1 General Principles

Let $S(t)$ be a 1-dimensional time signal (sound) to be compressed. It is discretized in time by sampling at regular frequency (example 8000 Hz): $t_0, t_1 \dots t_N$
$t_n = t_0 + n * \tau$, $\tau = 1/F$

Each signal measure $S_n = s(t_n)$ is converted from analogic to digital, so dicretized and encoded to a fixed integer precision (example: 16 bits).

1.1 different measures of residu

While decomposing a signal S into main components H_k: $S = \sum_k H_k + R$, the residual signal $R = S - \sum_k H_k$ must be as small as possible: $\lim_{k \to \infty} ||R|| = 0$
Signal Entropy:

$$E(S) = \sum_n p(S_n) log(S_n) \tag{1}$$

where $p(S_n)$ is the probability of obtaining value S_n, and $log(S_n)$ is the number of bits (log in base 2) for encoding value S_n.
Signal Quadratic Variance:

$$Var(S) = ||S|| = \sum_n S_n^2 \tag{2}$$

Var(S) represents the norm of the signal = distance to 0. It is linked to scalar product $< f, g >= \sum_n f_n g_n$.
Signal Absolute Area:

$$AbsArea(S) = \sum_n abs(S_n) \tag{3}$$

AbsArea(S) represents the total area of the signal with the x-axis.
Signal Maximum Range:

$$MaxRange(S) = \max_n S_n - \min_n S_n \tag{4}$$

MaxRange(S) represents the heigth of the horizontal band containing the signal (for a symmetric signal arround 0, it is the max distance to 0).

2 Linear Autocorrelation

$\tilde{(}X_t)$ linear prediction for X_t knowing X_{t-1}:

$$\tilde{(}X_t) = AX_{t-1} + B_t, X_t = \begin{pmatrix} x_t \\ x_{t-1} \\ \vdots \\ x_{t-P} \end{pmatrix}, A = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ 1 & 0 & \dots & & \\ 0 & 1 & 0 & & \\ \vdots & & \ddots & & \end{pmatrix} \tag{5}$$

The error between real and predicted value is $\tilde{(X_t)} - X_t = \begin{pmatrix} x_t - \sum_n a_n x_{t-n} - B_t \\ 0 \\ 0 \\ 0 \end{pmatrix}$

The quadratic error is

$$
\begin{aligned}
Q = Var(\tilde{(X)} - X) &= \sum_n (\tilde{(X_n)} - X_n)^2 \\
&= \sum_t (x_t - \sum_n a_n x_{t-n} - B_t).(x_t - \sum_n a_n x_{t-n} - B_t) \\
&= \sum_t \left((x_t - B_t)^2 - 2(x_t - B_t).(\sum_n a_n x_{t-n}) + (\sum_{n_1} a_{n_1} x_{t-n_1}).(\sum_{n_2} a_{n_2} x_{t-n_2}) \right) \\
&= \sum_{n_1,n_2} a_{n_1} a_{n_2} \left(\sum_t x_{t-n_1} x_{t-n_2} \right) \\
&\quad -2 \sum_n a_n \left(\sum_t (x_t - B_t) x_{t-n} \right) \\
&\quad + \left(\sum_t (x_t - B_t)^2 \right)
\end{aligned}
\tag{6}
$$

This is a quadratic form of term a_{ij}. It can be written as

$$
Q(a) = Q^0 - 2N^t a + a^t M a \tag{7}
$$

with $M = \begin{pmatrix} & \vdots \\ \dots & \sum_t x_{t-i} x_{t-j} \end{pmatrix}$, $N = \begin{pmatrix} \vdots \\ \sum_t (x_t - B_t) x_{t-i} \\ \vdots \end{pmatrix}$, $Q^0 = \sum_t (x_t - B_t)^2$

A simple linear calculation, analogue to quadratic scalar form $ax^2 + bx + c$ ...gives min for $x = -b/2a$, with value=$c - b^2/2a$.

If M is inversible, the minimum of $Q(a)$ is obtained for $a = M^{-1}N$, and has value $Q^0 - N^t M^{-1} N$ If M is not inversible, a similar result can be obtained with the pseudo inverse.

Note that the result can be interpreted by saying that the variance has reduced by $N^t M^{-1} N$ when applying the autocorrelation procedure.

3 Least Square Method for Harmonics Amplitude

The signal is decomposed in K main harmonics, with a residu: $S(t_n) = \sum_k S_k(t_n) + R(t)$

We supposed we know frequencies w_k and phases ϕ_k (example: by using zero-crossing algorithms, see next), we want to adjust optimal amplitude coefficient c_k to miminize quadratic errors.
For calculation with complex $H_k = c_k \cos(w_k t + \phi_k)$ will be replaced by $H_k = c_k e^{i(w_k t+\phi_k)}$, and $||x||^2 = x\bar{x}$

$$
\begin{aligned}
||R||^2 &= \sum_n ||S_n - \sum_k H_k(t_n)||^2 \\
&= \sum_n \left(S_n - \sum_{k_1} H_{k_1}(t_n)\right)\left(S_n - \sum_{k_2} \bar{H}_{k_2}(t_n)\right) \\
&= \underbrace{\left(\sum_n S_n^2\right)}_{c} - \left(\sum_k \underbrace{\left(\sum_n S_n(H_k + \bar{H}_k)(t_n)\right)}_{b_k}\right) + \left(\sum_{k_1,k_2} \underbrace{\sum_n H_{k_1}\bar{H}_{k_2}(t_n)}_{a_{k_1k_2}}\right)
\end{aligned}
$$

Using $H_k(t_n) = c_k e^{i(w_k t+\phi_k)}$
it follows $H_k + \bar{H}_k(t) = 2c_k \cos(w_k t + \phi_k)$ (is real, not complex), and

$$b_k = 2c_k \sum_n S_n \cos(w_k t + \phi_k)$$

Then for $a_{k_1k_2}$, $H_{k_1}\bar{H}_{k_2}(t) = c_{k_1}c_{k_2}e^{i((w_{k_1}-w_{k_2})t+(\phi_{k_1}-\phi_{k_2}))}$
summing twice, complex part of $\ldots(k_1 - k_2) + \ldots(k_2 - k_1)$ give zero
$\frac{1}{2}(H_{k_1}\bar{H}_{k_2} + H_{k_2}\bar{H}_{k_1})(t) = \frac{1}{2}c_{k_1}c_{k_2}2\cos((w_{k_1} - w_{k_2})t + (\phi_{k_1} - \phi_{k_2}))$
so

$$a_{k_1k_2} = c_{k_1}c_{k_2} \sum_n \cos\left((w_{k_1} - w_{k_2})t_n + \phi_{k_1} - \phi_{k_2}\right)$$

($a_{k_1k_2}$ is a real symmetric positive matrix)
Finally,

$$||R||^2 = \underbrace{\left(\sum_n S_n^2\right)}_{R^0} -2\sum_k c_k \underbrace{\left(\sum_n S_n \cos(w_k t + \phi_k)\right)}_{B_k} + \sum_{k_1,k_2} c_{k_1}c_{k_2} \underbrace{\left(\sum_n \cos\left((w_{k_1} - w_{k_2})t_n + \phi_{k_1} - \phi_{k_2}\right)\right)}_{A_{k_1k_2}} \tag{8}$$

This is a quadratic form on vector variable c_k :

$$||R||^2 = R^0 - 2B^t c + c^t Ac \tag{9}$$

The minimum is reached for $c = A^{-1}B$, and the minimum is $R^0 - B^t A^{-1}B$
Note that the result can be interpreted by saying that the variance has reduced by $B^t A^{-1}B$ when applying the least-square amplitude fitting.

4 Least Square Method for Harmonics Amplitude Linear Perturbation

The signal is decomposed in K main harmonics, with a residu: $S(t_n) = \sum_k S_k(t_n) + R(t)$
We have known approximations for frequencies w_k, phases ϕ_k and amplitudes c_k.
The K^{th} harmonic $S_k(t) = c_k \cos(w_k t + \phi_k)$ is replaced by modifying the constant amplitude c_k, to obtain a "non-periodic harmonic": $c_k(t) = c_k^0 + c_k^1(t - t_0)$
This model is usable only for short-time interval!
The calculation done in previous section is slightly modified.

$$
\begin{aligned}
||R||^2 = & \left(\sum_n S_n^2\right) \\
& -2\sum_k \left(\sum_n c_k(t_n) S_n \cos(w_k t_n + \phi_k)\right) \\
& + \sum_{k_1,k_2} \left(\sum_n c_{k_1}(t_n) c_{k_2}(t_n) \cos\left((w_{k_1} - w_{k_2})t_n + \phi_{k_1} - \phi_{k_2}\right)\right)
\end{aligned}
\tag{10}
$$

We want to expand $c_k(t) = c_k^0 + c_k^1(t - t_0)$, then factorize the variance as a quadratic form on vector term c_k^1
For ease of calculation, lets note $COS_{kn} = \cos(w_k t_n + \phi_k)$ and $COS\Delta_{k_1 k_2 n} = \cos\left((w_{k_1} - w_{k_2})t_n + \phi_{k_1} - \phi_{k_2}\right)$

$$
\begin{aligned}
||R||^2 = & \left(\sum_n S_n^2\right) \\
& -2\sum_k \sum_n \left(c_k^0 + c_k^1(t_n - t_0)) S_n COS_{kn}\right) \\
& + \sum_{k_1,k_2} \left(\sum_n (c_{k_1}^0 + c_{k_1}^1(t_n - t_0))(c_{k_2}^0 + c_{k_2}^1(t_n - t_0)) COS\Delta_{k_1 k_2 n}\right) \\
= & \sum_n S_n^2 \\
& -2\sum_k c_k^0 \sum_n S_n COS_{kn} - 2\sum_k c_k^1 \left(\sum_n (t_n - t_0) S_n COS_{kn}\right) \\
& + \sum_{k_1,k_2} \left(\sum_n \left(c_{k_1}^0 c_{k_2}^0 + (c_{k_1}^0 c_{k_2}^1 + c_{k_1}^1 c_{k_2}^0)(t_n - t_0) + c_{k_1}^1 c_{k_2}^1 (t_n - t_0)^2\right) COS\Delta_{k_1 k_2 n}\right) \\
= & \sum_n S_n^2
\end{aligned}
$$

$$
\begin{aligned}
&-2\sum_k c_k^0 \sum_n S_n COS_{kn} - 2\sum_k c_k^1 \left(\sum_n (t_n - t_0) S_n COS_{kn}\right) \\
&+ \sum_{k_1,k_2} \left(\sum_n \left(c_{k_1}^0 c_{k_2}^0\right) COS\Delta_{k_1 k_2 n}\right) \\
&+ \sum_{k_1,k_2} \left(\sum_n \underbrace{(c_{k_1}^0 c_{k_2}^1 + c_{k_1}^1 c_{k_2}^0)}_{=2c_{k_1}^1 c_{k_2}^0} (t_n - t_0) COS\Delta_{k_1 k_2 n}\right) \\
&+ \sum_{k_1,k_2} \left(\sum_n \left(c_{k_1}^1 c_{k_2}^1 (t_n - t_0)^2\right) COS\Delta_{k_1 k_2 n}\right)
\end{aligned}
\tag{11}
$$

Then. . .

$$
\begin{aligned}
||R||^2 = &\underbrace{\left(\sum_n S_n^2 - 2\sum_k c_k^0 \sum_n S_n COS_{kn} + \sum_{k_1,k_2} c_{k_1}^0 c_{k_2}^0 \sum_n COS\Delta_{k_1 k_2 n}\right)}_{R^0} \\
&-2\sum_k c_k^1 \underbrace{\left(\sum_n (t_n - t_0) S_n COS_{kn} - \sum_{k_2} c_{k_2}^0 \sum_n (t_n - t_0) COS\Delta_{k k_2 n}\right)}_{B_k} \\
&+ \sum_{k_1,k_2} c_{k_1}^1 c_{k_2}^1 \underbrace{\left(\sum_n \left((t_n - t_0)^2\right) COS\Delta_{k_1 k_2 n}\right)}_{A_{k_1 k_2}}
\end{aligned}
\tag{12}
$$

Finally,

$$||R||^2 = R^0 - 2B^t c^1 + (c^1)^t A(c^1) \tag{13}$$

The minimum is reached for $c^1 = A^{-1}B$, and the minimum is $R^0 - B^t A^{-1} B$
Note that the result can be interpreted by saying that the variance has reduced by $B^t A^{-1} B$ when applying the least-square amplitude linear perturbation procedure.

5 Least Square Method for Harmonics Frequency Linear Perturbation

The signal is decomposed in K main harmonics, with a residu: $S(t_n) = \sum_k S_k(t_n) + R(t)$

The K^{th} harmonic $S_k(t) = c_k cos(w_k t + \phi_k)$ with $c_k >= 0$, $\phi_k \in [-\pi, \pi]$ is replaced by modifying the constant frequency w_k, to obtain a "non-periodic harmonic": $w_k(t) = w_k^0 + w_k^1(t - t_0)$ (or ease of writing, $t_0 = 0$ ... which is a change in t origin).
This model is usable only for short-time interval!
The calculation done in previous section is slightly modified.

$$
\begin{aligned}
||R||^2 = &\left(\sum_n S_n^2\right) \\
&-2\sum_k \left(\sum_n c_k S_n \cos(w_k(t_n)t_n + \phi_k)\right) \\
&+\sum_{k_1,k_2} \left(\sum_n c_{k_1} c_{k_2} \cos\left((w_{k_1}(t_n) - w_{k_2}(t_n))t_n + \phi_{k_1} - \phi_{k_2}\right)\right)
\end{aligned}
\tag{14}
$$

Developping trigonometric formula $\cos(a+b) = \cos(a)cos(b) - \sin(a)\sin(b)$, for expanding $w_k(t)$, we get

$$
\begin{aligned}
\cos(w_k(t_n)t_n + \phi_k) &= \cos((w_k^0 + w_k^1 t_n)t_n + \phi_k) \\
&= \cos((w_k^0 t_n + \phi_k) + w_k^1 t_n) \\
&= \cos(w_k^0 t_n + \phi_k)\cos(w_k^1 t_n) - \sin(w_k^0 t_n + \phi_k)\sin(w_k^1 t_n) \\
&\approx \cos(w_k^0 t_n + \phi_k)(1 - 1/2(w_k^1 t_n)^2) - \sin(w_k^0 t_n + \phi_k)(w_k^1 t_n)
\end{aligned}
\tag{15}
$$

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6 Partial Derivative of Perturbation Residu

given P parameters,

$$S_k(t_n) = A_k(t_n, p_1, p_2..p_P) sin(w_k(t_n, p_1, p_2, ..p_P)t_n + \phi_k(t_n, p_1, p_2, ..p_P))$$

A_k, w_k and ϕ_k could have the form $A_k(t, p..) = p_0 + p_1 t + p_2 t^2$. For now we compute in the general case with p derivatives.

$$
\begin{aligned}
E = ||R||^2 = &\sum_n \left(S_n - \sum_k S_k(t_n, p_1, p_2, \ldots p_P)\right)^2 \\
\frac{\partial E}{\partial p_i} = &-2\sum_n \left(\sum_k \frac{\partial S_k}{\partial p_i}(t_n, p_1, p_2, \ldots p_P)\right)\left(S_n - \sum_k S_k(t_n, p_1, p_2, \ldots p_P)\right)
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
||R||^2 &= \sum_n \left(S_n - A_k(t_n, p_.) sin(w_k(t_n, p_.)t_n + \phi_k(t_n, p_.))\right)^2 \\
\frac{\partial ||R(p)||^2}{\partial p_i} &= -2 \sum_n \frac{\partial}{\partial p} \left(A_k(t_n, p_.) \sin(w_k(t_n, p_.)t_n + \phi_k(t_n, p_.))\right) . \left(S_n - A_k(t_n, p_.) \sin(w_k(t_n, p_.)t_n + \phi_k(t_n, p
\\
&= -2 \sum_n (\frac{\partial A_k}{\partial p}(t_n, p_.) \sin(w_k t_n + \phi_k) + A_k(t_n, p_.) \left(\frac{\partial w_k}{\partial p}(t_n, p_.)t_n + \frac{\partial \phi_k}{\partial p}(t_n, p_.) \right) \cos(w_k t_n + \phi_k) \\
&\quad) \left(S_n - A_k \sin(w_k t_n + \phi_k)\right)
\end{aligned}
$$