Tutorial on non-parametric estimation
TSIA202b

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Exercise 1 *(White noise & Periodogram)*

Let y_t be a (centered) Gaussian white noise of variance σ^2 and let

$$Y(\nu_k) = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} y_t e^{-2i\pi\nu_k t}; \quad \nu_k = \frac{k}{N} \quad k \in \{0, \ldots, N-1\}$$

where $Y(\nu_k)$ represents the discrete Fourier transform (DFT) normalized (and evaluated at the Fourier frequency ν_k).

1. Compute the covariances

$$\mathbb{E}\left(Y(\nu_k)\overline{Y(\nu_r)}\right), k, r = 0, \ldots, N-1$$

2. Use the result of the previous question to conclude that the periodogram $\hat{S}_{P,yy}(\nu_k) = |Y(\nu_k)|^2$ is an unbiased estimator of the power spectral density (PSD) of y_t.

3. Explain why this estimator remains unbiased, even in the case where $\nu \neq \nu_k$ (give an intuitive explanation).

Exercise 2 *(Reduction of the mean squared error (MSE))*

Let $\hat{x}$ be an estimate of an unknown parameter x. Suppose that $\hat{x}$ is unbiased (i.e. $\mathbb{E}(\hat{x}) = x$) and let $\sigma_{\hat{x}}^2$ be the mean square error (MSE) of $\hat{x}$:

$$MSE(\hat{x}) := \sigma_{\hat{x}}^2 := \mathbb{E}\left((\hat{x} - x)^2\right)$$

(it should be noted that since $\hat{x}$ is unbiased, $\sigma_{\hat{x}}^2$ is equal to the variance of $\hat{x}$). For a fixed constant ρ, we define:

$$\tilde{x} = \rho\hat{x}$$

another estimator of x. The coefficient ρ, often called the "reduction or regression coefficient", can be chosen in such a way as to minimize the MSE of $\tilde{x}$ (denoted $\sigma_{\tilde{x}}^2$) and to make the latter smaller than $\sigma_{\hat{x}}^2$.

1. *(General result)* Let $\hat{y}$ be an estimator (biased or not) of an unknown parameter y. Prove that:

$$MSE(\hat{y}) = \mathbb{E}\left((\hat{y} - \mathbb{E}(\hat{y}))^2\right) + (\mathbb{E}(\hat{y}) - y)^2$$

2. Deduce that the MSE of $\tilde{x}$ reaches its minimum value:

$$\sigma_{\tilde{x}_0}^2 = \rho_0 \sigma_{\hat{x}}^2$$

   for:

$$\rho_0 = \frac{x^2}{x^2 + \sigma_{\hat{x}}^2}$$

We will now apply this result to the case of the periodogram. In the course, we have seen that the spectral estimation based on the periodogram $\hat{S}_{P,XX}$ is asymptotically unbiased and has an MSE equal to the square of the value of the power spectral density (PSD) S_{XX}:

$$\mathbb{E}\left(\hat{S}_{P,XX}(\nu)\right) \underset{N\to\infty}{\longrightarrow} S_{XX}(\nu), \quad \mathbb{E}\left(\left(\hat{S}_{P,XX}(\nu) - S_{XX}(\nu)\right)^2\right) \underset{N\to\infty}{\longrightarrow} S_{XX}(\nu)^2$$

3. Show that the estimate of the "optimal periodogram" (in the sense mentioned above) is:

$$\tilde{S}_{P,XX}(\nu) = \hat{S}_{P,XX}(\nu)/2$$

and that the MSE of $\tilde{S}_{P,XX}(\nu)$ is equal to half the MSE of $\hat{S}_{P,XX}(\nu)$. Then propose a method to estimate the PSD of the signal from this new estimator minimizing the MSE.

Exercise 3 *(Estimation of the asymptotic maximum likelihood of $\hat{S}_{P,XX}(\nu)$ from $\hat{S}_{P,XX}(\nu)$)*

It has been proven in the course that, asymptotically w.r.t. N, the average of $\hat{S}_{P,XX}(\nu)$ is $S_{XX}(\nu)$ and its variance is $S_{XX}(\nu)^2$. In this exercise, we assume that the $\hat{S}_{P,XX}(\nu)$ are Gaussian. This results in the fact that the probability density of $\hat{S}_{P,XX}(\nu)$ is (for the sake of clarity, we omit the dependency in ν):

$$p_{S_{XX}}\left(\hat{S}_{P,XX}\right) = \frac{1}{\sqrt{2\pi S_{XX}^2}} \exp\left(-\frac{\left(\hat{S}_{P,XX} - S_{XX}\right)^2}{2S_{XX}^2}\right)$$

1. Show that the maximum likelihood estimate (MLE) of S_{XX} by $\hat{S}_{P,XX}$ (defined as the maximum argument S_{XX} of $p_{S_{XX}}\left(\hat{S}_{P,XX}\right)$) is given by:

$$\tilde{S}_{P,XX} = \frac{-1+\sqrt{5}}{2}\hat{S}_{P,XX}$$

2. Compare $\tilde{S}_{P,XX}$ with the estimate of S_{XX} presented in Exercise 2, in terms of bias and MSE.