

Correction des exercices de TD

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Exercice 1 (Logique propositionnelle)

Démontrer les propositions suivantes dans LJ et LK :

1. $A \Rightarrow B \Rightarrow A$
2. $(A \Rightarrow B \Rightarrow C) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C$
3. $A \wedge B \Rightarrow B$
4. $B \Rightarrow A \vee B$
5. $(A \vee B) \Rightarrow (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C$
6. $A \Rightarrow \perp \Rightarrow \neg A$
7. $\perp \Rightarrow A$
8. $(A \Leftrightarrow B) \Rightarrow A \Rightarrow B$
9. $(A \Leftrightarrow B) \Rightarrow B \Rightarrow A$
10. $(A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)$

1.	$\frac{\frac{\overline{A, B \vdash A}^{\text{ax}}}{A \vdash B \Rightarrow A} \Rightarrow_{\text{right}}}{\vdash A \Rightarrow B \Rightarrow A} \Rightarrow_{\text{right}}$	Ici, la preuve en LJ est la même que la preuve en LK.
2.	$\frac{\frac{\frac{\overline{B, A \vdash B}^{\text{ax}}}{A \Rightarrow B, A \vdash B} \Rightarrow_{\text{left}} \quad \frac{\overline{A \vdash A}^{\text{ax}}}{C, A \Rightarrow B, A \vdash C} \Rightarrow_{\text{left}} \quad \frac{\overline{A \Rightarrow B, A \vdash C}^{\text{ax}}}{A \Rightarrow B \Rightarrow C, A \Rightarrow B, A \vdash C} \Rightarrow_{\text{left}}}{\vdash (A \Rightarrow B \Rightarrow C) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C} \Rightarrow_{\text{right}} \times 3$ <p>Preuve dans LJ</p>	$\frac{\frac{\frac{\overline{B, A \vdash B, C}^{\text{ax}}}{A \Rightarrow B, A \vdash B, C} \Rightarrow_{\text{left}} \quad \frac{\overline{A \vdash A, B, C}^{\text{ax}}}{C, A \Rightarrow B, A \vdash C} \Rightarrow_{\text{left}} \quad \frac{\overline{A \Rightarrow B, A \vdash C}^{\text{ax}}}{A \Rightarrow B \Rightarrow C, A \Rightarrow B, A \vdash C} \Rightarrow_{\text{left}}}{\vdash (A \Rightarrow B \Rightarrow C) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C} \Rightarrow_{\text{right}} \times 3$ <p>Preuve dans LK</p>
3.	$\frac{\frac{\overline{A, B \vdash B}^{\text{ax}}}{A \wedge B \vdash B} \wedge_{\text{left}}}{\vdash A \wedge B \Rightarrow B} \Rightarrow_{\text{right}}$	Ici, la preuve en LJ est la même que la preuve en LK.
4.	$\frac{\frac{\overline{B \vdash B}^{\text{ax}}}{B \vdash A \vee B} \vee_{\text{right}}}{\vdash B \Rightarrow A \vee B} \Rightarrow_{\text{right}}$ <p>Preuve dans LJ</p>	$\frac{\frac{\overline{B \vdash A, B}^{\text{ax}}}{B \vdash A \vee B} \vee_{\text{right}}}{\vdash B \Rightarrow A \vee B} \Rightarrow_{\text{right}}$ <p>Preuve dans LK</p>
5.	$\frac{\frac{\frac{\overline{B, A \Rightarrow C, C \vdash C}^{\text{ax}}}{B, A \Rightarrow C, B \Rightarrow C \vdash C} \Rightarrow_{\text{left}} \quad \frac{\overline{B, A \Rightarrow C \vdash B}^{\text{ax}}}{A, C, B \Rightarrow C \vdash C} \Rightarrow_{\text{left}} \quad \frac{\overline{A, C, B \Rightarrow C \vdash C}^{\text{ax}}}{A, A \Rightarrow C, B \Rightarrow C \vdash C} \Rightarrow_{\text{left}} \quad \frac{\overline{A, B \Rightarrow C \vdash A}^{\text{ax}}}{A \vee B, A \Rightarrow C, B \Rightarrow C \vdash C} \vee_{\text{left}}}{\vdash (A \vee B) \Rightarrow (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C} \Rightarrow_{\text{right}} \times 3$ <p>Preuve dans LJ</p>	$\frac{\frac{\frac{\overline{B, A \Rightarrow C, C \vdash C}^{\text{ax}}}{B, A \Rightarrow C, B \Rightarrow C \vdash C} \Rightarrow_{\text{left}} \quad \frac{\overline{B, A \Rightarrow C \vdash C, B}^{\text{ax}}}{A, C, B \Rightarrow C \vdash C} \Rightarrow_{\text{left}} \quad \frac{\overline{A, C, B \Rightarrow C \vdash C}^{\text{ax}}}{A, A \Rightarrow C, B \Rightarrow C \vdash C} \Rightarrow_{\text{left}} \quad \frac{\overline{A, B \Rightarrow C \vdash C, A}^{\text{ax}}}{A \vee B, A \Rightarrow C, B \Rightarrow C \vdash C} \vee_{\text{left}}}{\vdash (A \vee B) \Rightarrow (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C} \Rightarrow_{\text{right}} \times 3$ <p>Preuve dans LK</p>

6.	$\frac{\overline{A, \perp \vdash \neg A} \perp_{\text{left}}}{\vdash A \Rightarrow \perp \Rightarrow \neg A} \Rightarrow_{\text{right}} \times 2$	Ici, la preuve en LJ est la même que la preuve en LK.
7.	$\frac{\overline{\perp \vdash A} \perp_{\text{left}}}{\vdash \perp \Rightarrow A} \Rightarrow_{\text{right}}$	Ici, la preuve en LJ est la même que la preuve en LK.
8.	$\frac{\overline{A \vdash A} \text{ax} \quad \overline{B, A \vdash B} \text{ax}}{A \Leftrightarrow B, A \vdash B} \Leftrightarrow_{\text{left}} \Rightarrow_{\text{right}} \times 2$ Preuve dans LJ	$\frac{\overline{A \vdash A, B} \text{ax} \quad \overline{A, B \vdash B} \text{ax}}{A \Leftrightarrow B, A \vdash B} \Leftrightarrow_{\text{left}} \Rightarrow_{\text{right}} \times 2$ Preuve dans LK
9.	$\frac{\overline{B \vdash B} \text{ax} \quad \overline{A, B \vdash A} \text{ax}}{A \Leftrightarrow B, B \vdash A} \Leftrightarrow_{\text{left}} \Rightarrow_{\text{right}} \times 2$ Preuve dans LJ	$\frac{\overline{B \vdash A, B} \text{ax} \quad \overline{A, B \vdash A} \text{ax}}{A \Leftrightarrow B, B \vdash A} \Leftrightarrow_{\text{left}} \Rightarrow_{\text{right}} \times 2$ Preuve dans LK
10.	$\frac{\overline{A \Rightarrow B, A, B \vdash A} \text{ax} \quad \overline{A \Rightarrow B, B \vdash B} \text{ax}}{A \Rightarrow B, B \Rightarrow A, B \vdash A} \Rightarrow_{\text{left}} \quad \frac{\overline{B \Rightarrow A, A \vdash A} \text{ax} \quad \overline{B \Rightarrow A, B, A \vdash B} \text{ax}}{A \Rightarrow B, B \Rightarrow A, A \vdash B} \Rightarrow_{\text{left}} \quad \frac{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)} \Leftrightarrow_{\text{right}} \Rightarrow_{\text{right}} \times 2$ Preuve dans LJ	$\frac{\overline{A \Rightarrow B, A, B \vdash A} \text{ax} \quad \overline{A \Rightarrow B, B \vdash A, B} \text{ax} \quad \overline{B \Rightarrow A, A \vdash A} \text{ax} \quad \overline{B \Rightarrow A, B, A \vdash B} \text{ax}}{A \Rightarrow B, B \Rightarrow A, B \vdash A} \Rightarrow_{\text{left}} \quad \frac{A \Rightarrow B, B \Rightarrow A, A \vdash B}{A \Rightarrow B, B \Rightarrow A, A \vdash B} \Leftrightarrow_{\text{right}} \Rightarrow_{\text{right}} \times 2$ Preuve dans LK

Exercice 2 (Logique du premier ordre)

Démontrer les propositions suivantes dans LJ et LK (si la proposition n'admet pas de preuve intuitionniste, démontrer la proposition dans LJ_(em)) :

1. $\forall x.P(x) \Rightarrow \exists y.P(y) \vee Q(y)$
2. $(\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))$
3. $(\forall x.P(x)) \wedge (\forall x.Q(x)) \Rightarrow \forall x.P(x) \wedge Q(x)$
4. $(\forall x.P(x) \wedge Q(x)) \Rightarrow (\forall x.P(x)) \wedge (\forall x.Q(x))$
5. $(\forall x.\neg P(x)) \Rightarrow \neg(\exists x.P(x))$
6. $\neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)$

1.	$\frac{\overline{P(x) \vdash P(x)} \text{ax}}{P(x) \vdash P(x) \vee Q(x)} \vee_{\text{right}} \Rightarrow_{\text{right}} \frac{P(x) \vdash \exists y.P(y) \vee Q(y)}{\vdash P(x) \Rightarrow \exists y.P(y) \vee Q(y)} \Rightarrow_{\text{right}} \forall_{\text{right}}$ Preuve dans LJ	$\frac{\overline{P(x) \vdash P(x), Q(x)} \text{ax}}{P(x) \vdash P(x) \vee Q(x)} \vee_{\text{right}} \Rightarrow_{\text{right}} \frac{P(x) \vdash \exists y.P(y) \vee Q(y)}{\vdash P(x) \Rightarrow \exists y.P(y) \vee Q(y)} \Rightarrow_{\text{right}} \forall_{\text{right}}$ Preuve dans LK
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<p>2.</p> $\frac{\frac{\frac{Q(x) \vdash Q(x)}{\vdash} \text{ax}}{Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))} \exists_{\text{right}} \quad \frac{\frac{\frac{P(x) \vdash P(x)}{\vdash} \text{ax}}{P(x) \vdash (\exists x.P(x))} \exists_{\text{right}}}{P(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))} \vee_{\text{right}}}{\frac{P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))} \exists_{\text{left}}}{\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))} \Rightarrow_{\text{right}}$ <p>Preuve dans LJ</p>	$\frac{\frac{\frac{Q(x) \vdash P(x), Q(x)}{\vdash} \text{ax} \quad \frac{P(x) \vdash P(x), Q(x)}{\vdash} \text{ax}}{P(x) \vee Q(x) \vdash P(x), Q(x)} \vee_{\text{left}} \times 2}{\frac{P(x) \vee Q(x) \vdash \exists x.P(x), \exists x.Q(x)}{P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))} \exists_{\text{right}}}{\frac{P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))} \exists_{\text{left}}}{\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))} \Rightarrow_{\text{right}}$ <p>Preuve dans LK</p>
<p>3.</p> $\frac{\frac{\frac{P(x), Q(x) \vdash Q(x)}{\vdash} \text{ax} \quad \frac{P(x), Q(x) \vdash P(x)}{\vdash} \text{ax}}{P(x), Q(x) \vdash P(x) \wedge Q(x)} \wedge_{\text{right}}}{\frac{P(x), Q(x) \vdash P(x) \wedge Q(x)}{\forall x.P(x), \forall x.Q(x) \vdash P(x) \wedge Q(x)} \forall_{\text{left}} \times 2}{\frac{\forall x.P(x), \forall x.Q(x) \vdash P(x) \wedge Q(x)}{\forall x.P(x), \forall x.Q(x) \vdash \forall x.P(x) \wedge Q(x)} \forall_{\text{right}}}{\frac{(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash \forall x.P(x) \wedge Q(x)}{\vdash (\forall x.P(x)) \wedge (\forall x.Q(x)) \Rightarrow \forall x.P(x) \wedge Q(x)} \Rightarrow_{\text{right}}$	<p>Ici, la preuve en LJ est la même que la preuve en LK.</p>
<p>4.</p> $\frac{\frac{\frac{P(x), Q(x) \vdash Q(x)}{\vdash} \text{ax}}{P(x) \wedge Q(x) \vdash Q(x)} \wedge_{\text{left}} \quad \frac{\frac{\frac{P(x), Q(x) \vdash P(x)}{\vdash} \text{ax}}{P(x) \wedge Q(x) \vdash P(x)} \wedge_{\text{left}}}{\frac{\forall x.P(x) \wedge Q(x) \vdash Q(x)}{\forall x.P(x) \wedge Q(x) \vdash \forall x.Q(x)} \forall_{\text{right}}}{\frac{\forall x.P(x) \wedge Q(x) \vdash \forall x.Q(x)}{\vdash (\forall x.P(x) \wedge Q(x)) \Rightarrow (\forall x.P(x)) \wedge (\forall x.Q(x))} \Rightarrow_{\text{right}}$	<p>Ici, la preuve en LJ est la même que la preuve en LK.</p>
<p>5.</p> $\frac{\frac{\frac{P(x) \vdash P(x)}{\vdash} \text{ax}}{\neg P(x), P(x) \vdash} \neg_{\text{left}}}{\frac{\neg P(x), P(x) \vdash}{\forall x. \neg P(x), P(x) \vdash} \forall_{\text{left}}}{\frac{\forall x. \neg P(x), P(x) \vdash}{\forall x. \neg P(x), \exists x.P(x) \vdash} \exists_{\text{left}}}{\frac{\forall x. \neg P(x), \exists x.P(x) \vdash}{\forall x. \neg P(x) \vdash \neg(\exists x.P(x))} \neg_{\text{right}}}{\vdash (\forall x. \neg P(x)) \Rightarrow \neg(\exists x.P(x))} \Rightarrow_{\text{right}}$	<p>Ici, la preuve en LJ est la même que la preuve en LK.</p>
<p>6.</p> $\frac{\frac{\frac{\neg P(x) \vdash \neg P(x)}{\vdash} \text{ax}}{\neg P(x) \vdash \exists x. \neg P(x)} \exists_{\text{right}}}{\frac{\neg P(x) \vdash \exists x. \neg P(x)}{\neg \exists x. \neg P(x), \neg P(x) \vdash} \neg_{\text{left}}}{\frac{\neg \exists x. \neg P(x), \neg P(x) \vdash}{\neg \exists x. \neg P(x) \vdash \neg \neg P(x)} \neg_{\text{right}}}{\frac{\neg \exists x. \neg P(x) \vdash \neg \neg P(x)}{\neg \exists x. \neg P(x) \vdash P(x)} \text{em}}{\frac{\neg \exists x. \neg P(x) \vdash P(x)}{\neg \exists x. \neg P(x) \vdash \forall x.P(x)} \forall_{\text{right}}}{\frac{\neg \exists x. \neg P(x) \vdash \forall x.P(x)}{\neg(\forall x.P(x)), \neg \exists x. \neg P(x) \vdash} \neg_{\text{left}}}{\frac{\neg(\forall x.P(x)), \neg \exists x. \neg P(x) \vdash}{\neg(\forall x.P(x)) \vdash \neg \neg \exists x. \neg P(x)} \neg_{\text{right}}}{\frac{\neg(\forall x.P(x)) \vdash \neg \neg \exists x. \neg P(x)}{\neg(\forall x.P(x)) \vdash \exists x. \neg P(x)} \text{em}}{\vdash \neg(\forall x.P(x)) \Rightarrow \exists x. \neg P(x)} \Rightarrow_{\text{right}}$ <p>Preuve dans LJ_(em) (pas de preuve dans LJ possible)</p>	$\frac{\frac{\frac{P(x) \vdash P(x)}{\vdash} \text{ax}}{\vdash \neg P(x), P(x)} \neg_{\text{right}}}{\frac{\vdash \neg P(x), P(x)}{\vdash \exists x. \neg P(x), P(x)} \exists_{\text{right}}}{\frac{\vdash \exists x. \neg P(x), P(x)}{\vdash \exists x. \neg P(x), \forall x.P(x)} \forall_{\text{right}}}{\frac{\vdash \exists x. \neg P(x), \forall x.P(x)}{\neg(\forall x.P(x)) \vdash \exists x. \neg P(x)} \neg_{\text{left}}}{\vdash \neg(\forall x.P(x)) \Rightarrow \exists x. \neg P(x)} \Rightarrow_{\text{right}}$ <p>Preuve dans LK</p>

Exercice 3 (Preuves en Coq)

Démontrer les propositions des exercices 1 et 2 en Coq.

On rappelle que pour lancer Coq, il suffit de se mettre dans un terminal et de taper la commande `coqide`, qui lance l'IDE de Coq.

► Voir TP1