INDENG 262A : Homework 2

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September 29, 2022

Problem 1.4

$\min(2x_1 + 3|x_2 - 10|)$ s.t. $|x_1 + 2| + |x_2| \leq 5$

Let $u_0 \in \mathbb{R}$ such that $u_0 \geq u_0 \geq x_2 - 10$ and $u_0 \geq 10 - x_2$
Then $u_0 \geq |x_2 - 10|$ so $2x_1 + 3u_0 \geq 2x_1 + 3|x_2 - 10|$ and $\min_{x_1,x_2,u_0}(2x_1 + 3u_0) \geq \min_{x_1,x_2,u_0}(2x_1 + 3|x_2 - 10|)$
But let $u_0 = |x_2 - 10|$ and $2x_1 + 3u_0 = 2x_1 + 3|x_2 - 10|$ so $\min_{x_1,x_2,u_0}(2x_1 + 3u_0) = \min_{x_1,x_2,u_0}(2x_1 + 3|x_2 - 10|)$.

To linearize $|x_1 + 2| + |x_2| \leq 5$:
Introduce $u_1, u_2 \in \mathbb{R}$ such that, $u_1 \geq x_1 + 2$ and $u_1 \geq -2 - x_1$ and $u_2 \geq x_2$ and $u_2 \geq -x_2$ and $u_1 + u_2 \leq 5$

The first problem which is non linear is equivalent to this linear problem :

$$\boxed{\min(2x_1 + 3u_0) \text{ s.t. } \left| \begin{array}{c} u_0 \geq x_2 - 10 \\ u_0 \geq 10 - x_2 \\ u_1 \geq x_1 + 2 \\ u_1 \geq -2 - x_1 \\ u_2 \geq x_2 \\ u_2 \geq -x_2 \\ u_1 + u_2 \leq 5 \end{array} \right.}$$

Problem 1.9

In this problem I introduced a tensor $T = (s_{igj})_{(i,g,j) \in [\![1;I]\!] \times [\![1;G]\!] \times [\![1;J]\!]}$ where s_{igj} represents the number of students from neighborhood i of grade g going to school j.
So that finding T gives an attribution of each student to a school.

Moreover : we can notice that $\sum_{j=1}^{J} s_{igj} = S_{ig}$ as a student has a school, no students has no school to say so. And $\sum_{i=1}^{I} s_{igj} \leq C_{jg}$ as the number of students in a school and of a certain grade can not exceed the school capacity.
The problem becomes :

$$\boxed{\min(\sum_{i,j} d_{ij} \sum_{g=1}^{G} s_{igj}) \text{ s.t. } \forall (i,g,j) \in [\![1;I]\!] \times [\![1;G]\!] \times [\![1;J]\!], \sum_{j=1}^{J} s_{igj} = S_{ig}, \sum_{i=1}^{I} s_{igj} \leq C_{jg}, s_{igj} \geq 0}$$

Problem 1.11

Define $\forall r \in [\![0; N-3]\!], S_r = \left\{(i_0, i_1, ..., i_r) \in [\![2; N-1]\!]^r \text{ s.t. } \forall p, q \in [\![0;r]\!]^2, p \neq q \implies i_p \neq i_q\right\}$
And define : $\forall k \in [\![2; N-1]\!], S_{rk} = \{s \in S_r, k \in s\}$
Now notice that a transaction from devise 1 to devise N is uniquely identified by the intermediate devise and the order in which they are exchanged.
Let $r \in [\![0; N-3]\!], p = (p_0, p_1, ..., p_r) \in S_r$, r is the number of intermediate devise used to change 1 in N, p is identifying which devise is being exchange from which divise so that $\forall i \in [\![0; r-1]\!], p_i$ is exchanged for p_{i+1} and if $r = 0$ it means 1 is directly exchanged to N.

Define $\forall r \in [\![0; N-3]\!], \forall p \in S_r x_p$ which is the amount changed using the path p from 1 to N.
The problem is :

$$\boxed{\max\left(\sum_{r=0}^{N-3} \sum_{(i_0,i_1,\dots,i_r)\in S_r} x_{i_0,i_1,\dots,i_r} r_{1i_0} r_{i_r N} \prod_{k=0}^{r} r_{i_k}\right) \text{ s.t. } \sum_{k=2}^{N-1} \sum_{p\in S_{rk}} x_p \leq u_r}$$

Problem 4.1

The corresponding dual problem is :

$$\boxed{\begin{array}{ll} \max(3y_2+6y_3) \text{ s.t.} & 2y_1+3y_2-y_3 \geq 1 \\ & 3y_1+y_2-y_3 \leq -1 \\ & -y_1+4y_2+2y_3 \leq 0 \\ & y_1-2y_2+y_3 = 0 \\ & y_1 \leq 0, y_2 \geq 0 \end{array}}$$

Problem 4.4

x^* is also a boundedness certificate : ${x^*}^T A = (A^T x^*)^T = (Ax^*)^T = c^T$ so ${x^*}^T A \leq c^T$ and $x^* \geq 0$ so x^* is a certificate of boundedness.
After the weak duality : $\min(c^T x) \geq {x^*}^T c = c^T x^*$ as x^* is also feasable then $\min(c^T x) \leq c^T x^*$ and finally combining the two inequalities we have :

$$\boxed{\min(c^T x) = c^T x^*,\ x^* \text{ is optimal}}$$

Problem 4.8

(a)

$\tilde{x}$ (resp. x^*) is optimal for $\bar{c}$ (resp. c) so $\forall x \in \mathbb{R}^n, \bar{c}^T\tilde{x} \leq \bar{c}^T x$ (resp. $c^T x^* \leq c^T x$)
So : $(\bar{c}-c)^T(\tilde{x}-x^*) = \bar{c}^T\tilde{x} - \bar{c}^T x^* - c^T\tilde{x} + c^T x^*$ using the two conditions above it is clear that $\bar{c}^T\tilde{x} \leq \bar{c}^T x^*$ and $c^T x^* \leq c^T\tilde{x}$
Finally summing the two inequalities and reorganizing the terms we have :

$$\boxed{(\bar{c}-c)^T(\tilde{x}-x^*) \leq 0}$$

(b)

Firstly : ${p^*}^T b = c^T x^*$.
As between the two problems only b has changed and it does not change the condition of feasability of the dual then p^* is a certificate of boundedness for the changed dual and we can conclude that ${p^*}^T\tilde{b} \leq c^T\tilde{x}$
So ${p^*}^T\tilde{b} - {p^*}^T b \leq c^T\tilde{x} - {p^*}^T b = c^T\tilde{x} - c^T x^*$ and we have obtained that : ${p^*}^T\tilde{b} - {p^*}^T b \leq c^T\tilde{x} - c^T x^*$ which is the same thing as :

$$\boxed{{p^*}^T(\tilde{b}-b) \leq c^T(\tilde{x}-x^*)}$$

Problem 4.26

Suppose both are true at the same time : then $\forall j \in [\![1;n]\!], (pA)_j = u_j > 0$ but $pAx = 0$ thus $\sum_{i=1}^{n} u_j x_j = 0$ with $x_j \geq 0$ thus $u_j x_j \geq 0$ so $\forall j \in [\![1;n]\!], u_j x_j = 0$ so $x_j = 0$ as $u_j \neq 0$ which contradicts $x \neq 0$.
So both can't be true at the same time.

Need to show that if one is false the other is true.
Let $u \in \mathbb{R}^n$ such that $\forall i \in [\![1;n]\!], u_i = 1$ now notice that the condition $pA > 0$ is equivalent to $pA \geq u$.

Write the optimization problem $(\mathcal{P})$: $\min(p^T 0_{\mathbb{R}^m})$ s.t. $pA \geq u^T$
Its dual is $(\mathcal{D}) : \max(u^T x)$ s.t. $Ax = 0, x \geq 0$.

Suppose not (b) which means : $\forall p \in \mathbb{R}^m, pA$ is not greater or equal to 0. So $(\mathcal{P})$ is unfeasable. Thus $(\mathcal{D})$ is either unfeasable either unbounded. But $x = 0$ is a certificate of feasability for $(\mathcal{D})$ so $(\mathcal{D})$ has to be unbounded.
Which means that $\exists w \in \mathbb{R}^n, w \geq 0, Aw = 0$ and $u^T w = \sum_{i=1}^{n} w_i > 0$ thus $w \neq 0$. Thus (a) is true.

So we have shown : [(b) $\Longrightarrow$ not (a)] and [not (b) $\Longrightarrow$ (a)] thus [(b) $\Leftrightarrow$ not (a)] or more simply either a is true or b is true.