IEOR 262 : Homework 3

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Problem 1.16 :

Let p_1, p_2, p_3 the number of time process 1, 2, 3 are processed. The problem is :

$$\boxed{\max(200p_1 + 60p_2 + 206p_3) \text{ s.t. } \begin{array}{r} 3p_1 + p_2 + 5p_3 \leq 8 * 10^6 \\ 5p_1 + p_2 + 3p_3 \leq 5 * 10^6 \\ p_1, p_2, p_3 \geq 0 \end{array}}$$

Problem 1.17 :

Let $\forall i \in [\![1; n]\!], x_i$ be the fraction of s_i sold. The problem is :

$$\boxed{\max\left(\sum_{i=1}^{n} r_i s_i (1 - x_i)\right) \text{ s.t. } \begin{array}{c} \sum\limits_{i=1}^{n} x_i s_i p_i \geq K \\ \forall i \in [\![1; n]\!], 0 \leq x_i \leq 1 \end{array}}$$

Problem 2.10 :

(a)

If $n = m + 1$ then we can define : $\forall i \in [\![1; m]\!], P_i = (x \in \mathbb{R}^n | a_i x = b_i)$ where a_i is the i-th row of A and b_i is the i-th coefficient of b.
We can notice that $P = \cap_{i=1}^{m} P_i$ where P_i is a hyperplane of $\mathbb{R}^n$ and you have $\dim(P_i) = n - 1$. Each a_i is perpendicular to P_i, and as each a_i are linearly independent the intersection of two hyperplanes will reduce the dimension of at least 1. So we have $\dim(P) = \dim(\cap_{i=1}^{m} P_i) \leq n - m = 1$. There can only be two extreme points on a line.

(b)

I will only consider the case where P is nonempty and related to a linear optimization problem that is bounded. In this case : an optimal solution is a basic point and there is only a finite number of basic points as there are only m constraints so at most $\binom{n}{m}$ basic points. As it is a finite set it is bounded.

(c)

This is false : let $(\mathcal{P})$: $\min(x_1 - x_2)$ s.t. $x_1 - x2 = 1$, $x_1, x_2 \geq 0$, then $x_1 = \frac{3}{2}, x_2 = \frac{1}{2}$ is optimal but not basic and more than 1 variable is non zero.

(d)

This is true : let $x_1, x_2 \in P$ be two optimal solution, let $\lambda \in [0; 1]$ and $x_\lambda = \lambda x_1 + (1 - \lambda) x_2 \geq 0$ then $A x_\lambda = \lambda A x_1 + (1 - \lambda) A x_2 = b$ thus $x_\lambda \in P$ and $c^T x_\lambda = \lambda c^T x_1 + (1 - \lambda) c^T x_2 = c^T x_1$ which is the optimal value. Therefore there are infinetely many optimal solutions.

(e)

This is false : let $(\mathcal{P})$: $\min(x1 - x2)$ s.t. $x_1 - x_2 = 0$, $x_1 \geq 0, x_2 \geq 0$. There are multiple optimal solutions but only one basic optimal solution.

(f)

Let $P = \{(x_1, x_2, x_3) \in \mathbb{R}^n | x_1+x_2+x_3 = 1\}$ and $(\mathcal{P}) : \min(\max(x_1-x_2+x_3, -x_1+x_2-x_3))$ s.t. $x_1, x_2, x_3 \in P$ then an optimal solution is $x_1 = \frac{1}{2}, x_2 = \frac{1}{2}, x_3 = 0$ and is not a basic solution as only two constraints are active.

Problem 3.12 :

(a)

The problem in standard form is :

$$\boxed{\min(-2x_1 - x_2) \text{ s.t. } \begin{aligned} x_1 - x_2 + s_1 &= 2 \\ x_1 + x_2 + s_2 &= 6 \\ x_1, x_2, s_1, s_2 &\geq 0 \end{aligned}}$$

A solution to this problem would be $x_1 = 0, x_2 = 0, s_1 = 2, s_2 = 6$

(b)

Starting from the point : $(x_1, x_2, s_1, s_2) = (0, 0, 2, 6)$.

0	-2	-1	0	0
2	1	-1	1	0
6	1	1	0	1

x_1 and x_2 reduced cost are both negative so I will choose that x_1 enters the basis, we compute $\theta = 2$ and s_1 leaves the basis.

The new array is :

4	0	-3	2	0
2	1	-1	1	0
4	0	2	-1	1

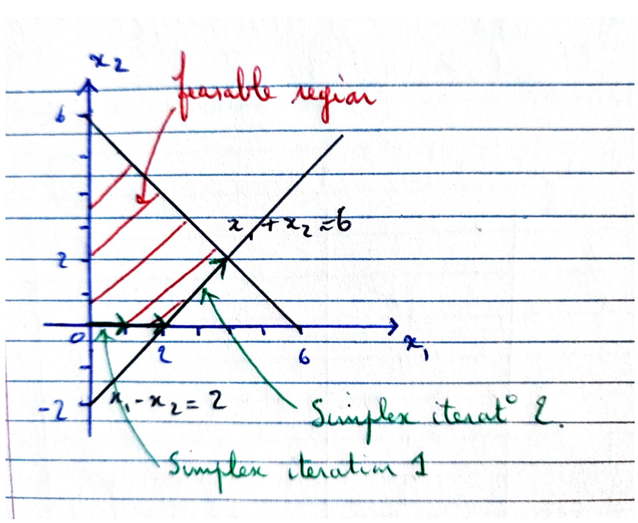
x_2 reduced cost is negative so x_2 enters the basis. Compute $\theta = 2$ and s_2 leaves the basis.

The new array is :

10	0	0	0.5	1.5
4	1	0	0.5	0.5
2	0	1	-0.5	0.5

No reduced costs are negative so the point $\boxed{x_1 = 4, x_2 = 2, s_1 = 0, s_2 = 0}$ is optimal.

(c)



Problem 3.17 :

The phase I :

Intialisation : as $B = I_3$, the reduced cost vector $\overline{c_N} = (0,0,0,0,0) - (1,1,1)N$ where $N = \begin{pmatrix} 1 & 3 & 0 & 4 & 1 \\ 1 & 2 & 0 & -3 & 1 \\ -1 & -4 & 3 & 0 & 0 \end{pmatrix}$,

5	−1	−1	−3	−1	0	0	0	0
2	1	3	0	4	1	1	0	0
2	1	2	0	−3	1	0	1	0
1	−1	−4	3	0	0	0	0	1

x_1 enters the basis and s_1 leaves gives the new array :

3	0	2	−3	3	0	1	0	0
2	1	3	0	4	1	1	0	0
0	0	−1	0	−7	0	−1	1	0
3	0	−1	3	4	1	1	0	1

x_3 enters the basis and s_3 leaves gives the new reduced cost vector : $\overline{c_N} = (1,7,1,2,1) \geq 0$ so the solution $x_1 = 2, x_3 = 1$ and the others equal to 0 is optimal. But as s_2 is still in the basis we apply a change of basis : s_2 leaves and x_2 enters.

The final tableau is :

3	0	2	−3	3	0	1	0	0
2	1	0	0	−17	1	−2	3	0
0	0	1	0	7	0	1	−1	0
1	0	0	3	11	1	2	−1	1

Phase II :

Taking x_1, x_2, x_3 as the basis, $B = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 2 & 0 \\ -1 & -4 & 3 \end{pmatrix}$ and $B^{-1} = \begin{pmatrix} -2 & 3 & 0 \\ 1 & -1 & 0 \\ 2/3 & -1/3 & 1/3 \end{pmatrix}$, reduced cost vector is (3,-5) and the array is

7	0	0	0	3	−5
2	1	0	0	−17	1
0	0	1	0	7	0
1	0	0	1	11/3	1/3

. Thus x_5 enters the basis and x_1 leaves and the new array is

−1	5	0	0	−82	0
2	1	0	0	−17	1
0	0	1	0	7	0
1/3	−1/3	0	1	28/3	0

and $\boxed{x_5 = 2, x_3 = 1/3 \text{ is an optimal degenerate solution of the pb}}$ as x_4 has to enter the base but $x_2 = 0$ has to leave the basis and you are going to exchange forever those two.

Problem 3.19 :

(a)

Let $\alpha = 0, \beta = 0, \gamma = 0, \eta = 0, \delta = -2$, there are mulitple solutions.

(b)

Let $\alpha = -1, \gamma = -1, \beta = 1, \eta = 1, \delta = -1$ the problem is unbounded as if x_1 is chosen to enter the basis then all coefficient are negative and you can choose any θ.

(c)

Let $\alpha = 1, \gamma = 1, \beta = 1, \eta = 1, \delta = -1$.

Problem 4.33 :

Let $p_S = S$ be the price of the stock, $p_B = 1$ the price of the bond and p_O the price of the option. $R = \begin{pmatrix} Su & r & \max(0, Su-K) \\ Sd & r & \max(0, Sd-K) \end{pmatrix}$, after theorem 4.8, the absence of arbitrage condition yields there exist $\delta, \gamma >= 0$ such that $p_S = \gamma Su + \delta Sd = S$ and $p_B = \gamma r + \delta r$ and $p_O = \gamma \max(0, Su-K) + \delta \max(0, Sd-K)$

So :

$$\boxed{p_O = \gamma \max(0, Su - K) + \delta \max(0, Sd - K) \text{ where } 1 = \gamma u + \delta d \text{ and } \frac{1}{r} = \delta + \gamma}$$

Problem 4.39 :

Let $A \in \mathcal{M}_{m,n}(\mathbb{R})$ without any loss of generality suppose n rows of A are linearly independent. We can suppose this because if there exist an dependent row we can drop it. Let $C = \{x \in \mathbb{R}^n | Ax \geq 0\}$.

Definition 1 : $d \in C$ is an extreme ray if there are n-1 constraints bounding.
Definition 2 : $d \in C$ is an extreme ray if $\forall (f, g) \in C^2, f + g = d \implies (f, g) \in \text{Vect}(d)^2$

Suppose $d \in C$ extreme ray after definition 1 ie. $\forall i \in [\![1; n-1]\!], \sum\limits_{j=1}^{n} a_{ij} d_j = 0$.
Let $f, g \in C$ such that $d = f + g$, if $f = 0 or g = 0$ then either $f = d$ or $g = d$ so we can suppose $f \neq 0$ and $g \neq 0$.
Thus $\sum\limits_{j=1}^{n} a_{ij} f_j + \sum\limits_{j=1}^{n} a_{ij} g_j = 0$ (1)
Moreover, $f \in C$ so $Af \geq 0$ so $\forall i [\![1; m]\!], \sum\limits_{j=1}^{n} a_{ij} f_j \geq 0$ and the same holds for g.
Both terms in (1) are positive and the sum is equal to 0 thus both terms have to be 0.
Thus $\forall i \in [\![1; n-1]\!], \sum\limits_{j=1}^{n} a_{ij} f_j = 0$ and $\sum\limits_{j=1}^{n} a_{ij} g_j = 0$.
Now suppose that $f \notin \text{Vect}(g)$ then $\dim(\text{Vect}(f, g)) = 2$ and $\text{Vect}(f, g)$ is orthogonal to $\text{Vect}(a_1, a_2, a_3, ..., a_{n-1})$.
By learly independence hypothesis : $\dim(\text{Vect}(a_1, ..., a_{n-1})) = n - 1$
Thus $\dim(\text{Vect}(a_1, a_2, ..., a_{n-1}) + \text{Vect}(f, g)) = n+1$ but $\text{Vect}(a_1, a_2, ..., a_{n-1}) + \text{Vect}(f, g) \subseteq \mathbb{R}^n$ and we have a contradiction.
So f is proportional to g ie. $\exists \lambda \in \mathbb{R}^n$ s.t. $f = \lambda g$, moreover $A(\lambda g) \geq 0$ so $\lambda \geq 0$ thus $d = f + g = (1 + \lambda) g = (1 + \frac{1}{\lambda}) f$ and both are proportional to d.

Now let d an extreme ray after definition 2 ie. $\forall (f, g) \in C^2, f + g = d \implies (f, g) \in \text{Vect}(d)^2$.
Let f be an extreme ray after definition 1 and $g = d - f$ we have $f = d - g$ and after proof 1 we have d verifies n-1 bounds.