Individual homework. All the answers have to be mathematically justified (unless specified explicitly). For

all these exercise, if X is a random variable that we can generate with a computer, we denote by X_i the ith realization of this variable. Assume that we do $M = 1,000$ simulation of the random variable X denoted by $(X_i)_{1\leq i\leq M}$, we approach the expectation by the empirical mean

$$\mathbb{E}[X] \approx \bar{X} := \frac{1}{M}\sum_{i=1}^{M} X_i.$$

Please use python for simulation. While doing the simulation, you can use the built-in function to generate random variable that follows uniform distribution on [0,1]. Please do NOT use any other built-in function besides the above one.

Exercise 1. Uniform law simulation.

1. Simulate a uniform distribution U on $[0,1]$.
2. By generating 1000 simulation, check that $\bar{U} \approx \mathbb{E}[U] = 1/2$.

Exercise 2. Uniform law on $[0,2\pi]$ simulation.

1. Prove that $V = 2\pi U$ is uniformly distributed on $[0,2\pi]$ where U is uniformly distributed on $[0,1]$ and simulate a uniform distribution on $[0,2\pi]$.
2. Check that $\bar{V} \approx \mathbb{E}[V] = \pi$.

Exercise 3. Exponential distribution.

1. Simulate an exponential distribution $\mathscr{E}$ with parameter $\frac{1}{2}$ by using the inverse transformation method.
2. By generating 1,000 simulation, check that $\bar{\mathscr{E}} \approx \mathbb{E}[\mathscr{E}]$.

Exercise 4. Independent normal random variables.

1. Simulate two independent normal random variable N_1 and N_2 from a joint simulation of V and $\mathscr{E}$.

   Hint: we have seen that $N_1 = \mathscr{E}\cos(V)$ and $N_2 = \mathscr{E}\sin(V)$ are two independent normally distributed random variables with zero mean and unit variance.
2. Check that $\bar{N}_1 \approx E[N_1] = 0$, $\bar{N}_2 \approx E[N_2] = 0$.

Exercise 5. Correlated normal random variables.

1. Use the Cholesky method to generate two normal random variable X and Y with correlation $\rho = 0.5$, respective mean $\mu_X = 1$ and $\mu_Y = 1.5$ and respective variance $\sigma_X^2 = 1$ and $\sigma_Y^2 = 4$.
2. Check that $\mathbb{E}[X] = \mu_X, \mathbb{E}[Y] = \mu_Y$ and

$$\frac{\sum_{i=1}^{1000}(X_i - \mu_X)(Y_i - \mu_Y)}{\sigma_X\sigma_Y} \approx \rho.$$