Homework 3

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Exercise 1

1.1

Suppose $E \searrow F$ ie. $\mathbb{P}(F|E) \leq \mathbb{P}(F)$ then $\frac{\mathbb{P}(F\cap E)}{\mathbb{P}(E)} \leq \mathbb{P}(F)$ so $\frac{\mathbb{P}(F\cap E)}{\mathbb{P}(F)} \leq \mathbb{P}(E)$ and $\mathbb{P}(E|F) \leq \mathbb{P}(E)$ so $F \searrow E$.

1.2

This proposition is false : Consider rolling a fair dice and observing the result, E :"Obtain an odd number", F:"Obtain a 6", G:"Obtain an even number".
$\mathbb{P}(E|F) = 0 \leq \mathbb{P}(E)$ so $F \searrow E$. Moreover $\mathbb{P}(G|E) = 0 \leq \mathbb{P}(G)$ so $E \searrow G$. But $\mathbb{P}(G|F) = 1$ and $\mathbb{P}(G) = \frac{1}{2}$, the proposition is false.

1.3

This proposition is also false : Consider rolling a fair dice and observing the result. Define E:"the dice fall on either 1 or 6", F:"the dice fall on 1 or 2 or 3" and G:"the dice fall on 1 or 4 or 5".
$\mathbb{P}(E) = \frac{2}{6} = \frac{1}{3}$, $\mathbb{P}(E|G) = \frac{1}{3}$ so $G \searrow E$. Moreover, $\mathbb{P}(E|F) = \frac{1}{3}$ so $F \searrow E$.
But $\mathbb{P}(E|F \cap G) = 1 \geq \mathbb{P}(E)$ so the proposition is false.

Exercise 2

2.1

Define two events : A:"machine M1 does not work" and B:"machine M2 does not work".
The probability that no machine work is : $\mathbb{P}(A \cap B) = \mathbb{P}(B|A)\mathbb{P}(A) = 0.4 * 0.01 = 0.004$.

2.2

The probability that at least one machine work is : $\mathbb{P}(\overline{A} \cup \overline{B}) = 1 - \mathbb{P}(A \cap B) = 0.996$.

Exercise 3

Event A happens with probability $\frac{1}{2}$. Event B happens with probability $(\frac{4}{5})^3 = 0.512$. Finally event C happens with probability $(\frac{9}{10})^7 = 0.48$.
B is the most likely to happen. The second most likely to happen is A and the less likely to happen is C.

Exercise 4

4.1

Define the event : D_n as "the day n is dry", $P_n = \mathbb{P}(D_n)$.
Then $\forall n \in \mathbb{N}, D_n \cup \overline{D_n} = \Omega$ with a disjoint union.
Thus $P_n = \mathbb{P}(D_n|D_{n-1})\mathbb{P}(D_{n-1}) + \mathbb{P}(D_n|\overline{D_{n-1}})\mathbb{P}(\overline{D_{n-1}}) = pP_{n-1} + (1-p)(1-P_{n-1}) = (2p-1)P_{n-1} + (1-p)$

4.2

$u_n = P_n - \frac{1}{2} = (2p-1)P_{n-1} + (1-p) - \frac{1}{2} = (2p-1)(P_{n-1} - \frac{1}{2} + \frac{1}{2}) + \frac{1}{2} - p$
$= (2p-1)u_{n-1} + \frac{1}{2}(2p-1) + \frac{1}{2} - p = (2p-1)u_{n-1}$

4.3

As $u_n = (2p-1)u_{n-1}$, u is a geometric sequence so $u_n = (2p-1)^n u_0 = \frac{1}{2}(2p-1)^n$
So $P_n = u_n + \frac{1}{2} = \frac{1}{2}((2p-1)^n + 1)$

Exercise 5

5.1

Define the Event A:"a coupon of each type is chosen". As a coupon of each type is chosen and there is as much coupon chosen than different types of coupon, there exist a bijection between the index of the coupon and the type of the coupon. It means that there is $n!$ possibilities for chosing a coupon of each type.
As there each choose has a probability of $\Pi_{i=1}^n p_i$:

$$\boxed{\mathbb{P}(A) = n!(\Pi_{i=1}^n p_i)}$$

5.2

Define E_i : "no coupons of type i is chosen". The Event A : "a coupon of each type is chosen"$= \cap_{i=1}^n \overline{E_i}$.
So : $\mathbb{P}(\cup_{i=1}^n E_i) = 1 - \mathbb{P}(\cap_{i=1}^n \overline{E_i}) = 1 - n!(\Pi_{i=1}^n p_i) = \frac{n^n - n!}{n^n}$ with $p_i = \frac{1}{n}$

Moreover : $\mathbb{P}(\cup_{i=1}^n E_i) = \sum_{k=1}^n (-1)^{k+1} \sum_{i_1 < i_2 < ... < i_k} \mathbb{P}(\cap_{r=1}^k E_{i_r})$ after the inclusion exclusion principle.

$\forall k \in \{1,2,...,n\}, \forall (i_1, i_2, ..., i_k) \in \{1,2,...,n\}^k | i_1 < i_2 < ... < i_k, \mathbb{P}(\cap_{r=1}^k E_{i_r}) = P_{n,k}$ because each coupon can be exchanged with another one, the index type of the coupon does not matter as long as you don't pick k coupons in the n available. So $\forall n \in \mathbb{N}, \forall k \in \{1,2,...,n\}, P_{n,k} = \frac{(n-k)^n}{n^n}$

And : $\forall k \in \{1,2,...,n\}, \forall (i_1, i_2, ..., i_k) \in \{1,2,...,n\}^k, \sum_{i_1 < i_2 < ... < i_k} 1 = \binom{n}{k}$ as each choice of k integer in $\{1,2,...,n\}$ is a good choice for $(i_1, i_2, ..., i_k)$ assigning the lowest to i_1, the second lower to i_2 and so on till i_k which is the biggest of the selection.

With those two results : $\mathbb{P}(\cup_{i=1}^n E_i) = \sum_{k=1}^n (-1)^{k+1} \sum_{i_1 < i_2 < ... < i_k} \mathbb{P}(\cap_{r=1}^k E_{i_r}) = \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} \frac{(n-k)^n}{n^n}$

We have obtained that : $\frac{n^n - n!}{n^n} = \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} \frac{(n-k)^n}{n^n}$. We can simplify multiplying both sides by n^n and rearranging each term on the good side we obtain : $n! = n^n - \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} (n-k)^n = n^n + \sum_{k=1}^n (-1)^k \binom{n}{k} (n-k)^n$ noticing that n^n is the term for $k = 0$ of the sum :

$$\boxed{n! = \sum_{k=0}^n (-1)^k \binom{n}{k} (n-k)^n}$$

Exercise 6

6.1

Define H_n : "No consecutive 3 heads appear in n tosses of a faire coin" ; and t_n :"the toss n result in head".
We can decompose : $H_n = (H_{n-3} \cap (\overline{t_{n-2}}, t_{n-1}, t_n)) \cup (H_{n-2} \cap (\overline{t_{n-1}}, t_n)) \cup (H_{n-1} \cap \overline{t_n})$ Which is a disjoint union.
Moreover $\forall i \in \mathbb{N}, \forall j \in \mathbb{N}, j > i : H_i$ is independent of t_j.
So : $q_n = \mathbb{P}(H_n) = \mathbb{P}(H_{n-3})\mathbb{P}(\overline{t_{n-2}}, t_{n-1}, t_n) + \mathbb{P}(H_{n-2})\mathbb{P}(\overline{t_{n-1}}, t_n) + \mathbb{P}(H_{n-1})\mathbb{P}(\overline{t_n})$
We obtain the final formula :

$$q_n = \frac{1}{8}q_{n-3} + \frac{1}{4}q_{n-2} + \frac{1}{2}q_{n-1}$$

6.2

$q_{10} = 0.49$