IEOR 263A : Homework 10

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Problem 7.13

a. Let T be the number of game played. Let $\forall i \in \mathbb{N}^*$, X_i be the outcome of the play i. We have $(X_i)_{i\in\mathbb{N}^*}$ iid. and T is a stopping time as it does not depend on the future games played. Let $X = \sum_{i=1}^{T} X_i$ be the winnings. Furthermore, $\mathbb{E}(T) \leq 3 < \infty$ thus after Wald's equation we have

$$\boxed{\mathbb{E}(X) = \mathbb{E}(T)\mathbb{E}(X_i) = 0 \text{ as } \mathbb{E}(X_i) = 0}$$

.

b. Let $I = \mathbb{1}(X_1 = -1)$ and notice that $T = 2I + 1$. Thus $\mathbb{P}(T = 1) = \mathbb{P}(I = 0) = \frac{1}{2}$, $\mathbb{P}(T = 2) = 0$ and $\mathbb{P}(T = 3) = \frac{1}{2}$. Moreover, $\mathbb{P}(X = -3) = \frac{1}{8}$, $\mathbb{P}(X = -1) = \frac{1}{4}$, $\mathbb{P}(X = 1) = \frac{5}{8}$.

$$\boxed{\mathbb{E}(X) = -3\frac{1}{8} - \frac{1}{4} + \frac{5}{8} = 0}$$

Problem 7.15

a. Let $\forall i \in \mathbb{N}^*$, $X_i \sim \mathcal{U}(\{2, 4, 6\})$ be the inter travel time before either reaching safety either going back to his room. Let $N = \min(n \in \mathbb{N}^* | X_n = 2)$ then N is a stopping time as $\mathbb{P}(N = n) = \mathbb{P}(\cap_{i=1}^{n-1}, X_i \neq 2, X_n = 2)$ which is independent from X_i, $i \geq n + 1$.

$$\boxed{T = \sum_{i=1}^{N} X_i}$$

b. We notice that $N \sim \mathcal{G}(\frac{1}{3})$ thus $\forall n \in \mathbb{N}^*$, $\mathbb{P}(N = n) = \frac{1}{3}(\frac{2}{3})^{n-1}$ and $\mathbb{E}(N) = 3$. After Wald's Equation :

$$\boxed{\mathbb{E}(T) = \mathbb{E}(N)\mathbb{E}(X) = 12}$$

c.

$$\boxed{\mathbb{E}\left(\sum_{i=1}^{N} X_i | N = n\right) = 2 + 5(n-1)}$$

This is different from

$$\boxed{\mathbb{E}\left(\sum_{i=1}^{n} X_i\right) = 4n}$$

d. We have $\mathbb{E}(T) = \mathbb{E}\left(\mathbb{E}(\sum_{i=1}^{N} X_i | N)\right) = \sum_{n=1}^{\infty} \mathbb{P}(N = n)\mathbb{E}\left(\sum_{i=1}^{N} X_i | N = n\right)$. Hence :

$$\boxed{\mathbb{E}(T) = 2 + 5\sum_{n=1}^{\infty}(n-1)p(1-p)^{n-1} = 2 + 5\left(\frac{\frac{2}{3}}{\frac{1}{3}}\right) = 12}$$

Problem 7.16

Wald's equation does not hold here as X_i are not iid. Let N_i be the number of aces discovered after having revealed i cards. Then $\forall i \in [\![1;48]\!]$, $\mathbb{P}(X_i = 1|N_{i-1} = 0) = \frac{4}{52-i}$ which is different for all i. Moreover N is not a stopping time as knowing $N = n$ yields $X_{n+1}, X_{n+2}, ..., X_{52}$ will be 0.

Problem 7.34

a. Let λ be the rate at which customers are coming in the system. The number of customers lost during service time is a non homogeneous poisson process $N_1(t)$ with rate $\lambda(t) = \begin{cases} 0 & \text{if } t \in [nT; nT + T/4], n \in \mathbb{N} \\ \lambda \int_t^T 1 - G(u)du & \text{if } t \in [nT + T/4; (n+1)T], n \in \mathbb{N} \end{cases}$.

Define the number of customers lost during cleaning : $N_2(t) = \begin{cases} \lambda & \text{if } t \in [nT + T/4; (n+1)T], n \in \mathbb{N} \\ 0 & \text{if } t \in [nT; nT + T/4], n \in \mathbb{N} \end{cases}$

$$\boxed{\lim_{t\to\infty} \frac{C_1 N_1(t) + C_2 N_2(t)}{t} = \frac{\mathbb{E}(N_1(T)C_1 + N_2(T)C_2)}{T} = \frac{C_1\mathbb{E}(N_1(T)) + C_2\mathbb{E}(N_2(T))}{T} = \frac{C_1 \int_0^T \lambda(s)ds + C_2\lambda T/4}{T}}$$

b. Let E be the time the system is being cleaned after the last clean. A cycle happens every T. The long run proportion of time the system is being cleaned is

$$\boxed{\frac{\mathbb{E}(E)}{T} = \frac{1}{4}}$$

Problem 7.37

a. Let T be the time before one machine fails. Let R be the time for repairation. With $\mathbb{E}(T) = \dfrac{1}{\lambda_1 + \lambda_2 + \lambda_3}$ and $\mathbb{E}(R) = 1 + \dfrac{\lambda_1}{5(\lambda_1 + \lambda_2 + \lambda_3)} + \dfrac{2\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} + \dfrac{3\lambda_3}{2(\lambda_1 + \lambda_2 + \lambda_3)}$. $C = T + R$ is the time of a cycle made of a failure and a repairation. The proportion of time the system is working is

$$\boxed{\frac{\mathbb{E}(T)}{\mathbb{E}(T+R)} = \frac{1}{1 + \frac{\lambda_1}{5} + 2\lambda_2 + \frac{3}{2}\lambda_3}}$$

b. Let R_i be the time the machine i is repaired in a cycle :

$$\boxed{\frac{\mathbb{E}(R_1)}{\mathbb{E}(C)} = \frac{\lambda_1}{5(1 + \frac{\lambda_1}{5} + 2\lambda_2 + \frac{3}{2}\lambda_3)}}$$

c. Let S_2 be the time the machine 2 is in suspended animation :

$$\boxed{\frac{\mathbb{E}(R_1 + R_3)}{\mathbb{E}(C)} = \frac{\lambda_1/5 + 3\lambda_3/2}{1 + \frac{\lambda_1}{5} + 2\lambda_2 + \frac{3}{2}\lambda_3}}$$

Additionnal Problem 1

a. Let $\forall n \in \mathbb{N}$, $u_n = m(n + 0.5) - m(n) = 0$ then $\lim_{n\to\infty} u_n = 0$.
Let $\forall n \in \mathbb{N}^*$, $v_n = m(n + 0.4) - m(n - 0.1) = 1$ and $\lim_{n\to\infty} v_n = 1$

$$\boxed{\text{There is no limits as two subsequences don't converge to the same limit}}$$

b. F is non lattice and we can apply the limit theorem for Renewal Process Mean, $\lim_{t\to\infty} \frac{m(t)}{t} - \frac{1}{\mu} = (c^2-1)/2$ where $\mu = \mathbb{E}(X) = \frac{3}{2}$. Thus we have $m(t+a) - m(t) = \dfrac{t+a}{\mu} - \dfrac{t}{\mu} + \dfrac{(c^2-1)}{2} - \dfrac{(c^2-1)}{2} + o_{t\to\infty}(1) =_{t\to\infty} \dfrac{a}{\mu} + o(1) \to_{t\to\infty} \dfrac{a}{\mu} = \dfrac{1}{3}$