COEP Technological University

A Unitary Public University of Government of Maharashtra

(MA-20001) Ordinary Differential Equations and Multivariate Calculus

Program : S.Y.B.Tech. Sem. I | Academic Year : 2023-24

Examination : Test 2 | Maximum Marks : 20

Date : 21/10/2023 | Time : 7:45 am - 8:45 am

Branch:

Student MIS Number :

Name and Signature of the Invigilator: ____________________

Q.1	Q.2	Q.3	Total	Signature

Attempt All the Questions.

Question [I](10 marks)

(1) Fill in the blanks: For the differential equation $x^2y'' + 0.6\,xy' + 0.05\,y = 0$,

(a) Auxiliary equation is $m^2 - 0.4m + 0.05 = 0$ — $\left(\frac{1}{2}\right)$

(b) General solution is $x^{0.2}\left(A\cos(0.1\ln|x|) + B\sin(0.1\ln|x|)\right)$ — (1) [CO2][1.5]

$(\because m = 0.2 \pm 0.1 i)$

(2) Using method of variation of parameters, find $y_p(x)$ of $(D^2+6D+9I)y = \dfrac{16\,e^{-3x}}{x^2+1}$ whose linearly independent solutions are $y_1(x) = e^{-3x}$ and $y_2(x) = x\,e^{-3x}$. [CO3][3]

Detailed Answer:

$$W = \begin{vmatrix} e^{-3x} & xe^{-3x} \\ -3e^{-3x} & e^{-3x} - 3xe^{-3x} \end{vmatrix} = e^{-6x} \neq 0 \quad \text{— } \left(\tfrac{1}{2}\right)$$

$$u(x) = -\int \frac{y_2 r}{W}\,dx = -\int \frac{xe^{-3x} \times 16 e^{-3x}}{(x^2+1)\times e^{-6x}}\,dx$$

$$= -\int \frac{16x}{x^2+1}\,dx = -8\ln(x^2+1)$$

$$\therefore\ u(x) = -8\ln(x^2+1) \quad \text{— } (1)$$

$$\& \ v(x) = \int \frac{y_1 r}{W}\,dx = \int \frac{e^{-3x}\times 16 e^{-3x}}{(x^2+1)\times e^{-6x}}\,dx$$

$$= 16\int \frac{1}{x^2+1}\,dx$$

$$\therefore\ v(x) = 16\tan^{-1}x \quad \text{— } \left(\tfrac{1}{2}\right)$$

$$\therefore\ y_p(x) = u(x)y_1 + v(x)y_2$$

$$= -8\ln(x^2+1)e^{-3x} + 16\tan^{-1}x\, x e^{-3x} \quad \text{— } (1)$$

$$y_p(x) = 8e^{-3x}\left(2x\tan^{-1}x - \ln(x^2+1)\right)$$

(3) Find the current $I(t)$ in an RLC circuit with $R = 12$ *ohms*, $L = 0.4$ *henry*, $C = \frac{1}{80}$ *farad*, which is connected to a source of EMF $E(t) = 220\ \sin 10t$. [CO5][3]

Detailed Answer:

By KVL, $LI' + RI + \frac{1}{C}\int I\,dt = E(t)$

This is an integro-diff. eqn. —$(\frac{1}{2})$

To remove the integral, we differentiate it,

$\therefore LI'' + RI' + \frac{1}{C} I = E'(t)$

$\Rightarrow 0.4 I'' + 12 I' + 80 I = 2200 \cos 10t$

Divide by 0.4, we get

$I'' + 30 I' + 200 I = 5500 \cos 10t$ —(*)

AE of corres. homo. eqn is

$\lambda^2 + 30\lambda + 200 = 0 \Rightarrow \lambda = -20, -10$

$\therefore I_h = c_1 e^{-20t} + c_2 e^{-10t}$ —$(\frac{1}{2})$

By the method of Undetermined coefficients,

choose $I_p = A\cos 10t + B\sin 10t$ —$(\frac{1}{2})$

$\therefore I_p' = -10A\sin 10t + 10B\cos 10t$ &

$I_p'' = -100A\cos 10t - 100B\sin 10t$

$\therefore (*) \Rightarrow -100A\cos 10t - 100B\sin 10t$
$+ 30(-10A\sin 10t + 10B\cos 10t)$
$+ 200(A\cos 10t + B\sin 10t) = 5500\cos 10t$

Equating coefficients, we get

$\sin 10t$: $-100B - 300A + 200B = 0$
$\Rightarrow -3A + B = 0$

$\cos 10t$: $-100A + 300B + 200A = 5500$
$\Rightarrow A + 3B = 55$

$\therefore A = 5.5,\ B = 16.5$

$\therefore I_p = 5.5\cos 10t + 16.5\sin 10t$ —(1)

$\therefore I = I_h + I_p$

$I = c_1 e^{-20t} + c_2 e^{-10t} + 5.5\cos 10t + 16.5\sin 10t$ —$(\frac{1}{2})$

OR

By KVL, $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E(t)$ —$(\frac{1}{2})$

$\Rightarrow 0.4\frac{d^2q}{dt^2} + 12\frac{dq}{dt} + 80q = 220\sin 10t$

$\Rightarrow \frac{d^2q}{dt^2} + 30\frac{dq}{dt} + 200q = 550\sin 10t$ —(*)

AE of corres. homo. eqn is

$\lambda^2 + 30\lambda + 200 = 0 \Rightarrow \lambda = -20, -10$

$\therefore q_h = c_1 e^{-20t} + c_2 e^{-10t}$ —$(\frac{1}{2})$

By the method of Undetermined coeffs,

choose $q_p = A\cos 10t + B\sin 10t$ —$(\frac{1}{2})$

$\Rightarrow q_p' = -10A\sin 10t + 10B\cos 10t$ &

$q_p'' = -100A\cos 10t - 100B\sin 10t$

$\therefore (*) \Rightarrow -100A\cos 10t - 100B\sin 10t$
$+ 30(-10A\sin 10t + 10B\cos 10t)$
$+ 200(A\cos 10t + B\sin 10t) = 550\sin 10t$

Equating coefficients,

$\sin 10t$: $-100B - 300A + 200B = 550$
$\Rightarrow -3A + B = 5.5$

$\cos 10t$; $-100A + 300B + 200A = 0$
$\Rightarrow A + 3B = 0$

$\Rightarrow A = -1.65$ & $B = 0.55$

$\therefore q_p = -1.65\cos 10t + 0.55\sin 10t$ —(1)

$\therefore q = q_h + q_p$

$\Rightarrow q = c_1 e^{-20t} + c_2 e^{-10t} - 1.65\cos 10t + 0.55\sin 10t$

$\therefore I = \frac{dq}{dt} = -20e^{-20t}c_1 - 10c_2 e^{-10t} + 16.5\sin 10t + 5.5\cos 10t$

$\therefore I = c_3 e^{-20t} + c_4 e^{-10t} + 5.5\cos 10t + 16.5\sin 10t$ —$(\frac{1}{2})$

(4) Solve $y''' + y' = 3x$ by the method of undetermined coefficients. [CO3][2.5]

Detailed Answer:

AE of corres. homo. eqn is

$\lambda^3 + \lambda = 0 \Rightarrow \lambda = 0, \pm i$ —$(\frac{1}{2})$

$\therefore y_h = c_1 + c_2\cos x + c_3\sin x$ —$(\frac{1}{2})$

soln set $= \{1, \cos x, \sin x\}$

choose $y_p = x(Ax + B)$ —$(\frac{1}{2})$

i.e. $y_p = Ax^2 + Bx$

$\Rightarrow y_p' = 2Ax,\ y_p'' = 2A,$

$y_p''' = 0$

$\therefore$ Given eqn becomes

$2Ax + B = 3x$

$\Rightarrow \quad 2A = 3 \quad , \quad B = 0$

$\Rightarrow \quad A = \frac{3}{2}$

$\therefore \quad y_p = \frac{3x^2}{2}$ — $\left(\frac{1}{2}\right)$

$\therefore \quad y = y_h + y_p$

$\Rightarrow \quad y = c_1 + c_2 \cos x + c_3 \sin x + \frac{3x^2}{2}$ — $\left(\frac{1}{2}\right)$

Question [II](5 marks)

(1) Using appropriate theorems/properties, find the Laplace transform of $e^{-2t}\int_0^t \frac{\sin 3u}{u}\,du$. [CO3][3]

Detailed Answer:

$L\left\{\int_0^t \frac{\sin 3u}{u}\,du\right\}$

$= \frac{1}{s}\int_s^\infty \frac{3\,du}{u^2+9}$ $\left\{\begin{array}{l}\text{L.T of int} \\ \text{Trans. of int}\end{array}\right.$ —— 01

$= \frac{\cancel{3}}{s} \times \frac{1}{\cancel{3}} \tan^{-1}\left(\frac{u}{3}\right)\Big|_{u=s}^{\infty}$ —— $\frac{1}{2}$

$= \frac{1}{s}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{3}\right)\right]$ —— $\frac{1}{2}$

$= \frac{1}{s}\cot^{-1}\left(\frac{s}{3}\right)$. —— $\frac{1}{2}$

$\therefore \boxed{L\left\{e^{-2t}\int_0^t \frac{\sin 3u}{u}\,du\right\} = \frac{1}{s+2}\cot^{-1}\left(\frac{s+2}{3}\right)}$ $\Big\}$ s-shifting —— $\frac{1}{2}$

(2) Let $f(t)$ be continuous and satisfies growth restriction for all $t \geq 0$. Further, $f'(t)$ be piecewise continuous on every finite subinterval of $t \geq 0$. Prove that $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$. [CO4][2]

Proof: $L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$ —— $\frac{1}{2}$

$= e^{-st} \cdot f(t)\Big|_{t=0}^{\infty} - \int_0^\infty (-s)\, e^{-st} f(t)\,dt$ —— $\frac{1}{2}$

$= -f(0) + s\int_0^\infty e^{-st} f(t)\,dt$ —— $\frac{1}{2}$

$\boxed{L\{f'(t)\} = sL\{f(t)\} - f(0)}$ —— $\frac{1}{2}$

Question [III](5 marks)

(1) Fill in the blanks: If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ then

(a) $\mathcal{L}^{-1}\{F(s+a)\} = e^{-at} f(t)$ (b) $\mathcal{L}^{-1}\left\{\frac{\pi F'(s)}{2}\right\} = -\frac{\pi}{2}\,t\,f(t)$. [CO1][2]

1 mrk each

(2) Find $\mathcal{L}^{-1}\left\{\left(1+e^{-2\pi(s+1)}\right)\frac{s+1}{(s+1)^2+1}\right\}$ [CO3][3]

Detailed Answer:

$F(s) = \underbrace{e^{-0s}\frac{s+1}{(s+1)^2+1}}_{①} + \underbrace{e^{-2\pi} \cdot e^{-2\pi s} \cdot \frac{s+1}{(s+1)^2+1}}_{(2)}$

Working:

$= e^{-0s} L\{e^{-t}.\cos t\} + e^{-2\pi} \cdot e^{-2\pi s} L\{e^{-t}\cos t\}$

Taking Inverse Laplace trans. on both sides.

$f(t) = u(t) \cdot e^{-t} \cdot \cos t + e^{-2\pi} \cdot u(t-2\pi) \cdot e^{-(t-2\pi)} \cdot \cos(t-2\pi)$

— t-shifting thm.

$\therefore f(t) = u(t) \cdot e^{-t}\cos t - u(t-2\pi) \cdot e^{-t}\cos t$

$= e^{-t}\cos t\,[u(t) - u(t-2\pi)]$

$$\boxed{f(t) = \begin{cases} e^{-t}\cos t & 0 < t < 2\pi \\ 0 & t > 2\pi \end{cases}}$$

ROUGH WORK (Will Not Be Assessed)