Individual homework. All the answers have to be mathematically justified (unless specified explicitly).

Exercise 1. An ambulance travels back and forth at a constant speed along a road of length L. At a certain moment of time, an accident occurs at a point uniformly distributed on the road. [That is, the distance of the point from one of the fixed ends of the road is uniformly distributed over $(0, L)$.] Assuming that the ambulance's location at the moment of the accident is also uniformly distributed, and assuming independence of the variables, compute the distribution of the distance of the ambulance from the accident.

Exercise 2. The joint density of X and Y is given by

$$f(x,y) = \begin{cases} xe^{-(x+y)} & x>0, y>0 \\ 0 & \text{otherwise} \end{cases}$$

Are X and Y independent? If, instead, $f(x,y)$ were given by

$$f(x,y) = \begin{cases} 2 & 0<x<y, 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

would X and Y be independent?

Exercise 3. The joint density function of X and Y is

$$f(x,y) = \begin{cases} x+y & 0<x<1, 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Are X and Y independent? (b) Find the density function of X. (c) Find $P\{X+Y<1\}$.

Exercise 4. If X_1 and X_2 are independent exponential random variables with respective parameters λ_1 and λ_2, find the distribution of $Z = X_1/X_2$. Also compute $P\{X_1 < X_2\}$

Exercise 5. Let X and Y denote the coordinates of a point uniformly chosen in the circle of radius 1 centered at the origin. That is, their joint density is

$$f(x,y) = \frac{1}{\pi} \quad x^2+y^2 \le 1$$

Find the joint density function of the polar coordinates $R = \left(X^2+Y^2\right)^{1/2}$ and $\Theta = \tan^{-1} Y/X$.

Exercise 6. X and Y have joint density function

$$f(x,y) = \frac{1}{x^2 y^2} \quad x \ge 1, y \ge 1$$

(a) Compute the joint density function of $U = XY, V = X/Y$. (b) What are the marginal densities?

Exercise 7. Let X and Y be continuous random variables with joint density function

$$f(x,y) = \begin{cases} \frac{x}{5} + cy & 0<x<1, 1<y<5 \\ 0 & \text{otherwise} \end{cases}$$

where c is a constant.

(a) What is the value of c ?

(b) Are X and Y independent?

(c) Find $P[X+Y>3\}$.