Homogeneous Linear ODEs

The functions $y = \cos x$ and $y(x) = \sin x$ are solutions of the homogeneous linear ODE

$$y'' + y = 0, \quad \forall x. \quad -(1)$$

Yes! we can verify this by differentiation and substitution.

As $y(x) = \cos x$

$\Rightarrow y'(x) = -\sin x$

$\Rightarrow y''(x) = -\cos x$

Hence $y'' + y = -\cos x + \cos x = 0$

Similarly for $y(x) = \sin x$, $y'(x) = \cos x$, $y'' = -\sin x$

Hence

$y'' + y = -\sin x + \sin x = 0$

Further, if you multiply $\cos x$ by any constant (say 4.5) and $\sin x$ by (say -2) and then take a sum of the results i.e $4.5\cos x - 2\sin x$.

Next we will show that $4.5\cos x - 2\sin x$ is also solution of the given ODE (1).

Consider l.h.s of (1)

$$y'' + y$$

$$= (4.5\cos x - 2\sin x)'' + (4.5\cos x - 2\sin x)$$

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$= -4.5\cos x + 2\sin x + (4.5\cos x - 2\sin x)$

$= 0$

$=$ r.h.s of (1).

This show's that $4.5\cos x - 2\sin x$ is also solution of (1).

So from above example we say that

$y = c_1 y_1 + c_2 y_2$ — (2)

where $y_1 = \cos x$ and $y_2 = \sin x$, and

c_1, c_2 arbitrary constants.

is also solution of (1).

Note that (2) called a linear combination of y_1 and y_2

In terms of this concept we can now formulate the result suggested by our example, often called the superposition principle or linearity principle.

Fundamental theorem for the homogeneous linear ODE of second order:

For a homogeneous linear ODE of 2^{nd} order
$$y'' + p(x)y' + q(x)y = 0, \quad x \in I = (a,b) \subseteq \mathbb{R} \quad -(1)$$
where p and q are any given functions of x.

Any linear combination of two solutions on an open interval I is again a solution of (1) on I.

In particular, for such an equation, sums and constant multiples of solutions are again solutions.

Proof

Let $y_1(x)$ and $y_2(x)$ be two solutions of (1) on I.

Claim: $y = c_1y_1 + c_2y_2$ is a solution of (1) on I, where c_1 and c_2 are constants.

Consider the l.h.s of (1)

$$y'' + py' + qy$$
$$= (c_1y_1 + c_2y_2)'' + p(c_1y_1 + c_2y_2)' + q(c_1y_1 + c_2y_2)$$
$$= (c_1y_1'' + c_2y_2'') + p(c_1y_1' + c_2y_2') + q(c_1y_1 + c_2y_2)$$
--- (by derivative rule)
$$= c_1(y_1'' + py_1' + qy_1) + c_2(y_2'' + py_2' + qy_2)$$
$$= c_1(0) + c_2(0) \quad \text{--- (Since } y_1 \text{ and } y_2 \text{ are sol}^{ns} \text{ of (1))}$$
$$= 0 = \text{r.s.h of (1).}$$

This implies $y = c_1y_1 + c_2y_2$ is sol^n of (1) on I

Hence proved.

Think On!

The extension of above theorem for the Homogeneous linear ODE of n^{th} order

For statement see page no 106 (section 3.1) of 10th edition

Write the proof of this theorem yourself.

Hint: Let $y_1(x), y_2(x), \ldots y_n(x)$ be solutions of n^{th} order homogeneous linear ODE.

claim: $y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$ is solution of n^{th} order homogeneous linear ODE, where $c_1, c_2, \ldots c_n$ are constants.

Try to prove it.

Remarks

(1) Above theorem (i.e superposition principle) is NOT valid for nonlinear ODEs.

For example-

$$y''y - xy' = 0 \quad \text{—} (1)$$

$y = x^2$ and $y = 1$ are solutions of above nonlinear ODE (verify!)

But their sum $1 + x^2$ is NOT a solution and constant multiplication $-x^2$ and -1 are NOT a solution of given nonlinear ODE (1).

(2) Above theorem (i.e superposition principle) is NOT valid for nonhomogeneous linear ODEs.

For example-

$y(x) = 1 + \cos x$ and $y = 1 + \sin x$ are

the solution of the nonhomogeneous linear ODE

$$y'' + y = 1, \quad \text{—} (2)$$

but their sum i.e $2 + \sin x + \cos x$ is NOT a solution and constant multiplication i.e

$$2(1 + \cos x) \quad \text{or} \quad 7(1 + \sin x)$$

are NOT solution of given ODE (2).

(3) The set $V_2 = \{h(x) \mid h$ is a sol^n of $y'' + p(x)y' + q(x)y = 0,\ x \in I\}$ of all solutions of homogeneous 2^{nd} order linear ODE is a vector space.

(4) What is the dimension of the solution space of homogeneous 2^{nd} order linear ODE?

Think ON!

(5) The set of all solutions of homogeneous n^{th} order linear ODE is a vector space.

i.e $V_n = \{h(x) \mid h$ is a sol^n of $y^{(n)} + P_{n-1}(x)y^{(n-1)} + \cdots + P_0(x)y = 0\}$

is a vector space.

Think ON!
(Why?)

(6) What is the dimension of the solution space of homogeneous n^{th} order linear ODE?

Think ON!

Linear Independence and Dependence

Linear Independent (LI) :

A set of functions $f_1(x), f_2(x), f_3(x), \cdots, f_n(x)$ is said to be linearly independent on an interval I if there exist constants $k_1, k_2, k_3, \cdots k_n$ all zero (ie $k_1 = k_2 = \cdots = k_n = 0$) such that

$$k_1 f_1(x) + k_2 f_2(x) + \cdots + k_n f_n(x) = 0$$

for every x in the interval I.

Linear Dependent (LD) :

A set of functions $f_1(x), f_2(x), \cdots, f_n(x)$ is said to be linearly dependent on an interval I if there exist constants $k_1, k_2, \cdots k_n$ NOT all zero such that

$$k_1 f_1(x) + k_2 f_2(x) + \cdots + k_n f_n(x) = 0, \text{ on } I.$$

To better understand of this concept consider for two functions.

A set of two functions $f_1(x), f_2(x)$ defined on some interval I and is said to be LI on an interval I if $\exists$ constants k_1, k_2 all zero such that

$$k_1 f_1(x) + k_2 f_2(x) = 0 \quad \forall\, x \in I. \quad \text{— (1)}$$

These functions are called LD on I if this equation (1) also holds on I for some k_1 (or k_2) not equal to zero.

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In other words, a set of functions is linearly dependent if at least one function can be expressed as a linear combination of the remaining functions.

For example

Show that $\{\sqrt{x}+5, \sqrt{x}+5x, x-1, x^2\}$

is linearly dependent on the interval $(0,\infty)$.

→

Consider
$$k_1(\sqrt{x}+5) + k_2(\sqrt{x}+5x) + k_3(x-1) + k_4 x^2 = 0$$

and this linear combination is zero for choice of $k_1 = 1$, $k_2 = -1$, $k_3 = 5$, $k_4 = 0$

So this set of function LD on $(0,\infty)$

A better (and faster) way to determine if function are dependent or independent is to put the functions in their derivatives in a matrix, and then find it's determinant.

This determinant is called the Wronskian of the functions [Named after Joseph Wronski, a Polish philosopher and mathematician]

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Wronskian :-

If $f_1(x), f_2(x), \ldots, f_n(x)$ have at least $(n-1)$ derivative on an open interval I. Then determinant

$$W(f_1, f_2, \cdots f_n) = \begin{vmatrix} f_1 & f_2 & f_3 & \cdots & f_n \\ f_1' & f_2' & f_3' & \cdots & f_n' \\ f_1'' & f_2'' & f_3'' & \cdots & f_n'' \\ \vdots & & & & \\ f_1^{(n-1)} & f_2^{(n-1)} & f_3^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is called the Wronskian of the functions.

For three functions.

$$W(f_1, f_2, f_3) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$$

For two functions

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

The value of the Wronskian can help us to determine whether functions are dependent or independent.

FACTS :-

(1) If $f_1(x)$ and $f_2(x)$ are linearly dependent functions on I then $W(f_1(x), f_2(x)) = 0 \ \forall x \in I$.

$$\boxed{\begin{array}{l} f_1(x), f_2(x) \\ \text{LD functions} \Rightarrow W(f_1(x), f_2(x)) = 0 \ \forall x \in I \\ \text{on } I \end{array}}$$

Proof :-

As $f_1(x)$ and $f_2(x)$ are linearly dependent function on I. Then by definition.

$$K_1 f_1(x) + K_2 f_2(x) = 0 \quad \text{on } I$$

for K_1, K_2 not all zero. In another way we can say that $f_1(x) = K f_2(x)$. *

where K some scalar.

Now

$$W(f_1(x), f_2(x)) = \begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix}$$

$$= \begin{vmatrix} K f_2(x) & f_2(x) \\ K f_2'(x) & f_2'(x) \end{vmatrix} \quad (\because \text{by } *)$$

$$= 0 \quad \text{on } I$$

Note that converse of above fact is NOT True

[$W(f_1(x), f_2(x)) = 0$ for all x on I does not mean that the functions are linearly dependent.]

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Example

$x|x|$, x^2 ; $x \in [-2, 2]$

→ Here $f_1(x) = x|x|$, $f_2(x) = x^2$, $I = [-2, 2]$

For $x > 0$, $f_1(x) = x^2$

$$W(f_1, f_2) = \begin{vmatrix} x^2 & x^2 \\ 2x & 2x \end{vmatrix} = 0$$

For $x < 0$, $f_1(x) = -x^2$

$$W(f_1, f_2) = \begin{vmatrix} -x^2 & x^2 \\ -2x & 2x \end{vmatrix} = 0$$

For $x = 0$, $f_1(x) = 0$

$$W = \begin{vmatrix} 0 & x^2 \\ 0 & 2x \end{vmatrix} = 0$$

Thus $W(f_1(x), f_2(x)) = 0 \;\forall\; x \in I$.

Here we can't conclude anything. So we use the definition.

Consider

$$k_1 f_1(x) + k_2 f_2(x) = 0 \;,\; x \in [-2, 2]$$

i.e $k_1 x|x| + k_2 x^2 = 0$, $x \in [-2, 2]$ — (1)

at $x = 1$ from (1)

$1|1| k_1 + 1 k_2 = 0$

$k_1 + k_2 = 0$ — (i)

at $x = -1$ from (1)

$-k_1 + k_2 = 0$ — (ii)

From (i) and (ii) we have $k_1 = 0$ and $k_2 = 0$

Thus $f_1(x) = x|x|$ and $f_2(x) = x^2$ $x \in [-2, 2]$ are linearly independent on $[-2, 2]$.

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(2) If $W(f_1(x), f_2(x)) \neq 0$ for some x in I then $f_1(x)$ and $f_2(x)$ are linearly independent functions on the interval I.

$W(f_1(x), f_2(x)) \neq 0$ for some $x \in I \Rightarrow$ $f_1(x)$ and $f_2(x)$ are L.I. functions on I

This is a contrapositive statement of fact ①.

Find the example Think on!

Converse of above fact is NOT true.

<u>Remark</u> :- Facts (1) and (2) also true for n- functions.

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Use the Wronskian to determine if the given functions are linearly dependent or independent.

(1) $\{\sin x, \cos x\}$; $x \in \mathbb{R}$

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

$$= \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1 \neq 0$$

Here Wronskian is NOT equal to zero for all $x \in \mathbb{R}$. Thus by fact (2), they are LI on $\mathbb{R}$.

(2) $\{x; x^2\}$

$$W(f_1, f_2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$$

$$= 2x^2 - x^2 = x^2 \neq 0 \text{ (except } x=0)$$

Remember, if the Wronskian is non-zero for some value of x in the interval, then functions are linearly independent

The functions are linearly independent on $\mathbb{R} - \{0\}$.

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(3) $\{1+x,\ x^3\}$

$$W(f_1, f_2) = \begin{vmatrix} 1+x & x^3 \\ 1 & 3x^2 \end{vmatrix}$$

$$= (1+x)\cdot 3x^2 - x^3$$

$$= 3x^2 + 3x^3 - x^3$$

$$= 3x^2 + 2x^3$$

?

Find the value of x for that Wronskian is zero $\xrightarrow{\text{Ans}}$

$$3x^2 + 2x^3 = 0$$

$$x^2(3+2x) = 0$$

$$x^2 = 0, \quad 3+2x = 0$$

$$x = 0, \quad x = -\frac{3}{2}$$

Thus functions are linearly independent on $\left(-\infty, -\frac{3}{2}\right) \cup \left(-\frac{3}{2}, 0\right) \cup (0, \infty)$.

(4) $\{e^{2x},\ e^{2(x-1)}\}$, $x \in (-\infty, \infty)$

$$W = \begin{vmatrix} e^{2x} & e^{2(x-1)} \\ 2e^{2x} & 2e^{2(x-1)} \end{vmatrix} = 0$$

We can't say anything about dependency. So use the definition / (or) observation

$$\frac{e^{2x} e^{-2}}{e^{2x}} = e^{-2} = \text{constant}$$

$\Rightarrow$ They are L.D on $(-\infty, \infty)$.

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④ $\{x, x\ln x\}$, $0 < x < 10$

$$W = \begin{vmatrix} x & x\ln x \\ 1 & \ln x + 1 \end{vmatrix}$$

$$= \cancel{x\ln x} + x - \cancel{x\ln x}$$

$$= x \neq 0 \quad \text{on } (0, 10)$$

Thus they are LI on $(0, 10)$

⑤ $\{\ln x, \ln x^2\}$, $(x>0)$

As $\ln x^2 = 2\ln x$.

$$(1)\ln x^2 + (-2)\ln x = 0$$

This linear combination is zero for $k_1 = 1$ and $k = -2$

Thus they are LD for $x>0$.

⑥ $\{x+1, x+2, x\}$, $(0, \infty)$

By observation.

$$(-2)(x+1) + (1)(x+2) + (1)x = 0$$

This linear combination is zero on $(0,\infty)$ for $k_1 = -2$, $k_2 = 1 = k_3$.

Thus they are LD on $(0,\infty)$

⑦ e^{2t}, e^{-2t}, $\cosh 2t$, on $\mathbb{R}$

→ They are LD on $\mathbb{R}$ since.

$$\left(\tfrac{1}{2}\right)e^{2t} + \left(\tfrac{1}{2}\right)e^{-2t} + (-1)\cosh 2t = 0.$$

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Determine if the following functions are dependent or independent. (Homework)

(1) $x^2+2x, \; x^2-2x$ (on $\mathbb{R}$)

(2) $\sin 2x, \; \cos 2x$ (on $\mathbb{R}$)

(3) $2^{x+3}, \; 2^{x}$ $(-\infty, \infty)$

(4) $e^{ax}, \; e^{-ax}$ (any interval)

(5) $\cos^2 x \quad \sin^2 x$ (any interval)

(6) $1, \sin^2 x, \cos^2 x$ on $(0,\infty)$

(7) $1, x+3, 2x$ on $(0,\infty)$

(8) $\ln x, \ln x^2, (\ln x)^2$ on $(0,\infty)$

(9) $x^2, x|x|, x$ on $(0,\infty)$

(10) $x, \frac{1}{x}, 0$ on $(0,\infty)$

(11) $1, e^{x}, e^{-x}$ on $(0,\infty)$

(12) $\sqrt{x}, x, x^2$ on $(0,\infty)$

(13) $\frac{1}{x}, x, \ln x$ on $(0,\infty)$

(14) $e^{2x}, e^{-2x}, \sinh 2x$ on $(-\infty,\infty)$

(15) $1, \tan^2 x, \sec^2 x$ on $(0,\infty)$

Basis :-

A basis is subset of set of all solutions of homogeneous linear 2^{nd} order ODE, which consists of 2 linearly independent solutions.

What is Basis for hom. linear n^{th} order ODE ?

General Solution :-

A second order homogeneous linear ODE

$$y'' + p(x)y' + q(x)y = 0 \quad , x \in I \quad - (1)$$

where p and q are continuous functions of x on I.

General solution of an ODE (1) on an open interval I is a solution

$$y = c_1 y_1 + c_2 y_2 \quad - \quad (2)$$

in which $y_1(x)$ and $y_2(x)$ are linearly independent solutions of (1). (that is y_1, y_2 are basis of sol^n of (1) on I)

and c_1, c_2 are arbitrary constants.

A particular solution of (1) on I is obtained if we assign specific values to c_1 and c_2 in (2)

∴ So (2) includes all solutions of (1) on I.

Note that (1) has no singular solutions

Think ON!

(1) Under what conditions, n^{th} order homo linear ODE has general solution?

(2) If n^{th}-order homo linear ODE has general solution then how to write?

Ref 106 page.

The following criterion of linear independence and dependence of solutions will be helpful.

Linear Dependence and Independence of Solutions:

Let the ODE $y'' + p(x)y' + q(x)y = 0$; $x \in I$ — (1) have continuous coefficients $p(x)$ and $q(x)$ on an open interval I. Then

Two solutions $y_1(x)$ and $y_2(x)$ of (1) on I are linearly dependent on I. $\Leftrightarrow W(y_1(x), y_2(x)) = 0, \forall x \in I$

If there is an x_1 in I at which $W(y_1(x_1), y_2(x_1)) \neq 0$, then $y_1(x), y_2(x)$ are Linearly independent on I.

{That is $W(y_1(x), y_2(x)) \neq 0$ for some $x \in I$. $\Rightarrow$ $y_1(x), y_2(x)$ are LI on I.}

$\boxed{\text{Example}}$

The functions $y_1(x) = \cos \omega x$ and $y_2(x) = \sin \omega x$ are solutions of $y'' + \omega^2 y = 0$

Their Wronskian is

$$W(\cos \omega x, \sin \omega x) = \begin{vmatrix} \cos \omega x & \sin \omega x \\ -\omega \sin \omega x & \omega \cos \omega x \end{vmatrix}$$

$$= \omega$$

Above shows these solutions are linearly independent if and only if $\omega \neq 0$.

For $w=0$, we have from (1)

$$y'' = 0 \quad -(2)$$

and $y_1 = 1$ and $y_2 = 0$ are two solutions of ODE (2).

These two solutions are linearly dependent by theorem (1), since

$$W(1,0) = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0.$$

Think on! Another way why? they are LD.

Remark:

Imp ① :- $y_1(x)$ and $y_2(x)$ are solutions of (1) on I, and theses solutions form a basis if and only if Wronskian is not equal to zero.

② The extension of above theorem for the Homogeneous linear ODE of n^{th} order. Think ON!

For the statement see page no. 109 (section 3.1) of 10th edition.

Verify by substitution that the given functions form a basis. Solve the given initial value problem.

① $y'' - 16y = 0$, e^{4x}, e^{-4x} , $y(0) = 3$, $y'(0) = 8$
$\quad$ └ (1)

→ Let $y_1(x) = e^{4x}$, $y_2(x) = e^{-4x}$

Now,

$$(e^{4x})'' - 16(e^{4x}) = 16e^{4x} - 16e^{4x} = 0 \Rightarrow e^{4x} \text{ is sol}^n \text{ of } ①$$

$$(e^{-4x})'' - 16(e^{-4x}) = 16e^{-4x} - 16e^{-4x} = 0 \Rightarrow e^{-4x} \text{ —}\|\text{—}$$

Thus they are solution of (1) $\forall x$.

Further

$$W = \begin{vmatrix} e^{4x} & e^{-4x} \\ 4e^{4x} & -4e^{-4x} \end{vmatrix}$$

$$= -4e^{4x} \cdot e^{-4x} - 4e^{4x} e^{-4x}$$

$$= -8e^{4x} e^{-4x} \neq 0 \ \forall x$$

These solutions are L.I, hence they form a basis.

Thus general solution.

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

$$y(x) = C_1 e^{4x} + C_2 e^{-4x}$$

$$y(0) = 3 = C_1 e^0 + C_2 e^0 = C_1 + C_2 \quad \text{— (I)}$$

$$y'(x) = 4C_1 e^{4x} - 4C_2 e^{-4x}$$

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$y'(0) = 8 = 4c_1 - 4c_2 \Rightarrow \quad 2 = c_1 - c_2$ —— (II)

Using (I) and (II) find the values of c_1 and c_2

$$c_1 + c_2 = 3$$
$$c_1 - c_2 = 2$$

$$2c_1 = 5$$

$\Rightarrow c_1 = \frac{5}{2}$ This gives $c_2 = \frac{1}{2}$

Thus
$$y(x) = \frac{5}{2} e^{4x} + \frac{1}{2} e^{-4x}$$ is

particular solution of IVP

(2) $x^2 y'' - xy' + y = 0$, x and $x \ln x$, $y(1) = 1$, $y'(1) = 2$
└ (1)

→ Here

$x^2(0) - x(1) + x = 0 \Rightarrow x$ is sol^n of (1)

$x^2 \cdot \left(\frac{1}{x}\right) - x\left[\ln x + x\right] + x \ln x = 0 \Rightarrow x \ln x$ is sol^n of (1)

Now.

$$W = \begin{vmatrix} x & x \ln x \\ 1 & \ln x + 1 \end{vmatrix} = x \neq 0 \text{ for } x > 0$$

Thus this are linearly independent sol^n of (1). Thus they form a basis.

The general sol^n of (1) is
$$y(x) = c_1 x + c_2 x \ln x.$$

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$y(1) = 1 \Rightarrow C_1 + C_2(0) \Rightarrow C_1 = 1$ — (I)

$y'(x) = C_1 + C_2[\ln x + 1]$

$y'(1) = 2 = C_1 + C_2[0+1] \Rightarrow 2 = C_1 + C_2$ — (II)

By (I) and (II) $C_2 = 1$.

Thus $y(x) = x + x\ln x$ is particular solution of (1) for $x > 0$.

(3) $x^2y'' + xy' - 4y = 0$ — (1), x^2, x^{-2}; $y(1) = 11$, $y'(1) = -6$

→

$x^2(2) + x(2x) - 4x^2 \Rightarrow x^2$ is a sol^n of (1)

$x^2\left(\frac{6}{x^4}\right) + x\left(\frac{-2}{x^3}\right) - 4x^{-2} \Rightarrow \frac{6}{x^2} - \frac{2}{x^2} - \frac{4}{x^2} = 0$

$\Rightarrow x^{-2}$ is a sol^n of (1)

Now

$$W = \begin{vmatrix} x^2 & x^{-2} \\ x & -2x^{-3} \end{vmatrix} = \frac{-4}{x}, \quad x > 0.$$

So $\{x^2, x^{-2}\}$ form a basis.

The general solution is

$$y(x) = C_1x^2 + C_2x^{2}$$

$y(1) = 11 = C_1 + C_2$ — (I)

$y'(x) = 2C_1x - 2C_2x^{-3}$

$y'(1) = -6 = 2C_1 - 2C_2$ — (II)

From (I) and II $c_1 = 4$ and $c_2 = 7$

Thus $y(x) = 4x^2 + 7x^{-2}$ is a particular sol^n of given IVP.

Homework.

(1) $x^2y'' - 7xy' + 15y = 0$, x^3, x^5,
$y(1) = 0.4$,
$y'(1) = 1.0$

(2) $y'' - 6y' + 9y = 0$, e^{3x}, xe^{3x}
$y(0) = -1.4$
$y'(0) = 4.6$.

Find a basis if one solution is known (Reduction of order)

It happens quite often that one solution can be found by inspection or in some other way.

Then a second linearly independent solution can be obtained by solving a first-order ODE.

This is called the method of reduction of order.

Suppose we have 2^{nd} order homogeneous linear ODE

$$y'' + p(x)y' + q(x)y = 0, \quad x \in I \quad -(1)$$

where $p(x)$ and $q(x)$ are continuous functions of x on I.

In addition we have (Given!) non-zero solution of (1) on I.

How to find 2^{nd} Linearly independent solution of (1) on I?

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Assume that 2^{nd} solution will be in the form

$$y(x) = y_2(x) = u(x) \cdot y_1(x)$$

i.e $y = y_2 = u y_1$ — (2)

$$\Rightarrow y' = y_2' = u' y_1 + u y_1' \quad \text{— (3)}$$

$$\Rightarrow y'' = y_2'' = u'' y_1 + u' y_1' + u' y_1' + u y_1''$$

$$= u'' y_1 + 2u' y_1' + u y_1'' \quad \text{— (4)}$$

Put (2), (3) and (4) in (1) we get.

$$u'' y_1 + 2u' y_1' + u y_1'' + p(u' y_1 + u y_1') + q u y_1 = 0$$

Collecting terms in u'', u' and u we have

$$u'' y_1 + u'(2y_1' + p y_1) + u(y_1'' + p y_1' + q y_1) = 0 \quad \text{— (5)}$$

Since y_1 is solution of (1), so

$$y_1'' + p y_1' + q y_1 = 0$$

$$\Rightarrow u(y_1'' + p y_1' + q y_1) = 0$$

From (5).

$$u'' y_1 + u'(2y_1' + p y_1) = 0$$

$$u'' + u'\left(\frac{2y_1' + p y_1}{y_1}\right) = 0 \quad \text{— (6)}$$

To solve the above 2^{nd} order ODE we set

$$u' = U$$

$$u'' = U'$$

Then eq^n (6) become

$$U' + \left(\frac{2y_1' + py_1}{y_1}\right) U = 0$$

This is the reduced ODE (i.e 1^{st} order ODE)

Using seperation of variables

$$\frac{dU}{dx} + \left(\frac{2y_1' + py_1}{y_1}\right) U = 0$$

$$\Rightarrow \frac{dU}{U} = -\left(\frac{2y_1'}{y_1} + p\right) dx$$

$$\Rightarrow \ln|U| = -2\ln|y_1| - \int p(x)\, dx$$

--- (On integration)

$$\Rightarrow \ln U = \ln y_1^{-2} - \int p(x)\, dx$$

$$\Rightarrow \ln U - \ln y_1^{-2} = -\int p(x)\, dx$$

$$\Rightarrow \frac{U}{y_1^{-2}} = e^{-\int p(x)\, dx}$$

$$\Rightarrow U = y_1^{-2} e^{-\int p(x)\, dx}$$

That is $U = \frac{1}{y_1^2} e^{-\int p(x)\, dx}$

Here $U = u'$, so that $u = \int U$

Hence the desired second sol^n is

$$y_2 = y_1 u = y_1 \int U \, dx$$

where $U = \frac{1}{y_1^2} e^{-\int p(x) dx}$

If we have one non-zero sol^n say (y_1) then other linearly independent solution is given by

$$y_2(x) = y_1(x) \int U \, dx,$$

where $U = \frac{1}{y_1^2} e^{-\int p(x) dx}$.

Example:- $xy'' + 2y' + xy = 0$, $y_1(x) = x^{-1}\cos x$, $x > 0$

Find the other LI sol^n of given ODE.

→ given ODE

$$xy'' + 2y' + xy = 0$$

$$\Rightarrow y'' + \frac{2}{x} y' + y = 0$$ (standard form of 2^{nd} hom. linear ODE)

Here $p(x) = \frac{2}{x}$, $x > 0$ (conts!)

$q(x) = 1$ (conts!)

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$$y_1(x) = \frac{\cos x}{x}$$

Now $y_2(x) = y_1(x) \int U$, where $U = \frac{1}{y_1^2} e^{-\int p(x)\,dx}$

So $U = \frac{1}{x^{-2}\cos^2 x} e^{-\int \frac{2}{x} dx}$

$$= \frac{e^{-2\ln x}}{x^{-2}\cos^2 x} = \frac{x^{-2}}{x^{-2}\cos^2 x} = \frac{1}{\cos^2 x}.$$

Thus

$$y_2(x) = y_1(x) \int \frac{1}{\cos^2 x} dx$$

$$= y_1(x) \int \sec^2 x \, dx$$

$$= y_1(x) \tan x$$

$$= \frac{\cancel{\cos x}}{x} \frac{\sin x}{\cancel{\cos x}}$$

$$= x^{-1} \sin x$$

Thus the general solution of (1) is

$$y(x) = c_1 x^{-1} \cos x + c_2 x^{-1} \sin x.$$

where c_1 and c_2 arbitary constants.

Example $(1-x^2)y'' - 2xy' + 2y = 0$, $y_1 = x$

Find the other L I sol^n of given ODE
(H.W)