INDENG241 : Homework 4

Arnaud Minondo

September 28, 2022

Problem : Convergence of the Binomial to Poisson

Let $n \in \mathbb{N}, \lambda \in \mathbb{R}, X_n \sim \mathcal{B}(n, \frac{\lambda}{n}), X \sim \mathcal{P}(\lambda)$

1.

Show that : $\forall k \in \mathbb{N}, \lim_{n\to\infty} \mathbb{P}(X_n = k) = \mathbb{P}(X = k)$.

2.

Show that : $\forall k \in \mathbb{N}, |P(X_n = k) - P(X = k)| \sim_{n\to\infty} \mathbb{P}(X = k)\frac{|2k\lambda - 2k(k-1) - \lambda^2|}{2n}$

3.

A fisherman fishes in a lake where there are two kinds of fishes : kind A and B. There are 10000 fishes among which are 1000 A and 9000 B. The fisherman releases all its catches just after having fished it and fishes 100 fishes.
Let X be the random variable counting the number of fish of kind A fished. Give $\mathbb{P}(X = 10)$

4.

Give an appraoched value using 1. and comment on the difference using 2.

Solution

1.

Let $k \in \mathbb{N}, \mathbb{P}(X_n = k) = \binom{n}{k}(\frac{\lambda}{n})^k(1-\frac{\lambda}{n})^{n-k} = \frac{\lambda^k}{k!}\left[\frac{n!}{(n-k)!n^k}(1-\frac{\lambda}{n})^{n-k}\right]$.
Moreover : $\frac{n!}{n^k(n-k)!} = \prod_{i=0}^{k-1}\frac{n-i}{n} \to_{n\to\infty} 1$
And $(1-\frac{\lambda}{n})^{n-k} = \exp((n-k)\log(1-\frac{\lambda}{n})) =_{n\to\infty} \exp(-\lambda)\exp(\frac{k}{n}) \to_{n\to\infty} \exp(-\lambda)$
Finally : $\forall k \in \mathbb{N}, \lim_{n\to\infty} \mathbb{P}(X_n = k) = \exp(-\lambda)\frac{\lambda^k}{k!} = \mathbb{P}(X = k)$

2.

After 1. we can write : $|\mathbb{P}(X_n = k) - \mathbb{P}(X = k)| = \exp(-\lambda)\frac{\lambda^k}{k!}\left|\left(\prod_{i=0}^{k-1}\frac{n-i}{n}\right)(1-\frac{\lambda}{n})^{n-k}\exp(\lambda) - 1\right|$
With : $\prod_{i=0}^{k-1}\frac{n-i}{n} = \prod_{i=0}^{k-1}(1-\frac{i}{n}) =_{n\to\infty} 1 - \frac{\sum_{i=0}^{k-1} i}{n} + o(\frac{1}{n}) =_{n\to\infty} 1 - \frac{k(k-1)}{n} + o(\frac{1}{n})$
And : $\left(1-\frac{\lambda}{n}\right)^{n-k}\exp(\lambda) = \exp((n-k)\log(1-\frac{\lambda}{n}) + 1) =_{n\to\infty} \exp(1 + (n-k)(-\frac{\lambda}{n} - \frac{\lambda^2}{2n^2} + o(\frac{1}{n^2})))$
$= \exp(\frac{2\lambda k - \lambda^2}{2n}) + o(\frac{1}{n}) = (1 + \frac{2k\lambda - \lambda^2}{2n}) + o(\frac{1}{n})$
Finally : $|\left(\prod_{i=0}^{k-1}\frac{n-i}{n}\right)\left(1-\frac{\lambda}{n}\right)^{n-k}\exp(\lambda) - 1| = |\left(1 + \frac{2k\lambda-\lambda^2}{2n}\right)\left(1 - \frac{k(k-1)}{n}\right) - 1| + o(\frac{1}{n})$
$= \frac{|2k\lambda - 2k(k-1) - \lambda^2|}{2n} + o(\frac{1}{n}) \sim \frac{|2k\lambda - 2k(k-1) - \lambda^2|}{2n}$
And :

$$\boxed{|\mathbb{P}(X_n = k) - \mathbb{P}(X = k)| \sim_{n\to\infty} \mathbb{P}(X = k)\frac{|2k\lambda - 2k(k-1) - \lambda^2|}{2n}}$$

3.

$\mathbb{P}(X = 10) = \binom{100}{10} \left(\frac{1}{10}\right)^{10} \left(\frac{9}{10}\right)^{90} = 0.131$

4.

As $\frac{\lambda}{100} = \frac{1}{10}$ then $\lambda = 10$ and let a Poisson random variable $Y \sim \mathcal{P}(10)$ then $\mathbb{P}(Y) = \exp(-10)\frac{10^1 0}{10!} = 0.125$.
After 2. the error between the two probabilities would be : $0.125 * 0.1 = 0.01$
We notice that $0.131 \in [0.125 - 0.1, 0.125 + 0.1]$