Filter bank methods

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Linear time series (part 2)
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The periodogram as a filter bank

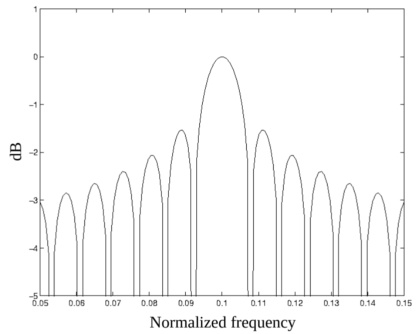
- The periodogram $\hat{S}_{XX}(\nu)$ of a centered WSS random process X_t is defined as $\hat{S}_{XX}(\nu) = \frac{1}{N}\left|\sum_{t=0}^{N-1} X_t e^{-i2\pi\nu t}\right|^2$
- It can also be expressed as

$$\hat{S}_{XX}(\nu) = N\left|\sum_{k\in\mathbb{Z}} w_k^\nu X_{N-1-k}\right|^2 = \frac{|Y_{N-1}|^2}{1/N},$$

where $Y_t = w^\nu * X_t$, $w_k^\nu = 0$ if $k \notin [0 \dots N-1]$, and $w_k^\nu = \frac{1}{N} e^{i2\pi\nu k}$ if $k \in [0 \dots N-1]$

The periodogram as a filter bank

- w^ν is such that $|W^\nu(\xi)| = \left|\sum_{k\in\mathbb{Z}} w_k^\nu e^{-i2\pi\xi k}\right| = \frac{1}{N}\left|\frac{\sin(\pi(\xi-\nu)N)}{\sin(\pi(\xi-\nu))}\right|$



Frequency response $\xi \mapsto |W^\nu(\xi)|$ with $\nu = 0.1$

The periodogram as a filter bank

- The periodogram is equivalent to filtering the signal X_t by a FIR filter, and computing the energy of the $(N-1)^{th}$ output sample.
- The resulting frequency response W^ν has a main lobe with a small width $(2/N)$, but side lobes with high amplitudes.
- Y has a non-zero power on a large frequency band around ν, and the estimation of $\hat{S}_{XX}(\nu)$ is biased.
- Capon's method consists in determining a filter w^ν such that $W^\nu(\nu) = 1$, which minimizes the energy of the output signal at frequencies other than ν.

Capon's method

- Let $Y = w^\nu * X$, where w^ν is a FIR filter of support $[0 \dots N-1]$, $\mathbf{w}^\nu = \{w_0^\nu \dots w_{N-1}^\nu\}$ and $\mathbf{X} = \{\overline{X_{N-1}} \dots \overline{X_0}\}$.
- Then $\mathbb{E}[|Y_{N-1}|^2] = \mathbf{w}^{\nu H} \mathbf{R}_{XX} \mathbf{w}^\nu$ where $\mathbf{R}_{XX} = \mathbb{E}[\mathbf{X}\mathbf{X}^H]$.
- Let $\mathbf{e}(\xi) = [1, e^{i2\pi\xi} \dots e^{i2\pi\xi(N-1)}]^\top$. Then $W^\nu(\xi) = \mathbf{e}(\xi)^H \mathbf{w}^\nu$.
- Let $\mathbf{w}_{\text{opt}}^\nu = \frac{\mathbf{R}_{XX}^{-1}\mathbf{e}(\nu)}{\mathbf{e}(\nu)^H \mathbf{R}_{XX}^{-1}\mathbf{e}(\nu)}$. Then for any $\mathbf{w}^\nu$ such that $W^\nu(\nu) = \mathbf{e}(\nu)^H \mathbf{w}^\nu = 1$,

$$\mathbb{E}[|Y_{N-1}|^2] = (\mathbf{w}^\nu - \mathbf{w}_{\text{opt}}^\nu)^H \mathbf{R}_{XX} (\mathbf{w}^\nu - \mathbf{w}_{\text{opt}}^\nu) + \frac{1}{\mathbf{e}(\nu)^H \mathbf{R}_{XX}^{-1}\mathbf{e}(\nu)}.$$

- The MVDR filter which minimizes $\mathbb{E}[|Y_{N-1}|^2]$ (*Minimum Variance*) subject to $\mathbf{e}(\nu)^H \mathbf{w}^\nu = 1$ (*Distortionless Response*) is $\mathbf{w}^\nu = \mathbf{w}_{\text{opt}}^\nu$
- The energy at the output is $\mathbb{E}[|Y_{N-1}|^2] = \frac{1}{\mathbf{e}(\nu)^H \mathbf{R}_{XX}^{-1}\mathbf{e}(\nu)}$.

Capon's method

- Capon's spectral estimator is

$$\hat{S}_{\text{CAP},XX}(\nu) = \frac{\mathbb{E}[|Y_{N-1}|^2]}{1/N} = \frac{\mathbf{w}_{\text{opt}}^{\nu\ H} \mathbf{R}_{XX} \mathbf{w}_{\text{opt}}^\nu}{1/N} = \frac{N}{\mathbf{e}(\nu)^H \mathbf{R}_{XX}^{-1}\mathbf{e}(\nu)}$$

- In practice, $\mathbf{R}_{XX}$ is unknown and has to be estimated.
- Example : if X_t is white noise, then $\mathbf{R}_{XX} = \sigma_X^2 \mathbf{I}_N$ and

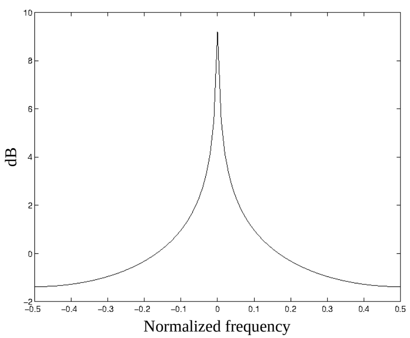
$$\mathbf{w}_{\text{opt}}^\nu = \frac{\mathbf{R}_{XX}^{-1}\mathbf{e}(\nu)}{\mathbf{e}(\nu)^H \mathbf{R}_{XX}^{-1}\mathbf{e}(\nu)} = \frac{1}{N}\mathbf{e}(\nu)$$

is the same filter as the one involved in the periodogram.

Example : AR(1) process

- $X_t = a_1 X_{t-1} + Z_t$ where $a_1 \in]0,1[$ and $Z_t \sim \mathrm{WN}(0, \sigma_Z^2)$
- Then $S_{XX}(\nu) = \frac{\sigma_Z^2}{|1 - a_1 e^{-i2\pi\nu}|^2}$
- $$\mathbf{R}_{XX} = \frac{\sigma_Z^2}{1-a_1^2} \begin{bmatrix} 1 & a_1 & \dots & a_1^{N-1} \\ a_1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_1 \\ a_1^{N-1} & \dots & a_1 & 1 \end{bmatrix}$$

Example : AR(1) process



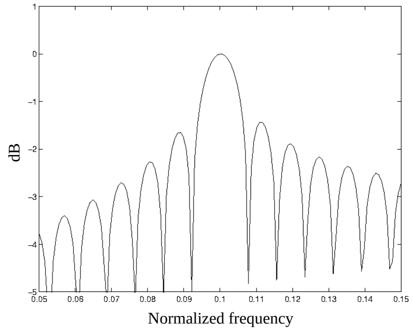
PSD of the AR process : $\nu \mapsto |S_{XX}(\nu)|$ with $a_1 = 0.99$ and $\sigma_Z^2 = 1$

Example : AR(1) process

- $\mathbf{R}_{XX}^{-1} = \frac{1}{\sigma_Z^2} \begin{bmatrix} 1 & -a_1 & 0 & \dots & 0 \\ -a_1 & 1+a_1^2 & -a_1 & \ddots & \vdots \\ 0 & -a_1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1+a_1^2 & -a_1 \\ 0 & \dots & 0 & -a_1 & 1 \end{bmatrix}$
- Therefore $\mathbf{R}_{XX}^{-1}\mathbf{e}(\nu) = \frac{\left|1-a_1 e^{-i2\pi\nu}\right|^2}{\sigma_Z^2}(\mathbf{e}(\nu)+\mathbf{v}(\nu))$
  where $\mathbf{v}(\nu) = a_1 \left[\frac{e^{-i2\pi\nu}}{1-a_1 e^{-i2\pi\nu}}, 0, \dots, 0, \frac{e^{+i2\pi N\nu}}{1-a_1 e^{+i2\pi\nu}}\right]^\top$
  and $\mathbf{w}_{\text{opt}}^{\nu} = \frac{\mathbf{R}_{XX}^{-1}\mathbf{e}(\nu)}{\mathbf{e}(\nu)^H\mathbf{R}_{XX}^{-1}\mathbf{e}(\nu)} = \frac{\mathbf{e}(\nu)+\mathbf{v}(\nu)}{N+2a_1\frac{\cos(2\pi\nu)-a_1}{1-2a_1\cos(2\pi\nu)+a_1^2}}$

Example : AR(1) process

- The *shape* of the filter depends on the estimated frequency ν :



Frequency response $\xi \mapsto |W^{\nu}(\xi)|$ with $\nu = 0.1$ and $a_1 = 0.99$

Statistical properties

- This estimator has a better resolution than the periodogram.
- Its variance is lower than that of autoregressive methods, but its spectral resolution is worse.
- Indeed, if $\hat{S}_{\text{CAP},XX}(\nu)$ is calculated with a $(p+1)\times(p+1)$ matrix $\hat{\mathbf{R}}_{XX}^{-1}$, then $\hat{S}_{\text{CAP},XX}(\nu)$ is related to AR estimators though

$$\frac{1}{\hat{S}_{\text{CAP},XX}(\nu)} = \frac{1}{p+1}\sum_{k=0}^{p}\frac{1}{\hat{S}_{\text{AR},XX}^{(k)}(\nu)}$$

  where $\hat{S}_{\text{AR},XX}^{(k)}(\nu)$ is the AR estimator of order k.
- The choice of order p is critical.

Variant (Lagunas)

- Compute the ratio between the obtained power and that of white noise filtered by $\mathbf{w}_{\text{opt}}^{\nu}$
- ${\mathbf{w}_{\text{opt}}^{\nu}}^H \mathbf{I}\mathbf{w}_{\text{opt}}^{\nu} = \frac{\mathbf{e}(\nu)^H\mathbf{R}_{XX}^{-1}}{\mathbf{e}(\nu)^H\mathbf{R}_{XX}^{-1}\mathbf{e}(\nu)}\frac{\mathbf{R}_{XX}^{-1}\mathbf{e}(\nu)}{\mathbf{e}(\nu)^H\mathbf{R}_{XX}^{-1}\mathbf{e}(\nu)} = \frac{\mathbf{e}(\nu)^H\mathbf{R}_{XX}^{-2}\mathbf{e}(\nu)}{(\mathbf{e}(\nu)^H\mathbf{R}_{XX}^{-1}\mathbf{e}(\nu))^2}$ and
  $\frac{{\mathbf{w}_{\text{opt}}^{\nu}}^H\mathbf{R}_{XX}^{-1}\mathbf{w}_{\text{opt}}^{\nu}}{{\mathbf{w}_{\text{opt}}^{\nu}}^H\mathbf{I}\mathbf{w}_{\text{opt}}^{\nu}} = \frac{1}{\mathbf{e}(\nu)^H\mathbf{R}_{XX}^{-1}\mathbf{e}(\nu)}\frac{(\mathbf{e}(\nu)^H\mathbf{R}_{XX}^{-1}\mathbf{e}(\nu))^2}{\mathbf{e}(\nu)^H\mathbf{R}_{XX}^{-2}\mathbf{e}(\nu)} = \frac{\mathbf{e}(\nu)^H\mathbf{R}_{XX}^{-1}\mathbf{e}(\nu)}{\mathbf{e}(\nu)^H\mathbf{R}_{XX}^{-2}\mathbf{e}(\nu)}$
- Hence

$$\hat{S}_{\text{LAG},XX}(\nu) = \frac{\mathbf{e}(\nu)^H\mathbf{R}_{XX}^{-1}\mathbf{e}(\nu)}{\mathbf{e}(\nu)^H\mathbf{R}_{XX}^{-2}\mathbf{e}(\nu)}$$