wednesday Sept 28th

(P) $\min c^T x$
st $Ax = b$
$x \geq 0$

(D) $\max(y^T b)$
st $y^T A + s^T = c^T$
$s \geq 0$

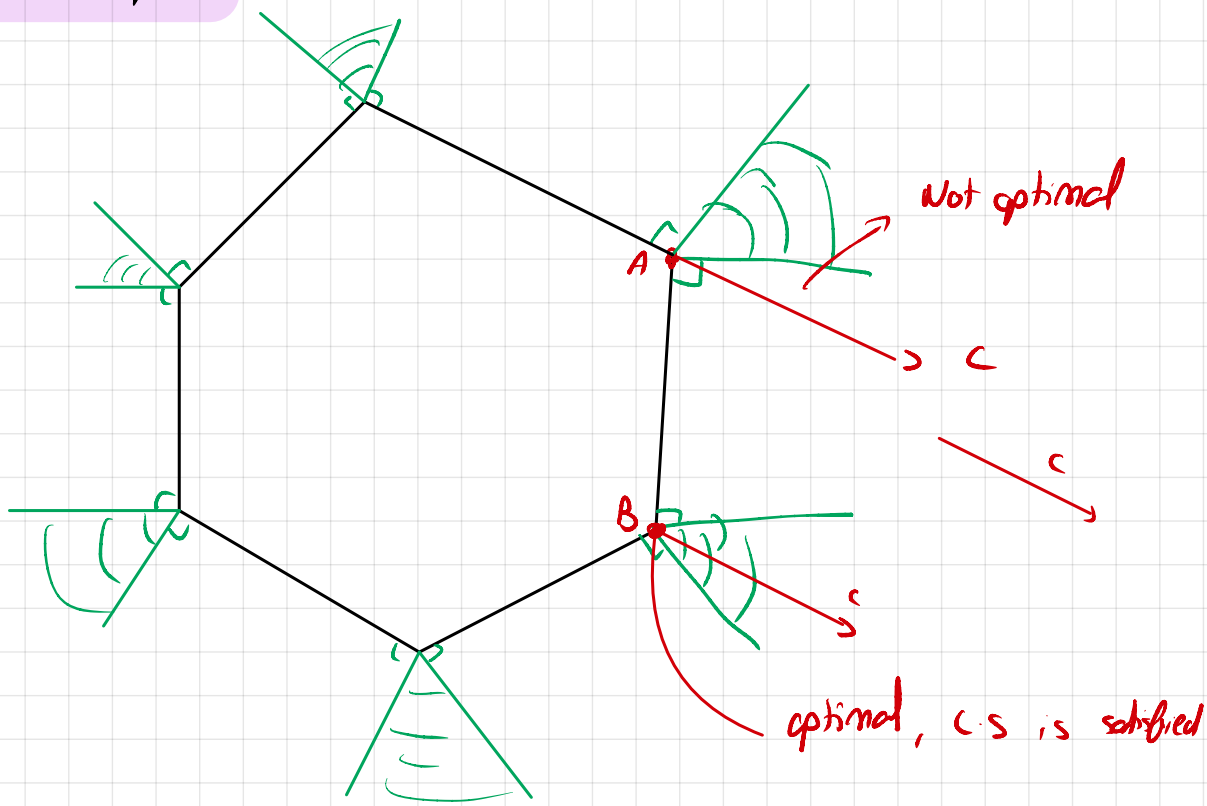
Complementary slackness $x_j s_j = 0$ $\forall j = 1, \dots, n$

(P) $\max(c^T x)$
st $Ax + s = b$
$s \geq 0$

(D) $\min(y^T b)$
st $y^T A = c^T$
$y \geq 0$

A — $m \times n$

C.S $s_i y_i = 0$, $i = 1, \dots, n$



Def: Constraint $a^T x \leq b$ is active (binding) at $\bar{x} \in \mathbb{R}^n$ iff $a^T \bar{x} = b$

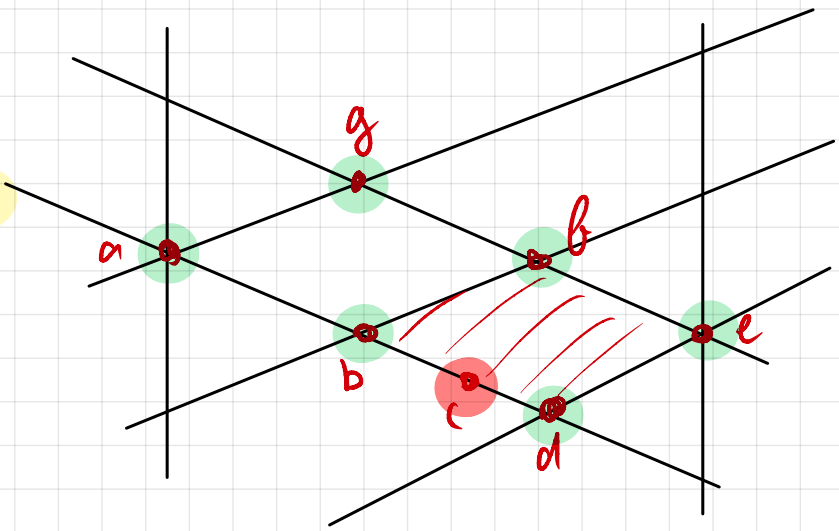
Def: Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$. $\bar{x}$ is a basic point of P if there are n linearly independent binding (active) constraints

at $\bar{x}$.

basic points
degenerate points

Ex: $\mathbb{R}^2$

a, e are degenerate points

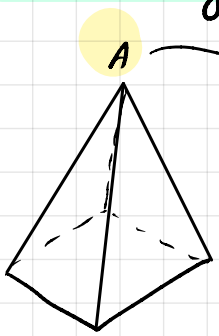


Def: $\bar{x}$ is a degenerate point iff there are more than n active constraints at $\bar{x}$

a constraint is redundant $\Leftrightarrow$ it does not change the feasible region if removed

⚠ Degeneracy $\not\Rightarrow$ Redundancy

In $\mathbb{R}^3$:



degenerate

but there are no redundant constraints because if we delete one it would completely change the shape.

Theorem:

Let $P = \{x \in \mathbb{R}^n, Ax \geq b\}$. $\bar{x}$ is a b.f.s (basic feasible solution) of P iff $\bar{x}$ is an extreme point of P.

Proof: $(\Rightarrow)$ Suppose $\bar{x}$ is a b.f.s of P. Then $\bar{x} \in P$.

Let $(A^=, b^=)$ be the set of active constraints on $\bar{x}$

$(\forall i \in A^=,\ A_i^{=T}\bar{x} = b_i^=)$

Let $\tilde{A}$ be a nonsingular submatrix of $A^=$. For contradiction,

Suppose $\bar{x}$ is not an extreme point of P.

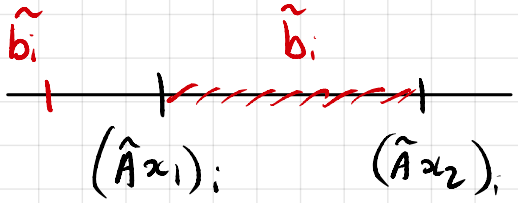
$\exists\, x_1, x_2 \in P$, $x_1 \neq x_2$, $\lambda \in (0,1)$, s.t

$$\bar{x} = \lambda x_1 + (1-\lambda) x_2$$

$$\tilde{b} = \tilde{A}\bar{x} = \underbrace{\lambda \tilde{A} x_1}_{\geq\, \hat{A}\bar{x} \,=\, \tilde{b}} + \underbrace{(1-\lambda)\tilde{A} x_2}_{\geq\, \hat{A}\bar{x} \,=\, \tilde{b}}$$

$$\boxed{Ax \geq b \Rightarrow \hat{A}x \geq \tilde{b}}$$

Passage délicat, $\hat{A}$ est une matrice extraite de A particulière, on supprime seulement des lignes, mais pas de colonne (car sinon $\hat{A}x$ n'aurait pas de sens)



$\Rightarrow \hat{A}x_1 = \hat{A}x_2 = \hat{A}\bar{x} = \tilde{b}$

The only possibility is $(\hat{A}x_1)_i = (\hat{A}x_2)_i \ \forall i, \Leftrightarrow \forall i, (x_1)_i = (x_2)_i$ (A nonsingular)

$\Leftrightarrow x_1 = x_2$ ↯

$(\Leftarrow)$ $\left(\bar{x} \in \mathrm{Extr}(P) \Rightarrow \bar{x} \text{ b.f.s}\right)$ or equivalently $(\bar{x} \text{ not b.f.s} \Rightarrow \bar{x} \notin \mathrm{Extr}(P))$

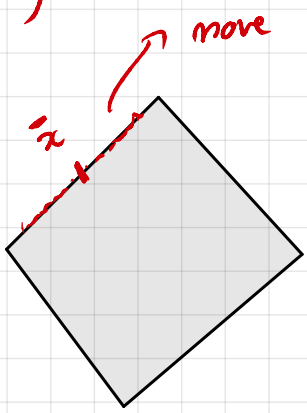
$\bar{x} \notin P$ clearly implies $\bar{x}$ not extreme

Suppose $\bar{x} \in P$ but not basic. Let $(A^=, b^=)$ be the active constraint.

Then $\mathrm{rank}(A^=) < n$ so $\dim(\mathrm{Null}(A^=)) \geq 1$

Let $y \in \mathrm{Null}(A^=)$, $y \neq 0$.

$$A^=(x + \varepsilon y) = A^= x + \underbrace{\varepsilon \overbrace{A^= y}^{=0 \text{ bc } y \in \mathrm{Ker}(A^=)}}_{\downarrow\ 0 \text{ for } \varepsilon \to 0} = b^=$$



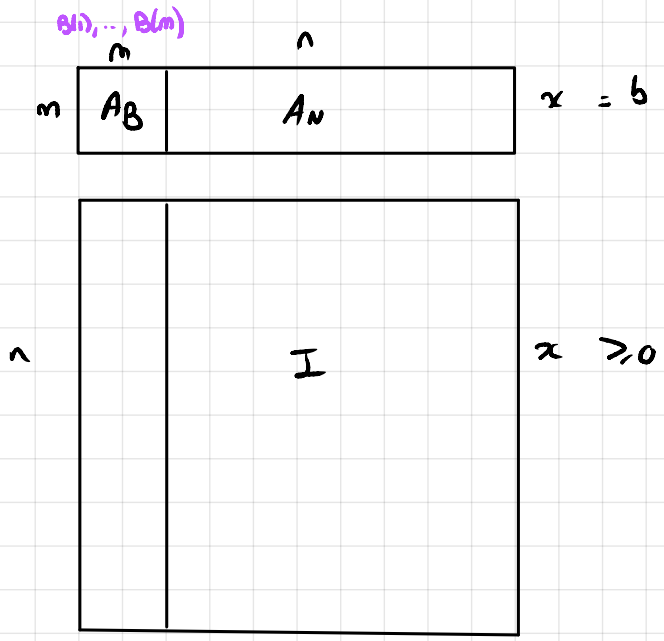
$$\bar{x} = \frac{1}{2}\underbrace{(\bar{x} + \varepsilon y)}_{P} + \frac{1}{2}\underbrace{(\bar{x} - \varepsilon y)}_{P}$$

$$A^{>}(\bar{x} \pm \varepsilon y) \geq b^{>} \qquad \square$$

In Standard form LP's have extreme points if feasible.

$Ax = b$
$x \geq 0$

WLOG assume $\text{rank}(A) = m$
without loss of generality



Prop : for $P = \{x \in \mathbb{R}^n, Ax = b, x \geq 0\}$ with $\text{rank}(A) = m$ $(A - m \times n)$

$\bar{x}$ is b.f.s iff 1) $A\bar{x} = b$

2) $\exists$ m linearly independants colums of A
$a^{B(1)}, \dots, a^{B(m)}$

3) $\bar{x}_j = 0 \ \forall j \notin \{B(1), \dots, B(m)\}$

4) $\bar{x}_B = A_B^{-1} b \geq 0$