All the answers have to be mathematically justified (unless specified explicitly).

Exercise 1.

An event F is said to carry negative information about an event E, and we write $F \searrow E$, if $\mathbb{P}(E|F) \leq \mathbb{P}(E)$

Prove or give counterexamples to the following assertions:

1. If $F \searrow E$ then $E \searrow F$
2. If $F \searrow E$ and $E \searrow G$ then $F \searrow G$
3. If $F \searrow E$ and $G \searrow E$ then $F \cap G \searrow E$

Exercise 2.

In a factory, two machines M1 and M2 are used together to produce cylindrical parts in series. cylindrical parts in series. For a given period, their probabilities of failure are 0.01 and 0.008 respectively. Moreover, the probability of the event"the machine M2 is down knowing that M1 is down" is is equal to 0.4.

1. Compute the probability that the two machines are down at the same time.
2. What is the probability to have at least one working machine?

Exercise 3.

Rank the following from most likely to least likely to occur:

a. A fair coin lands on heads.

b. Three independent trials, each of which is a success with probability .8, all result in successes.

c. Seven independent trials, each of which is a success with probability .9, all result in successes.

Exercise 4. Suppose that the weather (either wet or dry) tomorrow will be the same as the weather today with probability $p \in (0,1)$.

(a) Show that the weather is dry today, then the probability that it will be dry the next upcoming n days denoted by P_n, satisfies $P_n = (2p-1)P_{n-1} + (1-p)$, and $P_0 = 1$.

(b) By setting $u_n := P_n - \frac{1}{2}$ prove that $u_n = (2p-1)u_{n-1}$ and deduce u_n with respect to n and p.

(c) Deduce P_n from question b. as a function of n and p only.

Exercise 5. There are n distinct types of coupons, and each coupon obtained is, independently of prior types collected, of type i with probability p_i such that $\sum_{i=1}^n p_i = 1$.

a. If n coupons are collected, what is the probability that one of each type is obtained?

b. Now suppose that $p_1 = p_2 = \cdots = p_n = \frac{1}{n}$. Let E_i be the event that there are no type i coupons among the n collected. Apply the inclusion-exclusion identity for the probability of the union of events to $\mathbb{P}(\bigcup_i E_i)$ to prove the identity

$$n! = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^n.$$

Exercise 6.

Let q_n denote the probability that no run of 3 consecutive heads appears in n tosses of a fair coin, i.e. $\mathbb{P}(\text{“head}'') = \frac{1}{2} = \mathbb{P}(\text{“tail}'')$.

1. Show that
$$q_n = \frac{1}{2}q_{n-1} + \frac{1}{4}q_{n-2} + \frac{1}{8}q_{n-3}, \text{ with } q_0 = q_1 = q_2 = 1.$$
Hint : Condition on the first tail.
2. Compute q_{10}.