Parametric estimation of line spectra

Linear time series (part 2)
TSIA202b

Roland Badeau

Sinusoidal modeling of audio signals

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    - pairs or triplets of strings in a piano, plus coupling of the vertical and horizontal vibration modes

Part I

Parametric signal model

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- The observed signal x_t is modeled as the signal s_t plus a complex Gaussian white noise b_t of variance σ^2 (sequence of complex IID r.v. of PDF $p(b) = \frac{1}{\pi\sigma^2} \, e^{-\frac{|b|^2}{\sigma^2}}$)

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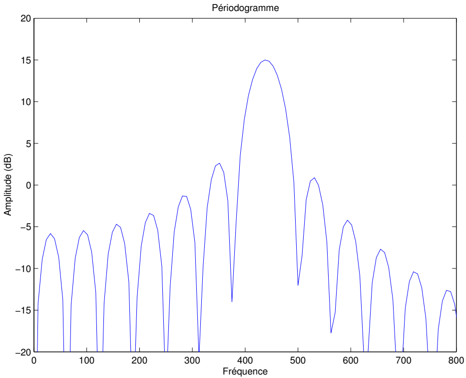
  - widening of the peak in case of exponential damping

Resolution problems

Test signal :

- Sampling frequency : 8000 Hz
- First sinusoid : 440 Hz (A)
- Second sinusoid : 415,3 Hz (G#)
- No damping, all amplitudes equal to 1
- Length of the rectangular window : $N = 128$ (16 ms)
- Length of the transform : 1024 samples

Resolution problems



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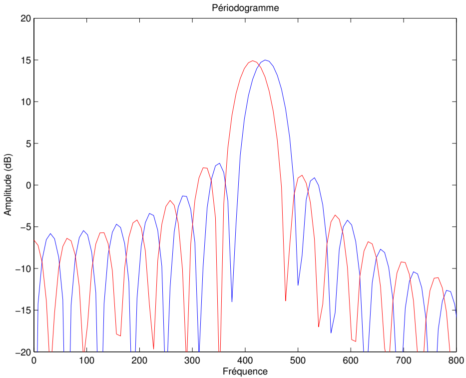

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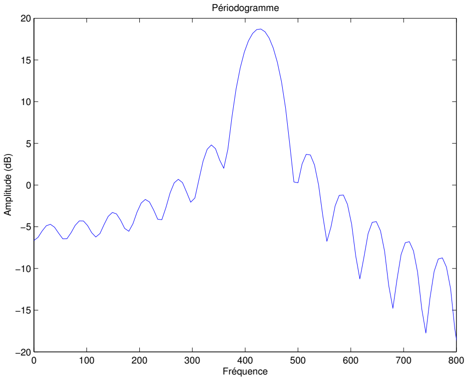

Part II

Maximum Likelihood Method

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- Maximization w.r.t. σ^2 : $\sigma^2 = \frac{1}{N}\left\|\boldsymbol{x}(t) - \boldsymbol{V}^N\alpha(t)\right\|^2$

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  $\mathcal{J}(z_0, \dots, z_{K-1}) = \boldsymbol{x}(t)^H \boldsymbol{V^N} \left(\boldsymbol{V^N}^H \boldsymbol{V^N} \right)^{-1} \boldsymbol{V^N}^H \boldsymbol{x}(t)$
- This optimization problem has to be solved numerically

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- High resolution parametric estimation methods overcome the limits of Fourier analysis

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- Limit of Fourier analysis : $\min\limits_{k_1 \neq k_2} \left| f_{k_2} - f_{k_1} \right| >> \frac{1}{N}$

Part III

High resolution methods based on linear prediction

Jean-Baptiste Joseph Fourier (1768-1830)

Gaspard-Marie Riche of Prony (1755-1839)

Linear prediction methods

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- Drawback : mediocre performance in presence of noise

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- The solution is $\boldsymbol{p} = \frac{1}{\boldsymbol{e}_1^H \widehat{\boldsymbol{R}}_{xx}(t)^{-1} \boldsymbol{e}_1}\, \widehat{\boldsymbol{R}}_{xx}(t)^{-1} \boldsymbol{e}_1$ where $\boldsymbol{e}_1 \triangleq [1,\, 0 \ldots 0]^\top$

Prony and Pisarenko methods

- Prony method :
  - Construct matrix $\boldsymbol{X}(t)$ and compute $\widehat{\boldsymbol{R}}_{xx}(t)$
  - Compute $\boldsymbol{p} = \frac{1}{\boldsymbol{e}_1^H \widehat{\boldsymbol{R}}_{xx}(t)^{-1} \boldsymbol{e}_1} \widehat{\boldsymbol{R}}_{xx}(t)^{-1} \boldsymbol{e}_1$
  - Determine the z_k's as the roots of $P[z] = \sum_{k=0}^{K} p_k \, z^{K-k}$

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  - Construct matrix $\boldsymbol{X}(t)$ and compute $\widehat{\boldsymbol{R}}_{xx}(t)$
  - Compute $\boldsymbol{p} = \frac{1}{\boldsymbol{e}_1^H \widehat{\boldsymbol{R}}_{xx}(t)^{-1} \boldsymbol{e}_1} \widehat{\boldsymbol{R}}_{xx}(t)^{-1} \boldsymbol{e}_1$
  - Determine the z_k's as the roots of $P[z] = \sum_{k=0}^{K} p_k \, z^{K-k}$
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Part IV

Subspace-based HR methods

Matrix representation of the signal

- Observation horizon : $t \in \{0 \dots N-1\}$, where $N > 2K$

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- For all $i \in \{0 \dots n-1\}$, $\boldsymbol{w}_i$ is also an eigenvector of $\boldsymbol{R}_{xx}$ corresponding to the eigenvalue $\lambda'_i = \lambda_i + \sigma^2$. Therefore,

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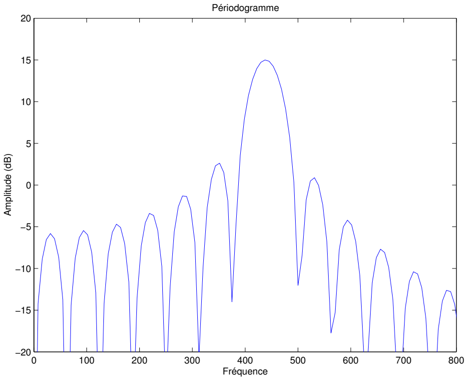
- The MUSIC method consists in solving this equation
- The Spectral-MUSIC method consists in detecting the K highest peaks in function $z \mapsto \dfrac{1}{\left\|\boldsymbol{W}_\perp^H \boldsymbol{v}(z)\right\|^2}$.

Spectral MUSIC method

Test signal :

- Sampling frequency : 8000 Hz
- First sinusoid : 440 Hz (A)
- Second sinusoid : 415,3 Hz (G#)
- No damping, all amplitudes equal to 1
- Length of the rectangular window : $N = 128$ (16 ms)
- Length of the transform : 1024 samples

Spectral *MUSIC* method



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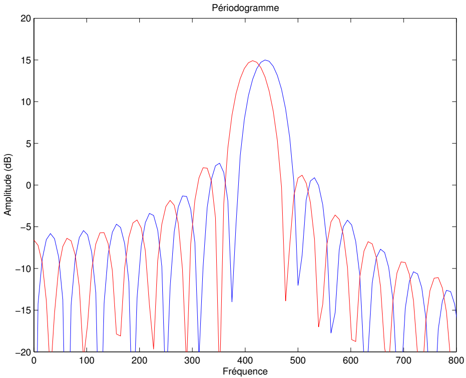

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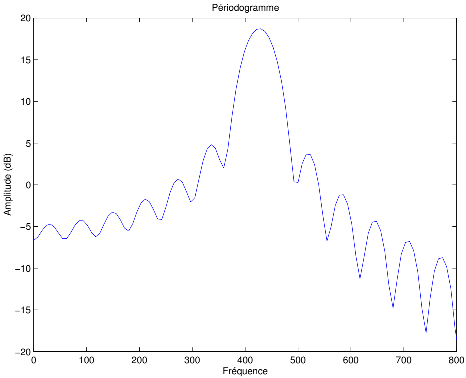

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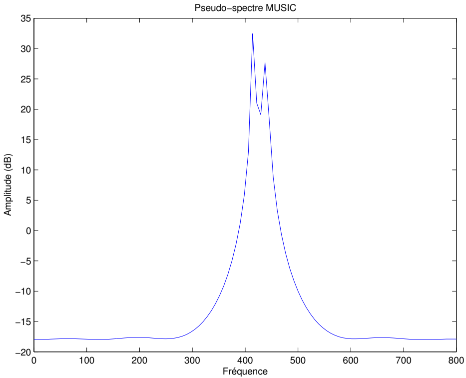

ESPRIT method

- Rotational invariance property of $\boldsymbol{V}^n$:

$$\underbrace{\begin{bmatrix} 1 & \dots & 1 \\ z_0 & \dots & z_{K-1} \\ \vdots & \dots & \vdots \\ {z_0}^{n-2} & \dots & {z_{K-1}}^{n-2} \\ {z_0}^{n-1} & \dots & {z_{K-1}}^{n-1} \end{bmatrix}}_{\substack{\boldsymbol{V}^n \\ n \times K}}$$

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Part V

Estimation of the other parameters

Estimation of the modeling order

- Information Theoretic Criteria (ITC) : we minimize

$$\mathrm{ITC}(p) = -(n-p)\,l\,\ln\left(\frac{\left(\prod_{q=p+1}^{n}\sigma_q^2\right)^{\frac{1}{n-p}}}{\frac{1}{n-p}\sum_{q=p+1}^{n}\sigma_q^2}\right) + p\,(2n-p)\,C(l)$$

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- We finally get $\widehat{a}_k = |\widehat{\alpha}_k|$ and $\widehat{\phi}_k = \arg(\widehat{\alpha}_k)$

Part VI

Performance of the estimators

Cramér-Rao bounds

- Regular statistical model
  - Consider a statistical model $p(\boldsymbol{x};\,\boldsymbol{\theta})$ parameterized by $\boldsymbol{\theta}$
  - Score function : $\boldsymbol{l}(\boldsymbol{x};\,\boldsymbol{\theta}) \triangleq \boldsymbol{\nabla}_{\boldsymbol{\theta}} \ln p(\boldsymbol{x};\,\boldsymbol{\theta})\, \mathbf{1}_{p(\boldsymbol{x};\,\boldsymbol{\theta})>0}$
  - The parameterization is said *regular* if :
    1. $p(\boldsymbol{x};\,\boldsymbol{\theta})$ is continuously differentiable w.r.t. $\boldsymbol{\theta}$.
    2. $\boldsymbol{F}(\boldsymbol{\theta}) \triangleq \int_H \boldsymbol{l}(\boldsymbol{x};\,\boldsymbol{\theta})\, \boldsymbol{l}(\boldsymbol{x};\,\boldsymbol{\theta})^\top p(\boldsymbol{x};\,\boldsymbol{\theta})\, \mathrm{d}\boldsymbol{x}$ (Fisher information matrix) is positive definite for all $\boldsymbol{\theta}$ and continuous w.r.t. $\boldsymbol{\theta}$

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- Cramér-Rao bounds
  - Consider a regular statistic model parameterized by $\boldsymbol{\theta}$
  - Let $\widehat{\boldsymbol{\theta}}$ be an unbiased estimator of $\boldsymbol{\theta}$ ($\forall \boldsymbol{\theta} \in \Theta,\, \mathbb{E}_{\boldsymbol{\theta}}[\widehat{\boldsymbol{\theta}}] = \boldsymbol{\theta}$)
  - Then the dispersion matrix $\boldsymbol{D}(\boldsymbol{\theta},\widehat{\boldsymbol{\theta}}) \triangleq \mathbb{E}_{\boldsymbol{\theta}}\left[\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)^\top\right]$ is such that matrix $\boldsymbol{D}(\boldsymbol{\theta},\,\widehat{\boldsymbol{\theta}}) - \boldsymbol{F}(\boldsymbol{\theta})^{-1}$ is positive semidefinite.

Cramér-Rao bounds

- For a family of complex Gaussian distributions of covariance $\boldsymbol{R}_{bb}(\boldsymbol{\theta}) \in \mathcal{C}^1(\Theta, \mathbb{C}^{N\times N})$ and of mean $\boldsymbol{s}(\boldsymbol{\theta}) \in \mathcal{C}^1(\Theta, \mathbb{C}^{N})$,
$$\boldsymbol{F}_{(i,j)}(\boldsymbol{\theta}) = \text{trace}\left(\boldsymbol{R}_{bb}^{-1}\frac{\partial \boldsymbol{R}_{bb}(\boldsymbol{\theta})}{\partial \theta_i}\boldsymbol{R}_{bb}^{-1}\frac{\partial \boldsymbol{R}_{bb}(\boldsymbol{\theta})}{\partial \theta_j}\right) + 2\mathcal{R}e\left(\frac{\partial \boldsymbol{s}(\boldsymbol{\theta})}{\partial \theta_i}^H \boldsymbol{R}_{bb}^{-1}\frac{\partial \boldsymbol{s}(\boldsymbol{\theta})}{\partial \theta_j}\right)$$

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- All bounds are independent of the phases ϕ_k and are unchanged by any translation of the set of frequencies f_k
- If $\forall k$, $\delta_k = 0$ and if $N \to +\infty$, then :
  - $\mathrm{CRB}\{\sigma\} = \frac{\sigma^2}{4N} + O\left(\frac{1}{N^2}\right)$
  - $\mathrm{CRB}\{f_k\} = \frac{6\sigma^2}{4\pi^2 N^3 a_k^2} + O\left(\frac{1}{N^4}\right)$
  - $\mathrm{CRB}\{a_k\} = \frac{2\sigma^2}{N} + O\left(\frac{1}{N^2}\right)$
  - $\mathrm{CRB}\{\phi_k\} = \frac{2\sigma^2}{Na_k^2} + O\left(\frac{1}{N^2}\right)$

Performance of HR methods

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  - Performance expressed in terms of bias and variance
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  - Efficiency : ratio between variance and Cramér-Rao bound
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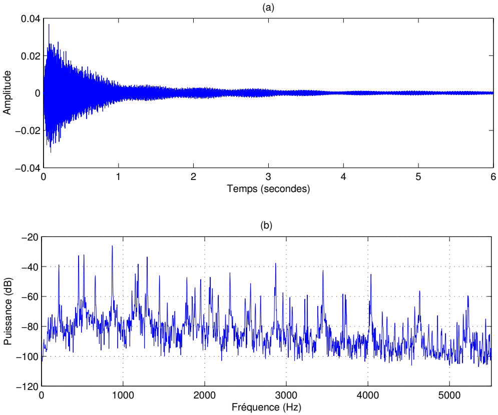
- Maximum likelihood : unbiased and asymptotically efficient ($N \to +\infty$)
- HR methods : results based on the perturbation theory
  - Assumptions : $N \to +\infty$ or SNR$\to +\infty$
  - All the HR methods are asymptotically unbiased
  - The Prony and Pisarenko methods are very inefficient : their variances are significantly greater than the Cramér-Rao bounds.
  - MUSIC and ESPRIT have an asymptotic efficiency close to 1

Part VII

Signals to be processed

Bell sound



(a) Signal waveform
(b) Power spectral density