Source: "Partial Differential Equations", Lawrence Ewans, American Mathematical Society, 1998.

Notations: ∂U = boundary of U, $\bar{U} = U \cup \partial U$ = closure of U.

Lipschitz continuity

Assume $U \subset \mathbb{R}^n$ is open and $0 < \gamma \leq 1$. We have previously considered the class of Lipschitz continuous functions $u : U \to \mathbb{R}$, which by definition satisfy the estimate

$$|u(x) - u(y)| \leq C|x - y| \quad (x, y \in U) \tag{1}$$

Norm in linear space

The proof is left as an exercise (Problem 1), but let us pause here to make clear what is being asserted. Recall from §D.1 that if X denotes a real linear space, then a mapping $\| \ \| : X \to [0, \infty)$ is called a *norm* provided

(i) $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in X$,
(ii) $\|\lambda u\| = |\lambda|\|u\|$ for all $u \in X$, $\lambda \in \mathbb{R}$,
(iii) $\|u\| = 0$ if and only if $u = 0$.

Banach space

A norm provides us with a notion of convergence: we say a sequence $\{u_k\}_{k=1}^\infty \subset X$ *converges* to $u \in X$, written $u_k \to u$, if $\lim_{k\to\infty} \|u_k - u\| = 0$. A *Banach space* is then a normed linear space which is *complete*, that is, within which each Cauchy sequence converges.

Sobolev Space

Fix $1 \leq p \leq \infty$ and let k be a nonnegative integer. We define now certain function spaces, whose members have weak derivatives of various orders lying in various L^p spaces.

DEFINITION. *The* Sobolev space

$$W^{k,p}(U)$$

consists of all locally summable functions $u : U \to \mathbb{R}$ such that for each multiindex α with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$.

Weak derivatives

Example 1. Let $n = 1$, $U = (0, 2)$, and

$$u(x) = \begin{cases} x & \text{if } 0 < x \leq 1 \\ 1 & \text{if } 1 \leq x < 2. \end{cases}$$

Define

$$v(x) = \begin{cases} 1 & \text{if } 0 < x \leq 1 \\ 0 & \text{if } 1 < x < 2. \end{cases}$$

Let us show $u' = v$ in the weak sense. To see this, choose any $\phi \in C_c^\infty(U)$. We must demonstrate

$$\int_0^2 u\phi' \, dx = -\int_0^2 v\phi \, dx.$$

functions in Sobolev space are not necessarily smooth: we must always rely solely upon the definition of weak derivatives.

Sobolev Space for p=2 ➔ Hilbert Space

Remarks. (i) If $p = 2$, we usually write

$$H^k(U) = W^{k,2}(U) \quad (k = 0, 1, \dots).$$

The letter H is used, since—as we will see—$H^k(U)$ is a Hilbert space. Note that $H^0(U) = L^2(U)$.

Sobolev Space ➔ Norm

DEFINITION. *If* $u \in W^{k,p}(U)$, *we define its* norm *to be*

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \left(\sum_{|\alpha| \le k} \int_U |D^\alpha u|^p \, dx \right)^{1/p} & (1 \le p < \infty) \\ \sum_{|\alpha| \le k} \operatorname{ess\,sup}_U |D^\alpha u| & (p = \infty). \end{cases}$$

Sobolev Space is a Banach space

THEOREM 2 (Sobolev spaces as function spaces). *For each* $k = 1, \dots$ *and* $1 \le p \le \infty$, *the Sobolev space* $W^{k,p}(U)$ *is a Banach space.*

Interior approximation by smooth functions

It is awkward to return continually to the definition of weak derivatives. In order to study the deeper properties of Sobolev spaces, we therefore need to develop some systematic procedures for approximating a function in a Sobolev space by smooth functions. The method of mollifiers, developed in §C.4, provides the tool.

Fix a positive integer k and $1 \le p < \infty$. Remember that $U_\varepsilon = \{x \in U \mid \operatorname{dist}(x, \partial U) > \varepsilon\}$.

$$u^\varepsilon = \eta_\varepsilon * u \quad \textit{in } U_\varepsilon.$$

Then

(i) $u^\varepsilon \in C^\infty(U_\varepsilon)$ *for each* $\varepsilon > 0$,

and

(ii) $u^\varepsilon \to u$ *in* $W^{k,p}_{\mathrm{loc}}(U)$, *as* $\varepsilon \to 0$.

Global approximation by smooth functions

THEOREM 3 (Global approximation by functions smooth up to the boundary). *Assume* U *is bounded and* ∂U *is* C^1. *Suppose* $u \in W^{k,p}(U)$ *for some* $1 \le p < \infty$. *Then there exist functions* $u_m \in C^\infty(\bar{U})$ *such that*

$$u_m \to u \quad \textit{in } W^{k,p}(U).$$

Calculus of variations (page 434)

(1) $$A[u] = 0.$$

PDE of the form (1), where the nonlinear operator $A[\cdot]$ is the "derivative" of an appropriate "energy" functional $I[\cdot]$. Symbolically we write

(2) $$A[\cdot] = I'[\cdot].$$

Then problem (1) reads

(3) $$I'[u] = 0.$$

The advantage of this new formulation is that we now can recognize solutions of (1) as being critical points of $I[\cdot]$. These in certain circumstances may be relatively easy to find; if, for instance, the functional $I[\cdot]$ has a minimum at u, then presumably (3) is valid and thus u is a solution of the original PDE (1). *The point is that whereas it is usually extremely difficult to solve (1) directly, it may be much easier to discover minimum (or maximum, or other critical) points of the functional* $I[\cdot]$.

In addition of course, many of the laws of physics and other scientific disciplines arise directly as variational principles.

This is the *Euler–Lagrange equation* associated with the energy functional $I[\cdot]$ defined by (4). Observe that (9) is a quasilinear, second-order PDE in divergence form.

In summary, any smooth minimizer of $I[\cdot]$ is a solution of the Euler–Lagrange partial differential equation (9), and thus—conversely—we can try to find a solution of (9) by searching for minimizers of (4).

Euler-Lagrange equation

Suppose now $U \subset \mathbb{R}^n$ is a bounded, open set with smooth boundary ∂U, and we are given a smooth function

$$L : \mathbb{R}^n \times \mathbb{R} \times \bar{U} \to \mathbb{R}.$$

We call L the *Lagrangian.*

Notation. We will write

$$L = L(p, z, x) = L(p_1, \dots, p_n, z, x_1, \dots, x_n)$$

for $p \in \mathbb{R}^n$, $z \in \mathbb{R}$, and $x \in U$. Thus "p" is the name of the variable for which we substitute $Dw(x)$ below, and "z" is the variable for which we substitute $w(x)$. We also set

$$\begin{cases} D_pL = (L_{p_1}, \dots, L_{p_n}) \\ D_zL = L_z \\ D_xL = (L_{x_1}, \dots, L_{x_n})\ . \end{cases}$$

Explicit form of I:

(4) $$I[w] = \int_U L(Dw(x), w(x), x)\, dx,$$

for smooth functions $w : \bar{U} \to \mathbb{R}$ satisfying, say, the boundary condition

(5) $$w = g \quad \text{on } \partial U.$$

(9) $$-\sum_{i=1}^{n} \left(L_{p_i}(Du, u, x)\right)_{x_i} + L_z(Du, u, x) = 0 \quad \text{in } U.$$