Tutorial on parametric estimation of rational spectra
TSIA202b

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Exercise 1: Maximum entropy distribution

Let X be a real random variable. Prove that the probability distribution p_X that maximizes the entropy of X, subject to fixed mean μ_X and variance σ_X^2, is the Gaussian distribution.

Exercise 2: Relationship between AR Modeling and Forward Linear Prediction

Suppose we have a zero-mean WSS process $\{Y_t\}$ (not necessarily AR) with autocovariance function $\{r_{YY}(k)\}_{k=-\infty}^{\infty}$. We wish to predict Y_t by a linear combination of its p past values: the predicted value is given by

$$\hat{Y}_t^f = \sum_{k=1}^{p} a_k Y_{t-k}$$

We define the forward prediction error as

$$Z_t^f = Y_t - \hat{Y}_t^f = Y_t - \sum_{k=1}^{p} a_k Y_{t-k}.$$

Show that the vector $\theta_f = [a_1 \dots a_p]^\top$ of prediction coefficients that minimizes the prediction-error variance $\sigma_f^2 \triangleq \mathbb{E}\{|Z_t^f|^2\}$ is the solution to

$$\begin{bmatrix} r_{YY}(0) & r_{YY}(-1) & \dots & r_{YY}(-p) \\ r_{YY}(1) & r_{YY}(0) & & \vdots \\ \vdots & & \ddots & r_{YY}(-1) \\ r_{YY}(p) & \dots & & r_{YY}(0) \end{bmatrix} \begin{bmatrix} 1 \\ -a_1 \\ \vdots \\ -a_p \end{bmatrix} = \begin{bmatrix} \sigma_p^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{1}$$

Show also that $\sigma_f^2 = \sigma_p^2$ (i.e., that σ_p^2 in (1) is the prediction-error variance).

Exercise 3: Relationship between AR Modeling and Backward Linear Prediction

Consider the signal $\{Y_t\}$, as in Exercise 2. This time, we will consider backward prediction: we will predict Y_t from its p immediate future values:

$$\hat{Y}_t^b = \sum_{k=1}^{p} b_k Y_{t+k}$$

with corresponding backward prediction error $Z_t^b = Y_t - \hat{Y}_t^b$. Such backward prediction is useful in applications where noncausal processing is permitted; for example, when the data has been prerecorded and is stored in memory or on a tape and we want to make inferences on samples that precede the observed ones. Find an expression similar to (1) for the backward prediction coefficient vector $\theta_b = [b_1 \dots b_p]^\top$. Find a relationship between the θ_b and the corresponding forward prediction coefficient vector θ_f. Relate the forward and backward prediction error variances.

Exercise 4: Prediction Filters and Smoothing Filters

The smoothing filter is a practically useful variation on the theme of linear prediction. A result of Exercises 2 and 3 should be that for the forward and backward prediction filters

$$A(z) = 1 + \sum_{k=1}^{p} a_k z^{-k} \text{ and } B(z) = 1 + \sum_{k=1}^{p} b_k z^{-k},$$

the prediction coefficients satisfy $a_k = \overline{b_k}$, and the prediction error variances are equal. Now consider the smoothing filter

$$Z_t^s = Y_t - \sum_{k=1}^{p} c_k Y_{t-k} - \sum_{k=1}^{p} d_k Y_{t+k}.$$

1. Derive a system of linear equations, similar to the forward and backward linear prediction equations, that relate the smoothing filter coefficients, the smoothing prediction error variance $\sigma_s^2 = \mathbb{E}\left\{|Z_t^s|^2\right\}$, and the autocovariance function of Y_t.

2. Show that the unconstrained minimum smoothing error variance solution satisfies $c_k = \overline{d_k}$.

3. Prove that the minimum smoothing prediction error variance is less than the minimum (forward or backward) prediction error variance.

Exercise 5: Generating the autocovariance function from ARMA parameters

In this lesson we have expressed the ARMA coefficients σ_Z^2, a_i, b_j in terms of the autocovariance function $\{r_{XX}(k)\}_{k=-\infty}^{\infty}$. Find the inverse map: given $\sigma_Z^2, a_1, \dots, a_p, b_1 \dots, b_q$, find equations to determine $\{r_{XX}(k)\}_{k=-\infty}^{\infty}$.