IEOR 241 : Homework 7

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Problem 1

If $\mathbb{E}(X) = \frac{3}{5} = \int_0^1 f(x)dx = \frac{a}{2} + \frac{b}{4}$. Moreover, $\int_0^1 f(x)dx = a + \frac{b}{3} = 1$
Solving the linear system we find :

$$\boxed{a = \frac{3}{5},\, b = \frac{6}{5}}$$

.

Problem 2

Let L be the lifetime of the tube : $\mathbb{E}(L) = \int_0^\infty xf(x)dx = \int_0^\infty x^2 f(x)dx$ integrating by part two times we find that

$$\boxed{\mathbb{E}(X) = 2}$$

Problem 3

Notice that : $\forall t \in \mathbb{R}+, t \in [0; 1] \iff e^t \in [1; e]$. Thus $\mathbb{P}(Y \leq t) = \mathbb{P}(e^X \leq t)$. If $t < 1$ then $\mathbb{P}(Y \leq t) = 0$ because $X \geq 0$. Moreover, $\forall t \in [1; \infty], \mathbb{P}(Y \leq t) = \mathbb{P}(X \leq \log(t)) = \begin{cases} \log(t) & \text{if } t \in [1; e[\\ 1 & \text{if } t \in [e; \infty[\end{cases}$ Thus,

$$\boxed{f_Y(t) = \begin{cases} 0 & \text{if } t \in]-\infty; 1] \cup]e, \infty[\\ \frac{1}{t} & \text{if } t \in [1, e] \end{cases}}$$

Problem 4

After problem 1 : $a = \frac{3}{5}, b = \frac{6}{5}$.
$\mathbb{P}(X < \frac{1}{2}) = \int_0^{\frac{1}{2}} \frac{3}{5} + \frac{6}{5}x^2 dx = \frac{7}{20}$
Moreover :

$$\boxed{\mathbb{V}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{11}{25} - \frac{9}{25} = \frac{2}{25}}$$

Problem 5

(a)

You can find the joint probability in the following table : $\mathbb{P}(X_1 = i, X_2 = j)$

$i \backslash j$	1	0
1	$\frac{5}{39}$	$\frac{10}{39}$
0	$\frac{10}{39}$	$\frac{14}{39}$

(b)

You can find the joint probability in the following table : $\mathbb{P}(X_1 = i, (X_2, X_3) = j)$

$i \backslash j$	$(1,1)$	$(1,0)$	$(0,1)$	$(0,0)$
1	0.03496503497	0.09324009324	0.09324009324	0.1631701632
0	0.09324009324	0.1631701632	0.1631701632	0.1958041958

Problem 6

$\mathbb{V}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$

$$\begin{aligned}
\mathbb{E}(X) &= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \int_{-\infty}^{\infty} \frac{x+\mu}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx \\
&= \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx + \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx \\
&= \mu + 0 \\
&= \mu
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}(X^2) &= \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \int_{-\infty}^{\infty} \frac{(x+\mu)^2}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx \\
&= \mu^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx + 2\mu \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx + \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx \\
&= \mu^2 + 0 + \sigma^2 \\
&= \mu^2 + \sigma^2
\end{aligned}$$

Thus

$$\boxed{\mathbb{E}(X) = \mu, \mathbb{V}(X) = \sigma^2}$$

Problem 7

Let $L =$ "A random person is chosen and is left-handed"
$\mathbb{P}(L) = \frac{12}{100} = \frac{3}{25}$
Let $n \in \mathbb{N}, A_n =$ number of left handed in a given population follows $\sim \mathcal{B}(n,p)$. We can approximate the binomial with a normal distribution. After the theorem of Moivre Laplace, $\frac{A_n - np}{\sqrt{np(1-p)}} \sim_{n\to\infty} \mathcal{N}(0,1)$
The approximation is good for $np(1-p) \geq 10$. In our case, $n = 200$, $p = \frac{3}{25}$, $np(1-p) = 21,12 \geq 10$ the approximation is :

$$\boxed{\mathbb{P}(A_{200} \geq 20) = \mathbb{P}(A_{200} > 19.5) \simeq \mathbb{P}(Z \geq -0.98) = 0.84}$$

where $Z \sim \mathcal{N}(0,1)$.