IEOR 240 : Homework 4

Arnaud Minondo

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Problem 1 :

Let $S = \{x \in \mathbb{R}^3 | x_1 + x_2 + x_3 \geq 1 \text{ and } x_1, x_2 \geq 0\}$
Let $S_l = \{(t, (x_1, x_2, x_3)) \in \mathbb{R} \times S \text{ s.t. } t \geq 2x_1, t \geq 3x_2, t \geq 4x_3 t \geq -4x_3\}$
By definition of S_l, $\forall (t, x) \in S_l, t \geq \max(2x_1, 3x_2, |4x_3|)$.

As a first result we have that $\min\limits_{(t,x)\in S_l}(t) \geq \min\limits_{(t,x)\in S_l}(\max(2x_1, 3x_2, |4x_3|)) = \min\limits_{x\in S}(\max(2x_1, 3x_2, |4x_3|))$ as the left side only depends on $x \in S$.

As a second result we can notice that $(\min\limits_{x\in S}(\max(2x_1, 3x_2, |4x_3|)), x^*) \in S_l$, so $\min\limits_{(t,x)\in S_l}(t) \leq \min\limits_{x\in S}(\max(2x_1, 3x_2, |4x_3|))$.

The results combined yield that the two min have to be equal. Moreover the problem where $(t, x) \in S_l$ is linear.

Problem 2 :

2.(a)

The reduced cost is 0 because of the complementary slackness theorem.

2.(b)

After the strong duality theorem : objective value are equal thus $0.5 * 12 + 2 * 10 + (b) * 8 = 2 * 7 + 4 * 3 = 26$ and conclude that $(b) = 0$.

2.(c)

The allowable decrease for constraint 3 is 2 as $8 - 6 = 2$ and we can decrease the constraint of two without changing the solution of the problem.

2.(d)

The optimal solution will not change, $x = (2, 0, 4, 0)$ is still optimal but the objective value changes and it is : 28.

2.(e)

If you decrease C1 to 10, the solution of the dual does not change, $\overline{y} = (0.5, 2, 0)$ is still optimal and the new objective value is : 25.

2.(f)

We need to compute the reduced cost of x_5, which is equal to $3 - 2 * 0.5 + 2 * 24 * 0 = -2$ which is negative so the objective function value can't be increased as it is a maximization problem. The solution is still optimal.

Problem 3 :

3.(a)

This is false : let $(\mathcal{P})$ $\max(x_1 - x_2)$ s.t. $x_1 + x_2 = 1$ and $x_1 + x_2 = -1$. Its dual constraints is the same and obviously both are infeasible.

3.(b)

This is false : Let the problem be $\min(x_1 + x_2)$ s.t. $x_1 + 2x_2 \geq 2$, $-x_1 - 2x_2 \geq -2$ and $x_1, x_2 \geq 0$ then after some computations you have the dual solution is $y_1 = \frac{1}{2}, y_2 = 0$.

3.(c)

This is false : consider the problem $\max(x_1 + x_2)$ s.t. $x_1 \leq 1$, $x_2 \leq 1$ and $x_1, x_2 \geq 0$, its dual is : $\min(-y_1 - y_2), y_1 \leq -1, y_2 \leq -1$ and the optimal sol is $y_1 = y_2 = -1$. Now consider the proble $\max(2x_1 + 2x_2)$ s.t. $x_1 \leq 1$, $x_2 \leq 1$ and $x_1, x_2 \geq 0$ which is the same as the original but only with the coefficient multiplied by two. The dual is : $\min(-y_1 - y_2)$ s.t. $y_1 \leq -2$, $y_2 \leq -2$ and $y_1, y_2 \leq 0$ with optimal solution $y_1 = y_2 = -2$. We can see that the solution of the dual changed.

3.(d)

As $x = 0_{\mathbb{R}^n}$ is feasible whatever are the value of a_1, a_2 the problem is feasible.

3.(e)

Suppose $\forall z \in \mathbb{R}^m, (\forall j \in [\![1; m]\!], \sum\limits_{i=1}^{n} a_{ij} z_i \leq 0, z_j \geq 0) \implies z = 0$.

Let w be a certificate of infeasibility. $w^T A \leq 0$ and we can notice that $\forall j \in [\![1; m]\!], (w^T A)_j = \sum\limits_{i=1}^{n} a_{ij} z_i \leq 0$. Moreover, $w \geq 0$ so w verifies the first two conditions so $w = 0$ which means that $w^T b = 0$ which is in contradiction with $w^T b < 0$ the last condition of the unfeasibility certificate.

So there does not exist any infeasibility certificate therefore after the theorem of alternatives, the problem is always feasible.