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Quadrature methods (I)

Exercice 1 (Lagrange interpolation).

Let f be a smooth function over $[a,b]$. Let $x_0,\dots,x_n \in [a,b]$ (all distinct). The Newton polynomial basis is defined as :

$$w_{n+1} = \prod_{i=0}^{n}(X - x_i)$$

with the convention that w_0 is the constant polynomial $w_0 = 1$. The Lagrange interpolation of f based on the nodes $x_0,\dots,x_n$ is defined as :

$$L_n f = \sum_{i=0}^{n} f(x_i) l_i \quad \text{where} \quad l_i(x) = \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

1. Assume that f is differentiable $(n+1)$ times on $[a,b]$. Show that for all $x \in [a,b]$ there exists $c_x \in [a,b]$ such that the interpolation error $e_n(x)$ can be written :

$$e_n(x) = f(x) - L_n f(x) = \frac{1}{(n+1!)} w_{n+1}(x) f^{(n+1)}(c_x)$$

   Hint : find the appropriate function on which to apply generalized Rolle's theorem

   To minimize the error, we can chose the nodes that minimize the values of $w_n(x)$.
2. Consider the equidistant nodes sequence $x_i = a + i\frac{b-a}{n}$. Show that :

$$\|w_n\|_\infty \leq (n+1)! \left(\frac{b-a}{n}\right)^n$$

3. Refine that upper-bound to obtain using Sterling's formula.

   The proof of the previous result highlights the fact that the amplitude of w_n is larger near the borders of the interval or when the nodes are too far from each other. To counteract this effect, we can pick nodes that get denser around the borders. Such nodes can be given by the roots of the Tchebychev polynomials defined by :

$$T_n(x) = \cos(n \arccos(x))$$

4. Show that T_n verify the recurrence :

$$T_0(x) = 1 \qquad T_1(x) = x \qquad T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x), \quad n \geq 1$$

   and deduce the degree, the roots and the coefficient of the leading term of T_n.
5. Let $u_0,\dots,u_{n+1}$ denote the roots of T_n. Let $\phi : [-1,1] \to [a,b]$ be the affine transformation $u \mapsto \frac{b-a}{2}u + \frac{a+b}{2}$. Define the Tchebychev interpolation nodes $x_k = \phi(u_k)$. Show that :

$$\|w_n^{Tch}\| \leq 2\left(\frac{b-a}{4}\right)^n$$

Exercice 2 (Cavalieri-Simpson).

Let f be a smooth (C^∞) function over $[a,b]$. We consider the problem of evaluating the integral $I(f) = \int_a^b f(x)\,dx$. We approximate $I(f)$ using a simple quadrature method of Newton-Cotes of order l with the nodes : $x_i = a + i\frac{b-a}{l}$, $i = 0,\dots,l$, and weights $\lambda_0,\dots,\lambda_l$, such that :

$$\hat{I}(f) = (b-a)\sum_{i=0}^{l} \lambda_i f(x_i)\,.$$

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1. Show that the weights λ_i are independent of (a, b). Without loss of generality, we can assume that $(a, b) = (-1, 1)$.
2. Find λ_i for $l = 2$.
3. Show that this quadrature method is of order 3.

Exercice 3 (Legendre-Gauss quadrature)**.**
Let $I(f) = \int_{-1}^{1} f$, where f is a smooth function over $[-1, 1]$. Let $u_0, u_1 \in [-1, 1]$. We are interested in a rule of the form :

$$\lambda_0 f(u_0) + \lambda_1 f(u_1).$$

In this exercise, the interpolation nodes are not fixed. Find $(u_0, u_1, \lambda_0, \lambda_1)$ so as to maximize the order of the method.

Exercice 4 (Gauss-Newton)**.**

Consider the problem of approximating $I_{a,b}(f) = \int_a^b f(x)\,\mathrm{d}x$, where f is in $\mathcal{C}^\infty$ via

$$\hat{I}_{a,b}(f) = (b - a)(\lambda_0 f(a) + \lambda_1 f(b) + \lambda_2 f'(a)),$$

with $(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{R}^3$.

1. For $a = 0$, $b = 1$, find $(\lambda_0, \lambda_1, \lambda_2)$ such that the order of the method is exact for polynomials of degree ≤ 2.
2. Deduce an expression of $(\lambda_0, \lambda_1, \lambda_2)$ as function of a and b such that the method is of order 2 for any interval $[a, b]$.