IEOR 263A : Homework 8

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November 3, 2022

Problem 6.1 :

$\forall (p,q) \in \mathbb{N}^2$ the states are : (p,q). $P_{(p,q)(p+1,q)} = \frac{1}{2}$ and $P_{(p,q)(p,q+1)} = \frac{1}{2}$ with a rate $v_{(p,q)} = (p+q)\lambda$

Problem 6.6 :

a. Let $\forall i \in \mathbb{N}$, T_i be the time to go from state i to state $i+1$, $\mathbb{E}(T_{04}) = \sum_{j=0}^{3} \mathbb{E}(T_j)$ with $\forall j \in \mathbb{N}^*$, $\mathbb{E}(T_j) = \frac{1}{\lambda_j} + \frac{\mu_j}{\lambda_j}\mathbb{E}(T_{j-1}) = \frac{1}{(j+1)\lambda} + \frac{j\mu}{(j+1)\lambda}\mathbb{E}(T_{j-1})$ and $\mathbb{E}(T_0) = \frac{1}{\lambda}$. Finally :

$$\boxed{\mathbb{E}(T_{04}) = \sum_{j=0}^{3} \mathbb{E}(T_j)}$$

b. Same as before :

$$\boxed{\mathbb{E}(T_{25}) = \sum_{j=2}^{4} \mathbb{E}(T_j)}$$

c. We use the formula : $\forall i \in \mathbb{N}^* \mathbb{V}(T_i) = \dfrac{1}{\lambda_i(\lambda_i + \mu_i)} + \dfrac{\mu_i}{\lambda_i}\mathbb{V}(T_{i-1}) + \dfrac{\mu_i}{\mu_i + \lambda_i}(\mathbb{E}(T_i) + \mathbb{E}(T_{i-1}))^2$ and $\mathbb{V}(T_0) = \frac{1}{\lambda^2}$. With the independence between T_i and T_j $\forall (i,j) \in \mathbb{N}^2, i \neq j$ we have :

$$\boxed{\mathbb{V}(T_{04}) = \sum_{j=0}^{3} \mathbb{V}(T_j) \text{ and } \mathbb{V}(T_{25}) = \sum_{j=2}^{5}\sum_{j=2}^{4} \mathbb{V}(T_j)}$$

d. This is quite different as you have infinite possible ways, we notice that $\mathbb{E}(T_{40}) = \mathbb{E}(T_{43}) + \mathbb{E}(T_{32}) + \mathbb{E}(T_{21}) + \mathbb{E}(T_{10}) = 4\mathbb{E}(T_{10})$ and as $\mathbb{E}(T_{10}) = \dfrac{1}{\mu - \lambda}$ then

$$\boxed{\mathbb{E}(T_{40}) = \frac{4}{\mu - \lambda}}$$

Problem 6.15 :

a. An equivalent Markov chain to the problem would be defined by the transition matrix $P = \begin{pmatrix} \frac{4}{7} & \frac{3}{7} & 0 & 0 \\ \frac{2}{7} & \frac{2}{7} & \frac{3}{7} & 0 \\ 0 & \frac{4}{7} & 0 & \frac{3}{7} \\ 0 & 0 & \frac{4}{7} & \frac{3}{7} \end{pmatrix}$

Let $\pi = (\pi_0, \pi_1, \pi_2, \pi_3)$ be the stationary distribution. On the long run the proportion of clients lost is equal

to $\dfrac{\pi_3\lambda}{\pi_0\lambda+\pi_1\lambda+\pi_2\lambda+\pi_3\lambda} = \pi_3$.
Moreover : $\pi P = \pi$ thus $\pi_3 = \frac{27}{143}$ thus the proportion of clients that enter the system is :

$$\boxed{1-\pi_3 = \frac{116}{143}}$$

b.

$$\boxed{1-\pi_3 = \frac{148}{175}}$$

Problem 6.20 :

a.

$$\boxed{\mathbb{E}(T_{02}) = \frac{2}{\lambda} + \frac{\mu}{\lambda^2}}$$

b.

$$\boxed{\mathbb{V}(T_{02}) = \frac{1}{\lambda^2} + \frac{1}{\lambda(\lambda+\mu)} + \frac{\mu}{\lambda^3} + \frac{\mu}{\mu+\lambda}(\frac{2}{\lambda} + \frac{\mu}{\lambda^2})^2}$$

c. With the equality of in-rates, out-rates for each state : $\pi_0\lambda = \pi_1\mu$, $\pi_1\lambda = \pi_2\mu$. Thus $\pi_0 = \frac{1}{1+\frac{\lambda}{\mu}+(\frac{\lambda}{\mu})^2}$ and the proportion of time there will be a working machine is

$$\boxed{\pi_0 + \pi_1 = \frac{1+\frac{\lambda}{\mu}}{1+\frac{\lambda}{\mu}+(\frac{\lambda}{\mu})^2}}$$

d.

$$\boxed{G = \begin{pmatrix} -\lambda & \lambda & 0 \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & \mu & -\mu \end{pmatrix}}$$

e.

$$\boxed{\nu = \lambda+\mu \text{ and } P^* = \begin{pmatrix} \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} & 0 \\ \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} \\ 0 & \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \end{pmatrix}}$$

Problem 7.2 :

a. S_n is a sum of n poisson thus

$$\boxed{\forall k \in \mathbb{N}, \mathbb{P}(S_n = k) = e^{-n\mu}\frac{(n\mu)^k}{k!}}$$

b.

$$\boxed{\mathbb{P}(N(t)=n) = \mathbb{P}(S_n \le t) - \mathbb{P}(S_{n+1} \le t) = \sum_{k=0}^{[t]} e^{-n\mu}\frac{(n\mu)^k}{k!} - \sum_{k=0}^{[t]} e^{-(n+1)\mu}\frac{((n+1)\mu)^k}{k!}}$$

Problem 7.4 :

a. No they can not independent between each other : consider the case where $N_1 \sim PP(1)$ and $N_2 \sim [t]$. Let $t \in [0;1[$ then $\mathbb{P}(T_2 > 1-t|T_1 = t) = 0$ but $\mathbb{P}(T_2 > 1-t) \neq 0$ so

$$\boxed{T_1 \text{ and } T_2 \text{ are not independent}}$$

b. No they can be not identically distributed take the same example. If T_1 arrives at $t_1 < 1$ then T_2 cannot have the same law because an event will ocure at $1 - t_1$.

c. No this is not a renewal process because you need independence and the identically distributed hypothesis.

Problem 7.9 :

Let T be the time that a project takes before completion. $E(T) = \int_0^\infty 1 - F(t)dt$

$$\boxed{\text{Projects are completed with rate : } \frac{1}{\lambda\mathbb{E}(T)+1}}$$

Problem 7.10 :

a. Let $N(t)$ be the renewal process with mean μ. Let F be the distribution of its interarival times. Then let $\forall i \in \mathbb{N}, I_i \sim \mathcal{B}(p)$ where I_i are iid. Let T_{Ci} be the interarival times for $N_C(t).T_{ci} = I_i T_i$ are independent between each other as T_i are independent and I_i also are. They are identically distributed as $\forall i \in \mathbb{N}, I_i \sim \mathcal{B}(p)$ and $T_i \sim F$. Finally $\{N_C(t)\}$ is a counting process thus

$$\boxed{\{N_C(t)\} \text{ is a renewal process.}}$$

b. We use the limit theorem for renewal process and

$$\boxed{\lim_{t\to\infty} \frac{N_C(t)}{t} = \frac{1}{p\mu}}$$

Problem 7.12 :

a. We use the limiting theorem for renewal process :

$$\boxed{\text{d-event occurs at rate : } (1 - e^{-\lambda d})\lambda}$$

b.

$$\boxed{\text{The proportion of event that are d-event is : } (1 - e^{-\lambda d})}$$

Additional Problem 1 :

The strategy repairing 2 before 1 correspond to the markov chain in the figure below where state 0 correspond to both machine are working, state 1 the machine 1 is not working, state 2 the machine 2 is not working and state 3 both machine are not working and we repair 2 before 1 :

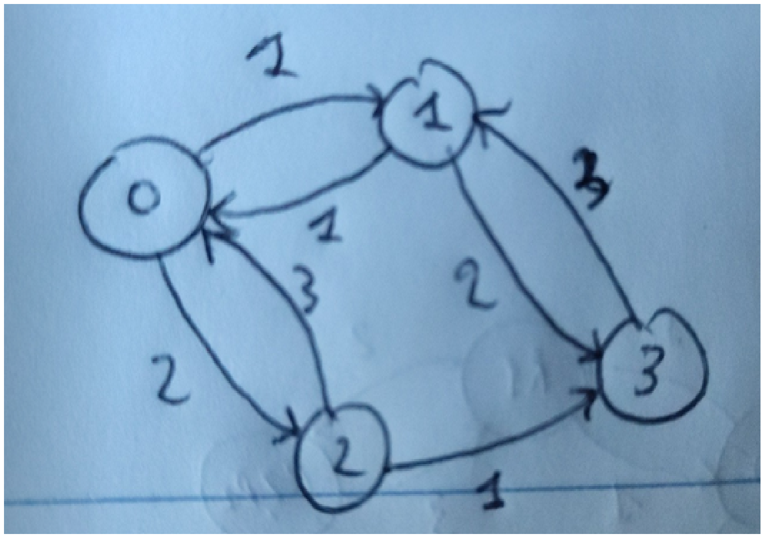

Figure 1: Strategy 1 : repairing machine 2 before machine 1

The stationary law is : $\pi = (\frac{6}{25}, \frac{9}{25}, \frac{3}{25}, \frac{6}{25})$ For another stratedy which is repairing machine 1 before machine 2 :$\pi = (\frac{36}{91}, \frac{12}{91}, \frac{23}{91}, \frac{20}{91})$ the markov chain is in the figure below :

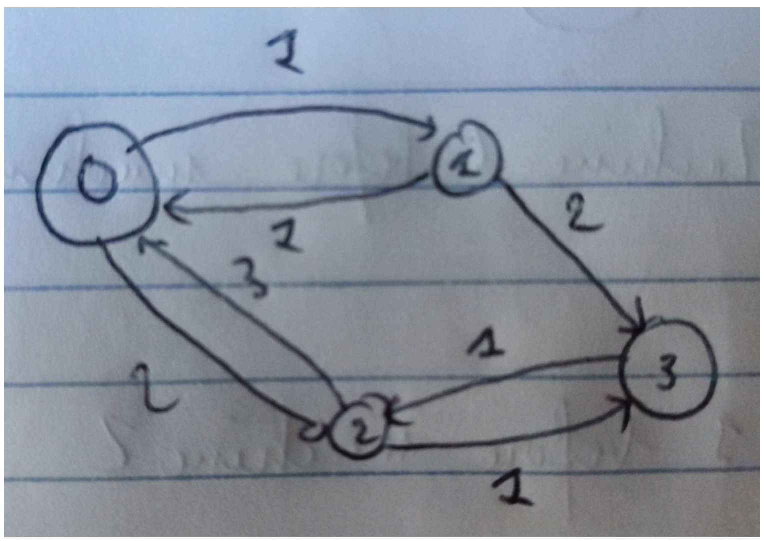

Figure 2: Strategy 2, repairing machine 1 before machine 2

a. To maximize the time both machines we need to choose the strategy corresponding to π_0 is maximized which is strategy 2 as $\frac{36}{91} \geq \frac{6}{25}$.

b. To minimize the time neither machine are working we choose the strategy that minimizes π_3 and it is strategy 1 that is to be prefered as $\frac{7}{25} \leq \frac{20}{91}$.

c. The long run production rate can be computed as $\sum_{i=0}^{3} \pi_i * \beta_i$ where $\beta_0 = \gamma_1 + \gamma_2 = 5$, $\beta_1 = \gamma_2$, $\beta_2 = \gamma_1$ and $\beta_3 = 0$.

$\boxed{\text{The strategy that maximizes the long-run production rate is Strategy 2, repairing machine 1 before machine 2.}}$

Additional Problem 2 :

As $m(t) = \frac{t}{2}$ then $1 = 2F'(t) + F(t)$ which is a differencial equation and solutions of this equations are $s \in \text{Vect}(e^{-\frac{t}{2}}) + 1$. With the conditions $F(0) = 0$ and $\lim_{t \to \infty} F(t) = 1$ we have $F(t) = 1 - e^{-\frac{t}{2}}$.

$$\boxed{\mathbb{P}(N(5) = 0) = 1 - F(5) = e^{-\frac{5}{2}}}$$