IEOR 263A Problem Set 5

5A: Write a good exam question based on the material so far, and provide the solution. This is to be uploaded separately from assignment 5B on gradescope, for an extra 10 points.

5B: Read Section 4.7 and Chapter 5. Do Chapter 4 problems 52, 63 (optional), 64, 66. Also:

1. (optional) Shooters A, B, and C are having a truel. The probability that shooter A hits his target is 1/3, the probability that B hits his target is 2/3, and the probability that C hits his target is 1. They take turns, A first, then B, then C, and repeating until only one shooter is left.
   (a) If A decides to shoot at C on his first turn, what is the probability that A wins? (Make reasonable assumptions about the behavior of B and C.)
   (b) If A decides to shoot into the ground on his first turn, what is the probability that A wins?
   (c) If A decides to shoot at C on his first turn, what is the expected number of shots for the entire truel?
2. Using the expression for the gambler's ruin probability, find expressions for f_{ij} for $j > i$, for $j < i$, and for $j = i$.
3. Using your answers to 1 above, find an expression for s_{jj} and s_{ij} for $i > j$, i.e., the number of times the gambler has \$$j$, starting from \$$i$, before she stops playing.
4. (optional) Consider the random walk on the integers with $P_{i,i+1} = p$ and $P_{i,i-1} = q$ (non-negative integers - no upper bound).
   (a) Using the ideas we developed in class (symmetry, skip-free, etc.), show that when $p = q = 1/2$ the random walk is null recurrent, and find a simple expression for ET_{i0} when $i > 0$ and $p < 1/2$.
   (b) Using these same ideas derive an expression for f_{i0} (you should get $f_{i0} = (q/p)^i$ when $p > 1/2$).
5. (optional) Set up a Markov chain to help you determine the number of flips to get the sequence HHH, and another to get the sequence HTH. (Hint: your starting state should be the null state, and the absorption state should be, e.g., HHH). Use your MC to determine the mean number of flips for each. Argue heuristically, based on the transition diagram, that the number of flips to get HHH is stochastically larger than the number to get HTH, THHH $\geq_{st}$THTH.

6. Consider a simple random walk bounded below by 0, where $P_{i,i+1} = p = 1 - P_{i,i-1}$ for $i >= 1$ and $P_{0,1} = p = 1 - P_{0,0}$. Show that the random process $\{X_n | X_0, n \geq 0\}$ is stochastically increasing in X_0, i.e., for any $i < j$, you can couple two random processes on the same probability space so that $[\tilde{X}_n | X_0 = i] \leq [\tilde{X}_n | X_0 = j]$, jointly for all n, with probability 1. Show that the random variable $[X_n | X_0 = 0]$ is stochastically increasing in n, i.e., $[X_n | X_0 = 0] \leq_{st} [X_{n+1} | X_0 = 0]$ for all $n \geq 0$.