INDENG263A : Homework 4

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September 29, 2022

Problem 52 : Taxi Driver

Let π be the stationary distribution of the location of the taxi.

π verifies : $\begin{matrix} \pi_0 = 0.6\pi_0 + 0.3\pi_1 \\ \pi_1 = 0.4\pi_0 + 0.7\pi_1 \end{matrix} \implies \pi_0 = \frac{3}{4}\pi_1$, with $\pi_0 + \pi_1 = 1$ we obtain : $\pi_0 = \frac{3}{7}, \pi_1 = \frac{4}{7}$

Let G be the profit of the taxi driver in one trip : $\mathbb{E}(G) = \pi_0(0.6 * 6 + 0.4 * 12) + \pi_1(0.3 * 12 + 0.7 * 8) = 8.86$

Problem 64 : Branching process, Total number of individuals

Branching process : $\forall n \in \mathbb{N} : Z_n \sim (P_0, P_1, P_2, ...)$, $\forall k \in \mathbb{N}$, $\mathbb{P}(Z_n = k) = P_k$, $X_{n+1} = \sum_{i=1}^{X_n} Z_i$
The total number of individuals that ever existed is : $N = \sum_{i=0}^{\infty} X_i$
$\mathbb{E}(N|X_0 = 1) = \sum_{i=0}^{\infty} \mathbb{E}(X_i|X_0 = 1)$
Moreover : $\forall i \in \mathbb{N}$, $\mathbb{E}(X_i|X_0 = 1) = \mathbb{E}(\mathbb{E}(X_i|X_{i-1}, X_0 = 1)) = \mathbb{E}(\mu X_{i-1}) = \mu\mathbb{E}(X_{i-1}) = \mu^i\mathbb{E}(X_0) = \mu^i$
So :

$$\boxed{\mathbb{E}(N|X_0 = 1) = \sum_{i=0}^{\infty} \mu^i = \frac{1}{1-\mu}}$$

If $X_0 = n$: $\mathbb{E}(X_i|X_0 = n) = \mu^i n$ so :

$$\boxed{\mathbb{E}(N|X_0 = n) = \frac{n}{1-\mu}}$$

Problem 66 : Branching process, π_0 derivation

In the branching process :

- if $\mu > 1$ then $\pi_0 < 1$ and $\pi_0 = \sum_{i=0}^{\infty} \pi_0^i P_i$.
- if $\mu = 1$ and $P_0 > 0$ then $\pi_0 = 1$
- if $\mu < 1$ then $\pi_0 = 1$

(a)

$\mu = \frac{3}{2}$ so $\pi_0 < 1$ and $\pi_0 = \frac{1}{4} + \frac{3}{4}\pi_0^2$ finally in the two possible solutions of this equation only one can be the value as $\pi_0 < 1$.

$$\boxed{\pi_0 = \frac{2}{3}}$$

(b)

$\mu = 1$ so

$$\boxed{\pi_0 = 1}$$

(c)

$\mu = \frac{3}{2}$, $\pi_0 = \frac{1}{6} + frac12\pi_0 + \frac{1}{3}\pi_0^3$ so $2\pi_0^3 - 3\pi_0 + 1 = 0$.
A trivial solution of this equation is $\pi_0 = 1$ which is impossible because $\mu > 1$. We can rewrite the equation as : $(\pi_0 - 1)(2\pi_0^2 + 2\pi_0 - 1) = 0$ and conclude that $2\pi_0^2 + 2\pi_0 - 1 = 0$. Finally :

$$\boxed{\pi_0 = \frac{\sqrt{3} - 1}{2}}$$

Problem Set

Question 1

(a)

Let A_w :"A wins the truel" I assumed that all the players know all of the strenght of all other players. So that A and B are first shooting on C and C is shooting on B first. There are three cases :

- A hits C and B v A, B starting
- A fails, B hits C and A v B, A starting
- A fails, B fails and A v C, A starting

In the case where A and B are in duel with B starting :
$\mathbb{P}(A_w|\text{A v B, B starts}) = 1 - \frac{2}{3}\sum_{i=0}^{\infty}(\frac{2}{3}\frac{1}{3})^i = \frac{1}{7}$

The same situation where A starts :
$\mathbb{P}(A_w|\text{A v B, A starts}) = \frac{1}{3}\sum_{i=0}^{\infty}(\frac{1}{3}\frac{2}{3})^i = \frac{3}{7}$

The situation where A and C are in duel with A starting :
$\mathbb{P}(A_w|\text{A v C, A starting}) = \frac{1}{3}$

$\mathbb{P}(A_w) = \frac{1}{7}\mathbb{P}(\text{A v B, Bstarts}) + \frac{3}{7}\mathbb{P}(\text{A v B, A starts}) + \frac{1}{3}\mathbb{P}(\text{A v C, A starts}) = \frac{1}{7}\frac{1}{3} + \frac{3}{7}(\frac{2}{3})^2 + \frac{1}{3}\frac{2}{3}\frac{1}{3} = \frac{59}{189}$

(b)

There is only two cases if A shoots into the ground on his first shot :

- B hits C and A versus B, A starting
- B fails and A versus C, A starting

So $\mathbb{P}(A_w|\text{A shoots into the ground on his first shot}) = \frac{2}{3}\frac{3}{7} + \frac{1}{3}\frac{1}{3} = \frac{25}{63}$

Question 2

In the gambler's ruin problem : $\forall i \in \{1, 2, ..., N-1\}, P_i = \begin{cases} \frac{1-(\frac{q}{p})^i}{1-(\frac{q}{p})^N} & \text{if } p \neq \frac{1}{2} \\ \frac{i}{N} & \text{if } p = \frac{1}{2} \end{cases}$ where P_i is the probability that the gambler reaches N before 0 starting with i.

Suppose : $i < j$ then f_{ij} is the probability that the gambler reaches j before reaching 0 starting from i then it is just the same game but with j instead of N.

$$\boxed{f_{ij} = \begin{cases} \frac{1-(\frac{q}{p})^i}{1-(\frac{q}{p})^j} & \text{if } p \neq \frac{1}{2} \\ \frac{i}{j} & \text{if } p = \frac{1}{2} \end{cases}}$$

Suppose : $j < i$ then loosing has probability : $1 - P_i = f_{ij}(1 - P_j)$ because you have to go from i to j

if you loose as $j < i$.
So :

$$\boxed{f_{ij} = \frac{1-P_i}{1-P_j} = \begin{array}{ll} \frac{(\frac{q}{p})^N-(\frac{q}{p})^i}{(\frac{q}{p})^N-(\frac{q}{p})^j} & \text{if } p \neq \frac{1}{2} \\ \frac{N-i}{N-j} & \text{if } p = \frac{1}{2} \end{array}}$$

Suppose : $i = j$:

$$\boxed{f_{ii} = pf_{i+1,i} + qf_{i-1,i} = \begin{array}{ll} p\frac{(\frac{q}{p})^N-(\frac{q}{p})^{i+1}}{(\frac{q}{p})^N-(\frac{q}{p})^i} + q\frac{1-(\frac{q}{p})^{i-1}}{1-(\frac{q}{p})^i} & \text{if } p \neq \frac{1}{2} \\ p\frac{N-i-1}{N-i} + q\frac{i-1}{i} & \text{if } p = \frac{1}{2} \end{array}}$$

Question 3

$s_{ii} = \frac{1}{1-f_{ii}}$ and $s_{ij} = f_{ij}s_{jj} = \frac{f_{ij}}{1-f_{jj}}$ where we know f_{ij} and f_{ii} after question 2.

Question 6

Let $k \in \mathbb{N}$ and $Y_k \sim B(p)$ be a bernoulli process. Y_k represents the outcome of a step, either back either forward.
Let $\forall n \in \mathbb{N} :\ X_{n+1} = \max(0, X_n + 2Y_n - 1)$.

Let $(i, j) \in \mathbb{N}^2, i < j$, let P_n :"$\forall k \in \mathbb{N}, \mathbb{P}(X_n \leq k|X_0 = i) \geq \mathbb{P}(X_n \leq k|X_0 = j)$".
As $\mathbb{P}(X_0 \leq k|X_0 = i) = I(i \leq k) = I(i \leq k < j) + I(j \leq k)$ and $\forall k \in \mathbb{N} : I(i \leq k < j) \geq 0$
Then $\mathbb{P}(X_0 \leq k|X_0 = i) \geq \mathbb{P}(X_0 \leq k|X_0 = j)$ so P_0 is true.

Let $n \in \mathbb{N}, P_n$ true : $\forall k \in \mathbb{N}\backslash\{0\}$:
$\mathbb{P}(X_{n+1} \leq k|X_0 = i) = \mathbb{P}(X_n \leq k-1|X_0 = i)\mathbb{P}(Y_n = 1) + \mathbb{P}(X_n \leq k+1|X_0 = i)\mathbb{P}(Y_n = -1)$
$\geq \mathbb{P}(X_n \leq k-1|X_0 = j)\mathbb{P}(Y_n = 1) + \mathbb{P}(X_n \leq k+1|X_0 = j)\mathbb{P}(Y_n = -1) = \mathbb{P}(X_{n+1} \leq k|X_0 = j)$
If $k = 0$ then both probability are 1 and the inequality holds.
So P_{n+1} is true. We conclude that $\forall n \in \mathbb{N}, P_n$ is true.
Our first intermediate result is : $X_n|X_0 = i \leq_{\text{st}} X_n|X_0 = j$.

Now, we notice that knowing the start at the origin, $X_0 = 0$ then $X_1 = 1$ or $X_1 = 0$
If $X_1 = 0 :\ X_{n+1}|X_1 = 0 \sim X_n|X_0 = 0$ and $X_{n+1}|(X_0 = 0) =_{\text{st}} X_n|(X_0 = 0)$
If $X_1 = 1 :\ X_{n+1}|X_1 = 1 \sim X_n|X_0 = 1$ and $X_{n+1}|X_1 = 1 \sim X_n|(X_0 = 1) \geq_{\text{st}} X_n|(X_0 = 0)$ because 1¿0 using the first result.
Finally :

$$\boxed{X_{n+1}|X_0 = 0 \geq_{\text{st}} X_n|X_0 = 0}$$