Shifted Data Problems :-

This means initial value problem with initial conditions given at some $t = t_0 > 0$ instead of $t = 0$.

For such problem set $t = \tilde{t} + t_0$, so that $t = t_0$ gives $\tilde{t} = 0$ and the Laplace transform can be applied.

Example (1)

$$y'' + y = 2t, \quad y\left(\frac{\pi}{4}\right) = \frac{\pi}{2}, \quad y'\left(\frac{\pi}{4}\right) = 2 - \sqrt{2}$$

Sol^n We have $t_0 = \frac{\pi}{4}$ and we set

$$t = \tilde{t} + \frac{\pi}{4} \quad \text{—— } (*)$$

Then the problem is (when $t = \pi/4$ by $(*)$ $\tilde{t} = 0$)

$$\tilde{y}'' + \tilde{y}' = 2\left(\tilde{t} + \frac{\pi}{4}\right), \quad \tilde{y}(0) = \frac{\pi}{2}, \quad \tilde{y}'(0) = 2 - \sqrt{2} \quad \text{—— (1)}$$

where $\tilde{y}(\tilde{t}) = y(t)$.

Taking Laplace transform (1) on both sides

$$L\{\tilde{y}''\} + L\{\tilde{y}'\} = 2L\{\tilde{t}\} + \frac{\pi}{2} L\{1\}$$

$$s^2 L\{\tilde{y}\} - s\tilde{y}(0) - \tilde{y}'(0) + L\{\tilde{y}(\tilde{t})\} = \frac{2}{s^2} + \frac{\pi}{2} \cdot \frac{1}{s}$$

$$s^2 L\{\tilde{y}\} - s\left(\frac{\pi}{2}\right) - (2 - \sqrt{2}) + L\{\tilde{y}(t)\} = \frac{2}{s^2} + \frac{\pi/2}{s}$$

$$L\{\tilde{y}(t)\}(s^2 + 1) = \frac{2}{s^2} + \frac{\pi/2}{s} + \frac{\pi}{2}s + (2 - \sqrt{2})$$

$$L\{\tilde{y}(t)\} = \frac{2}{s^2(s^2+1)} + \frac{\pi/2}{s(s^2+1)} + \frac{\pi/2\, s}{s^2+1} + \frac{(2-\sqrt{2})}{s^2+1}$$

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$$L\{\tilde{y}(\tilde{t})\} = 2\left[\frac{1}{s^2} - \frac{1}{s^2+1}\right] + \frac{\pi}{2}\left[\frac{1}{s} - \frac{s}{s^2+1}\right]$$

$$+ \frac{\pi}{2}\cdot\left[\frac{s}{s^2+1}\right] + \frac{2-\sqrt{2}}{s^2+1}$$

$$\tilde{y}(\tilde{t}) = 2\,L^{-1}\left\{\frac{1}{s^2}\right\} - 2\,L^{-1}\left\{\frac{1}{s^2+1}\right\} + \frac{\pi}{2}\,L^{-1}\left\{\frac{1}{s}\right\} \cancel{- \frac{\pi}{2} L^{-1}\left\{\frac{s}{s^2+1}\right\}}^{0}$$

$$\cancel{+ \frac{\pi}{2} L^{-1}\left[\frac{s}{s^2+1}\right]}^{0} + (2-\sqrt{2})\,L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= 2\tilde{t} - 2\sin\tilde{t} + \frac{\pi}{2} + (2-\sqrt{2})\sin\tilde{t}$$

$$= 2\tilde{t} - \cancel{2\sin\tilde{t}}^{0} + \frac{\pi}{2} + \cancel{2\sin\tilde{t}}^{0} - \sqrt{2}\sin\tilde{t}$$

$$= 2\tilde{t} + \frac{\pi}{2} - \sqrt{2}\sin\tilde{t}.$$

But $\tilde{t} = t - \frac{\pi}{4}$,

$$y(t) = 2\left(t - \frac{\pi}{4}\right) + \frac{\pi}{2} - \sqrt{2}\sin\left(t - \frac{\pi}{4}\right)$$

$$= 2t - \cancel{\frac{\pi}{2}} + \cancel{\frac{\pi}{2}} - \sqrt{2}\left[\sin t\cdot\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\cos t\right]$$

$$= 2t - \sqrt{2}\left[\sin t\cdot\left(\frac{1}{\sqrt{2}}\right) - \cos t\left(\frac{1}{\sqrt{2}}\right)\right]$$

$$= 2t - \frac{\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}}\left[\sin t - \cos t\right]$$

$$= 2t - \sin t + \cos t.$$

Example (2).

$$y' - 6y = 0, \quad y(2) = 4$$

→ Here $t_0 = 2$, set $t = \tilde{t} + 2$, and [*]

$\tilde{y}(\tilde{t}) = y(t)$. (when $t=2$ by [*] $\tilde{t}=0$)

$$\tilde{y}'(\tilde{t}) - 6\tilde{y}(\tilde{t}) = 0, \quad \tilde{y}(0) = 4.$$

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$L\{\tilde{y}'(\tilde{t})\} - 6L\{\tilde{y}(\tilde{t})\} = 0$

$s L\{\tilde{y}(\tilde{t})\} - \tilde{y}(0) - 6L\{\tilde{y}(\tilde{t})\} = 0$

$L\{\tilde{y}\}(s-6) = \tilde{y}(0)$

$L\{\tilde{y}\} = \dfrac{4}{s-6}$

$\tilde{y}(\tilde{t}) = 4e^{6\tilde{t}}$

But $\tilde{t} = t-2$

$\therefore y(t) = 4e^{6(t-2)} = 4e^{6t-12}$.

Homework $y'' + 3y' - 4y = 6e^{2t-2}$, $y(1)=4$, $y'(1)=5$

$y'' + 2y' + 5y = 50t - 150$, $y(3) = -4$, $y'(3) = 14$.

- Some problems on Laplace Transform of integrals.

① $L\left\{\int_0^t e^{2u}\sin(7u)\,du\right\}$

$= \frac{1}{s} L\{e^{2t}\sin(7t)\}$ -- by L.T of integral.

$= \frac{1}{s} \frac{7}{(s-2)^2+7^2}$ --- by s-shifting and $L\{\sin at\} = \frac{a}{s^2+a^2}$

② $y' + 8y = \int_0^t \sin(3u)\,du$, $y(0)=0$.

Taking L.T on bothe side.

$L\{y'(t)\} + 8L\{y(t)\} = L\left\{\int_0^t \sin(3u)\,du\right\}$

$sL\{y(t)\} - \cancel{y(0)}^{\,0} + 8L\{y(t)\} = \frac{1}{s} L\{\sin 3t\}$ -- by LT of derivative and $L\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}L\{f(t)\}$

$$L\{y(t)\}(s+8) = \frac{1}{s}\left(\frac{3}{s^2+3^2}\right)$$

$$L\{y(t)\} = \frac{3}{s(s+8)(s^2+3^2)}$$

Take Laplace inverse and find $\boxed{y(t) = ?}$ Do it.

Differentiation of Laplace Transfrom :-

Why this concept is Imp?

In last lecture, we saw

$$L\{f'(t)\} = s\,L\{f(t)\} - f(0)$$

$$L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)$$

$$L\{f'''(t)\} = s^3 L\{f(t)\} - s^2 f(0) - s f'(0) - f''(0)$$

$$\vdots$$

$$L\{f^{(n)}(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0).$$

* Now suppose $f(t)$, $t \geq 0$ (defined) piecewise continuous and exponential order, the by Existence theorem,

$L\{f(t)\}$ exists.

i.e. $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$. —— (1)

Then question $\frac{d}{ds}(F(s)) = ?$

or

What is Laplace transform of $tf(t)$

i.e $L\{t f(t)\} = ?$

Differentiating (1) with respect to s.

$$F'(s) = \frac{d}{ds}(F(s)) = \frac{d}{ds}\left(\int_0^\infty e^{-st} f(t)\,dt\right)$$

$$= \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st} f(t)\right) dt$$ ---- diff^n under integral sign

$$= \int_0^\infty e^{-st}\left[-t f(t)\right] dt$$

$$= L\{-t f(t)\}$$

$$= (-1)\, L\{t f(t)\}$$

$$\boxed{\begin{aligned} L\{t f(t)\} &= (-1)\frac{d}{ds}(F(s)) = (-1)\frac{d}{ds}\left(L\{f(t)\}\right) \\ L^{-1}\{F'(s)\} &= (-1)\, t\, f(t) = (-1)\, t\, L^{-1}\{F(s)\} \end{aligned}}$$

Similarly,

$$L\{t^2 f(t)\} = (-1)^2 F''(s) = F''(s)$$

$$L\{t^3 f(t)\} = (-1)^3 F'''(s) = -F'''(s)$$

In general,

$$\boxed{\begin{aligned} L\{t^n f(t)\} &= (-1)^n F^{(n)}(s) = (-1)^n \frac{d^n}{ds^n}(F(s)) = (-1)^n \frac{d^n}{ds^n}\left(L\{f(t)\}\right) \\ \text{Hence} \\ L^{-1}\{F^{(n)}(s)\} &= (-1)^n t^n f(t) = (-1)^n t^n L^{-1}\{F(s)\} \end{aligned}}$$

Find.

(1) $L\{t \sin at\}$

$\rightarrow$ $L\{t \sin at\} = (-1) \frac{d}{ds}(F(s))$ --- by Diff^n of LT.

$= (-1) \frac{d}{ds}\left(L\{\sin at\}\right)$

$= (-1) \frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)$

$= \frac{2as}{(s^2+a^2)^2}$

(2) $L\{t \sinh at\}$

$\rightarrow$ $L\{t \sinh at\} = (-1) \frac{d}{ds}\left(\frac{a}{s^2-a^2}\right)$ --- by diff^n of LT

$= (-1)\left(\frac{-2as}{(s^2-a^2)^2}\right)$

$= \frac{2as}{(s^2-a^2)^2}$.

(3) $L\{t^2 e^{2t}\} = (-1)^2 \frac{d^2}{ds^2}\left(L\{e^{2t}\}\right)$ --- by diff^n of LT

$= (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s-2}\right)$

$= \frac{d}{ds}\left[-(s-2)^{-2}\right]$

$= \frac{2}{(s-2)^3}$

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(4) $L\{(t^2-3t-2)\sin 3t\}$

$\rightarrow$ $L\{t^2\sin 3t - 3t\sin 3t - 2\sin 3t\}$

$= L\{t^2\sin 3t\} - 3L\{t\sin 3t\} - 2L\{\sin 3t\}$ — (1)

--- L is linear.

But,

$$L\{\sin 3t\} = \frac{3}{s^2+9}$$

$$L\{t\sin 3t\} = (-1)\frac{d}{ds}\left(\frac{3}{s^2+9}\right)$$

$$= \frac{6s}{(s^2+9)^2}$$

$$L\{t^2\sin 3t\} = \frac{18s^2-54}{(s^2+9)^3} \quad \text{(check it!)}$$

From (1),

$$\frac{18s^2-54}{(s^2+9)^3} - 3\left(\frac{6s}{(s^2+9)^2}\right) + 2\left(\frac{3}{s^2+9}\right)$$

(5) $L\{te^{-t}\sin t\}$

$\rightarrow$ We have $L\{\sin t\} = \frac{1}{s^2+1} = F(s)$ (say)

By s-shifting thm we have

$$L\{e^{-t}\sin t\} = F(s+1) = \frac{1}{(s+1)^2+1} = \frac{1}{s^2+2s+2}.$$

Now by diff^n of L.T.

$$L\{te^{-t}\sin t\} = (-1)\frac{d}{ds}\left(\frac{1}{s^2+2s+2}\right)$$

$$= \frac{2(s+1)}{(s^2+2s+2)^2}$$

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Homework :- ① $L\{te^{3t}\cos(4t)\}$

② $L\{t \cosh 3t\}$

Find the inverse Laplace transform

(i) $\bar{L}^1\left\{\frac{s}{(s^2+a^2)^2}\right\}$ (ii) $\bar{L}^1\left\{\log\left|\frac{1+s}{s}\right|\right\}$

(iii) $\bar{L}^1\left\{\cot^{-1}\left(\frac{s}{\omega}\right)\right\}$.

(i) → Let $F(s) = \frac{1}{s^2+a^2}$ — (#)

$\therefore \frac{d}{ds}(F(s)) = \frac{-2s}{(s^2+a^2)^2}$

Hence

$$\frac{s}{(s^2+a^2)^2} = -\frac{1}{2}\frac{d}{ds}(F(s))$$

$$\bar{L}^1\left\{\frac{s}{(s^2+a^2)^2}\right\} = -\frac{1}{2}\bar{L}^1\left\{\frac{d}{ds}F(s)\right\} \quad \text{— (1)}$$

You know $L\{t f(t)\} = -F'(s)$

$$\underline{\bar{L}^1\{F'(s)\}} = -t f(t) = \underline{-t\,\bar{L}^1\{F(s)\}} \quad *$$

From (1)

$$\bar{L}^1\left\{\frac{s}{(s^2+a^2)^2}\right\} = -\frac{1}{2}\left[-t\,\bar{L}^1\{F(s)\}\right] \text{--- by } (*)$$

$$= \frac{t}{2}\bar{L}^1\{F(s)\}$$

$$= \frac{t}{2}\bar{L}^1\left\{\frac{1}{s^2+a^2}\right\} \text{ by } (\#)$$

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t}{2a} L^{-1}\left\{\frac{a}{s^2+a^2}\right\}$$

$$= \frac{t}{2a} \sin at$$

(ii) Here $F(s) = \log\left(\frac{1+s}{s}\right) = \log(1+s) - \log(s)$

$$F'(s) = \frac{1}{1+s} - \frac{1}{s}$$

$$L^{-1}\{F'(s)\} = L^{-1}\left\{\frac{1}{1+s}\right\} - L^{-1}\left\{\frac{1}{s}\right\}$$

$$-t\, L^{-1}\{F(s)\} = e^{-t} - 1 \quad \left(\because L\{t f(t)\} = -F'(s) \quad L^{-1}\{F'(s)\} = -t f(t) = -t\, L^{-1}\{F(s)\} \quad \text{—(*)}\right)$$

$$L^{-1}\{F(s)\} = \frac{1-e^{-t}}{t}$$

(iii) Here

$$F(s) = \cot^{-1}\left(\frac{s}{\omega}\right)$$

$$F'(s) = \frac{d}{ds}\left(\cot^{-1}\left(\frac{s}{\omega}\right)\right)$$

$$= \frac{-1}{1+\frac{s^2}{\omega^2}}\left(\frac{1}{\omega}\right) = \frac{-\omega}{s^2+\omega^2}$$

Hence

$$L^{-1}\{F'(s)\} = L^{-1}\left\{\frac{-\omega}{s^2+\omega^2}\right\}$$

$$-t\, L^{-1}\{F(s)\} = L^{-1}\left\{\frac{-\omega}{s^2+\omega^2}\right\} \quad \left(\because L^{-1}\{F'(s)\} = -t\, L^{-1}\{F(s)\}\right)$$

$$L^{-1}\{F(s)\} = \frac{1}{t} \sin \omega t$$

(iv) $L^{-1}\left\{\dfrac{(s+1)}{(s^2+2s+2)^2}\right\}$

Let $\underline{F(s)} = \dfrac{1}{s^2+2s+2}$

$$F'(s) = \frac{-2s-2}{(s^2+2s+2)^2} = \frac{-2(s+1)}{(s^2+2s+2)^2}$$

Hence

$$\frac{s+1}{(s^2+2s+2)^2} = \frac{-1}{2}\left[F'(s)\right]$$

$$\therefore L^{-1}\left\{\frac{(s+1)}{(s^2+2s+2)^2}\right\} = \frac{-1}{2} L^{-1}\left\{F'(s)\right\}$$

$$= \frac{-1}{2}\left[-t\, \underline{L^{-1}\{F(s)\}}\right]$$

$$= \frac{t}{2} L^{-1}\left\{\frac{1}{s^2+2s+2}\right\}$$

$$= \frac{t}{2} L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= \frac{t}{2} e^{-t} L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \frac{t}{2} e^{-t} \sin t.$$

Homework :

① $L^{-1}\left\{\dfrac{2(s+4)}{(s+4)^2+1^2}\right\}$ ② $L^{-1}\left\{\log\left(\dfrac{s+a}{s+b}\right)\right\}$

③ $L^{-1}\left\{\tan^{-1}\left(\dfrac{s}{10}\right)\right\}$