Kalman Exercise Solution

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Exercise (Linear perdriction of an AR(1) observed with additive noise)

Consider an AR(1) real process Z_t[1] satisfying the following canonical equation:

$$\forall t \in \mathbb{Z},\ Z_{t+1} = \phi Z_t + \eta_t \tag{1}$$

where $(\eta_t)_{t\geq 0}$ is a centered white noise with known variance σ^2 and ϕ is a known constant. The process $(Z_t)_{t\geq 0}$ is not directly observed. Instead for all $t \geq 1$, one gets the following sequence of observations:

$$Y_t = Z_t + \epsilon_t \tag{2}$$

where $(\epsilon_t)_{t\geq 1}$ is a centered white noise with known variance ρ^2, that is uncorrelated with (η_t) and Z_0. We wish to solve the filtering problem, that is, to compute the orthogonal projection of Z_t on the space $H_t^Y = \text{span}\{Y_1, \ldots, Y_t\}$, iteatively in t.

We denote $\hat{Z}_{t|t} = \text{proj}\left(Z_t \mid H_t^Y\right)$ this projection and $P_{t|t} = \mathbb{E}\left[\left(Z_t - \hat{Z}_{t|t}\right)^2\right]$ the corresponding projection error variance[2] . Similarly, let $\hat{Z}_{t+1|t} = \text{proj}\left(Z_{t+1} \mid H_t^Y\right)$ be the best linear predictor and $P_{t+1|t} = \mathbb{E}\left[\left(Z_{t+1} - \hat{Z}_{t+1|t}\right)^2\right]$ the linear prediction error variance.

1. Show that Z_0 is a centered random variable and computes its variance σ_0^2 using the Corollary 3.1.3 and that Z_0 and $(\eta_t)_{t\geq 0}$ are uncorrelated. [3]

   Solution: From (1), we can write $(1 - \phi B) Z_{t+1} = \eta_t \Leftrightarrow Z_t = \sum_{n\geq 0} \phi^n B^n \eta_{t-1} = \sum_{n\geq 0} \phi^n \eta_{t-1-n}$

   it implies that $\mathbb{E}(Z_0) = \sum_{n\geq 0} \phi^n \mathbb{E}(\eta_{t-1-n}) = 0$ because $(\eta_t)_{t\geq 0}$ is centered.

   we have $Z_t = F_\psi(\eta_t)$ where $F_\psi = \sum_{n\in\mathbb{Z}} \psi_n B^n$. In our case, it means that $\psi_n = \phi^n$.

[1] the same exercise can be apply to a complex AR(1) process Z_t. Try by yourself to see what could be the slight difference in that case.

[2] in complex case: $P_{t|t} = \mathbb{E}\left[\left|Z_t - \hat{Z}_{t|t}\right|^2\right]$

[3] Hint: decompose Z_t as $F_\phi(B)\eta_t$ where $F_\phi(B)$ is a rational polynom fraction depends on the backshift operator and and then decompose $F_\phi(B)\eta_t$ as an infinite sum.

The corollary 3.1.3 states that the autocovariance is given by:

$$\begin{aligned}\gamma(h) &= \sigma^2 \sum_{n\in\mathbb{Z}} \psi_{n+h}\bar{\psi}_n \\ &= \sigma^2 \phi^h \sum_{n\in\mathbb{Z}} |\phi|^{2n} \\ &= \frac{\sigma^2\phi^h}{1-|\phi|^2}\end{aligned}$$

consequently: $\mathbb{E}\left[Z_0^2\right] = \gamma(0) = \frac{\sigma^2}{1-|\phi|^2}$
Finally $\mathbb{E}[Z_0\eta_t] = \sum_{n\geq 0}\phi^n \mathbb{E}[\eta_{-1-n}\eta_t] = 0$if $t \geq 0$ because (η_t) is a white noise.

2. Using the evolution (state) equation (1), show that

$$\hat{Z}_{t+1|t} = \phi\hat{Z}_{t|t} \quad \text{and} \quad P_{t+1|t} = \phi^2 P_{t|t} + \sigma^2$$

Solution:

$$\begin{aligned}\hat{Z}_{t+1|t} &= \text{proj}\left(Z_{t+1} \mid H_t^Y\right) \\ &= \phi\text{proj}\left(Z_t \mid H_t^Y\right) + \text{proj}\left(\eta_t \mid H_t^Y\right) \\ &= \phi\hat{Z}_{t|t} + \text{proj}\left(\eta_t \mid H_t^Y\right)\end{aligned}$$

However:
(i) $H_t^Y \subset H_t^Z \overset{\perp}{\oplus} H_t^\epsilon$ (the sum is due to (1) and the orthogonality because (ϵ_t) is uncorrelated with (η_t) and Z_0).
(ii) $\forall 1 \leq h \leq t, \mathbb{E}[Z_h\eta_t] = \sum_{n\geq 0}\phi^n\mathbb{E}[\eta_{h-n-1}\eta_t] = 0 \implies \eta_t \perp H_t^Z$ and $\eta_t \perp H_t^\epsilon$ by assumption.
Then we get $\text{proj}\left(\eta_t \mid H_t^Y\right) = 0$ and the result.

Now $P_{t+1|t} = \mathbb{E}\left(\left(Z_{t+1} - \hat{Z}_{t+1|t}\right)^2\right) = \mathbb{E}\left[\left(\phi\left[Z_{t+1} - \hat{Z}_{t|t}\right] + \eta_t\right)^2\right] = |\phi|^2 P_{t|t} + \sigma^2 + 2\phi\text{cov}\left(\eta_t, Z_t - \hat{Z}_{t|t}\right)$
But
(i) $Z_t - \hat{Z}_{t|t} \in H_t^Z \overset{\perp}{\oplus} H_t^\epsilon$ ($Z_t - \hat{Z}_{t|t} = Z_t - \sum_{h=1}^t \mu_h Z_h + \epsilon_h \in H_t^Z \overset{\perp}{\oplus} H_t^\epsilon$ (the orthogonality was already proven)
(ii) $\eta_t \perp H_t^Z$ and $\eta_t \perp H_t^\epsilon$
Then we get $\text{cov}\left(\eta_t, Z_t - \hat{Z}_{t|t}\right) = 0$ and the result

3. Let us define the innovation by $I_{t+1} = Y_{t+1} - \text{proj}\left(Y_{t+1} \mid H_t^Y\right)$. Using the observation equation (2), show that$I_{t+1} = Y_{t+1} - \hat{Z}_{t+1|t}$

Solution:

$$
\begin{aligned}
I_{t+1} &= Y_{t+1} - \text{proj}\left(Y_{t+1} \mid H_t^Y\right) \\
&= Y_{t+1} - \text{proj}\left(Z_{t+1} \mid H_t^Y\right) - \text{proj}\left(\epsilon_{t+1} \mid H_t^Y\right) \\
&= Y_{t+1} - \hat{Z}_{t+1|t}
\end{aligned}
$$

because

$$
\begin{aligned}
\mathbb{E}\left(\epsilon_{t+1} \sum_{h=1}^{t} \mu_h Y_h\right) &= \sum_{h=1}^{t} \mu_h \mathbb{E}\left(\epsilon_{t+1} Y_h\right) \\
&= \sum_{h=1}^{t} \mu_h \mathbb{E}\left(\epsilon_{t+1}\left[Z_h + \epsilon_h\right]\right) \\
&= \sum_{h=1}^{t} \mu_h \mathbb{E}\left(\epsilon_{t+1}\left[Z_h + \epsilon_h\right]\right) \\
&= \sum_{h=1}^{t} \mu_h \mathbb{E}\left[\epsilon_{t+1} Z_h\right] \\
&= \sum_{h=1}^{t} \mu_h \mathbb{E}\left[\epsilon_{t+1}\left[\phi Z_{h-1} + \eta_{h-1}\right]\right] \\
&= 0
\end{aligned}
$$

Because Z_h is an infinite sum of η_t and (η_t),(ϵ_t) are uncorellated.

4. Prove that $\mathbb{E}\left[I_{t+1}^2\right] = P_{t+1|t} + \rho^2$

   Solution:

$$
\begin{aligned}
\mathbb{E}\left[I_{t+1}^2\right] &= \mathbb{E}\left[\left(Z_{t+1} - \hat{Z}_{t+1|t} + \epsilon_{t+1}\right)^2\right] \\
&= P_{t+1|t} + \rho^2 + 2\text{cov}\left(\epsilon_{t+1}, Z_{t+1} - \hat{Z}_{t+1|t}\right)
\end{aligned}
$$

   But:
   (i) $Z_{t+1} - \hat{Z}_{t+1|t} = Z_{t+1} - \sum_{h=1}^{t} \mu_h Y_h = Z_{t+1} - \sum_{h=1}^{t} \mu_h \left(Z_h + \epsilon_h\right) \in H_{t+1}^Z + H_t^\epsilon$
   (ii): $\epsilon_{t+1} \perp H_t^\epsilon$ (white noise) and $\epsilon_{t+1} \perp H_{t+1}^Z$ (already proven in Q3) $\implies \epsilon_{t+1} \perp H_{t+1}^Z \overset{\perp}{\oplus} H_t^\epsilon$
   ie cov $\left(\epsilon_{t+1}, Z_{t+1} - \hat{Z}_{t+1|t}\right) = 0$

5. Give the arguments that shows

$$
\hat{Z}_{t+1|t+1} = \hat{Z}_{t+1|t} + k_{t+1} I_{t+1}
$$

where $k_{t+1} = \mathbb{E}\left[Z_{t+1} I_{t+1}\right] / \mathbb{E}\left[I_{t+1}^2\right]$ [4]

Solution: Because (I_t) is the innovative process of Y_t:
$\implies H_{t+1}^Y = H_t^Y \overset{\perp}{\oplus} \operatorname{vect}(I_{t+1})$
then

$$\begin{aligned}
\hat{Z}_{t+1|t+1} &= \operatorname{proj}\left(Z_{t+1} \mid H_{t+1}^Y\right) \\
&= \operatorname{proj}\left(Z_{t+1} \mid H_t^Y\right) + \operatorname{proj}\left(Z_{t+1} \mid \operatorname{vect}(I_{t+1})\right) \\
&= \hat{Z}_{t+1|t} + \underbrace{k_{t+1}}_{\in\mathbb{R},\text{Kalman gain}} I_{t+1}
\end{aligned}$$

if we calculate $\mathbb{E}\left[\hat{Z}_{t+1|t+1} I_{t+1}\right]$ we get:

$$\mathbb{E}\left[\hat{Z}_{t+1|t+1} I_{t+1}\right] = \mathbb{E}\left[\hat{Z}_{t+1|t} I_{t+1}\right] + k_{t+1}\mathbb{E}\left[I_{t+1}^2\right]$$

however
(i) $\hat{Z}_{t+1|t} \in H_t^Y$ and $H_{t+1}^Y = H_t^Y \overset{\perp}{\oplus} \operatorname{vect}(I_{t+1}) \implies \mathbb{E}\left[\hat{Z}_{t+1|t} I_{t+1}\right] = 0$
Moreover:

$$\mathbb{E}\left[\left[\hat{Z}_{t+1|t+1} - Z_{t+1}\right] I_{t+1}\right] = 0$$

because:

$$\begin{aligned}
\hat{Z}_{t+1|t+1} - Z_{t+1} &= \operatorname{proj}\left(Z_{t+1} \mid H_{t+1}^Y\right) - Z_{t+1} \\
&= -\operatorname{proj}\left(Z_{t+1} \mid \left(H_{t+1}^Y\right)^{\perp}\right)
\end{aligned}$$

as a reminder, $\operatorname{proj}\left(Z_{t+1} \mid H_{t+1}^Y\right) + \operatorname{proj}\left(Z_{t+1} \mid \left(H_{t+1}^Y\right)^{\perp}\right) = Z_{t+1}$. Then $\hat{Z}_{t+1|t+1} - Z_{t+1} \perp H_{t+1}^Y \supset \operatorname{vect}(I_{t+1})$ We finally get that:

$$\mathbb{E}\left[\hat{Z}_{t+1|t+1} I_{t+1}\right] = \mathbb{E}\left[Z_{t+1} I_{t+1}\right]$$

gathering together the equation we obtain the result.

6. Using the above expression of I_{t+1}, show that $\mathbb{E}\left[Z_{t+1} I_{t+1}\right] = P_{t+1|t}$

Solution:

$$\begin{aligned}
\mathbb{E}\left[Z_{t+1} I_{t+1}\right] &= \operatorname{Cov}\left(I_{t+1}, Z_{t+1}\right) \\
&\underset{\hat{Z}_{t+1,t} \in H_t^Y \perp I_{t+1} (Q5)}{=} \operatorname{Cov}\left(I_{t+1}, Z_{t+1} - \hat{Z}_{t+1|t}\right) \\
&= \operatorname{Cov}\left(Y_{t+1} - \hat{Z}_{t+1|t}, Z_{t+1} - \hat{Z}_{t+1|t}\right) \\
&= \operatorname{Cov}\left(Z_{t+1} + \epsilon_{t+1} - \hat{Z}_{t+1|t}, Z_{t+1} - \hat{Z}_{t+1|t}\right)
\end{aligned}$$

[4] k_{t+1} is the Kalman gain filter

we already proved that $\text{cov}\left(\epsilon_{t+1}, Z_{t+1}\right) = 0$ and $\text{cov}\left(\epsilon_{t+1}, \hat{Z}_{t+1|t}\right) = \sum_{h=1}^{t} \varphi_h \text{cov}\left(\epsilon_{t+1}, \epsilon_h\right) 0$ is easy to demonstrate.
$\implies \mathbb{E}\left[Z_{t+1} I_{t+1}\right] = \text{Cov}\left(Z_{t+1} - \hat{Z}_{t+1|t}, Z_{t+1} - \hat{Z}_{t+1|t}\right) = P_{t+1|t}$

7. Why is the following equation correct ?

$$P_{t+1|t+1} = P_{t+1|t} - \mathbb{E}\left[\left(k_{t+1} I_{t+1}\right)^2\right]$$

Deduce that $P_{t+1|t+1} = \left(1 - k_{t+1}\right) P_{t+1|t}$.

Solution:

$$\begin{aligned}
P_{t+1|t+1} &= \mathbb{E}\left[\left(Z_{t+1} - \hat{Z}_{t+1|t+1}\right)^2\right] \\
&\underset{(Q5)}{=} \mathbb{E}\left[\left(\left[Z_{t+1} - \hat{Z}_{t+1|t}\right] - k_{t+1} I_{t+1}\right)^2\right] \\
&= \mathbb{E}\left[\left(Z_{t+1} - \hat{Z}_{t+1|t}\right)^2\right] - 2\text{cov}\left(Z_{t+1} - \hat{Z}_{t+1|t}, k_{t+1} I_{t+1}\right) + \mathbb{E}\left(\left(k_{t+1} I_{t+1}\right)^2\right) \\
&= P_{t+1|t} + \underbrace{2k_{t+1}\text{cov}\left(\hat{Z}_{t+1|t}, I_{t+1}\right)}_{=0 (Q6)} - 2k_{t+1} \underbrace{\text{cov}\left(Z_{t+1}, I_{t+1}\right)}_{=k_{t+1}\mathbb{E}\left(I_{t+1}^2\right)(Q5)} + \mathbb{E}\left(\left(k_{t+1} I_{t+1}\right)^2\right) \\
&= P_{t+1|t} - \mathbb{E}\left[\left(k_{t+1} I_{t+1}\right)^2\right]
\end{aligned}$$

Then because:

$$k_{t+1} \underset{(Q5)}{=} \frac{\mathbb{E}\left[Z_{t+1} I_{t+1}\right]}{\mathbb{E}\left[I_{t+1}^2\right]} \underset{(Q6)}{=} \frac{P_{t+1,t}}{\mathbb{E}\left[I_{t+1}^2\right]} \implies \mathbb{E}\left[I_{t+1}^2\right] = \frac{P_{t+1,t}}{k_{t+1}}$$

Then by injecting the previous equality in $P_{t+1|t} - \mathbb{E}\left[\left(k_{t+1} I_{t+1}\right)^2\right]$ we get:

$$P_{t+1|t} - k_{t+1}^2 \mathbb{E}\left[I_{t+1}^2\right] = P_{t+1|t} - k_{t+1} P_{t+1,t} = \left(1 - k_{t+1}\right) P_{t+1,t}$$

8. Provide the complete set of equations for computing $\hat{Z}_{t|t}$ and $P_{t|t}$ iteratively for all $t \geq 1$ (Including the initial conditions.)

Solution: Init conditions $\hat{Z}_{0|0} = 0, P_{0|0} = \sigma_0^2$ Iterative procedure:

(a) $\hat{Z}_{t+1|t} = \phi \hat{Z}_{t|t}$ (Q2)
(b) $P_{t+1|t} = \phi^2 P_{t|t} + \sigma^2$ (Q2)
(c) $I_{t+1} = Y_{t+1} - \hat{Z}_{t+1|t}$ (Q3)
(d) $k_{t+1} = \frac{P_{t+1|t}}{P_{t+1|t} + \rho^2}$ (Q4 + Q5)
(e) $\hat{Z}_{t+1|t+1} = \hat{Z}_{t+1|t} + k_{t+1} I_{t+1}$ (Q5)
(f) $P_{t+1|t+1} = \left(1 - k_{t+1}\right) P_{t+1|t}$(Q7)

9. *Bonus*: Study the asymptotic behavior of $P_{t|t}$ as $t \to \infty$.

Solution: $0 \leq P_{t+1|t+1} = \left(1 - \frac{P_{t+1|t}}{P_{t+1|t}+\rho^2}\right) P_{t+1|t} = \frac{\rho^2 P_{t+1|t}}{\rho^2+P_{t+1|t}} = \rho^2 \frac{\phi^2 P_{t|t}+\sigma^2}{\phi^2 P_{t|t}+\sigma^2+\rho^2} \leq \rho^2$.

Then, the sequence $P_{t+1|t+1}$ is bounded. it admits a sub-sequence that converges. let P_∞ be that limit. if we let t tends to ∞ we get:

$$
\begin{aligned}
& P_\infty = \rho^2 \frac{\phi^2 P_\infty + \sigma^2}{\phi^2 P_\infty + \sigma^2 + \rho^2} \\
& \Longrightarrow \phi^2 P_\infty^2 + \left[\sigma^2 + \rho^2 \left(1 - \phi^2\right)\right] P_\infty - \rho^2 \sigma^2 = 0 \\
& \Longrightarrow P_\infty = \frac{-\left(\sigma^2 + \rho^2 \left(1 - \phi^2\right)\right) + \sqrt{\left[\sigma^2 + \rho^2 \left(1 - \phi^2\right)\right] + 4\phi^2 \rho^2 \sigma^2}}{2\phi^2}
\end{aligned}
$$

Remark: $P_{t|t}$ belongs to a compact and have only one accumulation point (valeur d'adhérence) it means it is all the sequence that tends to P_∞.