IEOR 263A : Homework 10

Arnaud Minondo

November 17, 2022

Problem 7.32

We want to determine : $\lim_{t\to\infty}\int_0^t \frac{\mathbb{I}(X_{N(s)+1} < c)ds}{t}$

Problem 7.42

a. Having an exponential with mean μ is the same as having an exponantial with rate $\frac{1}{\mu}$.

$\forall x \in \mathbb{R}+$, $F_e(x) = \frac{1}{\mu}\int_0^x (1 - F(y))dy = \frac{1}{\mu}\int_0^x e^{-\frac{y}{\mu}}dy = \dfrac{\mu - \mu e^{-\frac{x}{\mu}}}{\mu} = 1 - e^{-\frac{x}{\mu}}$

$$\boxed{\text{Hence } F_e = F}$$

b. We have that $\mu = \int_0^\infty 1 - F(t)dt = c$.
Thus : $\forall x \in \mathbb{R}+$, $F_e(x) = \frac{1}{c}\int_0^x 1 - F(t)dt = \frac{1}{c}\int_0^c 1 - F(t)dt + \frac{1}{c}\int_c^x 1 - F(t)dt$. If $x < c$ then $F_e(x) = \frac{x}{c}$ and if $x \geq c$ then $F_e(x) = 1$

$$\boxed{\text{Hence } F_e \sim \mathcal{U}([0;c])}$$

c. Suppose the time after which the officer marks the car is $T \sim \mathcal{U}([0;2])$ because the arrival time should follow the equilibrium distribution. The probability of event R :"you will receive a ticket" is

$$\boxed{\mathbb{P}(R) = \mathbb{P}(T < 1) = \frac{1}{2}}$$

Problem 7.43

a.Let $I \sim \mathcal{B}(\frac{1}{3})$.

$$\boxed{\text{The equilibrium distribution should be : } F_e \sim I\mathcal{E}(1) + (1-I)\mathcal{E}(\frac{1}{2})}$$

b.

$$\boxed{F_e(x) = \frac{2}{3}\int_0^x \frac{1}{2}e^{-x} + \frac{1}{2}e^{-\frac{x}{2}}dx = 1 - (\frac{1}{3}e^{-x} + \frac{2}{3}e^{-\frac{x}{2}})}$$

Problem 7.44

a. N is a stopping time thus we can apply Wald's formula. Thus $\dfrac{\mathbb{E}\left(\sum_{i=1}^N X_i\right)}{\mathbb{E}(N)} = \mathbb{E}(X_1) = \mathbb{P}(W_1 < x)$.
Moreover, after the weak law of large numbers : $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n X_i = \mathbb{P}(W_1 < x) = \pi$. Hence

$$\boxed{\pi = \frac{\mathbb{E}\left(\sum_{i=1}^N X_i\right)}{\mathbb{E}(N)}}$$

b. Let $R = \sum_{i=1}^{N} X_i$, $\mathbb{E}(R|T=t) = \mathbb{E}(\mathbb{E}(R|(N, T=t)))$

$$\boxed{\mathbb{E}(R|T=t) = \lambda \min(t,x)}$$

c. We have $\mathbb{E}(X_1 + X_2 + ... + X_N | T = t) = \lambda \min(t,x)$ thus

$$\boxed{\mathbb{E}(X_1 + X_2 + ... + X_N) = \int_0^{\infty} \lambda \min(t,x) dF(t) = \lambda \mathbb{E}(\min(T,x))}$$

d. $\pi = \dfrac{\mathbb{E}(X_1 + X_2 + ... + X_N)}{\mathbb{E}(N)} = \dfrac{\lambda\mathbb{E}(\min(T,x))}{\mathbb{E}(N)} = \dfrac{\lambda\mathbb{E}(\min(T,x))}{\mathbb{E}(\mathbb{E}(N|T))} = \dfrac{\mathbb{E}(\min(T,x))}{\mathbb{E}(T)}$.

With $\mathbb{E}(\min(T,x)) = \int_0^x t dF(t) + \int_x^{\infty} x dF(t) = xF(x) - \int_0^x F(t)dt + x(1 - F(x)) = x - \int_0^x 1 - \mathbb{P}(T > t)dt = \int_0^x \mathbb{P}(T > t)dt$ we have the result :

$$\boxed{\pi = \frac{\int_0^x \mathbb{P}(T > t)dt}{\mathbb{E}(T)}}$$