Individual homework. All the answers have to be mathematically justified (unless specified explicitly).

Exercise 1. The maximum height H of the annual flood of a river is carefully observed, because a flood greater than 6meters would be catastrophic. We model the distribution of the random variable H by a Rayleigh law whose probability density is given by

$$f(x) = \frac{x}{a} e^{-\frac{x^2}{2a}} \text{ if } x > 0 \text{ and } 0 \text{ otherwise,}$$

where a is an unknown parameter. Over an 8-year period, the following (assumed to be independent) flood heights in meters were observed: 2.1; 2.8; 1.7; 0.9; 1.8; 2.5; 2.2; 2.9.

1. Check that f defined a probability density function.
2. Give the MLE of the parameter a.
3. Give the cumulative distribution function of H.
4. Compute the probability of a disaster.
5. Give the probability to not have a flood during 100 years.

Exercise 2.

1. Let $S := \exp(X)$ where X is a normal distribution with parameters μ and σ^2. Prove that S has the following density function
$$f(s) = \frac{1}{\sqrt{2\pi} s\sigma} e^{-\frac{(\ln(s)-\mu)^2}{2\sigma^2}}, \text{ if } s > 0 \text{ and } 0 \text{ otherwise.}$$
2. Compute $\mathbb{E}[S]$.
3. We have a sample of the size of concrete particles in micrometer 3.1; 5.0; 8.9; 16.0; 25.6; 39.1; 109.2. Give an estimator of μ if we assume that the size of a particule concrete follows the distribution of S with a fix parameter $\sigma = 0.7$.

Exercise 3.
Consider N financial assets whose return are random variables R_i with the same expectation r. The risk (standard deviation) of each return is σ. Assume that you have one million dollar and you build a portfolio composed with these N asset. We denote by π_i the proportion of money invested in the asset i. **Note that necessarily $\sum_{i=1}^N \pi_i = 1$.**

Part A. Uncorrelated returns. We assume that the returns of each asset are not correlated.

1. Let X_T be the value of the portfolio at time T and $X(0)$ the initial value of the portfolio. Prove that the return rate of the portfolio $\frac{X(T)-X(0)}{X(0)}$ is
$$\sum_{i=1}^N \pi_i R_i$$
2. Prove that the risk (variance) associated to this portfolio is
$$\text{Var}(\sum_{i=1}^N \pi_i R_i) = \sigma \sum_{i=1}^N \pi_i^2.$$
3. Find the optimal allocation $(\pi_i^*)_{1\le i\le N}$ minimizing the risk of the portfolio under the constraint $\sum_{i=1}^N \pi_i = 1$.
4. Compute the value of the risk associated to the portfolio with the allocation π^* found above. What happened when N goes to $+\infty$?

Part B. Correlated returns. We now assume that the returns are correlated and the pairwise correlation coefficients are all equal to $\rho \in [-1, 1]$.

5. Prove that the risk associated to this portfolio is $\sigma \sum_{i=1}^{N} \pi_i^2 + \rho \sum_{i \neq j} \pi_i \pi_j$.
6. Find the optimal allocation $(\pi_i^*)_{1 \leq i \leq N}$ minimizing the risk associated to the portfolio.
7. Compute the value of the risk associated to the portfolio with the allocation π^* found above. What happened when N goes to $+\infty$?

Part C. Minimizing the risk with 2 assets

8. Consider 2 asset with return R_1 and R_2 such that $\mathrm{Var}(R_1) = 1$, $\mathrm{Var}(R_2) = 2$ and $\mathrm{Cov}(R_1, R_2) = 1$. What is the portfolio that minimizes the risk?