IEOR 263A : Homework 7

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Problem 51

Let $t \in \mathbb{R}+$, $A(t) \in \mathbb{N}$ be the number of accident.
Let $I(t) = \begin{cases} 1 \text{ if the person already had an accident by time } t \\ 0 \text{ otherwise} \end{cases}$
Let $P(t) \sim PP(\beta)$. We can notice that $A(t) = I(t)(1 + P(t))$. Let E be the time before the first accident. As the number of accident is a poisson process with rate α before the first accident we have that $\mathbb{P}(I(t) = 1) = \mathbb{P}(E \leq t) = 1 - e^{-\alpha t}$
Thus : $\mathbb{E}(A(t)) = \mathbb{E}(I(t)(1 + P(t))) = \mathbb{E}(I(t)) + \mathbb{E}(I(t)P(t)) = 1 - e^{-\alpha t} + \int_0^t \beta(t-u)\alpha e^{-\alpha u} du$
Hence :

$$\boxed{\mathbb{E}(A(t)) = 2(1 - e^{-\alpha t}) + \beta t(1 - e^{-\alpha t}) + te^{-\alpha t}}$$

Problem 52

Define $\forall i \in [\![-k, k]\!]$, $A_i =$ "Team 1 wins starting with i points advantage", $p_i = \mathbb{P}(A_i)$ so that $p_{-k} = 0$ and $p_k = 1$.
We notice that $p_i = \dfrac{\lambda_2}{\lambda_1 + \lambda_2} p_{i+1} + \dfrac{\lambda_1}{\lambda_1 + \lambda_2} p_{i-1}$.

Two solution for this recursive equation are : $u_i = 1$ or $v_i = \left(\dfrac{\lambda_1}{\lambda_2}\right)^i$

Thus $p_i = C_1 + C_2 \left(\dfrac{\lambda_1}{\lambda_2}\right)^i$, with $p_{-k} = 0$ and $p_k = 1$ it yields

$$\boxed{p_i = \frac{\lambda_2^{k-i}\lambda_1^{k+i} - \lambda_2^{2k}}{\lambda_1^{2k} - \lambda_2^{2k}}}$$

Problem 66

a. Define $\forall t \in \mathbb{R}$, $N(t) =$ number of accidents. Among those accidents some are reported at time t some are not. This is a poisson process that is being splitted. Thus $N(t) \sim PP(\lambda \int_0^t G(u)du)$ and

$$\boxed{\forall n \in \mathbb{N}, \mathbb{P}(N(t) = n) = \frac{(\lambda \int_0^t G(u)du)^n}{n!} e^{-\lambda \int_0^t G(u)du}}$$

b. Let $t \in \mathbb{R}+$, $A(t)$ be the amount of the accidents that have not been reported yet at time t :

$$\boxed{\mathbb{E}(A(t)) = \mathbb{E}(\mathbb{E}(A(t)|N(t))) = \mathbb{E}(N(t)\mathbb{E}(F)) = \mathbb{E}(N(t))\mathbb{E}(F) = \mathbb{E}(F)\lambda \int_0^t G(u)du}$$

Problem 70

a. Let $\lambda \in \mathbb{R}+$ be the rate of the poisson process $\{N(t)\}$. Let H = "The first client to arrive is also the first leaving". Let $i \in \mathbb{N}$, T_i is the service time for i-th server, t is the arrival time of the customer leaving second.

$\mathbb{P}(H|t) = \mathbb{P}(T_1 \leq t + T_2) = \mathbb{P}(T_1 > t, T_1 - t \leq T_2) + \mathbb{P}(T_1 \leq t)$.
Thus

$$\boxed{\mathbb{P}(H) = \int_0^\infty \lambda e^{-\lambda t}\left(G(t) + 1 - \int_t^\infty G(x-t)dG(x)\right)dt}$$

b. We can write : $S(t) = \sum_{i=1}^{N(t)} T_i - \sum_{j=1}^{M(t)} T_j$ where $\{M(t)\}$ is the poisson process counting the number of person that left the system. $S(t)$ is a compound poisson process as it is a linear combination of two CPP.

c. Let $N_1(t) = N(t) - M(t)$ be the number of customer in the system by time t :

$$\boxed{\mathbb{E}(S(t)) = \mathbb{E}(N_1(t))\mathbb{E}(T)}$$

where $T \sim G$.

d.

$$\boxed{\mathbb{V}(S(t)) = \mathbb{E}(\mathbb{V}(S(t)|N_1(t))) + \mathbb{V}(\mathbb{E}(S(t)|N_1(t))) = \mathbb{E}(N_1^2(t))\mathbb{V}(T) + \mathbb{E}(T)^2\mathbb{V}(N_1(t))}$$

Problem 80

i. $\forall i \in \mathbb{N}$, T_i are not independent between each other.

ii. They are not identically distributed because λ is a function of time.

iii. $T_1 \sim \mathcal{E}(\lambda(t))$ ie.

$$\boxed{\forall t \in \mathbb{R}+, \mathbb{P}(T_1 \leq t) = \int_0^t \lambda(u)e^{-\lambda(u)u}du}$$

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Problem 86

a. Let $I = \begin{cases} 0 & \text{if it is a bad year} \\ 1 & \text{otherwise} \end{cases}$

$$\boxed{\mathbb{P}(N(t) = n) = \mathbb{P}(N(t) = n|I = 1)\mathbb{P}(I = 1) + \mathbb{P}(N(t) = n|I = 0)\mathbb{P}(I = 0) = 0.3\frac{(3t)^n}{n!}e^{-3t} + 0.7\frac{(5t)^n}{n!}e^{-5t}}$$

b. $N(t)$ is not a poisson process because it does not verify the indepependent arrival times.

c.

$$\boxed{\mathbb{E}(N(t)) = \mathbb{E}(\mathbb{E}(N(t)|I)) = \mathbb{E}(N(t)|I = 1)\mathbb{P}(I = 1) + \mathbb{E}(N(t)|I = 0)\mathbb{P}(I = 0) = 0.3(3t) + 0.7(5t) = 4.4t}$$

d. We use the formula of the conditionnal variance : $\mathbb{V}(X) = \mathbb{E}(\mathbb{V}(X|Y)) + \mathbb{V}(\mathbb{E}(X|Y))$:

$$\boxed{\mathbb{V}(N(t)) = \mathbb{E}(\mathbb{V}(N(t)|I)) + \mathbb{V}(\mathbb{E}(N(t)|I))}$$

Problem 1

Let $\lambda(t) = \begin{cases} 1 & \text{if } t \in [0;1] \\ 2 & \text{if } t \in [1;+\infty[\end{cases}$ then if $T_1 > 1$, $\mathbb{P}(T_2 \geq t|T_1) = e^{-2t}$ and $\mathbb{P}(T_2 \geq t) \neq e^{-2t}$ so interarrival time can't be independent.
But increments are independent.

Problem 2

Consider a Markov Chain with three states : $P_{12} = P_{23} = P_{31} = 1$. Let $\forall n \in \mathbb{N}$, N_n count the number of time going by 1 starting at 1. You know the interarrival times will be 3 so all are independent as it is constant. Now suppose $N_{n+1} - N_n = 1$ then you know $N_{n+2} - N_{n+1} = 0$ and $N_{n+3} - N_{n+2} = 0$ so increments can't be independent.

Problem 3

Consider a Markov Chain with two states : $P_{11} = P_{12} = \frac{1}{2}$ and $P_{22} = 1$. Let N_n count the number of time going by one starting at 1 after n steps. Then the interarrival time are identically distributed : $T_i = \begin{cases} 1 & \text{if we were in 1 and steped at one with } p = \frac{1}{2} \\ \infty & \text{otherwise} \end{cases}$

Thus all interarrival time are identically distributed.

But suppose N_n stationnary : $\forall n, s \in \mathbb{N}$, $N_{n+s} - N_n \sim_{st.} N_s$. As $n \to \infty$ $N_{n+s} - N_n \to 0$ and $N_s \sim 0$ which is not true.

Problem 4

$X_i(t) \sim CPP(2i, \mathcal{U}(i, 3i))$ which means $\exists \{N_i(t)\} \sim PP(2i)$ and $\{T_{ij}\}_{j\in\mathbb{N}} \sim \mathcal{U}(i, 3i)$ such that $X_i(t) = \sum_{j=0}^{N_i(t)} T_{ij}$

$X(t) = X_1(t) + X_2(t) = \sum_{j=1}^{N_1(t)} T_{1j} + \sum_{j=1}^{N_2(t)} T_{2j}$ an event arrives each $\min(X, Y)$ where $X \sim \mathcal{E}(2), Y \sim \mathcal{E}(4)$ which implies $\min(X, Y) \sim \mathcal{E}(6)$ so the rate of the CPP has to be 6. The probability for $N_1(t)$ to increase before $N_2(t)$ is $\frac{2}{2+4} = \frac{1}{3}$ thus $N_2(t)$ to increases before $N_1(t)$ with probability $\frac{2}{3}$. Thus 1/3 of the time $X(t)$ gains $\mathcal{U}([1, 3])$ and 2/3 of the times gains $\mathcal{U}([3, 6])$ which is equivalent to always gaining $\mathcal{U}([1/3; 1]) + \mathcal{U}([4/3, 4])$

That's why :

$$\boxed{X(t) \sim CPP(6, \mathcal{U}([1/3; 1]) + \mathcal{U}([4/3; 4]))}$$

Problem 7

After the course notation : $L_{n+1} = L_n + I_n(L'_{n+1} - 1)$ so $\mathbb{V}(L_{n+1}) = \mathbb{V}(L_n) + \mathbb{V}(I_n L'_{n+1}) - \mathbb{V}(I_n)$.
Using the conditionnal variance formula :

$$\begin{aligned} \mathbb{V}(I_n L'_{n+1}) &= \mathbb{V}(\mathbb{E}(I_n L'_{n+1}|I_n)) + \mathbb{E}(\mathbb{V}(I_n L'_{n+1}|I_n)) \\ &= \mathbb{V}(I_n)\mathbb{E}(L_{n+1})^2 + \mathbb{V}(L_{n+1})\mathbb{E}(I_n) \\ &= p_n(1 - p_n) + \mathbb{V}(L_{n+1})p_n \end{aligned}$$

Thus :

$$\boxed{\mathbb{V}(L_{n+1}) = \frac{\mathbb{V}(L_n)}{1 - p_n}}$$