IEOR 241 : Homework 5

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October 5, 2022

Exercise 1

(a)

$\mathbb{P}(X=1) = \lim\limits_{t\to 1, t\leq 1}(F(1)-F(t)) = \frac{1}{4}$.
$\mathbb{P}(X=2) = \lim\limits_{t\to 2, t\leq 2}(F(2)-F(t)) = \frac{1}{6}$
$\mathbb{P}(X=3) = \lim\limits_{t\to 3, t\leq 3}(F(3)-F(t)) = \frac{1}{12}$

(b)

$\mathbb{P}(\frac{1}{2} < X < \frac{3}{2}) = F(\frac{3}{2}) - F(\frac{1}{2}) = \frac{5}{8} - \frac{1}{8} = \frac{1}{2}$

Exercise 2

Case $i = 2$:
Let $j \in \mathbb{N}$, A_j : "The j-th match is won by Team A", B_j:"The j-th match is won by Team B". Let S be the number of match played.
The probability that the game ends on the second match is :
$\mathbb{P}(S=2) = \mathbb{P}\left((A_1 \cap A_2) \cup (B_1 \cap B_2)\right) = p^2 + (1-p)^2 = 1 - 2p(1-p)$.
The probability that the game ends on the third match is :
$\mathbb{P}(S=3) = \mathbb{P}\left((A_1B_2A_3 \cup A_1B_2B_3 \cup B_1A_2B_3 \cup B_1A_2A_3)\right) = 2p^2(1-p) + 2p(1-p)^2 = 2p(1-p)$
So :

$$\boxed{\mathbb{E}(S) = 2\mathbb{P}(S=2) + 3\mathbb{P}(S=3) = 2(1-2p(1-p)) + 3(2p(1-p)) = 2 - 4p(1-p) + 6p(1-p) = 2 + 2p(1-p)}$$

Now let $f : x \to 2 + 2x(1-x)$, $f \in \mathcal{C}^1([0;1], \mathbb{R})$ as f is a polynomial function.
$\forall x \in [0;1], f'(x) = 2[(1-p)-p] = 2 - 4p$ as $\forall x \in [0; \frac{1}{2}], f'(x) \geq 0$ it is a maximum that is reached in $\frac{1}{2}$.
That's why $E(X) = f(p)$ is maximum when $p = \frac{1}{2}$ and in this case : $\mathbb{E}(X) = \frac{5}{2}$.

Case $i = 3$:
$\mathbb{P}(S=3) = p^3 + (1-p)^3$
$\mathbb{P}(S=4) = 3p^3(1-p) + 3p(1-p)^3$
$\mathbb{P}(S=5) = 5p^3(1-p)^2 + 5p^2(1-p)^3$

$\mathbb{E}(X) = 3\mathbb{P}(S=3) + 4\mathbb{P}(S=4) + 5\mathbb{P}(S=5) = 3 + 3p + 3p^2 - 12p^3 + 6p^4$

Let $f : x \to 3 + 3p + 3p^2 - 12p^3 + 6p^4 \in \mathcal{C}^2([0;1], \mathbb{R})$ because it is polynomial.
$\forall x \in [0;1] f'(x) = 3 + 6p - 36p^2 + 24p^3$ and $f''(x) = 6 - 72p + 72p^2 = 6(1 - 12p + 12p^2)$.
As A and B are playing symmetric roles in the problem you have :
$\forall x \in [0; \frac{1}{2}], f(x) = f(1-x)$ which implies : $f'(x) = -f'(1-x)$ and $f''(x) = f''(1-x)$. Finding the root of $f''(x^*) = 0$ in $[0; \frac{1}{2}]$ yields $x^* = \frac{12-4\sqrt{6}}{24}$.

As $\forall x \in [0; x^*], f''(x) \geq 0$ and $\forall x \in [x^*, \frac{1}{2}], f''(x) \leq 0$ it means $\forall x \in [0; \frac{1}{2}], f'(x) \geq 0$ and $f'(1-x) \leq 0$ and finally f is maximized on $x = \frac{1}{2}$.

As $\mathbb{E}(X) = f(p)$ then $\mathbb{E}(X)$ is maximized for $p = \frac{1}{2}$.

Exercise 3

Let $i \in \mathbb{N}, X_i = 1$ if the i-th person sits at an unoccupied table and 0 instead. Let also Y the number of table.
You can notice that : $\mathbb{P}(X_i = 1) = (1-p)^{i-1}$ and $Y = \sum_{i=1}^{N} X_i$

$$\boxed{\mathbb{E}(Y) = \sum_{i=1}^{N} \mathbb{E}(X_i) = \sum_{i=1}^{N} p^{i-1} = \frac{1-(1-p)^N}{p}}$$

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Exercise 4

In this case suppose $\forall i \in \mathbb{N}$, i-th person is aged i. Let X_i be 1 if the i-th person finds his hat and 0 otherwise. $\forall i \in [\![1; 1000]\!], \mathbb{P}(X_i = 1) = \frac{1}{1000}$ and $\mathbb{E}(X_i) = \frac{1}{1000}$.
Let $S = \sum_{i=1}^{1000} X_i$ be the number of people finding their hat.

$$\boxed{\mathbb{E}(S) = \sum_{i=1}^{1000} \mathbb{E}(X_i) = 1000\frac{1}{1000} = 1}$$

Exercise 5

Let M be the number of matched pairs and let $\forall (i,j) \in [\![1; N]\!], I_{ij}$ be the indicator of the i-th person finds hat j and j-th person finds hat i. $\mathbb{P}(I_{ij} = 1)\frac{1}{N(N-1)}$ and there are $\binom{N}{2}$ possible pairs so

$$\boxed{\mathbb{E}(M) = \frac{\binom{N}{2}}{N(N-1)} = \frac{1}{2}}$$

Exercise 6

(a)

$\mathbb{E}((2+X)^2) = 8 + \mathbb{E}(X^2)$. With $\mathbb{V}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \mathbb{E}(X^2) - 1$ and $\mathbb{V}(X) = 5$ you have $\mathbb{E}(X^2) = 6$ and

$$\boxed{\mathbb{E}((2+X)^2) = 14}$$

(b)

$\mathbb{V}(4+3X) = \mathbb{V}(3X) = 9\mathbb{V}(X) = 45$ so

$$\boxed{\mathbb{V}(4+3X) = 45}$$

Exercise 7

$$\boxed{\mathbb{E}((X-Y)^2) = \mathbb{E}(X^2 - 2XY + Y^2) = 2(\sigma^2 + \mu^2) - 2\mu^2}$$

Exercise 8

Let $i \in \mathbb{N}, X_i = \{ \begin{array}{cc} 1 & \text{if } i\text{-th roll is a 1} \\ 0 & \text{otherwise} \end{array}$ and $Y_i = \{ \begin{array}{cc} 1 & \text{if } i\text{-th roll is a 2} \\ 0 & \text{otherwise} \end{array}$

Notice that $X = \sum_{i=1}^{n} X_i$ and $Y = \sum_{i=1}^{n} Y_i$ then $\text{Cov}(X, Y) = \text{Cov}(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} Y_i) = \sum_{(i,j) \in [\![1;n]\!]^2} \text{Cov}(X_i, Y_j)$.

Now notice that $\forall i, j \in \mathbb{N}^2, i \neq j \implies X_i$ indep with Y_i. Thus $\forall i, j \in \mathbb{N}^2, i \neq j, \text{Cov}(X_i, Y_j) = 0$

Thus $\text{Cov}(X, Y) = \sum_{i=1}^{n} \text{Cov}(X_i, Y_i)$. Moreover $\text{Cov}(X_i, Y_i) = \mathbb{E}(X_i Y_i) - \mathbb{E}(X_i)\mathbb{E}(Y_i) = -(\frac{1}{6})^2$

Finally :

$$\boxed{\text{Cov}(X, Y) = -\frac{n}{36}}$$