IEOR 262A : Homework 4

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Problem 1.15 :

a. Let x_i be the number produced of type i. Then the problem can be formulated as follows :

$$\boxed{\begin{array}{ll} & \max(7.8x_1 + 7.2x_2) \\ \text{s.t.} & \frac{1}{4}x_1 + \frac{1}{3}x_2 \leq 90 \\ & \frac{1}{8}x_1 + \frac{1}{3}x_2 \leq 80 \\ & x_1, x_2 \geq 0 \end{array}}$$

b. The situation $ii.$ is easier to implement : let $d = \begin{cases} 1 & \text{if } 1.2x_1 + 0.9x_2 \geq 300 \\ 0 & \text{ortherwise} \end{cases}$ the problem becomes :

$$\boxed{\begin{array}{ll} & \max((7.8 + 0.12d)x_1 + (7.2 + 0.09d)x_2) \\ \text{s.t.} & \frac{1}{4}x_1 + \frac{1}{3}x_2 \leq 90 \\ & \frac{1}{8}x_1 + \frac{1}{3}x_2 \leq 80 \\ & 1.2x_1 + 0.8x_2 \geq 300d \\ & x_1, x_2 \geq 0 \end{array}}$$

For the situation $i.$ I would introduce a new variable l which is the number of hours above 90. The problem would become :

$$\boxed{\begin{array}{ll} & \max(7.8x_1 + 7.2x_2 - 7l) \\ \text{s.t.} & \frac{1}{4}x_1 + \frac{1}{3}x_2 - l \leq 90 \\ & \frac{1}{8}x_1 + \frac{1}{3}x_2 \leq 80 \\ & x_1, x_2 \geq 0 \\ & 0 \leq l \leq 50 \end{array}}$$

Problem 5.2 :

a. Let $G = I_m + \delta B^{-1}E$ notice that G is an inferior triangular matrix as $\delta B^{-1}E = \begin{pmatrix} \delta(B^{-1})_1 & 0_{\mathcal{M}_{(m,m-1)}} \end{pmatrix}$. Thus $\det(G) = 1 + \delta(B^{-1})_{1,1}$. If $(B^{-1})_{1,1} \neq 0$ then $\forall \delta \in \mathbb{R}$ such that $|\delta| < |1/(B^{-1})_{1,1}|$: $\det(G) \neq 0$. If $(B^{-1})_{1,1} = 0$ then $\det(G) = 1$. In both case, G is invertible.

Notice that $B + \delta E \in \mathcal{M}_{(m,m)}$ which means that $B + \delta E$ is a squared matrix and if $\exists A \in \mathcal{M}_{(m,m)}$ such that $A(B + \delta E) = I_m$ then A is the inverse of $B + \delta E$.
Moreover $G^{-1}B^{-1}(B + \delta E) = (I_m + \delta B^{-1}E)^{-1}B^{-1}(B + \delta E) = (I_m + \delta B^{-1}E)^{-1}(I_m + \delta B^{-1}E) = I_m$ then $B + \delta E$ is invertible with inverse $G^{-1}B^{-1}$ and

$$\boxed{B + \delta E \text{ is a basis matrix.}}$$

b. We have $(B+\delta E)x_B = b$ thus :

$$\boxed{x_B = G^{-1}B^{-1}b = (I_m + \delta B^{-1}E)^{-1}B^{-1}b}$$

c. For primal feasibility : let $d_\delta \in \mathbb{R}^m$ such that $d_\delta = \delta B^{-1}Ex_B$ we have $x_B = x^* - d_\delta$ where x^* is the original solution of the problem. $(B+\delta E)x_B = (Bx^* - Bd_\delta) + \delta Ex_B = b - Bd_\delta + \delta Ex_B = b$. As B is invertible $\lim_{\delta\to 0} d_\delta = 0$ ie. $\forall \epsilon > 0$, $\exists \alpha \in \mathbb{R}+$ such that $|\delta| < \alpha \implies d_\delta < \epsilon \begin{pmatrix} 1 \\ | \\ 1 \end{pmatrix}$. Which means that for $\epsilon = \min_{i\in[\![1,m]\!]}((B^{-1})_i b)$ then $\exists \alpha \in \mathbb{R}+$ such that $\forall |\delta| < \alpha$, $B^{-1}b - d_\delta \geq 0$ ie. $x_B \geq 0$ because $x_B = B^{-1}b - d_\delta$. Which means there exists δ small enough for the problem to stay feasible.

For the dual feasibility : we need to check the positivity of the reduced cost. In a similar way as the primal feasibility : the new reduced cost is somewhat a difference between the original one and a term multiplied by δ. As the old reduced cost were all strictly positive because it is a non degenerate solution then removing a small amount δ can not transform it to negative values or rather there always exist a δ such that the reduced cost is still positive.

Taking $\delta = \min(\delta_1, \delta_2)$ where δ_1 is given in primal feasibility proof and δ_2 is given in dual feasibility proof we have shown that

$$\boxed{\text{there exist } \delta \text{ such that the problem stays optimal}}$$

d. We use that $x^* = B^{-1}b$ and $c_B^T B^{-1} = p$ with the hint :

$$\boxed{\begin{aligned} c_B^T x_B &= c_B^T(I + \delta B^{-1}E)^{-1}B^{-1}b \\ &\approx c_B^T(I - \delta B^{-1}E)x^* \\ &\approx c^T x^* - \delta c_B^T B^{-1}Ex^* \\ &\approx c^T x^* - \delta p_1 x_1^* \end{aligned}}$$

Problem 5.5 :

a. Let $C = \overline{c_3}, \overline{c_5} \geq 0$. Suppose C then the simplex algorithm can't loop anymore and you can't imporve the objective value which means that the solution $x_2 = 1, x_4 = 2, x_1 = 3, x_3 = 0, x_5 = 0$ is optimal.

Now suppose that $x = (3, 1, 0, 2, 0)$ is optimal. It means that $\forall y \in \mathbb{R}^5$, y feasible $\implies$ objective value at y is greater or equal to objective value at x. It implies that all reduced cost for the choice of the basis is greater or equal to 0 ie. C is true.

$$\boxed{C \Leftrightarrow x \text{ optimal}}$$

b. If $\overline{c_3} = 0$ then we can let x_3 enters the basis as the objective value changes by the reduced cost $\overline{c_3} = 0$ ie. the objective value won't change. If x_3 enters the basis then x_1 has to leave the basis and the resulting tableau is :

	x_1	x_2	x_3	x_4	x_5
	$-\frac{\overline{c_3}}{4}$	0	0	0	$\overline{c_5} - \overline{c_3}\frac{\delta}{4}$
$x_2 = \frac{7}{4}$	$\frac{1}{4}$	1	0	0	$\beta + \frac{\delta}{4}$
$x_4 = \frac{1}{2}$	$-\frac{1}{2}$	0	0	1	$\gamma - \frac{\delta}{2}$
$x_3 = \frac{3}{4}$	$\frac{1}{4}$	0	1	0	$\frac{\delta}{4}$

Hence

$$\boxed{x_3 = \frac{3}{4} = 0.75, x_4 = 0.5 \text{ and } x_2 = 1.75 \text{ is another optimal solution.}}$$

c. Supose $\gamma > 0$: the problem is already feasible as $x = (3, 1, 0, 2, 0)$ is feasible and we can apply the simplex algorithm from this point. If the problem were infeasible then either column 3 or column 5 would have only negative value. It is not the case here as column 3 is $(-1, 2, 4)$ is not negative and column 5 is (β, γ, δ) is not negative as $\gamma > 0$. Hence the problem can't be unbounded which means that the problem has an optimal solution and finally

$$\boxed{\text{The problem has a optimal basic feasible solution.}}$$

d. This change only affects x as changing the right hand side constraints means changing the dual objective value and so it does not change the reduced costs which are positive as shown in question **a.** Thus we need to verify that x with this choice of basis is still feasible ie.

$$x = B^{-1}b + \epsilon B^{-1}e_1 \geq 0 \tag{1}$$

. In the exercise we have supposed that $A = \begin{pmatrix} a_{11} & a_{12} & 1 & 0 & 0 \\ a_{21} & a_{22} & 0 & 1 & 0 \\ a_{31} & a_{32} & 0 & 0 & 1 \end{pmatrix}$ and because x_1, x_2, x_4 are in the basis with $B(1) = 2, B(2) = 4$ and $B(3) = 1$ it means $B = \begin{pmatrix} a_{12} & 0 & a_{11} \\ a_{22} & 1 & a_{21} \\ a_{32} & 0 & a_{31} \end{pmatrix}$ and $N = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Using that $B^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = B^{-1}\begin{pmatrix} e_1 & e_3 \end{pmatrix} = \begin{pmatrix} -1 & \beta \\ 2 & \gamma \\ 4 & \delta \end{pmatrix}$ we can deduce that the first column of B^{-1} is $B^{-1}e_1 = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$. Hence the condition (1) yields $\begin{pmatrix} 3-\epsilon \\ 1+2\epsilon \\ 2+4\epsilon \end{pmatrix} \geq 0$ ie.

$$\boxed{-\frac{1}{2} \leq \epsilon \leq 3}$$

e. When c_1 become $c_1 + \epsilon$ as x_1 is a basic variable we need to check the dual feasibility. For the solution to stay optimal ϵ has to verify : $\forall i \in [\![2;5]\!]$, $c_i \geq \epsilon q_{3i}$ where q_{3i} is the i-th coefficient of the third row as x_1 is in the third row. This condition yields : $\epsilon 4 \leq \overline{c_3}$, $\epsilon\delta \leq \overline{c_3}$ as shown in **a.** $\overline{c_3} \geq 0$ and $\overline{c_5} \geq 0$ thus :

$$\boxed{\begin{cases} \frac{\overline{c_5}}{\delta} \leq \epsilon \leq \frac{\overline{c_3}}{4} & \text{if } \delta < 0 \\ \epsilon \leq \min(\frac{\overline{c_3}}{4}, \frac{\overline{c_5}}{\delta}) & \text{if } \delta > 0 \\ \epsilon \leq \frac{\overline{c_3}}{4} & \text{if } \delta = 0 \end{cases}}$$

Problem 5.15 :

a. Consider the problem $(\mathcal{P})$):

$$\begin{aligned} &\min(x_1 + 2x_2 + 3x_3) \\ \text{s.t. } &x_1 + x_2 = 1 \\ &x_1 + x_3 = 3 \\ &x_1, x_2, x_3 \geq 0 \end{aligned}$$

Its optimal tableau is

-7	0	4	0
$x_1 = 1$	1	1	0
$x_3 = 2$	0	-1	1

Consider adding θ to b_1 ie. change $(\mathcal{P})$ into $(\mathcal{P}(\theta))$:

$$\begin{aligned} &\min(x_1 + 2x_2 + 3x_3) \\ \text{s.t. } &x_1 + x_2 = 1 + \theta \\ &x_1 + x_3 = 3 \\ &x_1, x_2, x_3 \geq 0 \end{aligned}$$

We have that $x_B = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \theta B^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \geq 0$ thus $-1 \leq \theta \leq 2$ and $\left\{ \begin{pmatrix} 1+\theta \\ 0 \\ 2-\theta \end{pmatrix}, \forall \theta \in [-1;2] \right\} \subset X(-\infty, 2)$

If $\theta \in]-\infty;-1]$ then we have to apply the dual simplex algorithm to keep dual feasibility but find primal feasibility back. As $1+\theta<0$ then x_1 has to leave the basis but no any other one can enter it and the problem is infeasible.

If $\theta \in]2;\infty[$ then we apply the dual simplex algorithm and x_3 leaves the basis, x_2 enters and the resulting tableau, which is also optimal as it is primal and dual feasible, is :

$1-2\theta$	0	0	4
$x_1 = 3$	1	0	1
$x_2 = \theta - 2$	0	1	-1

Hence $\left\{ \begin{pmatrix} 3 \\ \theta-2 \\ 0 \end{pmatrix}, \forall \theta \in [2;\infty[\right\} \subset X(2;\infty)$

To conclude, look at $\left\{ u \in \mathbb{R}^3, \exists \theta \in [0;2], u = \begin{pmatrix} 1+\theta \\ 0 \\ 2-\theta \end{pmatrix} \text{ or } \exists \theta \in]2;3], u = \begin{pmatrix} 3 \\ \theta-2 \\ 0 \end{pmatrix} \right\} \subset X(0,3)$

Let $s_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \in X(0) \subset X(0,3)$ and $s_2 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \in X(3) \subset X(0,3)$. Let $\lambda = \frac{1}{2}$ and compute $\lambda s_1 + (1-\lambda)s_2 = \begin{pmatrix} 4 \\ 0.5 \\ 1 \end{pmatrix}$ and it cannot be in $X(0;3)$ as its cost is 7 whereas the optimal cost should be 4.

$$\boxed{X(0;3) \text{ is not convex}}$$

b. Removing the inequality constraints.
The general problem is $(\mathcal{G})$:

$$\min(c^T x)$$
$$\text{s.t. } Ax = b + \theta d$$

Where $A \in \mathcal{M}_{m,n}(\mathbb{R})$ without loss of generality $\text{rg}(A) = m$ because you can remove the linearly dependent conditions. Thus the system described by $Ax = b + \theta d$ has at least $n-m$ degrees of liberty. If $m > n$ then the problem is infeasible. If $m = n$ then there is an optimal solution and it is the only feasible point $x = A^{-1}(b+\theta d)$ and $\forall t \in \mathbb{R}, X(0,t)$ is convex.

If $m < n$: let $P_\theta = \{x \in \mathbb{R}^n \text{ such that } Ax = b + \theta d\}$. Suppose $A = \begin{pmatrix} B & N \end{pmatrix}$ and let $x_0 = \begin{pmatrix} B^{-1}(b+\theta d) \\ 0 \end{pmatrix}$ then $\forall x \in P_\theta$, $x - x_0 \in \ker(A)$ as $A(x-x_0) = b + \theta d - (b+\theta d) = 0$ this space is of dimension $n-m$ after the rank theorem. It means that $\exists (v_1, v_2, ..., v_{n-m}) \in (\mathbb{R}^n)^{n-m}$ such that $\forall p \in P_\theta$, $\exists (\lambda_1, \lambda_2, ..., \lambda_{n-m})$, $p = x_0 + \sum_{i=1}^{n-m} \lambda_i v_i$ and $(\mathcal{G})$ becomes $\min_{(\lambda_i)_{i \in [\![1;n-m]\!]}} (c^T x_0 + \sum_{i=1}^{n-m} \lambda_i c^T v_i)$. If $c \notin \text{Vect}(v_1, v_2, ..., v_{n-m})^\perp$ then the problem is unbounded letting $\lambda_j \to \pm\infty$ for a good choice of j. If $c \in \text{Vect}(v_1, v_2, ..., v_{n-m})^\perp$ then the set of optimal solution is P which is convex as it is a polyhedron. We have seen that the problem is either unbounded either $X(\theta) = P_\theta$. Now take $(\theta_1, \theta_2) \in \mathbb{R}^2$, $\lambda \in [0;1]$, let $(x_1, x_2) \in P_{\theta_1} \times P_{\theta_2}$, $A(\lambda x_1 + (1-\lambda)x_2) = b + (\lambda\theta_1 + (1-\lambda)\theta_2)d$ ie. $\lambda x_1 + (1-\lambda)x_2 \in P_{\lambda\theta_1 + (1-\lambda)\theta_2} = X(\lambda\theta_1 + (1-\lambda)\theta_2)$ hence

$$\boxed{\forall t \in \mathbb{R}, X(0,t) \text{ is convex.}}$$

c. In $(\mathcal{G})$: let B_θ be the optimal basis matrix for $(\mathcal{G})$ with parameter θ. $x_{B_\theta} = f(\theta) = B_\theta^{-1}(b+\theta d)$. Let $\theta_1 \in \mathbb{R}$ such that $x_{B_{\theta_1}}$ is a degenerate solution. Let $\epsilon \in \mathbb{R}$ small enough so that $\theta_1 + \epsilon$ has the same optimal basis then $x_{B_{\theta_1+\epsilon}} = B_{\theta_1}^{-1}(b + (\theta_1+\epsilon)d) = x_{B_{\theta_1}} + \epsilon B_{\theta_1}^{-1} d$ Thus if the basis does not change the function is continuous because linear. Now consider a degenerate solution so that the basis can change with respect to the variations of θ. The basis only changes when a coordinate goes to 0. The optimal degenerate point does not change with respect to those two basis as you are entering a 0 variable into the basis and a 0 varibale leaves the basis. Once the number left the basis you are in the case where g is linear and so is continuous. That's why in both cases :

$$\boxed{g \text{ is continuous}}$$

Additionnal Problem 1

Let $(x, y) \in \mathbb{R}^2$, $f \in \mathcal{C}^2(\mathbb{R}^2, \mathbb{R})$ as it is a polynomial form.

$$(x,y) \text{ is an extreme point} \Leftrightarrow \nabla f(x,y) = \begin{pmatrix} 2x+\beta y+1 \\ \beta x + 2y + 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \boxed{\begin{cases} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \dfrac{2-2\beta}{\beta^2-4} \\ \dfrac{4-\beta}{\beta^2-4} \end{pmatrix} & \text{if } \beta \neq \pm 2 \\ \text{impossible} & \text{otherwise} \end{cases}}$$

$\nabla^2 f(x,y) = \begin{pmatrix} 2 & \beta \\ \beta & 2 \end{pmatrix}$ we need to know when is it positive semi-definite.

Let $\lambda \in \mathbb{R}$, $\det(\nabla^2 f(x,y) - \lambda I_2) = (\lambda - (2+\beta))(\lambda - (2-\beta))$ thus the eigen values are $2+\beta$ and $2-\beta$ hence

$$\boxed{\forall \beta \in]-2;2[, \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \dfrac{2-2\beta}{\beta^2-4} \\ \dfrac{4-\beta}{\beta^2-4} \end{pmatrix} \text{ is a global minima}}$$

If $\beta \notin]-2;2[$ then two cases. First suppose $\beta < -2$, $f(x,y) = (x-y)^2 + (\beta+2)xy + x + 2y$ and $\lim_{x\to-\infty} f(x,x) = -\infty$ thus there can't be any minima. Second case suppose $\beta > 2$, $f(x,y) = (x+y)^2 + (\beta-2)xy + x + 2y$ and $\lim_{x\to\infty} f(x,-x) = -\infty$ thus there are no minima.

Additonnal problem 2

a. $f \in \mathcal{C}^2(\mathbb{R}^2, \mathbb{R})$ as f is a polynomial form. Thus $\nabla f(x,y) = \begin{pmatrix} 4x(x-2)(x+2) \\ 2y \end{pmatrix}$ and the stationary points are : $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$.

Moreover, $f(x,y) = (x^2-4)^2 + y^2 \geq 0$ and $f(2,0) = f(-2,0) = 0$ thus $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$ are global minimas.

$\nabla^2 f(0,0) = \begin{pmatrix} -16 & 0 \\ 0 & 2 \end{pmatrix}$ thus the eigen values are -16 and 2 which means that the hessian is indefinite and $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is neither a local minima nor a global minima.

b. $f(x,y) = \dfrac{x^2}{2} + x\cos(y)$, $f \in \mathcal{C}^2(\mathbb{R}^2, \mathbb{R})$ as it is a sum of function that are twice differentiable.
$\forall (x,y) \in \mathbb{R}^2, \nabla f(x,y) = \begin{pmatrix} x + \cos(y) \\ -x\sin(y) \end{pmatrix}$.
$\begin{pmatrix} x \\ y \end{pmatrix}$ is a stationary point $\Leftrightarrow \nabla f(x,y) = \begin{pmatrix} x+\cos(y) \\ -x\sin(y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ie. $\exists k \in \mathbb{Z}$ such that $y = k\frac{\pi}{2}$ and $x = \begin{cases} (-1)^{k/2} & \text{if } k \text{ even} \\ 0 & \text{otherwise} \end{cases}$

Moreover, $\forall (x,y) \in \mathbb{R}^2, \nabla^2 f(x,y) = \begin{pmatrix} 1 & -\sin(y) \\ -\sin(y) & -x\cos(y) \end{pmatrix}$ thus $\forall k \in \mathbb{Z}$, with k even :
$\nabla^2 f((-1)^{k/2+1}, k\frac{\pi}{2}) = I_2$ thus is definite positive and those points are local minimas.
For k odd : $\nabla^2 f(0, k\frac{\pi}{2}) = \begin{pmatrix} 1 & (-1)^{k/2+1} \\ (-1)^{k/2+1} & 0 \end{pmatrix}$ as $\det(\nabla^2 f(0, k\frac{\pi}{2})) = -1$ the matrix can't be positive semi-definite and the corresponding points can't be local minimas.

$$\boxed{\text{Local minimas are : } \forall n \in \mathbb{Z}, \begin{pmatrix} (-1)^{n+1} \\ n\pi \end{pmatrix}}$$

c.$f : x, y \mapsto \sin x + \sin y + \sin x + y$ let $\mathcal{A} = (0, 2\pi)^2$. f is twice differentiable and its gradient and hessian matrix are :

$$\forall x, y \in \mathcal{C},\ \nabla f(x,y) = \begin{pmatrix} \cos x + \cos x + y \\ \cos y + \cos x + y \end{pmatrix}$$

And :

$$\forall x, y \in \mathcal{C},\ \nabla^2 f(x,y) = \begin{pmatrix} -\sin x - \sin x + y & -\sin x + y \\ -\sin x + y & -\sin x - \sin x + y \end{pmatrix}$$

The stationary point condition yields : $\cos x + y = -\cos y = -\cos x$ then $\cos x = \cos y$ which means that $y = x$ or $y = 2\pi - x$.

If $y = 2\pi - x$: we have that $\cos x + y = -\cos x$. Then, $\cos x = -1$ which means $\boxed{x = y = \pi}$ since $x, y \in \mathcal{C}$.

Now if $y = x$: $\cos 2x = -\cos x$ which means that : $2\cos x^2 - 1 = -\cos x$. We have a second degree polynomial in $\cos x$ we know how to solve this type of equation : it gives us that : $\cos x = -1$ or $\cos x = \frac{1}{2}$. The first solution gives us that $\boxed{x = y = \pi}$. So we have that $\boxed{x = y = \dfrac{\pi}{3}}$ or $\boxed{x = y = \dfrac{5\pi}{3}}$. Now to get the character of these stationary points we have to look at the hessian for $x = y = \pi$:

$$\nabla^2 f(x,y) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence the hessian is indefinite and the point is just a stationary point. For $x = y = \frac{\pi}{3}$:

$$\nabla^2 f(x,y) = \begin{pmatrix} -\sqrt{3} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\sqrt{3} \end{pmatrix}$$

Then we have that the eigenvalues of the hessian are : $-\frac{\sqrt{3}}{2}$ and $-\frac{3\sqrt{3}}{2}$.
Then this points are maximums since the hessian definite negative.
we have for $x = y = \frac{5\pi}{3}$:

$$\nabla^2 f(x,y) = \begin{pmatrix} \sqrt{3} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \sqrt{3} \end{pmatrix}$$

Then we have that the eigenvalues of the hessian are : $\frac{\sqrt{3}}{2}$ and $\frac{3\sqrt{3}}{2}$.
Then this points are minimums since the hessian definite positive.

d. For this : $f : (x,y) \mapsto (y - x^2)^2 - x^2$ is twice differentiable as it is a polynomial form.
$\forall (x,y) \in \mathbb{R}^2, \nabla f(x,y) = \begin{pmatrix} -4x(y - x^2) - 2x \\ 2(y - x^2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies x = y = 0$ and $\nabla^2 f(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$ which is indefinite hence

$$\boxed{\text{The stationary point } \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ is neither a local minimum nor a local maximum.}}$$

e. The Karush-Kuhn-Tucker conditions are :

$$\begin{aligned} & -1 - y \leq 0 \\ & y - 1 \leq 0 \\ & \begin{pmatrix} 4x(y - x^2) + 2x \\ -2(y - x^2) \end{pmatrix} = \lambda_1 \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ & \lambda_1, \lambda_2 \geq 0 \\ & (-1 - y)\lambda_1 = 0 \\ & (y - 1)\lambda_2 = 0 \end{aligned}$$

We can deduce that $4x(y - x^2) + 2x = 0$ which yields $x = 0$ or $x^2 = y + 1/2$. If $x = 0$, $2y = -\lambda_1 + \lambda_2$. Suppose $\lambda_1 \neq 0$ it implies that $y = -1$ and $\lambda_2 = 0$ thus $\lambda_1 = 1$ first KKT point : $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ the objective value is 1. Suppose that $\lambda_1 = 0$, $-2y = \lambda_2$ which means that either $\lambda_2 = 0$ and $y = 0$ and $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a KKT point with objective value 0. Now suppose $x \neq 0$, which yields $x^2 = y + 1/2$ and $-\lambda_1 + \lambda_2 = 1$ so $\lambda_2 = \lambda_1 + 1$ ie. $\lambda_2 \geq 1$

thus $y = 1$ and $\lambda_2 = 0$ and $\lambda_1 = 1$. $\begin{pmatrix} \sqrt{3/2} \\ 1 \end{pmatrix}$ is another KKT point with objective value $-5/4$. If a min exist then it verifies the KKT conditions. Moreover the KKT conditions implies that $x \in [-\sqrt{3/2}; \sqrt{3/2}]$ and f is continuous on $[-\sqrt{3/2}; \sqrt{3/2}] \times [-1; 1]$ thus has a minimum value. A minimum exist and has been found it is $\begin{pmatrix} \sqrt{3/2} \\ 1 \end{pmatrix}$. Note that $f(-x, y) = f(x, y)$ thus $\begin{pmatrix} -\sqrt{3/2} \\ 1 \end{pmatrix}$ is another solution.