IEOR240 : Homework 2

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Problem 1 : True or False

For the following problem I considered the following linear optimization problem :

$$\min(c^T x), \text{ s.t. } x \in \mathbb{R}^n, A \in \mathcal{M}_{m,n}(\mathbb{R}), b \in \mathbb{R}^m, Ax \geq b, x \geq 0_{\mathbb{R}^n} \tag{1}$$

1.a

As the problem in symmetrical is equivalent to the original optimization problem then if there exist an unboundedness certificate on the symmetrical form then the problem is unbounded.

1.b

If a problem has an optimal solution then its dual also has and let y be the solution of the dual then y is a certificate of boundedness for the primal problem.

1.c

To know whether a problem is feasable or not you need to find an x verifying all the conditions, ie. $x \geq 0$, $Ax \geq b$
For example : $\max(x_1 + x_2)$, s.t. $x_1, x_2 \geq 0, x_1 + x_2 <= 4$ is feasable because $x = (0, 0)$ is a certificate of feasability for this problem.

1.d

This is true after the strong duality theorem, as $\mathcal{P}$ is feasable and bounded then $\mathcal{D}$ is feasable and bounded and both have optimal solutions.

1.e

This is not true, if the maximizing/minimizing direction goes in the direction of a border then the problem can be bounded.
For example : $\min x_1 + x_2, x_1 \geq 0, x_2 \geq 0$, the feasable region : $S = (\mathbb{R}^+)^2$ is unbounded but there is an optimal solution which is $x_1 = 0, x_2 = 0$.

1.f

This is true, as the set containing all feasable solution : $S = \{x \in \mathbb{R}^n, Ax \geq b\}$ is convex then if there are two optimal solution : x_0, x_1 then $\forall \lambda \in [0, 1], \lambda x_0 + (1 - \lambda)x_1 = \bar{x} \in S$ so is a feasable solution.
Moreover : $c^T x_1 = c^T x_0$ so $c^T(x_1 - x_0) = 0$ and $c^T \bar{x} = \lambda c^T x_0 + (1 - \lambda) c^T x_1 = c^T x_1$ so $\bar{x}$ is also optimal.

1.g

This is false for example : $\max(x_1 + x_2)$, s.t. $x_1, x_2 \geq 0, x_1 + x_2 <= 4$ is bounded because $y = (1)$ is a certificate of boundedness for this problem.

1.h

This is false because to be unbounded means that the objective function can be as big/small in case of max/min. But there are no any x that can be plugged into the objective function so the function can't be as big/small as you wish. Althought there might exist a certificate for both of boundedness and feasability.

1.i

It is false because of the Theorem "either bounded or unbounded".

1.j

This is true because : a max can be changed into a min, a $\leq$ can be changed by multiplying by -1, an $=$ can be changed into two inequalities, if a variable x_i is free you can say $x_i = x_i^+ - x_i^-$ with $x_i^+, x_i^- \geq 0$.

Problem 2 : Linear Program

2.a

The problem in symmetric form is :

$$\begin{gathered} \min(c^T x) \\ \text{s.t. } Ax \geq b, x \geq 0 \end{gathered} \tag{2}$$

Where $A = \begin{pmatrix} -1 & 1 & -1 & 0 & 0 \\ 0 & -2 & 2 & 3 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} -4 \\ 2 \\ 5 \end{pmatrix}$ and $c = \begin{pmatrix} 2 \\ -3 \\ 3 \\ 5 \\ 6 \end{pmatrix}$

2.b

Its dual is :

$$\begin{gathered} \max(b^T y) \\ \text{s.t. } y^T A \leq c^T \end{gathered} \tag{3}$$

2.c

My model file : pb2.mod

```
param n;
param p;

param c{i in 1..n};
param b{i in 1..p};

param A{j in 1..p, i in 1..n};

var u{i in 1..n};

minimize object: sum{i in 1..n}c[i]*u[i];

s.t. constraints1{j in 1..p}: sum{i in 1..n}A[j,i]*u[i]>=b[j];
s.t. constraints2{i in 1..n}: u[i]>=0;
```

My data file : pb2.dat

```
param n := 5;
param p := 3;

param b :=
1 -4
2 2
3 5;

param A :
   1  2  3  4  5=
1 -1  1 -1  0  0
2  0 -2  2  3  0
3  1  0  0  0  1;

param c:=
1 2
2 -3
3 3
4 5
5 6;
```

My run file : pb2.run

```
reset;

model pb2.mod;

data pb2.dat;

option solver cplex;
solve;

display object, u;
```

The solution for this problem is : $x_1 = 5, x_2 = 1, x_3 = 1.333, x_4 = 0$ with the min $= 13,6667$
The dual problem solution is : $y = (\frac{1}{3}, \frac{5}{3}, \frac{7}{3})$
The data file is the same. The model file : pb2_dual.mod

```
param p;
param n;

param A{i in 1..p,j in 1..n};

param b{i in 1..p};

param c{j in 1..n};

var y{i in 1..p};


maximize object: sum{i in 1..p}y[i]*b[i];

s.t. constraints{j in 1..n} :sum{i in 1..p}y[i]*A[i,j] <=c[j];
```

The run file : pb2_dual.run

```
reset;

model pb2_dual.mod;
data pb2_dual.dat;

option solver cplex;

solve;

display object, y;
```

Problem 3 : General Problem

(a)

$(\mathcal{P})$ is feasable because $x = 0_{\mathbb{R}^n}$ is a feasable solution. As $(\mathcal{P})$ is feasable its dual can't be unbounded, it is either unfeasable if $(\mathcal{P})$ is unbounded and is feasable in the other case.

(b)

$(\mathcal{D})$ objective function is : $y^T 0_{\mathbb{R}^m}$ so it's objective value will be 0. Moreover, $x = 0_{\mathbb{R}^n}$ verifies $c^T x = y^T 0_{\mathbb{R}^m}$ which means x is optimal. The objective value function is then 0.

Problem 4 :

The problem is feasable as $x = 0_{\mathbb{R}^n}$ is a feasable solution. Moreover it is bounded as $c \geq 0$ and $x \geq 0$ so $c^T x \geq 0$.
Thus $\min(c^T x) \geq 0$ and the problem is bounded. As it is feasable and bounded it must have an optimal solution.