# INDENG 241 Risk Modeling, Simulation, Data Analysis
# Week 2. Probability measure, independence and conditioning

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# Contents

# Sample space and events

Consider an experiment whose outcome is not predictable with certainty. Although the outcome of the experiment will not be known in advance, let us suppose that the set of all possible outcomes is known.

$\triangleright$ This set of all possible outcomes of an experiment is known as the sample space of the experiment and is denoted by $\Omega$.

Any subset $E$ of the sample space $\Omega$ is known as an event.
In other words, an event is a set consisting of possible outcomes of the experiment.

# Sample space and events: examples

1. If the experiment consists of flipping two coins, then the sample space consists of the following four points:

$$\Omega = \{(H, H), (H, T), (T, H), (T, T)\}$$

The outcome will be $(H, H)$ if both coins are heads, $(H, T)$ if the first coin is heads and the second tails, $(T, H)$ if the first is tails and the second heads, and $(T, T)$ if both coins are tails. An event is for example $(H, H)$ or $(T, H)$ or $E = \{(H, H), (H, T)\}$.

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2. If the experiment consists of measuring (in hours) the lifetime of a transistor, then the sample space consists of all nonnegative real numbers, that is,

$$\Omega = \{x : 0 \leqslant x < \infty\}.$$

   $E = \{x : 0 \leqslant x < 5\}$ is the event "the transistor does not last longer than 5 hours."

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3. If the outcome of an experiment is the order of finish in a race among 7 athletes having post positions 1, 2, 3, 4, 5, 6, and 7, then

$$\Omega = \{\text{ all } 7! \text{ permutations of } (1, 2, 3, 4, 5, 6, 7)\}$$

The outcome $(2, 3, 1, 6, 5, 4, 7)$ means, for instance, that the number 2 comes in first, then the number 3, then the number 1, and so on.

# Sample space and events

Let $E_1, E_2, \ldots$ be events of the sample space $\Omega$

- Union of events: denoted $\bigcup_{i=1}^{\infty} E_i$ and is defined to be all outcomes that are in $E_i$ for at least one value of $i = 1, 2, \ldots;$
- Intersection of the events: denoted $\bigcap_{i=1}^{\infty} E_i$ and is defined to be all outcomes that are in all the events $E_i$ for $i = 1, 2, \ldots;$
- Complement of an event $E$ denoted by $E^c$ consists of all outcomes in the sample space $\Omega$ that are not in $E$.

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**DeMorgan's laws**

$$\left(\bigcup_{i=1}^{\infty} E_i\right)^c = \bigcap_{i=1}^{\infty} E_i^c \text{ and } \left(\bigcap_{i=1}^{\infty} E_i\right)^c = \bigcup_{i=1}^{\infty} E_i^c.$$

# Probability measure

Associate to each event a 'weight' relatively to a sample space $\Omega$.
For example: If $\Omega$ is finite and each one point set is assumed to have equal probability then $\mathbb{P}(E) = \frac{|E|}{|\Omega|} = \frac{\#\text{' outcomes of interest in } E\text{'}}{\#\text{' total outcomes'}}$.

## Axioms of probability

For each event E of $\Omega$, we refer to $\mathbb{P}(E)$ as the probability of the event $E$ satisfying the following three axioms:

1. $0 \leqslant \mathbb{P}(E) \leqslant 1$;
2. $\mathbb{P}(\Omega) = 1$;
3. For any finite sequence of events $E_1, E_2, \ldots, E_n$ ,

$$\mathbb{P}\Big(\bigcup_{i=1}^{n} E_i\Big) = \sum_{i=1}^{n} \mathbb{P}(E_i) - \sum_{i<j} \mathbb{P}(E_i \cap E_j) + \sum_{i<j<k} \mathbb{P}(E_i \cap E_j \cap E_k) + \cdots + (-1)^{n+1}\mathbb{P}(E_1 \cap E_2 \cap \cdots \cap E_n).$$

In particular, if $E_1, \ldots, E_n$ are mutually exclusive (that is $E_i \cap E_j = \varnothing$ for $i \neq j$), then $\mathbb{P}\Big(\bigcup_{i=1}^{n} E_i\Big) = \sum_{i=1}^{n} \mathbb{P}(E_i)$.
No uniqueness of a probability measure!

# Probability measure

## Some additional properties

1. $\mathbb{P}(E^c) = 1 - \mathbb{P}(E)$,
2. If $E \subset F$ then $\mathbb{P}(E) \leqslant \mathbb{P}(F)$.

## Books

You are taking two books along on your holiday vacation. With probability .5, you will like the first book; with probability .4, you will like the second book; and with probability .3, you will like both books. What is the probability that you like neither book?

# Probability measure

### Committee selection

A committee of 5 is to be selected from a group of 6 men and 9 women. If the selection is made randomly, what is the probability that the committee consists of 3 men and 2 women?

### The matching problem

Suppose $N$ gentlemen threw hats into a room and hats are all mixed. Now each one randomly selects one hat.

What is the probability that no a single one gets his own hat?

# Independence

## Definition

Let $E$ and $F$ be two events of $\Omega$. $E$ and $F$ are independent if

$$\mathbb{P}(E \cap F) = \mathbb{P}(E)\mathbb{P}(F).$$

Notation: $E \perp\!\!\!\perp F$.

In particular

$$E \perp\!\!\!\perp F \Longrightarrow \mathbb{P}(E \cap F) = \mathbb{P}(F \cap E).$$

## Complementary and independence

If $E$ and $F$ are independent, then so are $E$ and $F^c$.

# Independence

## Cards

A standard 52-card deck comprises 13 ranks in each of the four French suits: clubs, diamonds, hearts and spades. Each suit includes three court cards (face cards), King, Queen and Jack, with reversible (double-headed) images. Each suit also includes ten numeral cards or pip cards, from one to ten. The card with one pip is known as an Ace.

A card is selected at random from an ordinary deck of 52 playing cards. If $E$ is the event that the selected card is an ace and $F$ is the event that it is a spade. Prove that $E$ and $F$ are independent.

## Flip a coin twice

We consider a fair coin. $E$ is the event 'the first toss is head' $F$ is the event 'the second toss is tail'.
Compute $\mathbb{P}(\text{'first toss is head and the second toss is tail'})$.

# Independence

### Dice roll

Suppose that we toss 2 fair dice. Let $E_1$ denote the event that the sum of the dice is 6 and $F$ denote the event that the first die equals 4. Are $E_1$ and $F$ independent?

Now, suppose that we let $E_2$ be the event that the sum of the dice equals 7. Is $E_2$ independent of $F$?

# Independence v.s. conditioning

Independence makes the calculation simple but it is often unrealistic.

- $F$: historical event of an experiment, of a scientific study, of a financial market...

  relevant to

- $E$: contemporary interest.

Independence is not helpful to learn.

Our interest: event $E$ given the information providing by observing $F$.

$\hookrightarrow$ Conditional probability $\mathbb{P}(E|F)$.

# Conditioning: definition

## Conditional probability

Let $E, F$ be two events in $\Omega$ such that $\mathbb{P}(F) > 0$.

$$\mathbb{P}(E|F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}.$$

Explanation with the uniform probability measure:
$\mathbb{P}(E) = \frac{|E|}{|\Omega|} = \frac{\#\text{ outcome of interest } E}{\#\text{ total outcomes}}$,

$$\mathbb{P}(E|F) = \frac{|E \cap F|}{|F|} = \frac{\frac{|E \cap F|}{|\Omega|}}{\frac{|F|}{|\Omega|}} = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}$$

# $\mathbb{P}(\cdot|F)$ is a probability measure

### Conditional probability as a probability measure

We fix an event $F \subset \Omega$ with $\mathbb{P}(F) > 0$. $\mathbb{P}(E|F)$ satifies the three axioms of a probability measures for any $E \subset \Omega$.

### Failure

Independent trials, each resulting in a success with probability $p$ or a failure with probability $q = 1 - p$, are performed. What is the probability that a run of $n$ consecutive successes occurs before a run of $m$ consecutive failures?

# Conditioning: example 1

## Cats and dogs owners

In a certain community:

- 36 percent of the families own a dog and 22 percent of the families that own a dog also own a cat.
- 30 percent of the families own a cat.

What is

1. the probability that a randomly selected family owns both a dog and a cat?
2. the conditional probability that a randomly selected family owns a dog given that it owns a cat?

# Conditioning: example 2

## Epidemiology and test policy

Go back to example A of Week 1.

- Percentage of the population having the disease: 0.1%.
- Diagnostic accuracy: a test enables health professional to detect it.
    - True-positive results: subjects positive to the detection test who have the disease. 99%.
    - False-positive results (*false alarm*): subjects positive to the detection test who do not have the disease. 1%.

If a person gets a positive test, what is the probability that this person has the disease?

# Bayes' Formula

**Bayes' formula**

Let $E, F$ be two events in $\Omega$.

$$\mathbb{P}(E|F) = \frac{\mathbb{P}(F|E)\mathbb{P}(E)}{\mathbb{P}(F)}.$$

Consequently:

$$\mathbb{P}(E|F) = \frac{\mathbb{P}(F|E)\mathbb{P}(E)}{\mathbb{P}(E \cap F) + \mathbb{P}(E^c \cap F)}$$

and

$$\mathbb{P}(E|F) = \frac{\mathbb{P}(F|E)\mathbb{P}(E)}{\mathbb{P}(F|E)\mathbb{P}(E) + \mathbb{P}(F|E^c)\mathbb{P}(E^c)}$$

# Example A

## Letters of recommendation

A student asked a professor for a letter of recommendation for a new job. The student estimates that there is an 80 percent chance to get the job with a strong recommendation, 40 percent chance with a moderately good recommendation, and a 10 percent chance with a weak recommendation.
The student further estimates that the probabilities that the recommendation will be strong, moderate, and weak are .7, .2, and .1, respectively.

1. What is the probability that the student will receive the new job offer?
2. Given that the student does receive the offer, how likely should this person feel that he/she/they received a strong recommendation?

# Example B

## Elections

In an election candidate $A$ receives $n$ votes candidate $B$ receives $m$ votes with $n > m$. We assume that votes are randomly mixed during the counting.

What is the probability that $A$ is always ahead of $B$ in the vote counting?

# Exercises

Exercises for the discussion session

# Exercise 1

## Electrical network

An electrical parallel system functions whenever at least one of its components works.

Consider a parallel system of $n$ components, and suppose that each component works independently with probability $\frac{1}{2}$.

Find the conditional probability that component 1 works given that the system is functioning.

# Exercise 2

## National League

On the morning of September 30, 1982, the won–lost records of the three leading baseball teams in the Western Division of the National League were as follows:

| Team | Won | Lost |
|---|---|---|
| Atlanta Braves | 87 | 72 |
| San Francisco Giants | 86 | 73 |
| Los Angeles Dodgers | 86 | 73 |

Each team had 3 games remaining. All 3 of the Giants' games were with the Dodgers, and the 3 remaining games of the Braves were against the San Diego Padres. Suppose that the outcomes of all remaining games are independent and each game is equally likely to be won by either participant.

1. For each team, what is the probability that it will win the division title?
2. If two teams tie for first place, they have a playoff game, which each team has an equal chance of winning.