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A Two-Stage Image Segmentation Method Using a Convex Variant of the Mumford–Shah Model and Thresholding*

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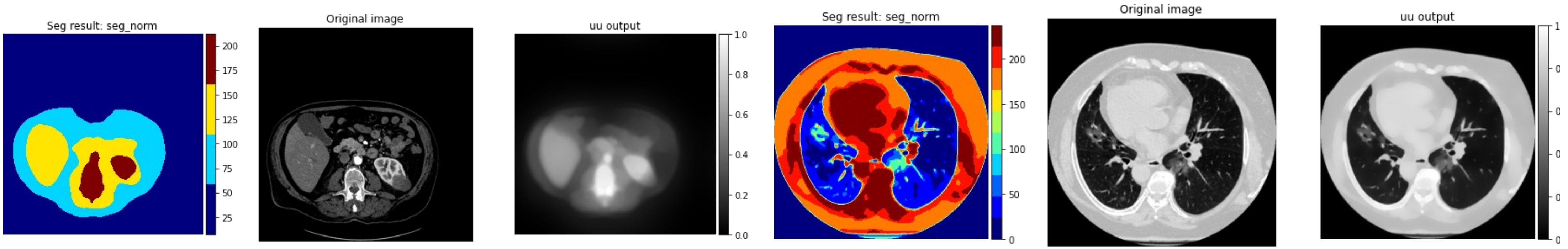
1. Introduction. Let $\Omega \subset \mathbb{R}^2$ be a bounded open connected set, Γ be a compact curve in Ω, and $f : \Omega \to \mathbb{R}$ be a given image.



Mumford-Shah framework

$$E_{\mathrm{MS}}(g,\Gamma) = \frac{\lambda}{2}\int_{\Omega}(f-g)^2dx + \frac{\mu}{2}\int_{\Omega\setminus\Gamma}|\nabla g|^2dx + \mathrm{Length}(\Gamma), \tag{1.1}$$

where λ and μ are positive parameters and $g : \Omega \to \mathbb{R}$ is continuous or even differentiable in $\Omega \setminus \Gamma$ but may be discontinuous across Γ. Here, the length of Γ can be written as $\mathcal{H}^1(\Gamma)$, the

Their solution

find a smooth image g that can facilitate the segmentation,

$$\inf_g \left\{ \frac{\lambda}{2}\int_{\Omega}(f-\mathcal{A}g)^2dx + \frac{\mu}{2}\int_{\Omega}|\nabla g|^2dx + \int_{\Omega}|\nabla g|dx \right\}, \tag{1.2}$$

$\mathcal{A}$ can be the identity operator or a blurring operator

threshold g to reveal different segmentation features.

convex with a unique smooth solution.

Simplification of the MS variational problem

$$E_{\mathrm{MS}}(g,\Gamma) = \frac{\lambda}{2}\int_{\Omega}(f-g)^2dx + \frac{\mu}{2}\int_{\Omega\backslash\Gamma}|\nabla g|^2dx + \mathrm{Length}(\Gamma), \tag{1.1}$$

where λ and μ are positive parameters and $g : \Omega \to \mathbb{R}$ is continuous or even differentiable in $\Omega \setminus \Gamma$ but may be discontinuous across Γ. Here, the length of Γ can be written as $\mathcal{H}^1(\Gamma)$, the

Assume that Γ is a Jordan curve. Let $\Sigma = \overline{\mathrm{Inside}(\Gamma)}$; then $\Gamma = \partial\Sigma$. Model (1.1) can be written as

$$\begin{aligned}\tilde{E}(\Sigma, g_1, g_2) := \frac{\lambda}{2}\int_{\Sigma\backslash\Gamma}(f-g_1)^2dx + \frac{\mu}{2}\int_{\Sigma\backslash\Gamma}|\nabla g_1|^2dx + \frac{\lambda}{2}\int_{\Omega\backslash\Sigma}(f-g_2)^2dx \\ + \frac{\mu}{2}\int_{\Omega\backslash\Sigma}|\nabla g_2|^2dx + \mathrm{Per}(\Sigma),\end{aligned} \tag{2.1}$$

where g_1 and g_2 are defined on $\Sigma \setminus \Gamma$ and $\Omega \setminus \Sigma$, respectively, and $\mathrm{Per}(\cdot)$ denotes the perimeter of Σ; i.e., $\mathrm{Per}(\Sigma) = \mathrm{Length}(\Gamma)$.

Left term

once Σ is fixed, then g_1 and g_2 are determined by the following two minimization problems:

$$\inf_{g_1 \in W^{1,2}(\Sigma\setminus\Gamma)} \left\{ \lambda \int_{\Sigma\setminus\Gamma} (f-g_1)^2 dx + \mu \int_{\Sigma\setminus\Gamma} |\nabla g_1|^2 dx \right\} \tag{2.2}$$

Extra slide

Sobolev space

and

$$\inf_{g_2 \in W^{1,2}(\Omega\setminus\Sigma)} \left\{ \lambda \int_{\Omega\setminus\Sigma} (f-g_2)^2 dx + \mu \int_{\Omega\setminus\Sigma} |\nabla g_2|^2 dx \right\}. \tag{2.3}$$

➔ Special case: Chan-Vese model

Proposition 2.1. *Let $f \in L^2(\Omega)$. Then the two minimization problems (2.2) and (2.3) have unique minimizers.*

right term

once g_1 and g_2 are fixed and smoothly extended to the whole Ω.

Theorem 2.2. *For any given fixed functions g_1 and $g_2 \in W^{1,2}(\Omega)$, a global minimizer for $\tilde{E}(\Sigma, g_1, g_2)$ in (2.1) can be found by carrying out the following convex minimization,*

$$\min_{0\le u\le 1}\left\{\int_\Omega |\nabla u| + \frac{1}{2}\int_\Omega \{\lambda(f-g_1)^2+\mu|\nabla g_1|^2-\lambda(f-g_2)^2-\mu|\nabla g_2|^2\}u(x)\right\}, \tag{2.4}$$

and setting $\Sigma = \{x : u(x) \ge \rho\}$ for almost every $\rho \in [0,1]$.

$0 \le u \le 1,$

$\int_\Omega |\nabla u| = \int_0^1 \mathrm{Per}(\{x : u(x) > \rho\})d\rho.$ $\quad u(x) = \int_0^1 \mathbf{1}_{[0,u(x)]}(\rho)d\rho dx$

Setting $\Sigma(\rho) = \overline{\{x : u(x) > \rho\}}$
$\Gamma(\rho) = \partial\Sigma(\rho)$

From Theorem 2.2, we see that the term $\mathrm{Per}(\Sigma)$ of (2.1) is replaced by a convex integral term $\int_\Omega |\nabla u|$. In other words, the boundary information of Γ in (1.1) can be extracted from the TV (total variation) term $\int_\Omega |\nabla u|$. This motivates us to use $\int_\Omega |\nabla g|$ to extract the boundary information Length(Γ) in (1.1).

middle term

Lemma 2.3. *If* $g \in W^{1,2}(\Omega)$ *and* Γ *is a closed curve with* $m(\Gamma) = 0$*, where* $m(\cdot)$ *is the Lebesgue measure, then* $\int_\Gamma |\nabla g|^2 dx = 0$*.*

Proof. Since $g \in W^{1,2}(\Omega)$, we have $\nabla g \in L^2(\Omega)$. Because of $m(\Gamma) = 0$, we get $\int_\Gamma |\nabla g|^2 dx = 0$ immediately. ■

Thus the middle term of model (1.1) becomes

$$\int_{\Omega\setminus\Gamma} |\nabla g|^2 dx = \int_\Omega |\nabla g|^2 dx - \int_\Gamma |\nabla g|^2 dx = \int_\Omega |\nabla g|^2 dx \quad \forall g \in W^{1,2}(\Omega). \tag{2.5}$$

$$E_{\mathrm{MS}}(g,\Gamma) = \frac{\lambda}{2}\int_{\Omega}(f-g)^2 dx + \frac{\mu}{2}\int_{\Omega\setminus\Gamma}|\nabla g|^2 dx + \mathrm{Length}(\Gamma), \tag{1.1}$$

where λ and μ are positive parameters and $g : \Omega \to \mathbb{R}$ is continuous or even differentiable in $\Omega \setminus \Gamma$ but may be discontinuous across Γ. Here, the length of Γ can be written as $\mathcal{H}^1(\Gamma)$, the

Final simplification

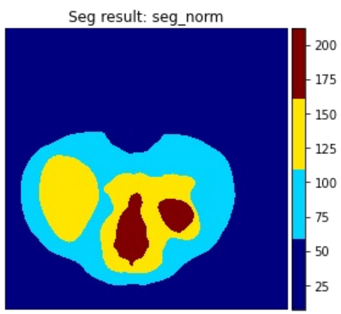
In view of Theorem 2.2 and (2.5), we propose our segmentation model as

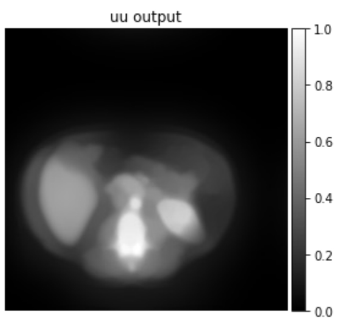
$$\inf_{g\in W^{1,2}(\Omega)} E(g) := \inf_{g\in W^{1,2}(\Omega)} \left\{ \frac{\lambda}{2}\int_{\Omega}(f-\mathcal{A}g)^2 dx + \frac{\mu}{2}\int_{\Omega}|\nabla g|^2 dx + \int_{\Omega}|\nabla g| dx \right\} \tag{2.6}$$

➔ A 2 stages segmentation problem:

We emphasize that model (2.6) can be minimized quickly by using currently available efficient algorithms such as the split-Bregman algorithm [26] or the Chambolle–Pock method [13, 44]. Once g is obtained, we enter into the second stage of our method, where we use thresholding to segment g into different phases. The thresholds can be determined by any clustering methods or be chosen by the users.

1^{st} stage: Numerical implementation





$$(3.1) \qquad \min_{g}\left\{\frac{\lambda}{2}\|f-\mathcal{A}g\|_2^2+\frac{\mu}{2}\|\nabla g\|_2^2+\|\nabla g\|_1\right\},$$

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$\|\nabla g\|_1 := \sum_{i\in\Omega}\sqrt{(\nabla_x g)_i^2+(\nabla_y g)_i^2}$ is the classical discrete TV seminorm.

$$(\nabla_x g)_i = \begin{cases} g(1,1)-g(1,n), & i=1,\\ g(1,i)-g(1,i-1), & i=2,\dots,n, \end{cases}$$

backward difference with periodic boundary condition

Set $d_x = \nabla_x g$ and $d_y = \nabla_y g$ in (3.1), and this yields the constrained problem

$$\min_{g}\left\{\frac{\lambda}{2}\|f-\mathcal{A}g\|_2^2+\frac{\mu}{2}\|\nabla g\|_2^2+\boxed{\|(d_x,d_y)\|_1}\right\} \quad \text{s.t. } d_x=\nabla_x g \text{ and } d_y=\nabla_y g.$$

Using the 2-norm to weakly enforce the above constraints, it becomes

1

$$\min_{g,d_x,d_y} \left\{ \frac{\lambda}{2}\|f - \mathcal{A}g\|_2^2 + \frac{\mu}{2}\|\nabla g\|_2^2 + \boxed{\|(d_x, d_y)\|_1 + \boxed{\frac{\sigma}{2}\|d_x - \nabla_x g\|_2^2} + \boxed{\frac{\sigma}{2}\|d_y - \nabla_y g\|_2^2}} \right\}.$$

1

Applying the split-Bregman iteration to strictly enforce the constraints, we have at step $(k+1)$

$$(g^{k+1}, d_x^{k+1}, d_y^{k+1}) = \arg\min_{g,d_x,d_y} \left\{ \frac{\lambda}{2}\|f - \mathcal{A}g\|_2^2 + \frac{\mu}{2}\|\nabla g\|_2^2 + \|(d_x, d_y)\|_1 \right.$$

$$\left. + \frac{\sigma}{2}\|d_x - \nabla_x g - b_x^k\|_2^2 + \frac{\sigma}{2}\|d_y - \nabla_y g - b_y^k\|_2^2 \right\}, \tag{3.2}$$

$$b_x^{k+1} = b_x^k + (\nabla_x g^{k+1} - d_x^{k+1}), \quad b_y^{k+1} = b_y^k + (\nabla_y g^{k+1} - d_y^{k+1}). \tag{3.3}$$

Extra slide

minimizing with respect to g and (d_x, d_y) alternatively.

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$$g^{k+1} = \arg\min_g \left\{ \frac{\lambda}{2}\|f - \mathcal{A}g\|_2^2 + \frac{\mu}{2}\|\nabla g\|_2^2 + \frac{\sigma}{2}\|d_x^k - \nabla_x g - b_x^k\|_2^2 \right.$$

$$\left. + \frac{\sigma}{2}\|d_y^k - \nabla_y g - b_y^k\|_2^2 \right\}, \tag{3.4}$$

differentiable

g^{k+1} satisfies

$$(\lambda\mathcal{A}^*\mathcal{A} - (\mu + \sigma)\Delta)g = \lambda\mathcal{A}^* f + \sigma\nabla_x^T(d_x^k - b_x^k) + \sigma\nabla_y^T(d_y^k - b_y^k), \tag{3.6}$$

$$(d_x^{k+1}, d_y^{k+1}) = \arg\min_{d_x, d_y} \left\{ \|(d_x, d_y)\|_1 + \frac{\sigma}{2}\|d_x - \nabla_x g^{k+1} - b_x^k\|_2^2 \right.$$

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$$\left. + \frac{\sigma}{2}\|d_y - \nabla_y g^{k+1} - b_y^k\|_2^2 \right\}. \tag{3.5}$$

Shrinkage operator

➔ L1 sparse approximation

➔ Keep strongest “coeff”

$$d_x^{k+1} = \max\left(s^k - \frac{1}{\sigma}, 0\right)\frac{s_x^k}{s^k}, \quad d_y^{k+1} = \max\left(s^k - \frac{1}{\sigma}, 0\right)\frac{s_y^k}{s^k}, \tag{3.7}$$

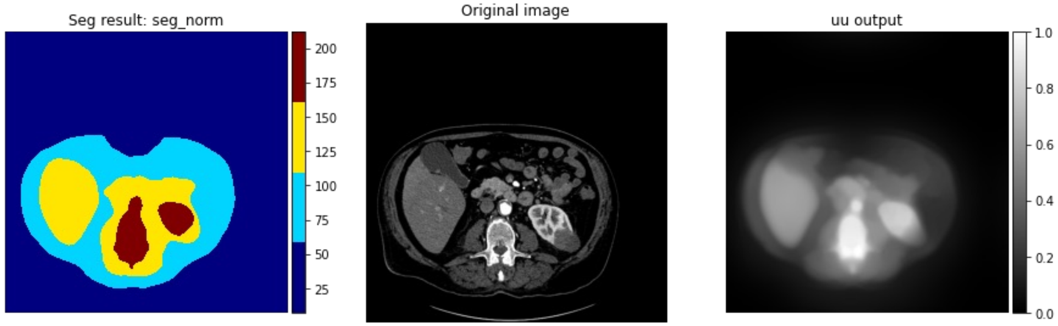
where $s_x^k = \nabla_x g^{k+1} + b_x^k$, $s_y^k = \nabla_y g^{k+1} + b_y^k$ and $s^k = \sqrt{(s_x^k)^2 + (s_y^k)^2}$. The followi

$$(3.1)\qquad \min_g \left\{ \frac{\lambda}{2}\|f - \mathcal{A}g\|_2^2 + \frac{\mu}{2}\|\nabla g\|_2^2 + \|\nabla g\|_1 \right\},$$

Algorithm 1. Solving (3.1) by the split-Bregman algorithm.

1. Initialize: $g^0 = f, d_x^0 = d_y^0 = b_x^0 = b_y^0 = 0$.
2. Do $k = 0, 1, \ldots,$ until $\frac{\|g^k - g^{k+1}\|_F}{\|g^{k+1}\|_F} < \epsilon$.
   (a) Compute g^{k+1} by solving (3.6).
   (b) Compute d_x^{k+1} and d_y^{k+1} by the shrinkage formula (3.7).
   (c) Update b_x^{k+1} and b_y^{k+1} by the formula (3.3).
3. Output: g.

2^{nd} stage: Numerical implementation



$$\bar{g} = \frac{g - g_{\min}}{g_{\max} - g_{\min}},$$

Suppose we want to segment $\bar{g}$ into K segments, $K \geq 2$. We use the K-means method to classify the set of pixel values of $\bar{g}$ into K clusters. Let the mean value of each cluster be $\hat{\rho}_1, \hat{\rho}_2, \ldots, \hat{\rho}_K$, and without loss of generality, let $\hat{\rho}_1 \leq \hat{\rho}_2 \leq \cdots \leq \hat{\rho}_K$. Then we define the $(K-1)$ thresholds as

$$\rho_i = \frac{\hat{\rho}_i + \hat{\rho}_{i+1}}{2}, \quad i = 1, 2, \ldots, K-1. \tag{3.9}$$

The ith phase of $\bar{g}$, $1 \leq i \leq K$, is then given by $\{x \in \Omega : \rho_{i-1} < \bar{g}(x) \leq \rho_i\}$. To obtain the