Mixing Problem :- (Single tank problem)

This is one more application of 1^{st} order differential equation.

Suppose your working in chemical factory or industry, and you have a large container which contains some solution and you are mixing that solution with some chemical and the process is happing and suppose you want to determine at a given time, what is the amount chemical in that solution?

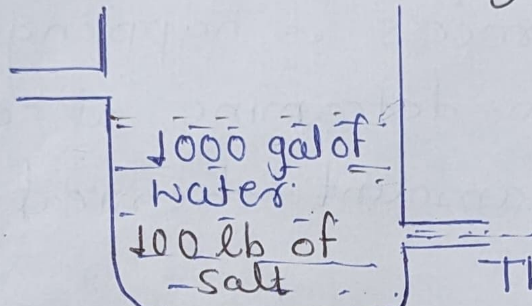
So here I am going to answer to above question with the help of the basic model involving a single tank. If you have more than one tank (2 or 3) in that case we have more than one differential equation (2 or 3), that is we have system of ordinary differential equations.

Problem :-

A tank contains 1000 gal of water in which 100 lb (pounds) of salt is dissolved. Brine runs in at a rate of 10 gal/min, and each gallon contains 5 lb of dissoved salt. The mixture in the tank is kept uniform by stirring. Brine runs out at 10 gal/min. Find the abount of salt in the tank at any time t.

→

This is a inlet



This is a outlet, and mixture is flowing with the same rate i.e 10 gal/min

From this inlet water is comming at the rate of 10 gal/min and each gallon contains 5 lb of salt.

Let y(t) denote the abount of salt present in the tank at time t.

So what is $\frac{dy}{dt}$ or y'? or what we $\frac{dy}{dt}$ represen

It represent the rate at which the amount of salt in water (or tank) is changing and how do you find the rate at which the amount of salt is changing, for that you see at what rate the salt is coming in and you see at what rate the salt goes out

i.e y' = Rate at which salt comes in (salt inflow rate) − Rate at which ~~salt~~ salt goes out (salt outflow rate) —— (1)

This rate of change of salt in the tank.

Let's try to find the inflow rate, so I know that per ~~minute~~ 10 gallons of water is coming in from the inlet and per gallon contain how much amount of salt 5 pounds, so we have.

(rate of mixture goes inside the salt)

$$\text{Inflow rate} = 5\,\frac{lb}{gal} \times \left(10\,\frac{gal}{min}\right) = 50\ lb/min$$

So salt is coming in at the rate of 50 pounds per minute.

Oky. try to find the outflow rate, i.e at what rate salt is going out.

$$\text{outflow rate} = \left(\frac{y}{1000}\,\frac{lb}{gal}\right) \times 10\,\frac{gal}{min}$$

$$= 0.01y\ lb/min$$

(How much mixture is going out?)

{And for each gallon what is the amount of salt present for that see that tank contains

1000 gallons of mixture in the tank. ⇝ $y(t)$ is amount of salt present in the tank

1 gallon of mixture ⇝ x (say)
[How mush amount of salt will be present in 1 gal of mixture]

i.e $x = \frac{y(t)}{1000}$ ↳ This a amount of salt present per gal.}

Thus from (1)

$y' = 50 - 0.01y$, i.e $\frac{dy}{dt} = 50 - 0.01y$

by variable seperable methode

$$\int \frac{dy}{50 - 0.01y} = \int dt$$

$$\Rightarrow \frac{\ln|50 - 0.01y|}{-0.01} = t + d$$ some constant

$$\Rightarrow \ln|50 - 0.01y| = (t+d)(-0.01)$$

$$\Rightarrow 50 - 0.01y = Ce^{(-0.01)t}$$

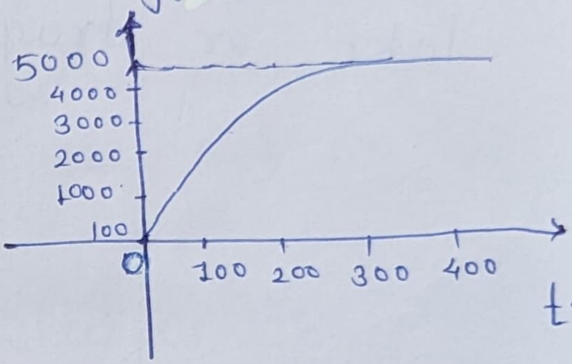
$$y = \frac{50 - C e^{(-0.01)t}}{0.01}$$

$$= \frac{50}{0.01} - \frac{C}{0.01} e^{-0.01t}$$

$$y(t) = 5000 - 100C e^{-0.01t} \quad \text{—— (2)}$$

Initially the tank contains 100 lb of salt. Hence

$y(0) = 100$, is initial condition.



So from (2).

$$100 = y(0) = 5000 - 100C e^{0}$$

$$100 = 5000 - 100C$$

$$\Rightarrow 100C = 4900$$

put in eqn (2)

$$y(t) = 5000 - 4900 e^{-0.01t}$$ (unit of time - min)

What happen as $t \to \infty$?

$y(t) = 5000$, so

This is a limit of salt. 5000 lb.

How mush salt is left in the tank after 1 hour?

$t = 1\ hr = 60\ min$

$\therefore y(t) = 5000 - 4900\, e^{-0.01t}$

$\Rightarrow y(t) = 5000 - 4900\, e^{(-0.01)60}$

This gives your answer.

The model discussed become more realistic in problem on pollutants in lake or drugs in orgain.

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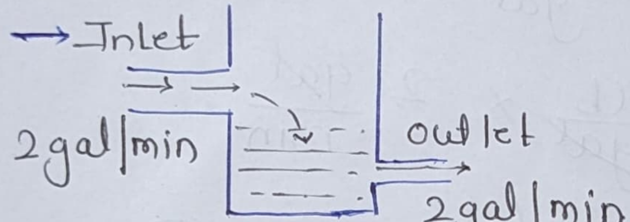
A tank contains 400 gal of brine in which 100 lb of salt are dissolved. Fresh water runs into the tank at a rate of 2 gal/min. The mixture kept practically uniform by stirring, runs out at the same rate. How much salt will be in the tank at the end of 1 hour?



At $t = 0$ we have

Brine - 400 gal
salt - 100 lb (pound)

Let $y(t)$ be amount of salt in the tank at any time t.
Then, what is $\frac{dy}{dx}$?
It represent the rate at which the about of salt is changing in the tank (How to calculate?)

By Balance law,

$$y' = \left(\begin{matrix}\text{Salt inflow} \\ \text{rate}\end{matrix}\right) - \left(\begin{matrix}\text{salt outflow} \\ \text{rate}\end{matrix}\right) \quad \text{--- (1)}$$

But in question, the fresh water runs into the tank (Here is no salt).

$\therefore$ salt inflow rate $= 0$ --- (2)

Outflow rate : $\begin{pmatrix}\text{Amount of salt present (in 1 gal) of mixture}\end{pmatrix} \cdot \begin{pmatrix}\text{Rate of mixture goes out}\end{pmatrix}$

(say x) — How to find?

(given. 2 gal/min)

For that, see.

400 gal $\longrightarrow$ $y(t)$ lb (pound) salt

1 gal $\longrightarrow$ say x

$\therefore x = \frac{y(t)}{400} \frac{\text{lb}}{\text{gal}}$

$\therefore$ outflow rate : $\frac{y(t)}{400} \cdot \frac{\text{lb}}{\cancel{\text{gal}}} \times 2 \frac{\cancel{\text{gal}}}{\text{min}}$

$= \frac{y(t)}{200}$ lb/min —③

Using ② and ③ in ① we get

$\Rightarrow \frac{dy}{dt} = 0 - \frac{y}{200}$

$\Rightarrow \int \frac{dy}{y} = -\frac{1}{200} \int dt$ — by variable seperable

$\Rightarrow \log|y| = -\frac{1}{200} t + \log c$

$\Rightarrow \ln\left(\frac{y}{c}\right) = -0.005t$

$y(t) = c\, e^{-0.005t}$ — This is a general soln

But, given $y(0) = 100$, i.e at $t=0$, the salt is 100.

$$100 = y(0) = ce^{0} = c$$

$$\boxed{c = 100}$$

Thus we have.

$$y(t) = 100\, e^{-0.005t}$$, (here t is in min. why? Think of!)

This is particular solⁿ.

Further

$t = 1\text{ hr} = 60\text{ min}.$

$$y(t) = y(60) = 100\, e^{(-0.005)(60)}$$

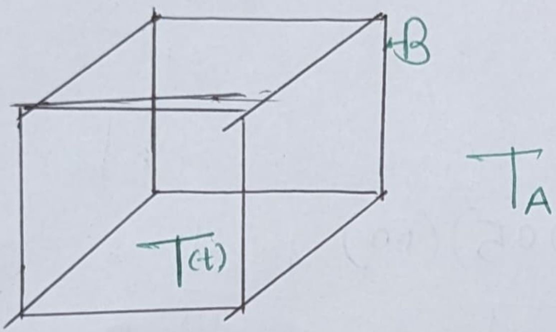
$$= 100 \times 0.7408$$

$$= 74.08\text{ lb.}$$

So after 1 hr salt will be 74.08 lb.

Newton's Law of Cooling :-

Expriments show that the time rate of change of the temperature $T(t)$ of the body B is proportional to difference between $T(t)$ and temperature of the surrounding mediam T_A



$$\frac{dT}{dt} \propto (T - T_A), \quad T_A \text{-constant}$$

Example : A body originally at 80°C cools down to 60°C in 20 minutes. The tep. of the air being 40°C. What will be the tep. of the body after 40 min from the original ?

Solⁿ Given data

		$T(t)$	T_A
1st IC	$t = 0$ min.	$80^\circ C$	$40^\circ C$
2nd IC	$t = 20$ (min)	$60^\circ C$	$40^\circ C$
	$t = 40$ (min)	$\boxed{?}$	$40^\circ C$

$$\frac{dT}{dt} = k(T - T_A)$$

$$\int \frac{dT}{T-40} = \int k\,dt$$

$$\log(T-40) - \log c = kt$$

$$\log\left(\frac{T-40}{c}\right) = kt$$

$$T - 40 = c e^{kt}$$

$$T(t) = 40 + c e^{kt} \quad \text{--- (1)}$$

By using 1st IC i.e. $T(0) = 80^\circ C$, so from (1)

$$80 = 40 + c e^{0k}$$

$$\Rightarrow \boxed{40 = c}$$

So (1) become $T(t) = 40 + 40 e^{kt}$ --- (2)

Now by 2^{nd} IC i.e $T(20) = 60°C$. so from ②, we have

$$60 = 40 + 40e^{k(20)}$$

$$20 = 40e^{20k}$$

$$0.5 = e^{20k}$$

$$\ln(0.5) = 20k$$

$$k = -0.0346$$

Thus ② become

$$T(t) = 40 + 40e^{-0.0346t} \quad \text{③}$$

Further

$$T(40) = 40 + 40e^{-0.0346(40)}$$

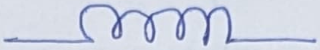
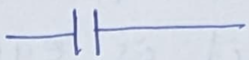
$$T(40) = 40 + 40 \times 0.25$$

$$= 40 + 10$$

$$T(40) = 50°C.$$

Application to Electric Circuit :-

Notation

Term	Notation	Diagram
Current	I	
charge	q	
resistance	R	
Inductance	L	
Capacitance	C	
Electromotive force (e.m.f) or Voltage	E or V	 (battery)

Note

(i) Voltage drop across inductor := $L\dfrac{dI}{dt}$

(ii) Voltage drop across resistor $= RI$

(iii) Voltage drop across capacitor $= \dfrac{q}{C}$

(iv) Also note that $I = \dfrac{dq}{dt} \Rightarrow q = \int I\, dt$.

(v) Ohm's law formula

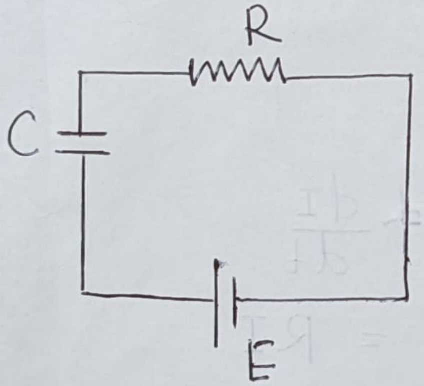
$$V = IR$$

By using the Kirchhoff's law (Current and Voltage) you can setup the differential equation of any circuit.

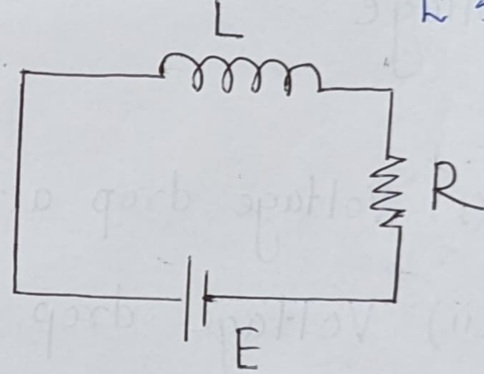
To set up the differential equation :-

If in the circuit containing inductance (L) then add $L\frac{dI}{dt}$ (i.e voltage drop across the inductor), if containing resistance (R) then add RI (i.e voltage drop across resistor), if containing capacitance (C) then add $\frac{q}{C}$ (i.e Voltage drop across capacitor) to L.H.S and it is equal to given voltage or e.m.f (E) to the circuit.

RC Circuit (Contain R & C)



LR Circuit (Contain L & R)



E = Voltage drop across R + Voltage drop across C

$$= RI + \frac{q}{C}$$

$$\boxed{E = R\frac{dq}{dt} + \frac{q}{C}}$$

First order ODE.

E = Voltage drop across L + Voltage drop across R

$$\boxed{E = L\frac{dI}{dt} + RI}$$

Example

Two 9-volt batteries are connected to a series in which the inductance is $\frac{1}{4}$ henry and the resistance is 8 ohms. Determine the current $I(t)$ if the initial current is zero.

→ Here equation of circuit is

$$L\frac{dI}{dt} + RI = E$$

$$\Rightarrow \quad \frac{dI}{dt} + \frac{R}{L}I = \frac{E}{L} \quad \text{—} \ (1)$$

given data $R = 8$ ohms, $L = \frac{1}{4}$ henry

$E = 18$ volt

So $\frac{R}{L} = \frac{8}{1/4} = 32$

$\frac{E}{L} = \frac{18}{1/4} = 72$

From (1),

$$\frac{dI}{dt} + 32I = 72 \quad (\text{Linear } 1^{st} \text{ order ODE})$$

Here integrating facter $I.F = e^{\int 32\,dt} = e^{32t}$

It's sol^n is

$$I(t)(I.F) = \int 72\,(I.F)\,dt + a$$

$$I(t).e^{32t} = \int 72\,e^{32t}\,dt + a$$

$$I(t).e^{32t} = 72\,\frac{e^{32t}}{32} + a$$

$$e^{32t}\, I(t) = 72\left(\frac{e^{32t}}{32}\right) + a$$

$$\Rightarrow I(t) = \frac{72}{32}\frac{e^{32t}}{e^{32t}} + \frac{a}{e^{32t}}$$

$$I(t) = \frac{9}{4} + a e^{-32t} \quad \text{--- (1)}$$

As given $I(0) = 0 \Rightarrow \frac{9}{4} + a = 0 \Rightarrow a = -\frac{9}{4}$

$$\therefore \boxed{I(t) = \frac{9}{4} - \frac{9}{4} e^{-32t}}$$

Here $-\frac{9}{4} e^{-32t}$ is the response to initial data. $\frac{9}{4}$ amp. current to the input 72

This tell us the current at any time t.

Notice as $t \to \infty$, $i(t) \to \frac{9}{4}$ because

as $t \to \infty$, $-\frac{9}{4} e^{-32t} \longrightarrow 0$.

$-\frac{9}{4} e^{-32t}$ is called the transient term

$\frac{9}{4}$ amperes is the steady rate current.

Ex In an electric circuit containing inductance and resistance in series with constant e.m.f E, if initial current is zero. Show that, the current builds up half of it's theoretical maximum in $\frac{L\log(2)}{R}$ seconds. (H.W.)