KALMAN EXERCISE

Exercise (Linear prerdiction of an AR(1) observed with additive noise)

Consider an AR(1) real process Z_t[1] satisfying the following canonical equation:

$$\forall t \in \mathbb{N},\ Z_{t+1} = \phi Z_t + \eta_t \tag{1}$$

where $(\eta_t)_{t\geq 0}$ is a centered white noise with known variance σ^2 and ϕ is a known constant. The process $(Z_t)_{t\geq 0}$ is not directly observed. Instead for all $t \geq 1$, one gets the following sequence of observations:

$$Y_t = Z_t + \epsilon_t \tag{2}$$

where $(\epsilon_t)_{t\geq 1}$ is a centered white noise with known variance ρ^2, that is uncorrelated with (η_t) and Z_0. We wish to solve the filtering problem, that is, to compute the orthogonal projection of Z_t on the space $H_t^Y = \text{span}\{Y_1, \dots, Y_t\}$, iteatively in t.

We denote $\hat{Z}_{t|t} = \text{proj}\left(Z_t \mid H_t^Y\right)$ this projection and $P_{t|t} = \mathbb{E}\left[\left(Z_t - \hat{Z}_{t|t}\right)^2\right]$ the corresponding projection error variance[2] . Similarly, let $\hat{Z}_{t+1|t} = \text{proj}\left(Z_{t+1} \mid H_t^Y\right)$ be the best linear predictor and $P_{t+1|t} = \mathbb{E}\left[\left(Z_{t+1} - \hat{Z}_{t+1|t}\right)^2\right]$ the linear prediction error variance.

1. Show that Z_0 is a centered random variable and computes its variance σ_0^2 using the Corollary 3.1.3 and that Z_0 and $(\eta_t)_{t\geq 0}$ are uncorrelated. [3]

2. Using the evolution (state) equation (1), show that

$$\hat{Z}_{t+1|t} = \phi \hat{Z}_{t|t} \quad \text{and} \quad P_{t+1|t} = \phi^2 P_{t|t} + \sigma^2$$

3. Let us define the innovation by $I_{t+1} = Y_{t+1} - \text{proj}\left(Y_{t+1} \mid H_t^Y\right)$. Using the observation equation (2), show that$I_{t+1} = Y_{t+1} - \hat{Z}_{t+1|t}$

4. Prove that $\mathbb{E}\left[I_{t+1}^2\right] = P_{t+1|t} + \rho^2$

5. Give the arguments that shows

$$\hat{Z}_{t+1|t+1} = \hat{Z}_{t+1|t} + k_{t+1} I_{t+1}$$

   where $k_{t+1} = \mathbb{E}\left[Z_{t+1} I_{t+1}\right] / \mathbb{E}\left[I_{t+1}^2\right]$ [4]

6. Using the above expression of I_{t+1}, show that $\mathbb{E}\left[Z_{t+1} I_{t+1}\right] = P_{t+1|t}$

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[1] the same exercise can be apply to a complex AR(1) process Z_t. Try by yourself to see what could be the slight difference in that case.

[2] in complex case: $P_{t|t} = \mathbb{E}\left[\left|Z_t - \hat{Z}_{t|t}\right|^2\right]$

[3] Hint: decompose Z_t as $F_\phi(B)\,\eta_t$ where $F_\phi(B)$ is a rational polynom fraction depends on the backshift operator and and then decompose $F_\phi(B)\,\eta_t$ as an infinite sum.

[4] k_{t+1} is the Kalman gain filter

7. Why is the following equation correct ?

$$P_{t+1|t+1} = P_{t+1|t} - \mathbb{E}\left[\left(k_{t+1} I_{t+1}\right)^2\right]$$

   Deduce that $P_{t+1|t+1} = (1 - k_{t+1})\, P_{t+1|t}$.

8. Provide the complete set of equations for computing $\hat{Z}_{t|t}$ and $P_{t|t}$ iteratively for all $t \geq 1$ (Including the initial conditions.)

9. *Bonus*: Study the asymptotic behavior of $P_{t|t}$ as $t \to \infty$.