ESE

KVD

(a) $x^2y'' + xy' - y = \frac{x^3}{x^2+1}$

AE is $m^2 - 1 = 0 \Rightarrow m = \pm 1$ $\therefore y_h(x) = c_1 x + c_2 x^{-1}$ — (1)

$W(x) = \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix} = -x^{-1} - x^{-1} = -\frac{2}{x}$ — $(\frac{1}{2})$

$W_1(x) = \begin{vmatrix} 0 & x^{-1} \\ 1 & -x^{-2} \end{vmatrix} = -\frac{1}{x} = -y_2$, $W_2(x) = \begin{vmatrix} x & 0 \\ 1 & 1 \end{vmatrix} = x = y_1$ — $(\frac{1}{2})$ $(\frac{1}{2})$

$r(x) = \frac{x}{x^2+1}$, $y_1(x) = x$, $y_2(x) = x^{-1}$

$u(x) = -\int \frac{y_2 r}{W} dx$

$= -\int x^{-1} \times \left(-\frac{x}{2}\right) \times \frac{x}{x^2+1} dx$

$u(x) = \frac{1}{4} \ln|x^2+1|$ — $(\frac{1}{2})$

$v(x) = \int \frac{y_1 r}{W} dx$

$= \int x \times \left(-\frac{x}{2}\right) \times \frac{x}{x^2+1} dx$

$= -\frac{1}{2} \int \frac{x^3}{x^2+1} dx$

$= -\frac{1}{2} \int \left[x - \frac{x}{x^2+1}\right] dx$ → by actual division

$v(x) = -\frac{x^2}{4} + \frac{1}{4} \ln|x^2+1|$ — $(\frac{1}{2})$

$\therefore y_p(x) = u(x) y_1 + v(x) y_2$

$= \frac{x}{4} \ln|x^2+1| - \frac{x}{4} + \frac{1}{4x} \ln|x^2+1|$

$\therefore$ GS is $y = y_h + y_p$ — $(\frac{1}{2})$

i.e. $y = c_1 x + c_2 x^{-1} + \frac{x}{4} \ln|x^2+1| - \frac{x}{4} + \frac{1}{4x} \ln|x^2+1|$

(b) Substitute $y = c_1 y_1 + c_2 y_2$ & it's derivatives in the LHS of $y'' + p(x) y' + q(x) y = 0$, we get — $(\frac{1}{2})$

$y'' + p(x) y' + q(x) y = (c_1 y_1 + c_2 y_2)'' + p(x)(c_1 y_1 + c_2 y_2)' + q(x)(c_1 y_1 + c_2 y_2)$

$= c_1 y_1'' + c_2 y_2'' + p(x)(c_1 y_1' + c_2 y_2') q(x)$
$+ c_1 y_1 + c_2 y_2$ — $(\frac{1}{2})$

$= c_1 (y_1'' + p(x) y_1' + y_1 q(x)) + c_2 (y_2'' + p(x) y_2' + q(x) y_2)$ — $(\frac{1}{2})$

$= 0$ — $(\frac{1}{2})$ $\because y_1$ & y_2 are solns of $y'' + p(x) y' + q(x) y = 0$

$\Rightarrow y = c_1 y_1 + c_2 y_2$ is a soln of $y'' + p(x) y' + q(x) y = 0$ on I.

(c) $P = \sin y \cos y + x \cos^2 y$, $Q = x$

$\frac{\partial P}{\partial y} = \cos^2 y - \sin^2 y - 2x \sin y \cos y$, $\frac{\partial Q}{\partial x} = 1$

$\therefore$ Not exact as $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ — $(\frac{1}{2})$

Now, $R^*(y) = \dfrac{Q_x - P_y}{P} = \dfrac{1-(\cos^2 y - \sin^2 y - 2x\sin y\cos y)}{\sin y\cos y + x\cos^2 y}$

$= \dfrac{2\sin^2 y + 2x\sin y\cos y}{\sin y\cos y + x\cos^2 y}$

$= \dfrac{2\sin y\,\cancel{(\sin y + x\cos y)}}{\cos y\,\cancel{(\sin y + x\cos y)}}$

$= 2\tan y$, a f^n of y only. — $(\frac{1}{2})$

$\therefore$ IF $= e^{\int R^*(y)dy} = e^{\int 2\tan y\,dy} = e^{2\ln\sec y} = e^{\ln\sec^2 y} = \sec^2 y = \dfrac{1}{\cos^2 y}$ — $(\frac{1}{2})$

$\therefore$ resulting exact DE is $(\tan y + x)dx + x\sec^2 y\,dy = 0$

$\therefore$ Sol^n is $u = \int_{y=\text{const}} M\,dx + K(y) = \int_{y=\text{const}} (\tan y + x)\,dx + K(y)$ — $(\frac{1}{2})$

$\Rightarrow u = x\tan y + \dfrac{x^2}{2} + K(y)$

To find $K(y)$, $u_y = \cancel{x\sec^2 y} + \dfrac{dK}{dy} = N = \cancel{x\sec^2 y} \Rightarrow \dfrac{dK}{dy} = 0 \Rightarrow K(y) = c_1$ — $(\frac{1}{2})$

$\therefore$ Ans is $x\tan y + \dfrac{x^2}{2} = c$ — $(\frac{1}{2})$

OR $(D^4 + K^4)y = 0$ — $(\frac{1}{2})$

AE is $D^4 + K^4 = 0 \Rightarrow D^4 + 2D^2K^2 + K^4 - 2D^2K^2 = 0$

$\Rightarrow (D^2 + K^2)^2 - (\sqrt{2}DK)^2 = 0$

$\Rightarrow D^2 + \sqrt{2}DK + K^2 = 0$, $D^2 - \sqrt{2}DK + K^2 = 0$

$\Rightarrow D = \dfrac{-\sqrt{2}K \pm \sqrt{2K^2 - 4K^2}}{2}$, $D = \dfrac{\sqrt{2}K \pm \sqrt{2K^2 - 4K^2}}{2}$

$\Rightarrow D = \dfrac{-\sqrt{2}K \pm \sqrt{-2K^2}}{2}$, $D = \dfrac{\sqrt{2}K \pm \sqrt{-2K^2}}{2}$

$\Rightarrow D = -\dfrac{K}{\sqrt{2}} \pm \dfrac{K}{\sqrt{2}}i$, $D = \dfrac{K}{\sqrt{2}} \pm \dfrac{K}{\sqrt{2}}i$ — $(1\frac{1}{2})$

$\therefore$ GS is $y = \left(c_1\cos\left(\dfrac{Kx}{\sqrt{2}}\right) + c_2\sin\left(\dfrac{Kx}{\sqrt{2}}\right)\right)e^{-\frac{Kx}{\sqrt{2}}}$
$+ \left(c_3\cos\left(\dfrac{Kx}{\sqrt{2}}\right) + c_4\sin\left(\dfrac{Kx}{\sqrt{2}}\right)\right)e^{\frac{Kx}{\sqrt{2}}}$ — (1)

(d) $y'' + 4y' + 3y = 0$

$\Rightarrow y'' = -4y' - 3y$

Let $y_1 = y$ & $y_2 = y'$ — $(\frac{1}{2})$

$\therefore y_1' = y' = y_2$ & $y_2' = y'' = -4y' - 3y = -4y_2 - 3y_1$

$\therefore$ System of equ^{ns} is $y_1' = y_2$, $y_2' = -3y_1 - 4y_2$ — $(\frac{1}{2})$

ESE - NOV 2023

Question [II] (20 marks) (Solution by Dr. Pallavi Shikhare)

(1) Write true or false for the following statements. [CO1] [2]

(a) Every function $f(t), t \geq 0$ has a Laplace transform.

$\frac{1}{2}$ → False. (for eg. $f(t) = e^{t^2}$, $f(t) = t^{-1/2}$, $t \geq 0$.)

(b) The Laplace transform of the product of two functions is the product of their Laplace transforms.

$\frac{1}{2}$ → False. (for eg. $f(t) = 1$, $g(t) = t$.

$L\{1\} \cdot L\{t\} = \frac{1}{s} \frac{1}{s^2} \neq L\{f \cdot g\} = L\{t\} = \frac{1}{s^2}$.)

(c) The function $f(t) = e^{4t^2} \sin t$ is of exponential order.

$\frac{1}{2}$ → False. Since $|f(t)| \leq |e^{4t^2}| \, |\sin t| \leq e^{4t^2} \nleq M e^{\alpha t}, \forall t \geq 0$

(Think on! why?) For any α and M.

(d) $L^{-1}\{2023 e^{-s}\} = 2023 \, u(t-1)$.

$\frac{1}{2}$ → False. Since $L^{-1}\{2023 e^{-s}\} = 2023 L^{-1}\{e^{-s}\} = 2023 \, \delta(t-1) u(t-1)$

(by t-shifting

$(\because L^{-1}\{1\} = \delta(t))$

---

(2) Proof of Existence Theorem for Laplace transform: [CO4] [2]

As $f(t)$ is piecewise continuous, $e^{-st} f(t)$ is integrable over any finite interval on $t \geq 0$. Also assume that growth restriction condition satisfies for $s > k$ (to be needed for the existence of the last of the integrals). We obtain the proof of the existence of $L\{f(t)\}$ form.

$$|F(s)| = |L\{f(t)\}| = \left| \int_0^\infty e^{-st} f(t) \, dt \right| \leq \int_0^\infty |f(t)| e^{-st} dt \leq \int_0^\infty M e^{kt} e^{-st} dt = \frac{M}{s-k}, \quad s > k.$$

$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$

$\Rightarrow |F(s)| \leq \frac{M}{s-k}, \; s > k. \rightarrow \frac{1}{2}$

Hence $L\{f(t)\}$ exists for all $s > k$.

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(3) Attempt ANY TWO. [CO3] [6]

(a) $L\{3\cos(4t+7) + t^{3/2}\}$ (3 marks)

$\rightarrow L\{3\cos(4t+7) + t^{3/2}\}$

$= 3L\{\cos 4t \cdot \cos 7 - \sin 4t \cdot \sin 7\} + L\{t^{3/2}\}$ — $1\frac{1}{2}$

L is linear.
$\cos(a+b) = \cos(a)\cdot\cos(b) - \sin(a)\sin(b)$

$= 3\cos 7 \, L\{\cos 4t\} - 3\sin 7 \, L\{\sin 4t\} + L\{t^{3/2}\}$

$= 3\cos 7\left(\frac{s}{s^2+4^2}\right) - 3\sin 7\left(\frac{4}{s^2+4^2}\right) + \frac{\sqrt{\frac{5}{2}}}{s^{5/2}}$ $\longrightarrow 1\frac{1}{2}$

(b) $L\left\{\int_0^t e^{-3\tau}\frac{1}{\sqrt{\tau}}\,d\tau\right\}$ (3 marks)

As we know, $L\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s} = \frac{L\{f(t)\}}{s}$

$1\frac{1}{2} \rightarrow L\left\{\int_0^t e^{-3\tau}\frac{1}{\sqrt{\tau}}\,d\tau\right\} = \frac{1}{s}\,L\left\{e^{-3t}(t)^{-\frac{1}{2}}\right\}$ ... by Laplace Transform of integral

$1\frac{1}{2} \longrightarrow = \frac{1}{s}\,\frac{\sqrt{\frac{1}{2}}}{(s+3)^{1/2}}$ ... (by $L\left\{\frac{1}{\sqrt{t}}\right\} = \frac{\sqrt{1/2}}{s^{1/2}} = \sqrt{\frac{\pi}{s}}$ and 1^{st} shifting th^{m})

$= \frac{1}{s}\sqrt{\frac{\pi}{s+3}}$

(c) $L\left\{\frac{d}{dt}\left(\frac{\sin t}{t}\right)\right\}$ (3 marks)

As we have $L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{u^2+1}\,du$ ... by integration of Laplace transform. — 1

$= \lim_{T\to\infty}\int_s^T \frac{1}{u^2+1}\,du = \lim_{T\to\infty}\left(\tan^{-1}u\right)_s^T$

$= \frac{\pi}{2} - \tan^{-1}s$

$= \cot^{-1}s$ — 1

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$$\left\{\frac{d}{dt}\left(\frac{\sin t}{t}\right)\right\} = s\, L\left\{\frac{\sin t}{t}\right\} - f(0) \quad \because \text{ by } L\{f'(t)\} = s L\{f(t)\} - f(0)$$

$$= s\, L\left\{\frac{\sin t}{t}\right\} - \lim_{t\to 0} \frac{\sin t}{t}$$

$$= s \cot^{-1} s - 1 \quad \text{—— 1} \quad \text{ i.e. } s\left(\frac{\pi}{2} - \tan^{-1}(s)\right) - 1.$$

(4) Find the inverse Laplace transform of the following functions. (Any Three).

(a) $\displaystyle\sum_{p=0}^{2} \frac{(p+1)^2}{s^2+p^2}$ (2 marks)

$$= \frac{(0+1)^2}{s^2+0^2} + \frac{(1+1)^2}{s^2+1} + \frac{(2+1)^2}{s^2+4} \quad \text{—— 1}$$

$$= \frac{1}{s^2} + \frac{4}{s^2+1} + \frac{9}{s^2+4}.$$

Let $F(s) = \frac{1}{s^2} + \frac{4}{s^2+1} + \frac{9}{s^2+4}$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^2}\right\} + 4\, L^{-1}\left\{\frac{1}{s^2+1}\right\} + \frac{9}{2} L^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$= t + 4\sin t + \frac{9}{2}\sin 2t \quad \text{—— 1}$$

(b) $\ln\sqrt{\left(\frac{s^2-a^2}{s^2}\right)} = \ln\left(\frac{s^2-a^2}{s^2}\right)^{\frac{1}{2}} = \frac{1}{2}\cdot\ln\left(\frac{s^2-a^2}{2}\right)$ (2 marks)

Let $F(s) = \frac{1}{2}\ln\left(\frac{s^2-a^2}{s^2}\right) = \frac{1}{2}\left[\ln(s^2-a^2) - \ln s^2\right].$ $\hookrightarrow \frac{1}{2}$

Then

$$F'(s) = \frac{1}{2}\left[\frac{1}{s^2-a^2}(2s) - \frac{1}{s^2}(2s)\right]$$

$$F'(s) = \frac{s}{s^2-a^2} - \frac{1}{s} \quad \text{— (1) —— } \frac{1}{2}$$

As we know $L\{t f(t)\} = -F'(s) \Rightarrow L^{-1}\{F'(s)\} = -t f(t)$

(Differentiation of Laplace transform).

Taking inverse Laplace of (1)

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s}{s^2-a^2}\right\} - L^{-1}\left\{\frac{1}{s}\right\}$$

$$-t f(t) = \cosh at - 1 \quad — \tfrac{1}{2}$$

$$f(t) = \frac{1-\cosh at}{t} \quad — \tfrac{1}{2}$$

(c) $e^{-11s}\left(\frac{s-1}{s^2-2s+2}\right)$ (2 marks)

Let $F(s) = e^{-11s}\left(\frac{s-1}{s^2-2s+2}\right) = e^{-11}\left[e^{-11s+11}\frac{s-1}{(s-1)^2+1}\right]$

Taking inverse Laplace transform.

$$f(t) = e^{-11} L^{-1}\left\{e^{-11(s-1)}\frac{s-1}{(s-1)^2+1}\right\} \quad — 1$$

$$\tfrac{1}{2} \leftarrow = e^{-11} e^{t} L^{-1}\left\{e^{-11s}\frac{s}{s^2+1}\right\} \text{ -- by s shifting}$$

$$\tfrac{1}{2} \leftarrow = e^{t-11}\cos(t-11)\,u(t-11) \text{ -- by t shifting.}$$

(d) $\frac{2s}{(s^2-4)^2}$ (2 marks)

Let $L^{-1}\left\{\frac{2s}{(s^2-4)^2}\right\} = f(t)$ --- (integration of Laplace transform.)

$$\therefore L^{-1}\left\{\int_s^{\infty}\frac{2u}{(u^2-4)^2}\,du\right\} = \frac{f(t)}{t} \quad — \tfrac{1}{2}$$

$$\therefore L^{-1}\left\{\lim_{T\to\infty}\left(-\frac{1}{u^2-4}\right)_s^{\infty}\right\} = \frac{f(t)}{t}$$

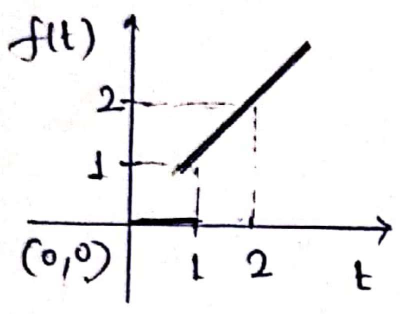
$$\therefore L^{-1}\left\{\frac{1}{s^2-4}\right\} = \frac{f(t)}{t} \quad — \tfrac{1}{2}$$

$$\therefore \frac{1}{2} L^{-1}\left\{\frac{2}{s^2-4}\right\} = \frac{f(t)}{t}$$

$$\Rightarrow \frac{\sinh 2t}{2} = \frac{f(t)}{t} \quad \frac{1}{2}$$

$$\Rightarrow f(t) = L^{-1}\left\{\frac{2s}{(s^2-4)^2}\right\} = \frac{t \sinh 2t}{2} \quad - \frac{1}{2}$$

(5) Write the piecewise continuous function $f(t)$ from the following graph. Further, determine the Laplace transform of $f(t)$ by expressing the function in terms of the unit step function. [CO3] [2]



$$f(t) = \begin{cases} 0; & 0 \le t < 1 \\ t; & t \geq 1 \end{cases} \quad - \frac{1}{2}$$

$$f(t) = t\, u(t-1) \quad - \frac{1}{2}$$

$$L\{f(t)\} = L\{t\, u(t-1)\}$$
$$= e^{-s} L\{t+1\} \quad - \frac{1}{2}$$
$$= e^{-s}\left[\frac{1}{s^2} + \frac{1}{s}\right] \quad - \frac{1}{2}$$

(6) Solve the integral equation $y(t) + 2e^{t}\int_0^t y(\tau)\, e^{-\tau} d\tau = te^{t}$ [CO3] [2]

$$\rightarrow y(t) + 2e^{t}\int_0^t y(\tau)\, e^{-\tau} d\tau = te^{t}$$

$$y(t) + 2\int_0^t y(\tau)\, e^{t-\tau} d\tau = te^{t} \quad - \frac{1}{2}$$

$\frac{1}{2} \leftarrow y(t) + 2\left(y(t) * e^{t}\right) = te^{t}$ --- by def^n of convolution

$L\{y(t)\} + 2L\{y(t) * e^{t}\} = L\{te^{t}\}$ ... Taking Laplace transform on both sides and it is linear.

$\Rightarrow L\{y(t)\} + 2\, L\{y(t)\}\, L\{e^{t}\} = L\{te^{t}\}$ ·· by Convolution Theorem,

$\Rightarrow L\{y(t)\} + \frac{2}{s-1} L\{y(t)\} = -\frac{d}{ds}\left(\frac{1}{s-1}\right)$ ······ $L\{t\,f(t)\} = -\frac{d}{ds}[F(s)]$

$L\{e^{t}\} = \frac{1}{s-1}$

$\Rightarrow L\{y(t)\}\left(\frac{s+1}{s-1}\right) = \frac{1}{(s-1)^2}$ $\quad \frac{1}{2}$

$\Rightarrow L\{y(t)\} = \frac{s-1}{(s-1)^2(s+1)}$

$\Rightarrow y(t) = L^{-1}\left\{\frac{1}{(s-1)(s+1)}\right\}$ ··· by taking inverse transform.

$\Rightarrow y(t) = \frac{1}{2} L^{-1}\left\{\frac{1}{s-1} - \frac{1}{s+1}\right\}$

$\Rightarrow y(t) = \frac{1}{2}\left(e^{t} - e^{-t}\right)$ $\quad — \frac{1}{2}$

$= \sinh t$

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I).

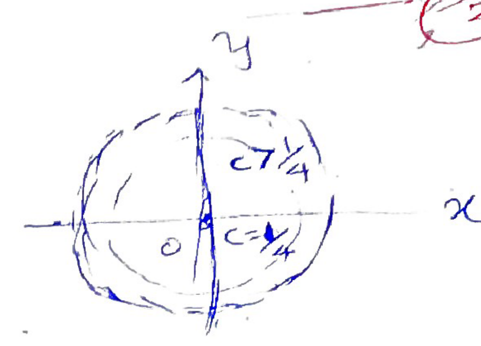
a). $f(x,y) = \dfrac{1}{\sqrt{16-x^2-y^2}}$ (3)

a) Domain $= \{(x,y) \in \mathbb{R}^2 \mid 0 \le x^2+y^2 < 16 \; , x,y \in \mathbb{R}\}$ (1)

b) Range $= \{f(x,y) \in \mathbb{R} \mid \frac{1}{4} \le f(x,y) < \infty\}$ ($\frac{1}{2}$)

c) Level curves:

$$x^2+y^2 = 16 - \frac{1}{c^2}$$



$\therefore c = \frac{1}{4} \Rightarrow (0,0)$ Origin.

$c > \frac{1}{4} \Rightarrow$ family of circles centered at $(0,0)$ & $r < 4$. ($\frac{1}{2}$)

d) Boundary of f^{ns} domain: $x^2+y^2=16$ ($\frac{1}{2}$)

e). f^{ns} domain is an Open region $\because$ Bpts not included. ($\frac{1}{2}$)

(2)

II). $f(x,y) = \dfrac{2xy}{x^2+y^2}$

$$\therefore \lim_{\substack{(x,y)\to(0,0) \\ y=x}} \frac{2xy}{x^2+y^2} = \lim_{x\to 0} \frac{2x^2}{2x^2} = 1 \quad (\tfrac{1}{2})$$

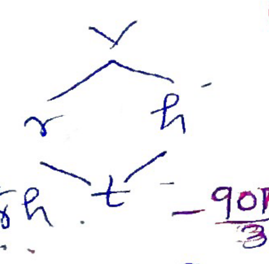
$$\text{lly} \lim_{\substack{(x,y)\to(0,0) \\ y=-x}} \frac{2xy}{x^2+y^2} = \lim_{x\to 0} \frac{-2x^2}{2x^2} = -1 \quad (\tfrac{1}{2})$$

$\therefore$ By 2 path test ($\frac{1}{2}$), limit doesnot exits ($\frac{1}{2}$).

(3)

III). $h = 15$ cms, $r = 10$ cms

$\frac{dh}{dt} = 0.2$ cms/sec, $\frac{dr}{dt} = -0.3$ cm/sec.



where $V = \frac{1}{3}\pi r^2 h$ (1)

$-\frac{90\pi}{3} + \frac{20}{3}$

$$\therefore \frac{dV}{dt} = \frac{\partial V}{\partial r}\cdot\frac{dr}{dt} + \frac{\partial V}{\partial h}\cdot\frac{dh}{dt} \quad (1)$$

$$= \frac{2\pi r h}{3}\frac{dr}{dt} + \frac{\pi r^2}{3}\frac{dh}{dt} = \frac{300}{3}\pi \times (-0.3) + \frac{100}{3}\pi \times (0.2) \quad (1)$$

$$= -\frac{70\pi}{3} \text{ cm}^3/\text{sec} = (-23.3)\pi \text{ cm}^3/\text{sec} = -73.3038$$

IV) $f(x,y) = xe^{y} + \cos(xy)$. , $(2,0)$.

(3)

$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]$$

$$= \left[e^{y} - y\sin(xy),\ xe^{y} - x\sin(xy)\right]$$ —— (1)

$\therefore$ $\boxed{\nabla f(2,0) = [1, 2]}$ —— ($\frac{1}{2}$)

a) Dir^{n} in which f increases rapidly at $(2,0)$ $= \dfrac{\nabla f_P}{|\nabla f_P|}$

$$= \left[\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right]$$ —— ($\frac{1}{2}$)

i.e $[0.447, 0.894]$

b) Dir^{n} in which f decreases rapidly at $(2,0)$ $= \dfrac{-\nabla f_P}{|\nabla f_P|} = \left[-\dfrac{1}{\sqrt{5}}, \dfrac{-2}{\sqrt{5}}\right]$ —— ($\frac{1}{2}$)

c) Dir^{ns} of zero change at $(2,0)$ are

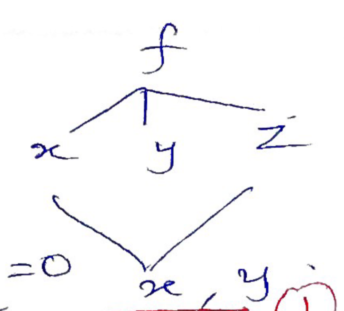
$$\nabla f_P \cdot \hat{n} = 0$$

$$[1, 2] \cdot [a, b] = 0$$

$$\left[\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right] \ \& \ \left[\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right]$$ —— ($\frac{1}{2}$)

V) Proof:- $f(x,y,z) = 0$. (3)

Diff. w.r.t. x partially.



$$\frac{\partial F}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial F}{\partial y}\cdot\cancel{\frac{\partial y}{\partial x}}^{0} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} = 0$$ (1)

~~where~~

(x, y indep. varis.)

$$F_x + F_z \cdot \frac{\partial z}{\partial x} = 0$$ (1)

$$\therefore \boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}}$$

given $f(x,y,z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$.

$$\therefore \frac{\partial z}{\partial x} = -\left(\frac{-1/x^2}{-1/z^2}\right)$$ —— ($\frac{1}{2}$)

$$\boxed{\frac{\partial z}{\partial x}(1,1,1) = -1}$$ —— ($\frac{1}{2}$)

VI). $f(x,y) = x^3 + 3xy + y^3$ ESE (3)

Step I: $f_x = 3x^2 + 3y$, $f_y = 3y^2 + 3x$.

solving $f_x = 0$ & $f_y = 0$.

$\therefore$ $x^2 + y = 0 \Rightarrow \boxed{y = -x^2}$

$x + y^2 = 0$

$\therefore$ $x + x^4 = 0$ $\therefore$ $x(x+1)(x^2 - x + 1) = 0$

$\therefore$ $x = 0,\ x = -1$.

$\therefore$ $\boxed{\text{Critical pts. } (0,0) \ \& \ (-1,-1)}$ — 1.

Step II:

$f_{xx} = 6x$ $f_{xy} = 3$

$f_{yx} = 3$ $f_{yy} = 6y$.

Case-I: (0,0).

$\Delta_2 = \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} = -9 < 0$. $\left\{ \begin{array}{l} \therefore f^n \text{ neither} \\ \text{attains local max} \\ \text{nor local mini.} \end{array} \right.$

$\therefore$ $\boxed{(0,0) \text{ saddle pt}}$ — 1

case-II: (-1,-1).

$\Delta_2 = \begin{vmatrix} -6 & 3 \\ 3 & -6 \end{vmatrix} = 27 > 0$ $\left\{ \begin{array}{l} \therefore f^n \text{ attains} \\ \text{local extrema} \\ \text{at } (-1,-1) \end{array} \right.$

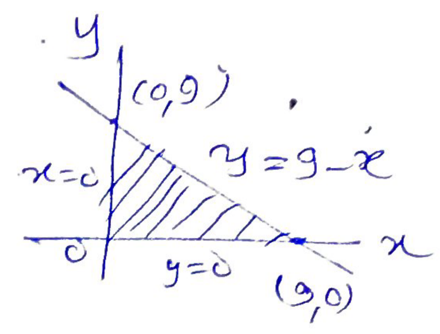
Since $\Delta_2 > 0$ & $\Delta_1 = -6 < 0$.

$\therefore$ $\boxed{f^n \text{ attains local maxima at } (-1,-1)}$

$\therefore$ $f_{max}(-1,-1) = -1 + 3 - 1 = 1$ Ans

— 1.

VII). $f(x,y) = 2 + 2x + 2y - x^2 - y^2$



f $\quad f_x = 2 - 2x \quad ; \quad f_y = 2 - 2y$

Solving $f_x = 0$. & $f_y = 0$

$\therefore \boxed{(1,1) \text{ . critical pt}}$ interior pt

$f(1,1) = 4 \longrightarrow$ abs max .

Edge : $x = 0$.

$f(0,y) = 2 + 2y - y^2 \quad , \quad 0 \le y \le 9$.

$f_y = 2 - 2y = 0 \quad . \quad \therefore \boxed{y = 1}$

$f(0,1) = 3$;

Edge : $y = 0$.

$f(x,0) = 2 + 2x - x^2 \; ; \; 0 \le x \le 9$

$f_x = 2 - 2x = 0 . \quad \therefore \boxed{x = 1.}$

$f(1,0) = 3$.

Edge : . $y = 9 - x$.

$f(x, 9-x) = 2 + 2x + 2(9-x) - x^2 - (9-x)^2$

$= -2x^2 + 18x - 61$.

$f_x = -4x + 18 = 0 . \quad \therefore \boxed{x = \frac{9}{2}}$

$f(9/2, 9/2) = -20.5$

But $f(0,9) = f(9,0) = -61 . \longrightarrow$ abs. mini

$\sqrt{\text{VII}}b$ (OR) ESE (3)

Minimize : $d = \sqrt{x^2+y^2+z^2}$ i.e $d^2 = x^2+y^2+z^2$

$f(x,y,z) = x^2+y^2+z^2$

constraints : $x-2y-2z = 1$

$g(x,y,z) = x-2y-2z-1 = 0$

Lagrange's f^n : $\boxed{\nabla f = \lambda \nabla g}$ ———— 1/2

$[2x, 2y, 2z] = \lambda[1, -2, -2]$ ———— 1/2

On comparing the components

$\therefore x = \frac{\lambda}{2}$, $y = -\lambda$, $z = -\lambda$ ———— 1/2

substituting x, y, z in $g(x,y,z) = 0$

$\therefore \frac{\lambda}{2} + 2\lambda + 2\lambda - 1 = 0$ ———— 1/2

$\therefore \lambda = \cancel{\ } \ 2/9$

$\therefore \boxed{x = 1/9, \quad y = -\frac{2}{9}, \quad z = -\frac{2}{9}}$ ———— 1/2

$\therefore$ ~~(1/, -2/, -2/) is cat~~

$\therefore d^2 = x^2+y^2+z^2$

$= \frac{1}{9}$

$\therefore d = \pm\frac{1}{3}$ ———— 1/2

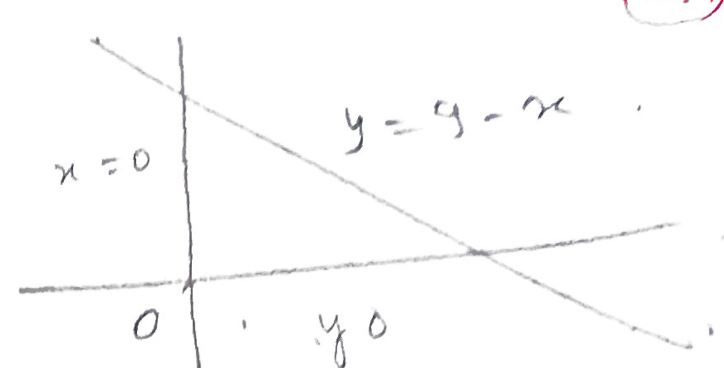
$\therefore$ shortest dist. from (0,0) to plane $x-2y-2z = 1$ is $(+1/3)$ units. {$\because$ dist always +ve.

VII $f(x,y) = 2 + x + y$

$f_x = 1$.

$f_y = 1$

No.



1

Edge : $x = 0$, $f(0, y) = 2 + y$ $\quad f_y = 1 = 0$
$0 \le y \le 9$. $\quad$ No

$y = 0$, $f(x, 0) = 2 + x$. $f_x = 1$
$0 \le x \le 9$ $\quad$ No

$y = 9 - x$. $\quad f(x, 9-x) = 2 + \cancel{x} + 9 - \cancel{x}$. $\quad f_x = 0$
$= 11$ $\quad$ 1 No

$(0,0)$: $f(0,0) = 2$. abs mini.

$(0,9)$ $\quad f(0,9) = 11$ $\Big\}$ abs max
$(9,0)$. $\quad f(9,0) = 11$.

1 .

ODEMVC - 2023-24

ESE Solution

① $\omega = mg = 9.8 \times 10 = 98$

$k = \frac{\omega}{l} = \frac{98}{0.7} = 140$ — $\frac{1}{2}$

$\therefore\ 10y'' + 90y' + 140y = 0$
with $y'(0) = -1$ m/sec
$y(0) = 0$ — $\frac{1}{2}$ m

$y'' + 9y' + 14y = 0$

A.E. $\rightarrow \lambda^2 + 9\lambda + 14 = 0$

$(\lambda+2)(\lambda+7)$

$\therefore$ G.S. $\rightarrow y = C_1 e^{-2t} + C_2 e^{-7t}$ — 1 M

$y(0) = 0 \Rightarrow C_1 + C_2 = 0$

$y'(0) = -1 \Rightarrow -2C_1 - 7C_2 = -1$

by solving, $C_1 = -\frac{1}{5}$, $C_2 = \frac{1}{5}$

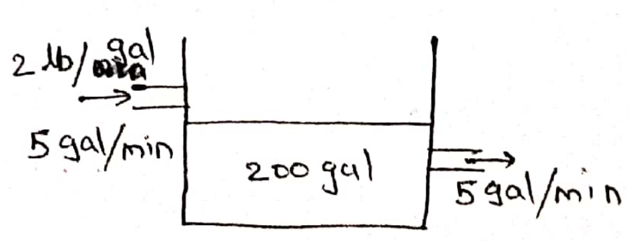
$\therefore\ y(t) = \frac{1}{5}\left(e^{-7t} - e^{-2t}\right)$ — 1 m

$\boxed{\text{OR}}$

②

$\frac{dy}{dt} = 10 - \frac{5y}{200}$

$y(0) = 40$



$\frac{dy}{dt} + \frac{5y}{200} = 10$ $\qquad y(0) = 40$

$\frac{dy}{dt} + \frac{y}{40} = 10$ — 1 m

I.F. $= e^{t/40}$

solⁿ is, $y e^{t/40} = \int e^{t/40} \times 10\,dt + C$

$= 400 e^{t/40} + C$

$y = 400 + C e^{-t/40}$ — 1 m

$\because y(0) = 40 \rightarrow C = 40 - 400 = -360$

$y = 400 - 360 e^{-t/40}$ ~~$y = 400(1 + e^{-t/40})$~~ — 1 m

(2)

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

ch. eqⁿ :- $\lambda^2 - 12\lambda + 27 = 0$

eigen values - $(\lambda-3)(\lambda-9) = 0$

$\lambda_1 = 3, \lambda_2 = 9$ — $\frac{1}{2}$ m

$\lambda_1 = 3 \to \begin{bmatrix} 3 & 9 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 + 3x_2 = 0$

Let $x_2 = t' \Rightarrow x_1 = -3t'$

$\therefore X_1 = t'\begin{bmatrix} -3 \\ 1 \end{bmatrix} \Rightarrow Y_1 = e^{3t}\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ — 1m

$\lambda_2 = 9 \to \begin{bmatrix} -3 & 9 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 - 3x_2 = 0$

$X_2 = t'\begin{bmatrix} 3 \\ 1 \end{bmatrix} \Rightarrow Y_2 = e^{9t}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ — 1m

$\therefore Y = c_1 Y_1 + c_2 Y_2$

$= c_1 e^{3t}\begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2 e^{9t}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ — $\frac{1}{2}$

(3) $y'' + 4y = \delta(t-\pi) - \delta(t-2\pi)$

$s^2 Y(s) - s(0) - 1 + 4Y(s) = e^{-\pi s} - e^{-2\pi s}$ — 1m

$(s^2+4) Y(s) = \dfrac{e^{-\pi s} - e^{-2\pi s} + 1}{s^2+4}$ — 1m

$\therefore y(t) = u(t-\pi)\dfrac{\sin 2(t-\pi)}{2} - u(t-2\pi)\dfrac{\sin 2(t-2\pi)}{2} + \dfrac{\sin 2t}{2}$

$= \dfrac{-\sin 2t\, u(t-\pi) - \sin 2t\, u(t-2\pi) + \sin 2t}{2}$

$= \dfrac{\sin 2t}{2}\left(1 - u(t-\pi) - u(t-2\pi)\right)$ — 2m