1 Eigenvalue method

1. We are interested here in the synthesis of linear phase FIR filters. We consider the particular case of a type I filter, of odd length N and symmetrical impulse response, whose transfer function is denoted $H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$. Let $M = \frac{N-1}{2}$. Verify that we can write

$$H\left(e^{i2\pi\nu}\right) = e^{-i2\pi\nu M}\, H_R\left(e^{i2\pi\nu}\right) \tag{1}$$

where $H_R\left(e^{i2\pi\nu}\right)$ is a real-valued function, called the amplitude response of filter H, defined by the equality $H_R\left(e^{i2\pi\nu}\right) = \boldsymbol{a}^T \boldsymbol{c}(\nu)$, where $\boldsymbol{c}(\nu) = [1, \cos(2\pi\nu), \ldots, \cos(2\pi M\nu)]^T$, and where the coefficients of vector $\boldsymbol{a} = [a_0, a_1, \ldots, a_M]^T$ are to be expressed in terms of $h(n)$.

We have

$$\begin{aligned} H(z) &= z^{-M}\left(\sum_{n=0}^{M-1} h(n)\, z^{-n+M} + h(M) + \sum_{n=M+1}^{N-1} h(n)\, z^{-n+M}\right) \\ &= z^{-M}\left(\sum_{n=0}^{M-1} h(n)\, z^{-n+M} + h(M) + \sum_{m=0}^{M-1} h(N-1-m)\, z^{m-M}\right) \\ &= z^{-M}\left(h(M) + \sum_{n=0}^{M-1} h(n)\,(z^{-n+M} + z^{n-M})\right). \end{aligned}$$

At $z = e^{i2\pi\nu}$, we get

$$H\left(e^{i2\pi\nu}\right) = e^{-i2\pi\nu M}\left(H(M) + 2\sum_{n=0}^{M-1} h(n)\cos(2\pi\nu(M-n))\right).$$

We thus retrieve (1) with $a_0 = h(M)$ and $a_m = 2h(M-m)$ $\forall m \in [[1, M]]$.

2. We wish to synthesize a low-pass filter with cutoff frequency $\nu_c \in \left]0, \frac{1}{2}\right[$ and whose stop-band starts at $\nu_a \in \left]\nu_c, \frac{1}{2}\right[$. The energy in the stop-band is $E_a = 2\int_{\nu_a}^{\frac{1}{2}} \left(H_R(e^{i2\pi\nu})\right)^2 \mathrm{d}\nu$. Show that we can write $E_a = \boldsymbol{a}^T \boldsymbol{P}\, \boldsymbol{a}$, where $\boldsymbol{P}$ is a positive semidefinite matrix, whose coefficients $\{\boldsymbol{P}_{(m,n)}\}_{(m,n)\in[[0,M]]^2}$ are to be determined in function of ν_a.

We have $E_a = 2\int_{\nu_a}^{\frac{1}{2}} \left(\boldsymbol{a}^T \boldsymbol{c}(\nu)\right)^2 \mathrm{d}\nu = \boldsymbol{a}^T \boldsymbol{P}\, \boldsymbol{a}$ with $\boldsymbol{P} = 2\int_{\nu_a}^{\frac{1}{2}} \boldsymbol{c}(\nu)\boldsymbol{c}(\nu)^T \mathrm{d}\nu$. Matrix $\boldsymbol{P}$ is positive semidefinite, as a sum of rank-1 positive semidefinite matrices. Moreover,

$$\begin{aligned} \boldsymbol{P}_{(m,n)} &= 2\int_{\nu_a}^{\frac{1}{2}} \cos(2\pi\nu m)\cos(2\pi\nu n)\mathrm{d}\nu \\ &= \int_{\nu_a}^{\frac{1}{2}} \cos(2\pi\nu(m+n)) + \cos(2\pi\nu(m-n))\mathrm{d}\nu \\ &= \left[\frac{\sin(2\pi\nu(m+n))}{2\pi(m+n)} + \frac{\sin(2\pi\nu(m-n))}{2\pi(m-n)}\right]_{\nu_a}^{\frac{1}{2}} \\ &= \frac{\delta(m+n)+\delta(m-n)}{2} - \nu_a\left(\mathrm{sinc}(2\pi\nu_a(m+n)) + \mathrm{sinc}(2\pi\nu_a(m-n))\right). \end{aligned}$$

3. Ideally, the amplitude response $H_R\left(e^{i2\pi\nu}\right)$ is equal to $H_R(1)$ in the bandwidth $[0, \nu_c]$. We therefore define the square error in the bandwidth as follows:

$$E_c = 2\int_0^{\nu_c} \left(H_R(e^{i2\pi\nu}) - H_R(1)\right)^2 \mathrm{d}\nu$$

Show that we can write $E_c = \boldsymbol{a}^T \boldsymbol{Q}\,\boldsymbol{a}$, where $\boldsymbol{Q}$ is a positive semidefinite matrix, whose coefficients $\left\{\boldsymbol{Q}_{(m,n)}\right\}_{(m,n)\in[[0,M]]^2}$ are to be determined in function of ν_c.

We have $E_c = 2\int_0^{\nu_c}\left(\boldsymbol{a}^T(\boldsymbol{c}(\nu)-\boldsymbol{c}(0))\right)^2 \mathrm{d}\nu = \boldsymbol{a}^T \boldsymbol{Q}\,\boldsymbol{a}$ *with* $\boldsymbol{Q} = 2\int_0^{\nu_c}(\boldsymbol{c}(\nu)-\boldsymbol{c}(0))(\boldsymbol{c}(\nu)-\boldsymbol{c}(0))^T \mathrm{d}\nu$. *Matrix* $\boldsymbol{Q}$ *is positive semidefinite, as a sum of rank-1 positive semidefinite matrices. Moreover,*

$$\begin{aligned}\boldsymbol{Q}_{(m,n)} &= 2\int_0^{\nu_c}(\cos(2\pi\nu m)-1)(\cos(2\pi\nu n)-1)\mathrm{d}\nu\\ &= \int_0^{\nu_c}\cos(2\pi\nu(m+n))+\cos(2\pi\nu(m-n))-2\cos(2\pi\nu m)-2\cos(2\pi\nu n)+2\,\mathrm{d}\nu.\end{aligned}$$

The end of the calculation is left to the reader.

4. The FIR filter synthesis method called *eigenvalue method* consists in minimizing with respect to $\boldsymbol{a}$ the cost function $E(\boldsymbol{a}) = \alpha E_c + (1-\alpha)E_a$, where $\alpha \in\,]0,1[$ is a trade-off parameter between pass-band and stop-band. We thus obtain $E(\boldsymbol{a}) = \boldsymbol{a}^T\boldsymbol{R}\,\boldsymbol{a}$, where $\boldsymbol{R} = \alpha\boldsymbol{Q} + (1-\alpha)\boldsymbol{P}$ is a positive semidefinite matrix. Show that vector $\boldsymbol{a}$ minimizes function E under unit norm constraint if and only if it is an eigenvector of $\boldsymbol{R}$, associated to the lowest eigenvalue (*Rayleigh's principle*).

   The Lagrangian of this optimization problem is

$$\mathcal{L}(\boldsymbol{a},\lambda) = \boldsymbol{a}^T\boldsymbol{R}\,\boldsymbol{a} + \lambda(1-\|\boldsymbol{a}\|^2).$$

   Its gradient w.r.t. $\boldsymbol{a}$ *is* $2\boldsymbol{R}\,\boldsymbol{a} - 2\lambda\boldsymbol{a}$. *This gradient is zero when* $\boldsymbol{R}\,\boldsymbol{a} = \lambda\boldsymbol{a}$ *(therefore* $\boldsymbol{a}$ *is an eigenvector of* $\boldsymbol{R}$*, associated to the eigenvalue* λ*), in which case* $E(\boldsymbol{a}) = \lambda$ *when* $\|\boldsymbol{a}\| = 1$. *Therefore vector* $\boldsymbol{a}$ *minimizes function* E *under unit norm constraint if and only if it is an eigenvector of* $\boldsymbol{R}$ *associated to the lowest eigenvalue* λ.

2 Synthesis of an integrator filter

We consider a digital signal $x(n)$, defined from an analog signal $x^a(t)$ sampled at sampling rate T: $x(n) = x^a(nT)$. This exercise aims at synthesizing a digital filter which allows to obtain, from the discrete signal $x(n)$, a sampled version of the integrated signal $y^a(t) = \int_{-\infty}^{t} x^a(u)\,du$.

Question 1 Show that the integrated signal $y^a(t)$ can be written as the convolution product between the signal $x^a(t)$ and the analog filter $h^a(t) = 1$ if $t \geq 0$ and $h^a(t) = 0$ otherwise (Heaviside function). Is this filter causal? Is it stable? (reminder : the filter is stable if and only if $\int_{-\infty}^{+\infty}|h^a(t)|\,dt < +\infty$). Compute the transfer function $H^a(p) = \int_{-\infty}^{+\infty} h^a(t)\,e^{-pt}\,dt$ (Laplace transform of h^a, with $p \in \mathbb{C}$), and specify its domain of definition.

We have $(h^a * x^a)(t) = \int_{v\in\mathbb{R}} h^a(v)x^a(t-v)dv = \int_{v=0}^{+\infty} x^a(t-v)dv = \int_{u=-\infty}^{t} x^a(u)\,du$ *with the change of variable* $v = t-u$. *Filter* h^a *is causal by definition, and it is unstable because* $h^a \notin L^1(\mathbb{R})$. *We get* $H^a(p) = \int_0^{+\infty} e^{-pt}\,dt = \frac{1}{p}$, *which is defined for* $Re(p) > 0$.

2.1 Approximation by the rectangle method

We wish to approximate the integral of the signal $x^a(t)$ by the rectangle method (an example is given on Figure 1-(a)), which amounts to computing the integral of the interpolated signal

$$x_0^a(t) = \sum_{m=-\infty}^{+\infty} x^a(mT)\,f_0(t-mT)$$

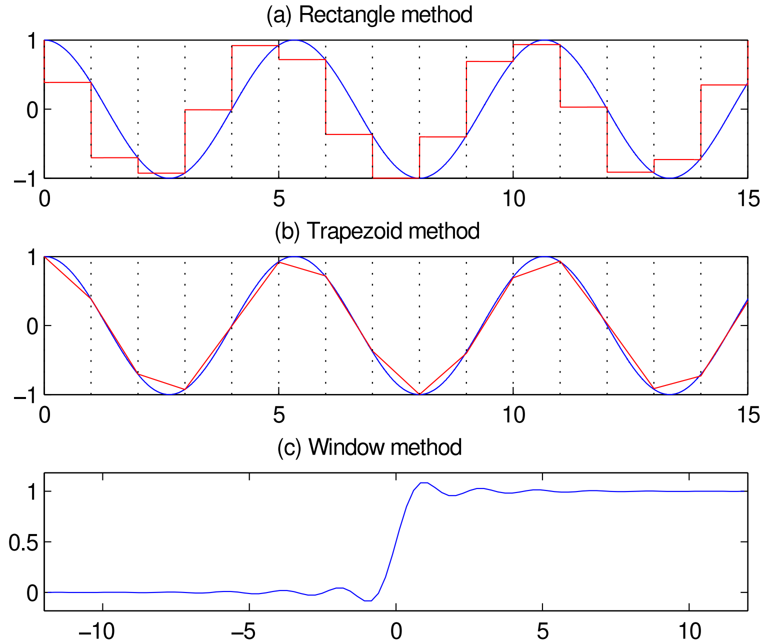

Figure 1: Three synthesis methods of an integrator filter

where $f_0(t) = 1$ if $t \in [-T, 0]$ and $f_0(t) = 0$ otherwise (rectangle function). We define the discrete-time integrated signal $y_0(n) = \int_{-\infty}^{nT} x_0^a(t)\, dt$.

Question 2 Show that $y_0(n)$ can be written as the convolution product between the signal $x(n)$ and a digital filter $h_0(n)$, and give the expression of its impulse response. Is this filter causal? Is it stable? Calculate the transfer function $H_0(z)$, and specify its domain of definition.

We have $y_0(n) = \sum\limits_{m=-\infty}^{+\infty} x(m) \int_{-\infty}^{nT} f_0(t-mT)\, dt = T \sum\limits_{m=-\infty}^{n} x(m) = \sum\limits_{m\in\mathbb{Z}} h_0(n-m)x(m)$ *with* $h_0(m) = T\, 1_{\mathbb{N}}(m)$. *Filter* h_0 *is causal by definition, and it is unstable because* $h_0 \notin l^1(\mathbb{Z})$. *We get* $H_0(z) = T \sum_{m\in\mathbb{N}} z^{-m} = \frac{T}{1-z^{-1}}$, *which is defined for* $|z| > 1$.

2.2 Approximation by the trapezoid method

We wish to approximate the integral of signal $x^a(t)$ by the trapezoid method (an example is given in Figure 1-(b)), which amounts to computing the integral of the interpolated signal

$$x_1^a(t) = \sum_{m=-\infty}^{+\infty} x^a(mT)\, f_1(t - mT)$$

where $f_1(t) = 1 - |t|/T$ if $t \in [-T, T]$ and $f_1(t) = 0$ elsewhere (triangle function). We define the discrete-time integrated signal $y_1(n) = \int_{-\infty}^{nT} x_1^a(t)\, dt$.

Question 3 Show that $y_1(n)$ can be written as the convolution product between the signal $x(n)$ and a digital filter $h_1(n)$, and give the expression of its impulse response. Is this filter causal? Is it stable?

We have $y_1(n) = \sum_{m=-\infty}^{+\infty} x(m) \int_{-\infty}^{nT} f_1(t - mT)\,dt = T(\frac{1}{2}x(n) + \sum_{m=-\infty}^{n-1} x(m)) = \sum_{m\in\mathbb{Z}} h_1(n-m)x(m)$ *with* $h_1(m) = T\,(\frac{1}{2}\delta_0(m) + 1_{\mathbb{N}^*}(m))$. *Filter* h_1 *is causal by definition, and it is unstable because* $h_1 \notin l^1(\mathbb{Z})$.

Question 4 Show that this method is equivalent to determining the digital filter from the analog filter of Question 1 by using the bilinear transformation (hint: we can identify the two transfer functions).
We get $H_1(z) = \frac{1}{2} + \sum_{m\in\mathbb{N}^*} z^{-m} = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}}$, *which is defined for* $|z| > 1$. *With the bilinear transform, we retrieve* $H_1(z) = H^a(p) = \frac{1}{p}$ *with* $p = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$.

2.3 Synthesis by the window method

We now wish to determine the integral of the signal $x^a(t)$ exactly. To do this, we assume that $x^a(t)$ satisfies the assumptions of the Shannon-Nyquist's theorem. It can then be reconstructed exactly from its samples:

$$x^a(t) = \sum_{m=-\infty}^{+\infty} x^a(mT)\,f(t - mT)$$

where $f(t) = \text{sinc}\left(\frac{t}{T}\right)$ avec $\text{sinc}(u) = \frac{\sin(\pi u)}{\pi u}$. We define the discrete time integrated signal $y(n) = \int_{-\infty}^{nT} x^a(t)\,dt$.

Question 5 Show that $y(n)$ can be written as the convolution product between the signal $x(n)$ and the digital filter $h(n) = T\int_{-\infty}^{n} \text{sinc}(u)\,du$ (hint: we will assume that $x^a(t)$ satisfies strong enough assumptions to be able to switch $\int$ and $\sum$).
We have $y(n) = \sum_{m=-\infty}^{+\infty} x(m) \int_{-\infty}^{nT} f(t - mT)\,dt = \sum_{m=-\infty}^{+\infty} x(m) \int_{-\infty}^{(n-m)T} f(t)\,dt = \sum_{m\in\mathbb{Z}} h(n-m)x(m)$ *with* $h(m) = \int_{-\infty}^{mT} f(t)\,dt = \int_{-\infty}^{mT} \text{sinc}\left(\frac{t}{T}\right) dt = T\int_{-\infty}^{m} \text{sinc}(u)\,du$ *with the change of variable* $t = uT$.

Question 6 The impulse response of filter h is represented in Figure 1-(c) (for $T = 1$). What phenomenon can be observed compared to the impulse responses calculated previously? Is this filter causal? Is it stable ? (hint: $h(n) \underset{n\to+\infty}{\longrightarrow} T$)
In Figure 1-(c), we observe a Gibbs phenomenon in the time domain. This filter is non longer causal, nor stable because $h \notin l^1(\mathbb{Z})$.
Since $h(n) \underset{n\to+\infty}{\longrightarrow} T$, it does not seem reasonable to synthesize filter h by directly applying the window method, which consists in truncating the impulse response. Instead, we define filter $G(z) = (1 - z^{-1})\,H(z)$, whose impulse response decreases towards 0 at infinity. This filter $G(z)$ can be synthesized by the window method. We can then deduce an integrating filter $H(z) = \frac{G(z)}{1-z^{-1}}$.

Question 7 Show that the impulse response of the filter $g(n)$ is symmetrical with respect to $\frac{1}{2}$, and is upper bounded in absolute value by $O\left(\frac{1}{n}\right)$ (the proof is simple but it may be useful to make a drawing).
We have $G(z) = (1 - z^{-1})H(z)$, *therefore* $g(n) = h(n) - h(n-1) = T\int_{n-1}^{n} \text{sinc}(u)\,du = O(\frac{1}{n})$. *Moreover,* $g(1-n) = T\int_{-n}^{1-n} \text{sinc}(u)\,du = g(n)$, *thus the impulse response* $g(n)$ *is symmetrical with respect to* $\frac{1}{2}$.

Question 8 Since the impulse response of filter g tends to 0 at infinity, it seems reasonable to synthesize this filter by the window method. In order for the resulting filter to be linear phase, should we choose an even or odd filter length N? What type of filter does this correspond to? (I, II, III or IV)
The symmetry of $g(n)$ w.r.t. $\frac{1}{2}$ imposes a type II filter with even filter length N.

Question 9 Is the resulting filter $H(z)$ stable? What would you suggest to remedy this?
$H(z)$ is unstable because of its pole at $z = 1$. Stability can be enforced by moving this pole inside the unit circle (e.g. at $z = 1 - \varepsilon$ with a small $\varepsilon > 0$).

This tutorial aims to carry out filtering in a multirate system (with an application to the conversion of sampling frequency), and to understand a perfect reconstruction filter bank for audio equalization.

1 Conversion of sampling frequency

We want to achieve the conversion of sampling frequency from $F_s = 48kHz$ to $F_s = 32kHz$.

1. Describe and draw the digital processing chain that will permit you to achieve such a conversion (in particular you need to specify the characteristics of the ideal filter $H(e^{2i\pi\nu})$ that you will have to use).

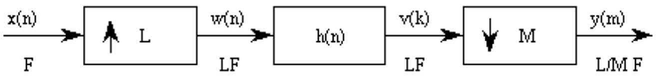

Figure 1: Resampling diagram

The block diagram of resampling is depicted in Figure 1. Here we have $F = F_s$, $L = 2$, and $M = 3$. The frequency response of the anti-aliasing, low-pass filter h is

$$\begin{cases} H(e^{i2\pi\nu}) = L = 2 & \forall |\nu| < \min\left(\frac{1}{2L}, \frac{1}{2M}\right) = \frac{1}{6} \\ H(e^{i2\pi\nu}) = 0 & \forall \min\left(\frac{1}{2L}, \frac{1}{2M}\right) = \frac{1}{6} < |\nu| < \frac{1}{2} \end{cases}.$$

The purpose of the rest of this section is to get an efficient implementation of this conversion of sampling frequency and to compare its performance with that of a direct implementation.
The efficient implementation will be achieved by means of polyphase decompositions (of type I and type II). The use of noble identities and of the equivalence below (Figure 2) will permit you to apply the filtering operations to signals at a lower sampling frequency.

2. Check the equivalence between the two diagrams in Figure 2.

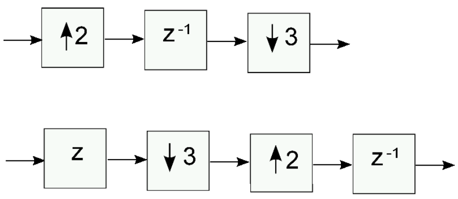

Figure 2: Equivalence

The equivalence in Figure 2 is proved in Figure 3.

3. Find the efficient implementation of this conversion of sampling rate by using two successive polyphase decompositions and the equivalence in Figure 2.

Filter h in Figure 1 has to be decomposed into its type 2 polyphase components at order $L = 2$. Then each of the two polyphase components obtained in this way have to be decomposed into their type 1 polyphase components at order $M = 3$ (actually, the two polyphase decompositions could

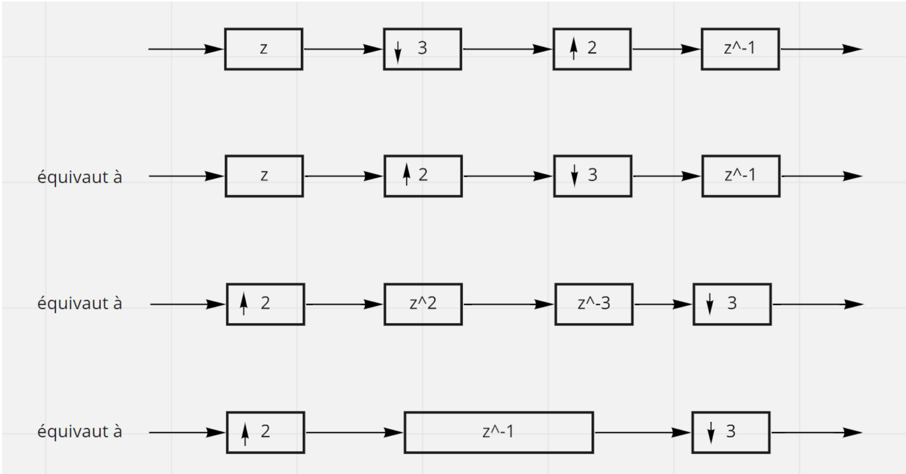

Figure 3: Proof of the equivalence

also be performed in the reverse order, but starting with the type 2 polyphase decomposition makes the derivation more compact). In the end, we get $2 \times 3 = 6$ polyphase components, which can be switched with the decimators and the insertions of zeros in Figure 1 by using the noble identities and the equivalence of Figure 2, in order to get the efficient implementation of resampling.

2 STFT audio equalization

2.1 STFT analysis

The definition of the STFT that is referred to as "low-pass convention" is given in discrete time by :

$$W_x(\lambda, b) = \sum_{n\in\mathbb{Z}} x(n)w(n-b)e^{-j2\pi\lambda n}, \tag{1}$$

where $w(n)$ is the analysis window in discrete time, which is supposed summable, real and symmetric.

1. We choose $w(n)$ as a Hann (Hanning) window of length N_w. What is the width of the main lobe as a function of N_w ?

   The width of the main lobe of the Hann window is $\frac{4}{N_w}$ (in normalized frequency scale).

2. Note that the expression (1), taken at fixed λ, can be written as a convolution and deduce an interpretation of the STFT in terms of filtering. Explain the role of the corresponding filter (low-pass? band-pass? high-pass?). As a linear phase FIR filter, specify its type (type 1,2,3,4?) according to its length (even or odd).

   Equation (1) is equivalent to $W_x(\lambda, b) = (\tilde{h} * \tilde{x})(b)$ with $\tilde{x}(n) = x(n)e^{-j2\pi\lambda n}$ and $\tilde{h}(n) = w(-n)$. Filter h is a low-pass filter of cut-off frequency $\frac{2}{N_w}$. If the length of h is odd, then h if a type 1 FIR filter, and if the length of h is even, then h if a type 2 FIR filter.

3. Another definition of the STFT, referred to as "band-pass convention", is given by

$$\tilde{X}(\lambda, b) = \sum_{n\in\mathbb{Z}} x(n+b)w(n)e^{-j2\pi\lambda n}. \tag{2}$$

Explain this latter designation and express $\tilde{X}$ as a function of W_x.

Equation (2) *is equivalent to* $\tilde{X}(\lambda, b) = (h * x)(b)$ *with* $h(n) = w(-n)e^{j2\pi\lambda n}$. *Filter* h *is a band-pass filter of center frequency* λ *and bandwith* $\frac{4}{N_w}$, *hence the "band-pass" designation. Moreover, with* $m = n + b$, *we get* $\tilde{X}(\lambda, b) = \sum_{m\in\mathbb{Z}} x(m)w(m-b)e^{-j2\pi\lambda(m-b)} = e^{j2\pi\lambda b}W_x(\lambda, b)$.

2.2 Reconstruction

The reconstruction is achieved by an *overlap-add* operation, which will be written as:

$$y(n) = \sum_{u\in\mathbb{Z}} y_s(u, n - uR),$$

where $y_s(u, n) = \mathrm{DFT}^{-1}[\tilde{X}(k, u)](n)\, w_s(n) = \dfrac{1}{M} \sum_{k=0}^{M-1} \tilde{X}(k, u)e^{j2\pi\frac{kn}{M}} \times w_s(n)$

4. Show that a sufficient condition for perfect reconstruction is $f(n) = 1\ \forall n$ where $f(n) = \sum_{u\in\mathbb{Z}} w(n-uR)w_s(n-uR)$.

   The proof is in the course handout.

1 CQF filter bank

A two-channel filter bank is defined by the diagram in Figure 1.

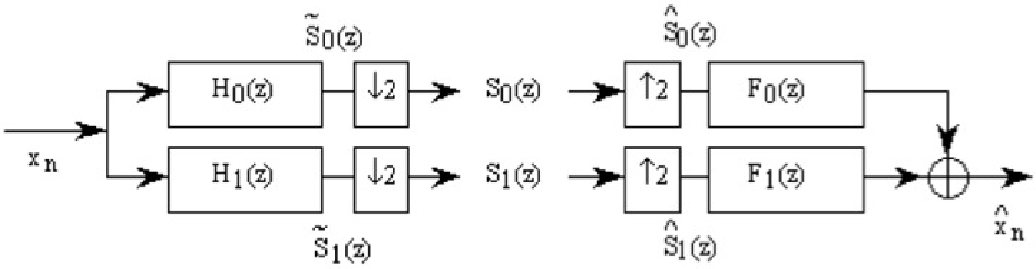

Figure 1: General diagram of a two-channel filter bank.

1. Express $\hat{X}(z)$ as a function of $X(z)$.

   The calculation is done in the course handout. We get

   $$\widehat{X}(z) = T(z)X(z) + A(z)X(-z) \tag{1}$$

   where

   $$T(z) = \frac{1}{2}(H_0(z)F_0(z) + H_1(z)F_1(z)), \tag{2}$$
   $$A(z) = \frac{1}{2}(H_0(-z)F_0(z) + H_1(-z)F_1(z)). \tag{3}$$

2. Deduce that the aliasing cancellation (AC) conditions of a 2-channel filter bank are $F_0(z) = H_1(-z)$ and $F_1(z) = -H_0(-z)$, and that its transfer function (TF) is $T(z) = \frac{1}{2}(H_0(z)F_0(z) + H_1(z)F_1(z))$.

   Indeed, if $F_0(z) = H_1(-z)$ and $F_1(z) = -H_0(-z)$, then $A(z) = 0$, therefore $\widehat{X}(z) = T(z)X(z)$, which proved that the transfer function is $T(z)$.

3. Now we assume that H_0 and H_1 are *conjugate quadrature filters* (CQF): $H_1(z) = -z^{-(N-1)}\widetilde{H}_0(-z)$ where $\tilde{H}_0(z) = H_0^*(\frac{1}{z^*})$ and N is even. Prove that equations (AC) and (CQF) imply that $\forall k \in \{0, 1\}$, $F_k(z) = z^{-(N-1)}\widetilde{H}_k(z)$: we say that the analysis and synthesis filters are *paraconjugate* (PC).

   The AC condition yields $F_0(z) = H_1(-z)$, then the CQF condition yields $H_1(-z) = z^{-(N-1)}\widetilde{H}_0(z)$ because N is even, therefore $F_0(z) = z^{-(N-1)}\widetilde{H}_0(z)$. In the same way, the AC condition yields $F_1(z) = -H_0(-z)$, then the CQF condition yields $H_0(-z) = -z^{-(N-1)}\widetilde{H}_1(z)$ because N is even, therefore $F_1(z) = z^{-(N-1)}\widetilde{H}_1(z)$.

4. Finally we assume that $H_0(z)$ is a *symmetric power* (SP) filter: $\widetilde{H}_0(z)H_0(z) + \widetilde{H}_0(-z)H_0(-z) = 2c$. Prove that equations (TF), (CQF), (PC) et (SP) imply that $T(z) = cz^{-(N-1)}$: the CQF filter bank guarantees perfect reconstruction.

   Substituting equations PC into TF yields $T(z) = \frac{z^{-(N-1)}}{2}(H_0(z)\widetilde{H}_0(z) + H_1(z)\widetilde{H}_1(z))$. Then substituting equation CQF into this last equation yields $T(z) = \frac{z^{-(N-1)}}{2}(H_0(z)\widetilde{H}_0(z)+\widetilde{H}_0(-z)H_0(-z))$. Finally, substituting equation SP into this last equation yields $T(z) = cz^{-(N-1)}$, which proves the perfect reconstruction of the CQF filter bank.

2 Transmultiplexer

We implement the transmultiplexer represented in Figure 2 by means of the filters defined in Exercise 1.

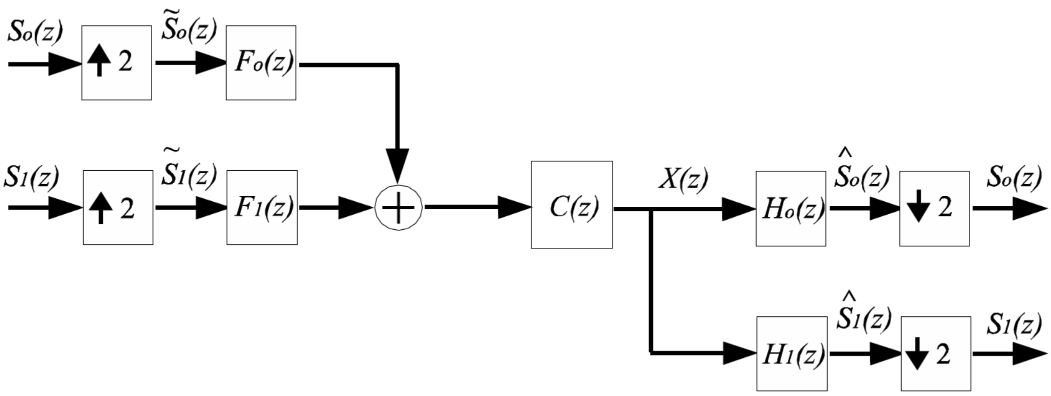

Figure 2: Transmultiplexer.

1. Prove that the transmultiplexer guarantees perfect reconstruction at the output when $C(z) = z^{-1}$.

   We already know that the diagram in Figure 1 is equivalent to the transfer function $cz^{-(N-1)}$ where N is even. This proves that the left block in Figure 1, preceded by $cz^{-(N-1)}$, defines a linear transform which is exactly the inverse of the linear transform defined by the right block in Figure 1. Besides, we notice that Figure 2 is obtained by switching the two blocks in Figure 1 and inserting filter $C(z)$ between them. Consequently, if we choose $C(z) = cz^{-(N-1)}$, then the transmultiplixer is equivalent to the identity transform, which is a particular case of perfect reconstruction. Equivalently, another perfect reconstruction solution is obtained by simply choosing $C(z) = z^{-1}$, because N is even, and because the transmultiplexer involves decimations and insertions of zeros of order 2.

2. From now on, filter $C(z)$ will represent the transfer function of a transmission channel between the encoder and the decoder. It is uniformly equal to 1 if the channel is transparent, but in general, the transmission channel is imperfect, and its transfer function $C(z)$ is not constant. In order to simplify, let us assume that $C(z)$ is of the form $1 - \alpha z^{-1}$. In order to keep the perfect reconstruction property at the output of the transmultiplexer, we will have to introduce just after $C(z)$ a causal filter $D(z)$ such that $C(z)D(z) = dz^{-n_0}$, where n_0 is an odd number.

   (a) How to choose $D(z)$ if $\alpha = 0.9$?

   If $\alpha = 0.9$, then we can simply set $D(z) = \frac{dz^{-n_0}}{1-\alpha z^{-1}}$, because the causal implementation of this filter is stable.

   (b) What problem do we encounter if $\alpha = 1.2$? Propose an approximate solution.

   If $\alpha = 1.2$, then the causal implementation of the filter $D(z) = \frac{dz^{-n_0}}{1-\alpha z^{-1}}$ is unstable. The stable implementation cannot be used either, because it is IIR and anti-causal. A solution would consist in approximating this filter by a causal FIR filter. This can be achieved by truncating the anti-causal, infinite impulse response of the stable implementation of $D(z)$ to a given finite length, and translating it over time, so as to make it causal. The quality of this approximation can be improved by increasing the length of the causal FIR filter obtained in this way.

3. If the channel transfer function $C(z)$ is unknown, propose a method for estimating it from the output signals, by choosing appropriate filter $D(z)$ and input signals s_0 and s_1.

   Let s_0 and s_1 be the two outputs of the left block in Figure 1, when the input signal is $x(n) = \delta_0(n)$. Then the signal entering $C(z)$ in Figure 2 is $c\delta_{N-1}(n)$. Let $D(z) = \frac{1}{c^2}z^{2N-2}$. Then the input of the right block in Figure 2 is the signal of Z-transform $\frac{1}{c}z^{N-1}C(z)$. Since this right block is the same as the left block in Figure 1, it suffices to take the two output signals of the transmultiplexer as input signals of the right block in Figure 1, to retrieve a signal of Z-transform $C(z)$. Thus the channel transfer function $C(z)$ is identified.

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