IEOR 241 : Homework 8

Arnaud Minondo

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Exercise 1

Let X be the location of the car at the moment of the accident. Let P be the location of the accident. With the problem statement assumptions we have : $X \sim \mathcal{U}([0;L])$ and $P \sim \mathcal{U}([0;L])$.
We are trying to search $\forall t \in [0;L]$, $\mathbb{P}(|X-P| \leq t)$ the cumulative distribution of $|X-P|$.
Let $t \in [0;L]$, $\mathbb{P}(|X-P| \leq t) = \mathbb{P}(|X-P| \leq t, X > P) + \mathbb{P}(|X-P| \leq t, X \leq P)$
By symmetry of P and X we have $\mathbb{P}(|X-P| \leq t, X > P) = \mathbb{P}(|X-P| \leq t, X \leq P)$
That's why :

$$\begin{aligned}
\mathbb{P}(|X-P| \leq t) &= 2\mathbb{P}(X \leq P+t, X > P) \\
&= 2\int_0^L \int_u^{u+t} f_X(x)dx f_P(u)du \\
&= 2\int_0^{L-t} \int_u^{u+t} f_X(x)dx f_P(u)du + 2\int_{L-t}^L \int_u^L f_X(x)dx f_P(u)du \\
&= 2\int_0^{L-t} \frac{t}{L^2}du + \frac{t^2}{L^2} \\
&= 2\frac{(L-t)t}{L^2} + \frac{t^2}{L^2} \\
&= \frac{2tL-t^2}{L^2}
\end{aligned}$$

Finally the cumulative function distribution of $|X-P|$ is F:

$$\boxed{\forall t \in \mathbb{R}, F(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ \dfrac{2tL-t^2}{L^2} & \text{if } t \in [0;L] \\ 1 & \text{if } t \geq [L;\infty] \end{cases}}$$

Exercise 2

$f(x,y) = xe^{-(x+y)}I(x \in [0;+\infty[)I(x \in [0;+\infty[) = g(x)h(y)$
where $g(x) = xe^{-x}I(x \in [0;+\infty[)$ and $h(y) = e^{-y}I(y \in [0;\infty[)$ so Y and X are independent.

If $f(x,y) = 2I(0 < x < y)I(y \in [0;1[)$ then $\mathbb{P}(x > \frac{1}{2}|y < \frac{1}{2}) = 0$ and $\mathbb{P}(x > \frac{1}{2}) = 1/4$ so X and Y can't be independent.

Exercise 3

a. X and Y are not independent.

b. Let $t \in [0;1]$, $\mathbb{P}(X \leq t) = \int_0^t \int_0^1 f(x,y)dydx = \int_0^t x + \frac{1}{2}dx = \frac{t(1+t)}{2}$

c. $\mathbb{P}(X+Y<1) = \int_0^1 \int_0^{1-y} f(x,y)dxdy = \int_0^1 (1-y)y + \frac{(1-y)^2}{2}dy = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$

Exercise 4

Let $t \in \mathbb{R}+$, $\mathbb{P}(Z \leq t) = \mathbb{P}(X_1 \leq tX_2) = \int_0^\infty \lambda_2 e^{-\lambda_2 y} \int_0^{ty} \lambda_1 e^{-\lambda_1 x} dx dy = \int_0^\infty \lambda_2 e^{-\lambda_2 y}(1 - e^{-\lambda_1 t y}) dy = 1 - \dfrac{\lambda_2}{\lambda_1 t + \lambda_2}$. As a result we have:

$$\boxed{F_Z(t) = \begin{cases} 0 & \text{if } t \in]-\infty; 0] \\ \dfrac{\lambda_1 t}{\lambda_1 t + \lambda_2} & \text{otherwise} \end{cases}}$$

Moreover, $\mathbb{P}(X < Y) = \int_0^\infty \lambda_1 e^{-\lambda_1 x} \int_x^\infty \lambda_2 e^{-\lambda_2 y} dy dx = \int_0^\infty \lambda_1 e^{-(\lambda_1+\lambda_2)x} dx$ so:

$$\boxed{\mathbb{P}(X < Y) = \frac{\lambda_1}{\lambda_1 + \lambda_2}}$$

Exercise 5

Let $(x, y, r, \theta) \in [0; 1]^2 \times \mathbb{R}^2$: $\begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \arctan\left(\dfrac{x}{y}\right) \end{cases} \Leftrightarrow \begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ (r, \theta) \in [0; 1] \times [0; 2\pi] \end{cases}$.

Now let $\forall (x, y) \in [0; 1]^2$, $\varphi(x, y) = \begin{pmatrix} \sqrt{x^2 + y^2} \\ \arctan(\frac{x}{y}) \end{pmatrix}$, $|J_\varphi(x, y)| = \dfrac{1}{r}$ and finally

$$\boxed{f(r, \theta) = \frac{f(x, y)}{|J_\varphi(x, y)|} = \frac{r}{\pi} I(r \in [0; 1], \theta \in [0; 2\pi])}$$

Exercise 6

Let $\varphi : (x, y) \to \begin{pmatrix} xy \\ x/y \end{pmatrix}$ the jacobian of φ is : $J(x, y) = \det \begin{pmatrix} y & x \\ \dfrac{1}{y} & \dfrac{-x}{y^2} \end{pmatrix} = \frac{-2x}{y}$ and the new density is : $f_{U,V}(u, v) = \dfrac{f_{X,Y}(x, y)}{J(x, y)}$ where $(u, v) = \varphi(x, y)$ ie. $(x, y) = \varphi^{-1}(u, v) = (\sqrt{uv}, \sqrt{\frac{u}{v}})$ hence :

$$\boxed{f_{U,V}(u, v) = \frac{1}{2u^2 v} I(u \in [1; \infty[) I\left(v \in \left[\frac{1}{u}; u\right]\right)}$$

We notice that : $u \geq v$ and $u \geq \frac{1}{v}$ thus $f_V(v) = \begin{cases} \int_v^\infty f_{U,V}(u, v) du & \text{if } v > 1 \\ \int_{\frac{1}{v}}^\infty f_{U,V}(u, v) du & \text{if } 1 \geq v > 0 \\ 0 & \text{otherwise} \end{cases}$

That's why :

$$\boxed{f_U(u) = \frac{\log(u)}{u^2} I(u \in [1; \infty[) \text{ and } f_V(v) = \begin{cases} \frac{1}{2v^2} & \text{if } v > 1 \\ \frac{1}{2} & \text{if } 1 \geq v > 0 \\ 0 & \text{otherwise} \end{cases}}$$

Exercise 7

a. We have that $\int_{-\infty}^\infty \int_{-\infty}^\infty f(x, y) dx dy = 1$ so $\int_1^5 \int_0^1 \frac{x}{5} + cy dx dy = 1$
Thus $\int_1^5 \frac{1}{10} + cy \, dy = 1$ so $\dfrac{2}{5} + 12c = 1$ and finally:

$$\boxed{c = \frac{1}{20}}$$

b. If X and Y were independent then we would have $f(x, y) = f(x)f(y)$ which is not the case. So X

and Y are not independent.

c. $\mathbb{P}(X+Y<3)=\mathbb{P}(Y<3-X)=\int_0^1\int_1^{3-x}\frac{x}{5}+\frac{y}{20}dydx=\int_0^1\frac{(3-x)x}{5}+\frac{(3-x)^2}{40}-\frac{1}{40}dx=\frac{3}{10}-\frac{1}{15}+\frac{9}{40}-\frac{3}{40}+\frac{1}{120}-\frac{1}{40}$

$$\boxed{\mathbb{P}(X+Y<3)=\frac{11}{30}}$$