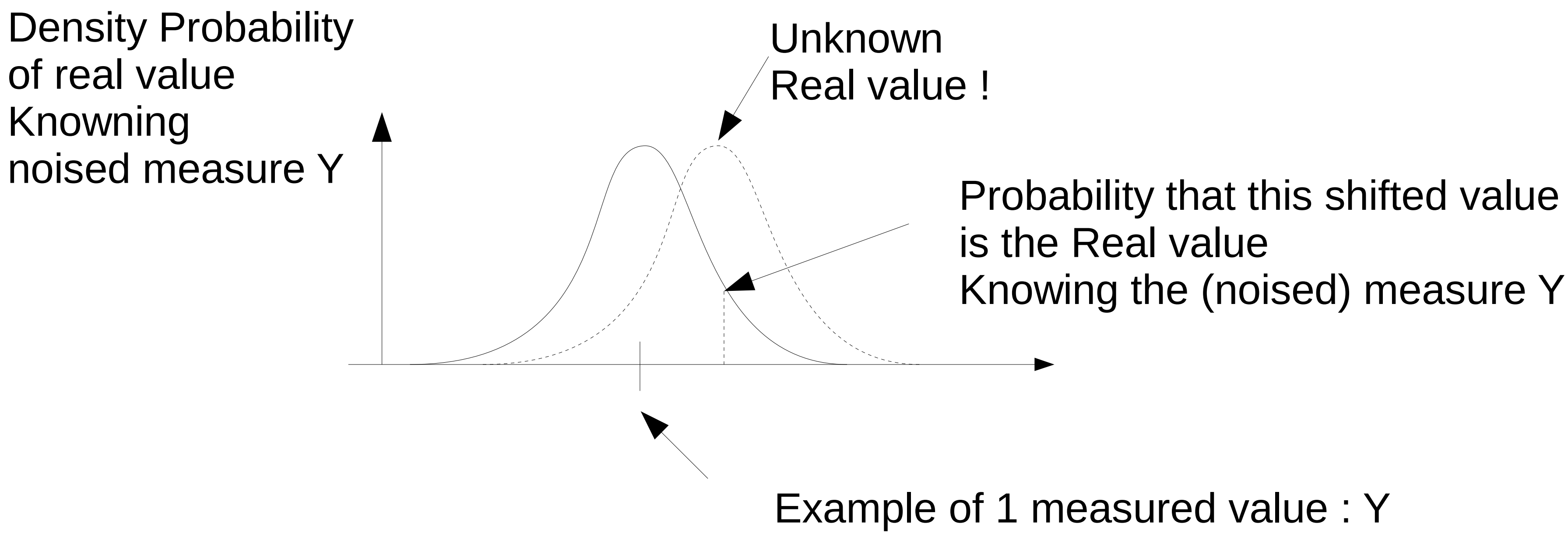
Measures Noise term: E Wn



(White) Noise =
repeating N measures give N different values following a Gaussian distribution

Biased = when the average is shifted

Noise Measure => Real Value?



Merge N noised measures : Average

$$Average(Y_{1..n}) = \frac{1}{N}\left(Y_1 + Y_2 + \ldots Y_n\right)$$

If all measures Yi have precisions $\pm\sigma$

Then Average(Y) has precision $\pm\frac{\sigma}{\sqrt{N}}$

(Theorem called “Law of Big Numbers”, for gaussians distributions)

Probability Espectation (synonym: Average, Mean)

E(Y) = Expected of Y = average value of Y
… for all random events
weigthed by their probabilities

Discrete case: given (Y1,P1), (Y2,P2)...(Yn,Pn)
where Yi : value of event i with probability Pi,

$$E(Y) = \sum_i P_i Y_i$$

Continuous case: given density probability (Y(x),P(x)dx)
where Yx value for x<.<x+dx with probability P(x)

$$E(Y) = \int_{x=-\infty}^{+\infty} Y(x)\, P(x)\, dx$$

Probability Variance (=centered moment of order 2)

V(Y) = Variance of Y = weighted average value of
square of centered (Y-E(Y))
… for all random events

$$Var(Y) = \sum_i P_i \bar{Y}_i^2 \ldots where\ \bar{Y}_i = Y_i - E(Y)$$

$$Var(Y) = \int_{x=-\infty}^{+\infty} \bar{Y}(x)^2 P(x) dx$$

Var is moment of order 2
E(Y) was moment of order 1
… we can also define moment 3, 4 ...

Standard Deviation
(= square root of variance)

By definition:

$$Var = \sigma^2$$

Equivalent:

$$\sigma = \sqrt{Var}$$

Standard Deviation of X has the same unit as X, E(X)
(if X is a distance, stddev is also distance)

Standard Deviation for Normal Distribution (Gaussian)

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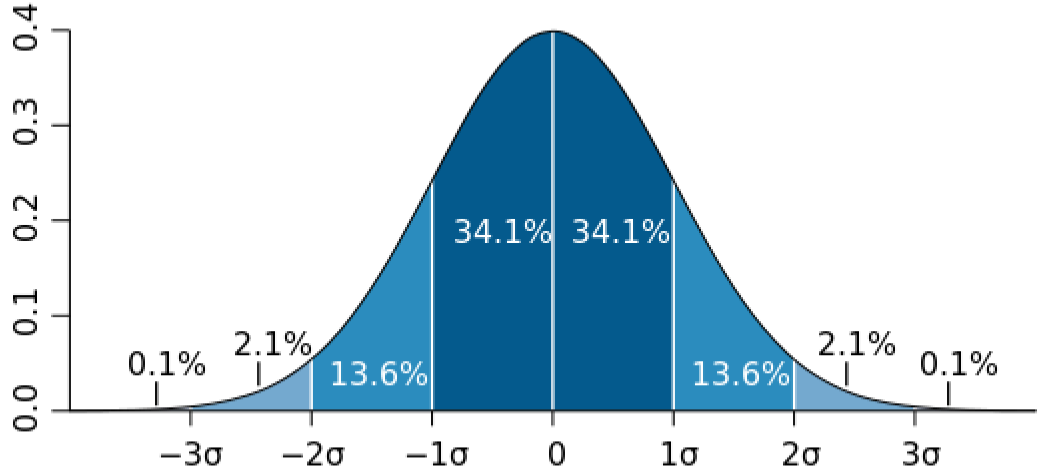

Probability ~68% that value in $[y-\sigma, y+\sigma]$

Probability ~95% that value in $[y-2\sigma, y+2\sigma]$

Probability ~99.7% that value in $[y-3\sigma, y+3\sigma]$

Gaussian Distribution
… defined by (Expect,StdDev)
Normale when StdDev=1

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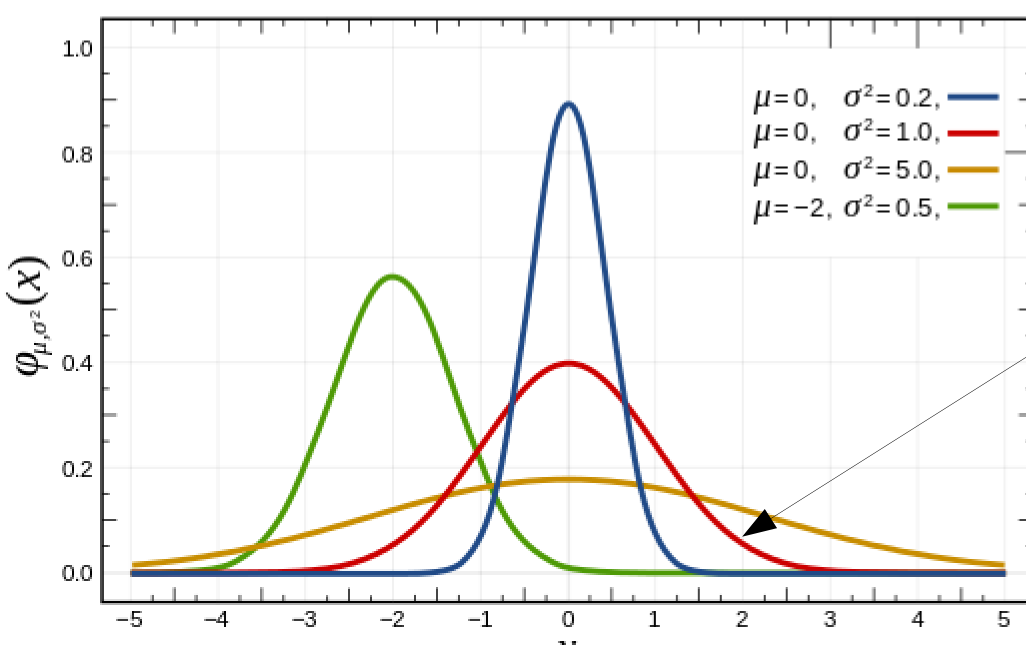
$$f(x;\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$



Normale when
$\sigma = 1$

Central Limit Theorem
(~ Law of Big Numbers …)

Theorem:
Given N random independent measures Y1,Y2...Yn

Then centered average multiplied by $\sqrt{N}$
Converges to gaussian distribution $N\left(0,\sigma^2\right)$

$$\sqrt{n}\,\bar{Y}_{1..n} \rightarrow N\left(0,\sigma^2\right)$$

... For large enough N,
the N-average follow a gaussian $N\left(\mu\,,\frac{\sigma}{\sqrt{N}}\right)$

Back To Measures Merge...

Sensor Merge point-of-view:

Suppose you have 2 measures Y1, Y2
with different stddeviations

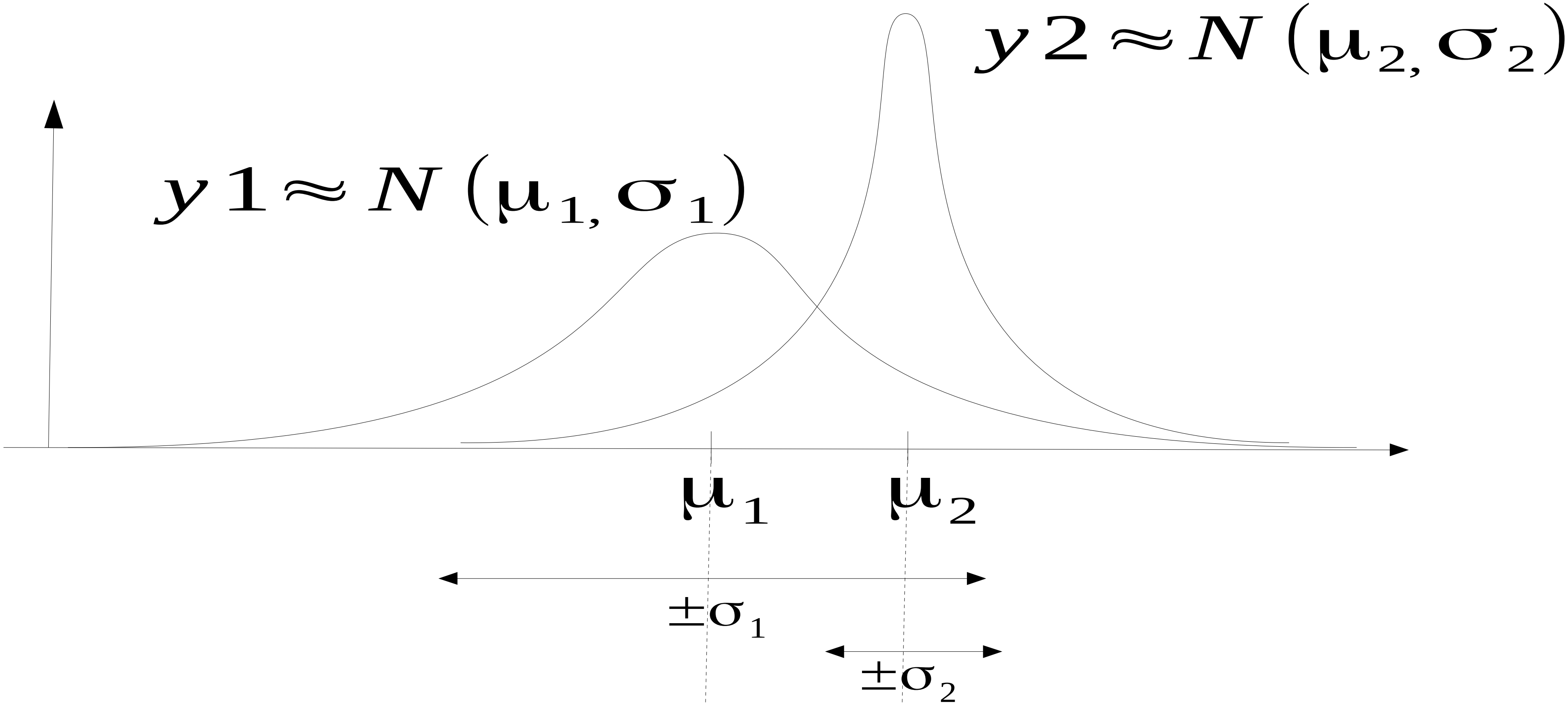
=> how to “weight-average” the 2 values ?

Kalman Filter point-of-view:

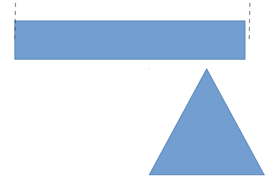
Suppose you have a-priori estimated prediction Y1
and a measure Y2

=> how to “weight-average” the 2 values ?

Merge 2 Gaussians ...



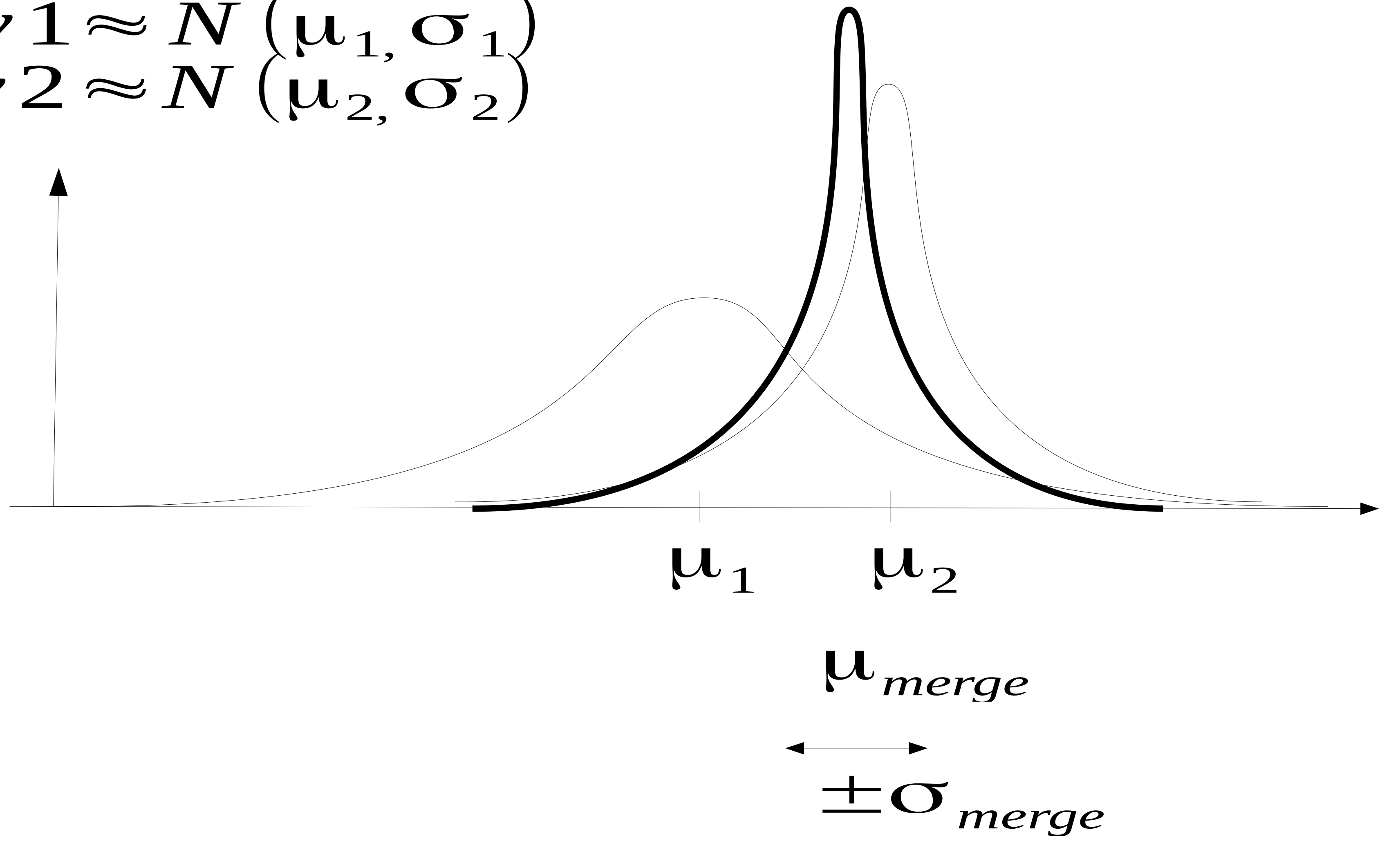
Merge Value MUST be in range $[\mu_{1,} \mu_2]$



Closest to the more accurate (smallest var)

Merge (multiply) 2 Gaussians => proportial to a Gaussian

$$y1 \approx N(\mu_{1,} \sigma_1)$$
$$y2 \approx N(\mu_{2,} \sigma_2)$$



Probability of 2 independent Events => Product of Probability

If A and B are two INDEPENDENT random events,
Then the probability of simultaneous event “A and B” is

$$P(A \wedge B) = P(A) . P(B)$$

Conditional probability (not independent): $P(A|B) = \frac{P(A \wedge B)}{P(B)}$

Y1 and Y2 are both dependent of Yreal

$$P(Y_1 \wedge Y_2 | Y_{real}) = P(Y_1 | Y_{real}) . P(Y_2 | Y_{real})$$

Computing Weigth for product of 2 Gaussians

Remember...

$$e^n = \underbrace{e \,.\, e \,.\, ..\, e}_{n} \qquad e^a . e^b = e^{a+b}$$

So

$$Y_1(x) . Y\,2(x) = \left(\frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}}\right)\left(\frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(x-\mu_2)^2}{2\sigma_2^2}}\right)$$

$$Y = k\, e^{-\frac{1}{2\sigma_1^2\sigma_2^2}\left(\sigma_2^2(x-\mu_1)^2+\sigma_1^2(x-\mu_2)^2\right)}$$

Quadratic form ax^2+bx+c

$$= (\sigma_1^2+\sigma_2^2)x^2 - 2(\sigma_2^2\mu_1+\sigma_1^2\mu_2)x + \ldots$$

Rewrite a(x+b/2a)² + ...

$$= (\sigma_1^2+\sigma_2^2)\left(x - \frac{\sigma_2^2\mu_1+\sigma_1^2\mu_2}{\sigma_1^2+\sigma_2^2}\right)^2 + \ldots$$

Result Gaussian Product

$$Y_1(\mu_{1,}\sigma_1).Y_2(\mu_{2,}\sigma_2)=cst.e^{-\frac{1}{2}\frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2\sigma_2^2}\left(x-\frac{\sigma_2^2\mu_1+\sigma_1^2\mu_2}{\sigma_1^2+\sigma_2^2}\right)^2}$$

$$=cst.Y(\mu,\sigma)$$

With $$\mu=\frac{\sigma_2^2\mu_1+\sigma_1^2\mu_2}{\sigma_1^2+\sigma_2^2}$$

$$\frac{1}{\sigma^2}=\frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2\sigma_2^2}$$

Re-interpret Expectation Weighted Sum

$$\mu = \frac{\sigma_2^2 \mu_1 + \sigma_1^2 \mu_2}{\sigma_1^2 + \sigma_2^2} \quad => \quad \mu = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \mu_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \mu_2$$

weight for u1

Complementary weight for u2

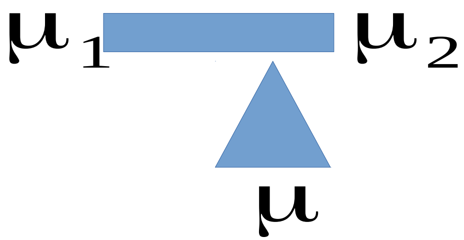
Weigths are in [0, 1]
(0% to 100%)

$$0 \leq \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \leq 1 \qquad 0 \leq \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \leq 1$$

Sum of both Weigths = 1
(sum is 100%)

$$\frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} = 1$$

Result value is in range $[\mu_1, \mu_2]$

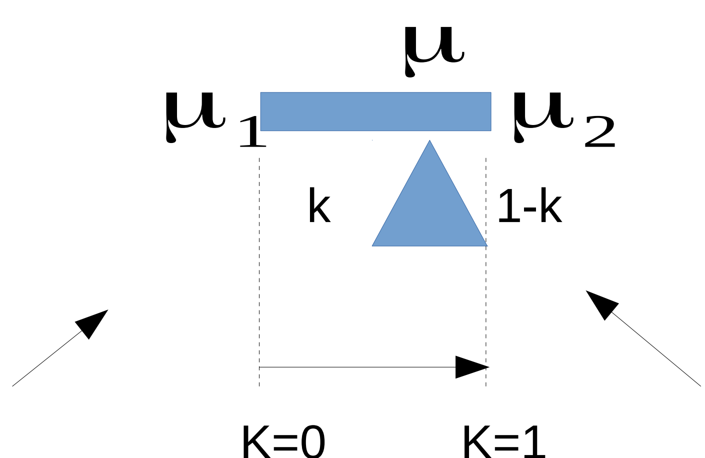


Closest to the more accurate (smallest var)

Re-interpret weight as Kalman Gain

posing $K=\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$ $\mu=K\mu_1+(1-K)\mu_2$

Choose Letter “K” in loving memory of Rudolph E Kalman



When $\sigma_1 \ll \sigma_2$

Y2 not Accurate... Y1 better

=> more weight on Y2
K ~ 1 1-K ~ 0
=> closest to u2

When $\sigma_1 \gg \sigma_2$

Y1 not Accurate... Y2 better

=> more weight on Y2
K ~ 0 1-K ~1
=> closest to u2

K = remember Y1 / correct Y2 ratio

Re-interpret StandardDeviation (variance)

$$\frac{1}{\sigma^2}=\frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2\sigma_2^2} \quad \Rightarrow \quad \sigma^2=\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}$$

Also see as Sum of inverse of square

$$\frac{1}{\sigma^2}=\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}$$

Remarks:

If $\sigma_1=\sigma_2=\sigma_i$ then $\frac{1}{\sigma^2}=\frac{2}{\sigma_i^2}$ so $\sigma=\frac{\sigma_i}{\sqrt{2}}$

Remember... Big Numbers Law (with N=2!) : Average(Y1...YN) → $N\left(\mu,\frac{\sigma}{\sqrt{N}}\right)$

Re-interpret StdDev with K

$$\sigma^2 = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

$$K = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

$$\sigma^2 = K \sigma_1^2 = (1 - K) \sigma_2^2$$

If Y1 is the a-priori estimation
and Y2 is a measurment value
Then the new variance of a-postiori estimation
 is reduced by a factor (1-K)