Tutorial on filter bank methods
TSIA202b

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Exercise 1: Capon Estimate of the Parameters of a Single Sine Wave

Assume that the data under study consists of a sinusoidal signal observed in white noise: $\forall t \in \mathbb{Z}$,

$$X_t = \alpha e^{i(2\pi\nu_0 t+\varphi)} + Z_t$$

where $\alpha \in \mathbb{C}$, $\nu_0 \in \mathbb{R}$, φ is a random variable uniformly distributed on $[0, 2\pi]$, and $Z_t \sim BB(0, \sigma^2)$ is independent from φ.

1. Prove that X_t is a WSS process and give the mathematical expression of its mean μ_X and its autocovariance function $r_{XX}(k)$.
2. Explain the expression *line spectrum* that is used to name the spectral measure of X_t.
3. Show that the $N \times N$ covariance matrix R_{XX} of $X_1 \dots X_N$ is given by:

   $$R_{XX} = |\alpha|^2 e(\nu_0) e(\nu_0)^H + \sigma^2 I_N$$

   where $e(\xi) = [1, e^{i2\pi\xi} \dots e^{i2\pi\xi(N-1)}]^\top$ and I_N denotes the $N \times N$ identity matrix.
4. Check that $\left\{ \frac{e(\nu_0 + \frac{k}{N})}{\sqrt{N}} \right\}_{k \in \{0 \dots N-1\}}$ forms a unitary basis of eigenvectors of R_{XX} and give the expressions of the corresponding eigenvalues.
5. Prove that the Capon spectrum $\hat{S}_{\text{CAP},XX}(\nu) = \frac{N}{e(\nu)^H R_{XX}^{-1} e(\nu)}$ peaks at $\nu = \nu_0$, and show that the height of the peak is $N|\alpha|^2 + \sigma^2$.