IEOR 241 : Homework 10

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Exercise 1

1. To verify if f is a density we just need to check that $\int_0^\infty f(t)dt = 1$:

$$\boxed{\int_0^\infty f(x)dx = \int_0^\infty \frac{x}{a} e^{-\frac{x^2}{2a}} dx = \left[-e^{-\frac{x^2}{2a}}\right]_0^\infty = 1}$$

2. Define $\forall i \in [\![1;8]\!], h_i$ to be the height of year i. The likelihood of h is : $\mathcal{L}(a) = \prod_{i=1}^{8} f(h_i)$. We will try to find $\hat{a} = \arg\max(\mathcal{L}(a))$.

$$\begin{aligned}
\hat{a} &= \arg\max(\mathcal{L}(a)) \\
&= \arg\max(\log(\mathcal{L}(a))) \\
&= \arg\max(\sum_{i=1}^{8} \log(h_i) - \frac{h_i^2}{2a} - \log(a)) \\
&= \arg\max\left(\left[\sum_{i=1}^{8} \log(h_i)\right] - \frac{\sum_{i=1}^{8} h_i^2}{2a} - 8\log(a)\right)
\end{aligned}$$

As $\hat{a}$ is a maximum it implies that $\frac{\sum_{i=1}^{8} h_i^2}{2\hat{a}^2} - \frac{8}{\hat{a}} = 0$ thus

$$\boxed{\hat{a} = \frac{\sum_{i=1}^{8} h_i^2}{16} = 2.42}$$

3. Let $t \in \mathbb{R}+$, $\mathbb{P}(H \leq t) = \int_0^t \frac{x}{a} e^{-\frac{x^2}{2a}} dx = 1 - e^{\frac{-t^2}{2a}}$. Hence :

$$\boxed{\forall t \in \mathbb{R}, F(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 - e^{\frac{-t^2}{2\hat{a}}} & \text{otherwise} \end{cases}}$$

4. The probability of a disaster is the probability of the event that $H > 6$.

$$\boxed{\mathbb{P}(H > 6) = 1 - F(6) = e^{\frac{-6^2}{2\hat{a}}=0.0006}}$$

5. Let N be the number of flood in 100 years. Under the assumption that all years are independent $N \sim \mathcal{B}(100, \mathbb{P}(H > 6))$. We have that

$$\boxed{\mathbb{P}(N = 0) = F(6)^{100} = 0.94}$$

Exercise 2

1. Let $t \in \mathbb{R}$. If $t < 0$ then $\mathbb{P}(S \leq t) = 0$ as exp takes only positive values.
If $t \geq 0$ then $\mathbb{P}(S \leq t) = \mathbb{P}(X \leq ln(t)) = \int_{-\infty}^{\log(t)} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$. Diffenrentiating both sides with respect to t we have :

$$\boxed{f_S(t) = \begin{cases} \frac{1}{t\sigma\sqrt{2\pi}} e^{-\frac{(\log(t)-\mu)^2}{2\sigma^2}} & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}}$$

2.

$$\boxed{\mathbb{E}(S) = \int_0^\infty \frac{1}{\sigma\sqrt{2\pi}} e^{t - \frac{(t-\mu)^2}{2\sigma^2}} dt = e^{\frac{\sigma^2}{2}(1+\frac{1}{\sigma^2})^2 - \frac{\mu^2}{2\sigma^2}}}$$

3. The means should match :

Exercise 3

Part A

1. Let $\forall i \in [\![1; N]\!]$, $p_i : t \mapsto p_i(t)$ be the price of the asset i as a function of time.
We have that $\pi_i = \frac{p_i(0)}{\sum_{i=1}^{N} p_i(0)} = \frac{p_i(0)}{X(0)}$ and $X(T) = \sum_{i=1}^{N} p_i(T)$.

Thus

$$\boxed{\frac{X(T) - X(0)}{X(0)} = \frac{\sum_{i=1}^N p_i(T) - p_i(0)}{X(0)} = \frac{\sum_{i=1}^N p_i(0)(1+R_i) - p_i(0)}{X(0)} = \sum_{i=1}^{N} \pi_i R_i}$$

2. Let $R = \frac{X(T)-X(0)}{X(0)} = \sum_{i=1}^{N} \pi_i R_i$ then $\mathbb{V}(R) \left(\sum_{i=1}^{N} \pi_i R_i \right) = \sum_{i=1}^{N} \pi^2 \mathbb{V}(R_i) = \sum_{i=1}^{N} \pi_i^2 \sigma^2$. Hence :

$$\boxed{\mathbb{V}(R) = \sigma^2 \sum_{i=1}^{N} \pi_i^2}$$

3. We use the Lagrange Multipliers Method : define $\forall \lambda, \pi,\ L(\lambda, \pi) = \sigma^2 \sum_{i=1}^N \pi_i^2 + \lambda \left(1 - \sum_{i=1}^{N} \pi_i\right)$ to minimize L is equivalent to minimizing $\mathbb{V}(R)$. Diffenrentiating with respect to π we have : $\nabla_\pi L(\lambda, \pi) = \begin{pmatrix} 2\sigma^2\pi_1 - \lambda \\ 2\sigma^2\pi_2 - \lambda \\ \ldots \\ 2\sigma^2\pi_N - \lambda \end{pmatrix}$. Let π^* be the optimal distribution, the optimality condition yields $\nabla_\pi L(\lambda, \pi^*) = 0$ thus

$$\boxed{\pi_1^* = \pi_2^* = \ldots = \pi_N^* = \frac{1}{N}}$$

4. $\mathbb{V}(R) = \sigma^2 \sum_{i=1}^{N} (\frac{1}{N})^2 = \frac{\sigma^2}{N}$ hence

$$\boxed{\mathbb{V}(R) \to_{N\to\infty} 0}$$

Part B

5.

$$\boxed{\begin{aligned} \mathbb{V}(R) = \mathrm{Cov}(R, R) = \mathrm{Cov}(\sum_{i=1}^{N} \pi_i R_i, \sum_{j=1}^{N} \pi_j R_j) = \sum_{i=1}^{N} \pi_i^2 \mathrm{Cov}(R_i, R_i) + \sum_{i \neq j} \pi_i \pi_j \mathrm{Cov}(R_i, R_j) \\ = \sigma^2 \sum_{i=1}^{N} \pi_i^2 + \rho\sigma^2 \sum_{i \neq j} \pi_i \pi_j \end{aligned}}$$

6. Same as question **3.** we use Lagrange Multipliers Method.

Define $L(\lambda,\pi) = \sigma^2 \sum_{i=1}^{N} \pi_i^2 + \rho\sigma^2 \sum_{i\neq j} \pi_i\pi_j + \lambda\left(1 - \sum_{i=1}^{N} \pi_i\right)$ we have $\nabla_\pi L(\lambda,\pi) = \begin{pmatrix} 2\sigma^2\pi_1 + 2\sigma^2\rho \sum_{1\neq j} \pi_j - \lambda \\ 2\sigma^2\pi_2 + 2\sigma^2\rho \sum_{2\neq j} \pi_j - \lambda \\ \ldots \\ 2\sigma^2\pi_N + 2\sigma^2\rho \sum_{N\neq j} \pi_j - \lambda \end{pmatrix}$

We can notice that for each term i fixed, $2\sigma^2\rho \sum_{i\neq j} \pi_j = 2\sigma^2\rho \sum_{i\neq j} \pi_j + 2\sigma^2\rho\pi_i - 2\sigma^2\rho\pi_i = 1 - 2\sigma^2\rho\pi_i$. Hence we have the same solution :

$$\boxed{\pi_1^* = \pi_2^* = \ldots = \pi_N^* = \frac{1}{N}}$$

7. $\mathbb{V}(R) = \sigma^2 \sum_{i=1}^{N} \pi_i^2 + \rho\sigma^2 \sum_{i\neq j} \pi_i\pi_j = \frac{\sigma^2}{N} + \sigma^2\rho\frac{N-1}{N}$

$$\boxed{\mathbb{V}(R) \to_{N\to\infty} \sigma^2\rho}$$

Part C

8. $\mathbb{V}(R) = \pi_1^2 + 2\pi_2^2 + 4\pi_1\pi_2 = \pi_1^2 + 2(1-\pi_1)^2 + 4\pi_1(1-\pi_1) = \pi_1^2 + 2 - 4\pi_1 + 2\pi_1^2 + 4\pi_1 - 4\pi_1^2 = 2 - \pi_1^2$ which is a decreasing function in π_1 as $\pi_1 > 0$ hence

$$\boxed{\text{The optimal portfolio minimizing the risk is : } \pi_1 = 1, \pi_2 = 0}$$