TD N°4

Exercice 1 :

Soit l’interprétation I telle que :
— DI = {0, 1, 2} ;
— I(P) = {(0, F), (1, T), (2, T)} ;
Version ensembliste : I(P) = {1, 2}
— I(Q)(x, y) = T si x < y, F sinon ;
Version ensembliste :
— I(Q) = {(0,1), (1,2), (0,2)}
— I(f) = {(0, 1), (1, 2), (2, 0)}.

1. ∀x.P(f(x))

⟦ ∀x.P(f(x))⟧ρ =
⋀ (v ∈ DI) ⟦ P(f(x))⟧ρ[v/x] =
⋀ (v ∈ DI) I(P) (⟦ (f(x))⟧ρ[v/x]) =
⋀ (v ∈ DI) I(P) (I(f)⟦x⟧ρ[v/x]) =
⋀ (v ∈ DI) I(P) (I(f)(v)) = F1

F1 = I(P)(I(f)(0)) ∧_B I(P)(I(f)(1)) ∧_B I(P)(I(f(2))
= I(P)(1) ∧_B I(P)(2) ∧_B I(P)(0)
= T ∧_B T ∧_B F = F

2. ∀x.Q(x,f(x))

⟦ ∀x.Q(x,f(x)⟧ρ
= ⋀ (v ∈ DI) ⟦Q(x,f(x))⟧ρ[v/x]
= ⋀ (v ∈ DI) I(Q) (⟦x⟧ρ[v/x],⟦f(x)⟧ρ[v/x])
= ⋀ (v ∈ DI) I(Q) (⟦x⟧ ρ[v/x] I(f)⟦x⟧ρ[v/x])
= ⋀ (v ∈ DI) I(Q) (v, I(f)(v)) = F2

F2 = I(Q)(0,I(f)(0)) ∧_B I(Q)(1,I(f)(1)) ∧_B I(Q)(2,I(f)(2))
= I(Q)(0,1) ∧_B I(Q)(1,2) ∧_B I(Q)(2,0)
= T ∧_B T ∧_B F
= F

3. ∀x.∃y.Q(f(x),y)

⟦ ∀x.∃y.Q(f(x), y)⟧ρ
= ⋀ (v ∈ DI) ⟦∃y.Q(f(x), y)⟧ρ[v/x]
= ⋀ (v ∈ DI) (V (v'∈ DI) ⟦Q(f(x), y)⟧ρ[v/x][v'/y]
= ⋀ (v ∈ DI) (V (v'∈ DI) I(Q)(⟦(f(x)⟧ρ[v/x][v'/y], ⟦y⟧ρ[v/x][v'/y]))
= ⋀ (v ∈ DI) (V (v'∈ DI) I(Q)(I(f)⟦x⟧ρ[v/x][v'/y], v'))
= ⋀ (v ∈ DI) (V (v'∈ DI) I(Q)(I(f)(v), v')) = F3

F3 = (I(Q)(I(f(0)),0) v_B I(Q)(I(f(0)),1) v_B I(Q)(I(f(0)), 2))
∧_B (I(Q)(I(f(1),0) v_B I(Q)(I(f(1),1) v_B I(Q)(I(f(1), 2))
∧_B (I(Q)(I(f(2),0) v_B I(Q)(I(f(2),1) v_B I(Q)(I(f(2), 2))

= (I(Q)(1,0) v_B I(Q)(1,1) v_B I(Q)(1, 2))
∧_B (I(Q)(2,0) v_B I(Q)(2,1) v_B I(Q)(I(2, 2))

∧_B (I(Q)(I(0,0) v_B I(Q)(0,1) v_B I(Q)(0, 2))

= (F v_B F v_B T) **∧_B** (F v_B F v_B F) **∧_B** (F v_B T v_B T)
= T **∧_B** F **∧_B** T
= F

4. ∀x, y.Q(x, y) ⇒ Q(f(x), f(y))

⟦ ∀x, y.Q(x, y) ⇒ Q(f(x), f(y)) ⟧ρ = ⋀ (v, v' ∈ DI) (I(Q)(v, v') ⇒_B I(Q)(I(f)(v), I(f)(v'))) = F4

On ne prend pas en compte que les cas où I(Q)(v,v') = T (v < v').

F4 = (I(Q)(0, 1) ⇒_B I(Q)(I(f)(0), I(f)(1))) **∧_B**
(I(Q)(1, 2) ⇒_B I(Q)(I(f)(1), I(f)(2))) **∧_B**
(I(Q)(0, 2) ⇒_B I(Q)(I(f)(0), I(f)(2)))
= (I(Q)(0, 1) ⇒_B I(Q)(1, 2)) **∧_B**
(I(Q)(1, 2) ⇒_B I(Q)(2, 0)) **∧_B**
(I(Q)(0, 2) ⇒_B I(Q)(1, 0))
= T ⇒_B T **∧_B** T ⇒_B F **∧_B** T ⇒_B F = F

En développant tout :

F4 = (I(Q)(0, 0) ⇒_B I(Q)(I(f)(0), I(f)(0))) **∧_B**
(I(Q)(0, 1) ⇒_B I(Q)(I(f)(0), I(f)(1))) **∧_B**
(I(Q)(0, 2) ⇒_B I(Q)(I(f)(0), I(f)(2))) **∧_B**
(I(Q)(1, 0) ⇒_B I(Q)(I(f)(1), I(f)(0))) **∧_B**
(I(Q)(1, 1) ⇒_B I(Q)(I(f)(1), I(f)(1))) **∧_B**
(I(Q)(1, 2) ⇒_B I(Q)(I(f)(1), I(f)(2))) **∧_B**
(I(Q)(2, 0) ⇒_B I(Q)(I(f)(2), I(f)(0))) **∧_B**
(I(Q)(2, 1) ⇒_B I(Q)(I(f)(2), I(f)(1))) **∧_B**
(I(Q)(2, 2) ⇒_B I(Q)(I(f)(2), I(f)(2)))
= T **∧_B** (T ⇒_B I(Q)(1, 2)) **∧_B** (T ⇒_B I(Q)(1, 0))) **∧_B T ∧_B T ∧_B** (T ⇒_B I(Q)(2, 0))
∧_B T ∧_B T ∧_B T == T **∧_B** T **∧_B F ∧_BT ∧_B T ∧_B F ∧_B T ∧_B T ∧_B T = F**

5. ∀x.P(x) ⇒ ∃y.Q(f(y), x)

⟦ ∀x.P(x) ⇒ ∃y.Q(f(y), x)⟧ρ
= ⋀ (v ∈ DI) **(**I(P)(v) ⇒_B V(v' ∈ DI) I(Q)(I(f)(v'), v)**)**
= F5

On ne considère que les cas où I(P)(v) = T

F5= (I(P)(1) ⇒_B (I(Q) (f(0), 1) v_B I(Q) (f(1), 1) v_B I(Q) (f(2), 1))

- **∧_B**
- (I(P)(2) ⇒_B (I(Q) (f(0), 2) v_B I(Q) (f(1), 2) v_B I(Q) (f(2), 2))
- = (T ⇒_B (I(Q) (1, 1) v_B I(Q) (2, 1) v_B I(Q) (0, 1)))

∧_B
(T ⇒_B (I(Q) (1, 2) v_B I(Q) (2, 2) v_B I(Q) (0, 2)))
= (T ⇒_B (F v_B F v_B T))
∧_B

$(T \Rightarrow_B (T \lor_B F \lor_B T))$
$= (T \Rightarrow_B T)$ **$\land_B$** $(T \Rightarrow_B T)$
$= T$

Exercice 2 :

— $F1 = \forall x.\exists y.P(x, y)$
— $F2 = \forall x.P(x, f(x))$

1. $F1 \Rightarrow F2$.
$(\forall x.\exists y.P(x, y)) \Rightarrow (\forall x.P(x, f(x)))$

contre modèle:
$DI = \{0,1\}$
$I(P)(x,y) = T$ si $x=y$, F sinon
$I(f) = \{(0,1), (1,0)\}$

$[\![(\forall x.\exists y.P(x, y)) \Rightarrow (\forall x.P(x, f(x)))]\!]\rho$

$= (\bigwedge (v \in DI) (\bigvee (v' \in DI) I(P)(v,v')) \Rightarrow_B (\bigwedge (u \in DI) I(P)(u, I(f)(u))) = A$

$A = (I(P)(0,0) \lor_B I(P) (0,1)$ **$\bigwedge_B$** $I(P) (1,0) \lor_B I(P) (1,1))$
$\Rightarrow_B$
$(I(P) (0,1)$ **$\bigwedge_B$** $I(P) (1,0))$

$= (T \lor_B F)$ **$\bigwedge_B$** $(F \lor_B T) \Rightarrow_B F$ **$\bigwedge_B$** F

$= T \Rightarrow_B F = F$

2. $F2 \Rightarrow F1$.
Valide.
$[\![(\forall x.P(x, f(x))) \Rightarrow (\forall x.\exists y.P(x, y))]\!]\rho$
$=(\bigwedge (u \in DI) I(P)(u, I(f)(u))) \Rightarrow_B (\bigwedge (v \in DI) (\bigvee (v' \in DI) I(P)(v,v')) = B$

Deux cas :

- $\bigwedge (u \in DI) I(P)(u, I(f)(u)) = F$: $B = T$
- **$\bigwedge (u \in DI) I(P)(u, I(f)(u)) = T$** : $B = F1$

    - $B = \bigwedge (v \in DI) (\bigvee (v' \in DI) I(P)(v,v'))$

$= (I(P)(v0, v0) \lor_B I(P)(v0, v1) \lor_B I(P)(v0, v2) \lor_B ...) \bigwedge_B$
$(I(P)(v1, v0) \lor_B I(P)(v1, v1) \lor_B I(P)(v1, v2) \lor_B ...) \bigwedge_B ...$

(Dans $I(P)(v0,vi)$, il existe une valeur vf telle que $vf = I(f)(v0)$ et on sait (par hypothèse) que :
$I(P)(v0,I(f)(v0)) = T$.)

$= (... \lor_B I(P)(v0,I(f)(v0)) \lor_B ...) \bigwedge_B (... \lor_B I(P)(v1,I(f)(v1)) \lor_B ...) \bigwedge_B ...$
$= T \bigwedge_B T \bigwedge_B ... = T$

Preuve dans LK :

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x, f(x)) \vdash P(x, f(x))}\ ax}{P(x, f(x)) \vdash \exists y.P(x, y)}\ \exists\ right}{(\forall x.P(x, f(x))) \vdash \exists y.P(x, y)}\ \forall\ left}{(\forall x.P(x, f(x))) \vdash (\forall x.\exists y.P(x, y))}\ \forall\ right}{\vdash (\forall x.P(x, f(x))) \Rightarrow (\forall x.\exists y.P(x, y))}\ \Rightarrow\ right
$$