Feedback Control System

Mathematical Model

UNIT- 1

Develop a Mathematical Model

- Once the schematic is drawn, the designer uses physical laws, such as Kirchhoff's laws for electrical networks and Newton's law for mechanical systems, along with simplifying assumptions, to model the system mathematically. These laws are
  - Kirchhoff's voltage law The sum of voltages around a closed path equals zero.
  - Kirchhoff's current law The sum of electric currents flowing from a node equals zero.
  - Newton's laws The sum of forces on a body equals zero;3 the sum of moments on a body equals zero.

Laplace Transform Review

- A system represented by a differential equation is difficult to model as a block diagram. Thus, we now lay the groundwork for the Laplace transform, with which we can represent the input, output, and system as separate entities. Further, their interrelationship will be simply algebraic.
- The Laplace transform is defined as $$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$$
- where $s = \sigma + j\omega$, a complex variable. Thus, knowing f(t) and that the integral in eq. exists, we can find a function, F(s), that is called the Laplace transform of f(t).
- The notation for the lower limit means that even if f(t) is discontinuous at t=0, we can start the integration prior to the discontinuity as long as the integral converges. Thus, we can find the Laplace transform of impulse functions

- The inverse Laplace transform, which allows us to find f(t) given F(s), is

$$\mathscr{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st}\,ds = f(t)u(t)$$

- Where u=1 for t > 0 and

  =0 for t < 0

is the unit step function. Multiplication of f(t) by u(t) yields a time function that is zero for t < 0.

TABLE 1.1 Test waveforms used in control systems

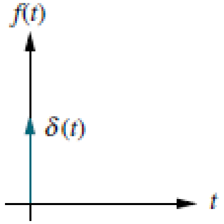
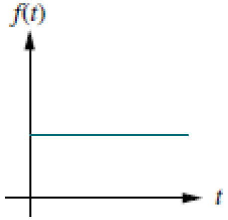
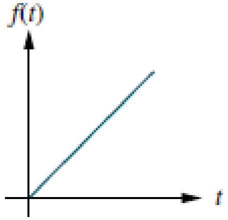
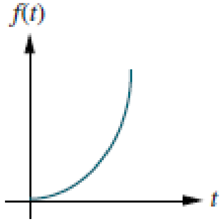
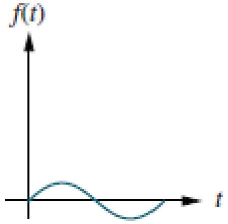
Input	Function	Description	Sketch	Use
Impulse	$\delta(t)$	$\delta(t) = \infty$ for $0- < t < 0+$ $= 0$ elsewhere $\int_{0-}^{0+} \delta(t)dt = 1$		Transient response Modeling
Step	$u(t)$	$u(t) = 1$ for $t > 0$ $= 0$ for $t < 0$		Transient response Steady-state error
Ramp	$tu(t)$	$tu(t) = t$ for $t \geq 0$ $= 0$ elsewhere		Steady-state error
Parabola	$\frac{1}{2}t^2u(t)$	$\frac{1}{2}t^2u(t) = \frac{1}{2}t^2$ for $t \geq 0$ $= 0$ elsewhere		Steady-state error
Sinusoid	$\sin \omega t$			Transient response Modeling Steady-state error

TABLE 2.1 Laplace transform table

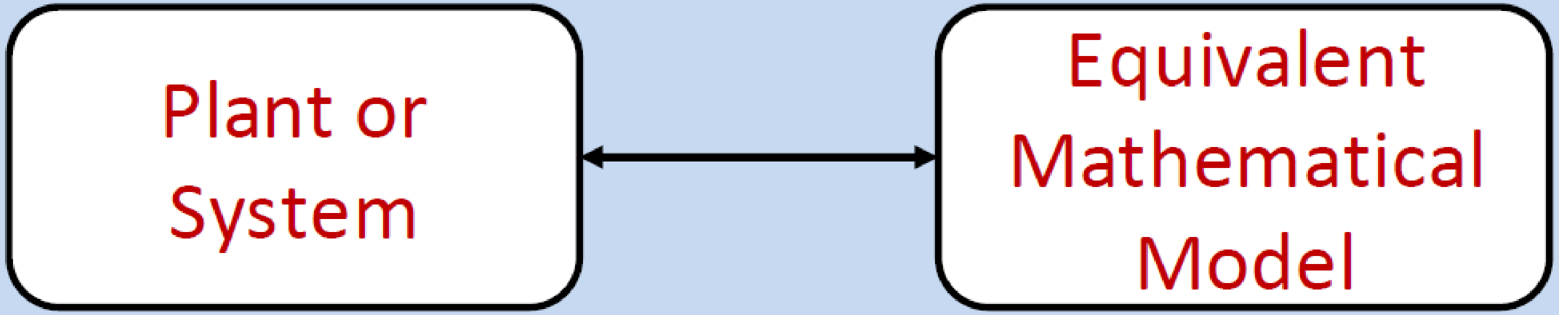
Item no.	$f(t)$	$F(s)$
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^n + 1}$
5.	$e^{-at}u(t)$	$\frac{1}{s + a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2 + \omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2 + \omega^2}$

TABLE 2.2 Laplace transform theorems

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)] = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s\to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) = \lim_{s\to\infty} sF(s)$	Initial value theorem[2]

What is a Model?

- An elemental or mathematical representation of a plant or system.
- Model helps in the analysis (input-output) of the system.
- Captures the dynamics of a system.
- Dynamics refers to evolution of system variables.
  - The change in the room temperature when an AC is switched ON
  - The change in the speed of car when accelerator is pushed by certain angle
  - The change in the current flowing through an inductor when an AC voltage is applied



Types of Mathematical Models

Differential equation model

- Dynamics of the system represented in terms of differential equations
- Time domain representation of the system

Transfer function model

- Dynamics represented in terms of a Laplace transform expression
- Frequency domain representation of the system

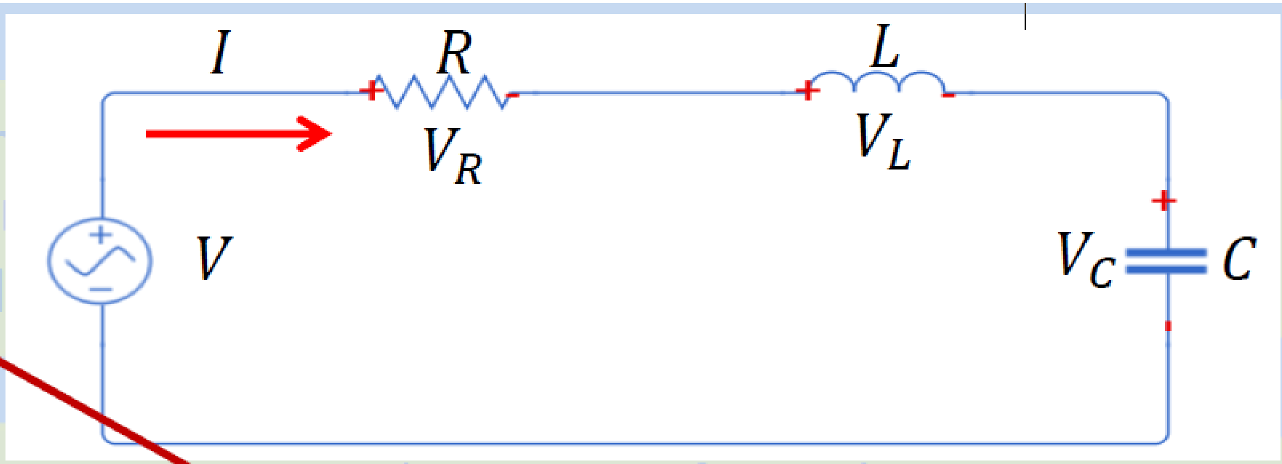
State space model

- State is a set of variables that describes the system behaviour in conjunction with the system inputs
- Dynamics are represented by a set of first order differential equations using these state variables

Types of Mathematical Models

Differential equation model

- Dynam…
  differe…
- Time d…



$$\frac{dV}{dt} = R\frac{dI}{dt} + L\frac{d^2 I}{dt^2} + \frac{I}{C}$$

Transfer function model

- Dynamics represented in terms of a Laplace transform expression
- Frequency domain representation of the system

$$\frac{I(s)}{V(s)} = \frac{1}{R + Ls + \frac{1}{Cs}}$$

State space model

- State is a set of variables that describes the system behaviour in conjunction with the system inputs
- Dynamics represented by a set of first order differential equations using state variables

$$\begin{bmatrix} \frac{dI_L}{dt} \\ \frac{dV_C}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & \frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} I_L \\ V_C \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} V$$

Modelling a System

- Two methods of modelling systems:

Analytical modelling :

–Involves systematic application of basic physical laws to system components and their interconnections

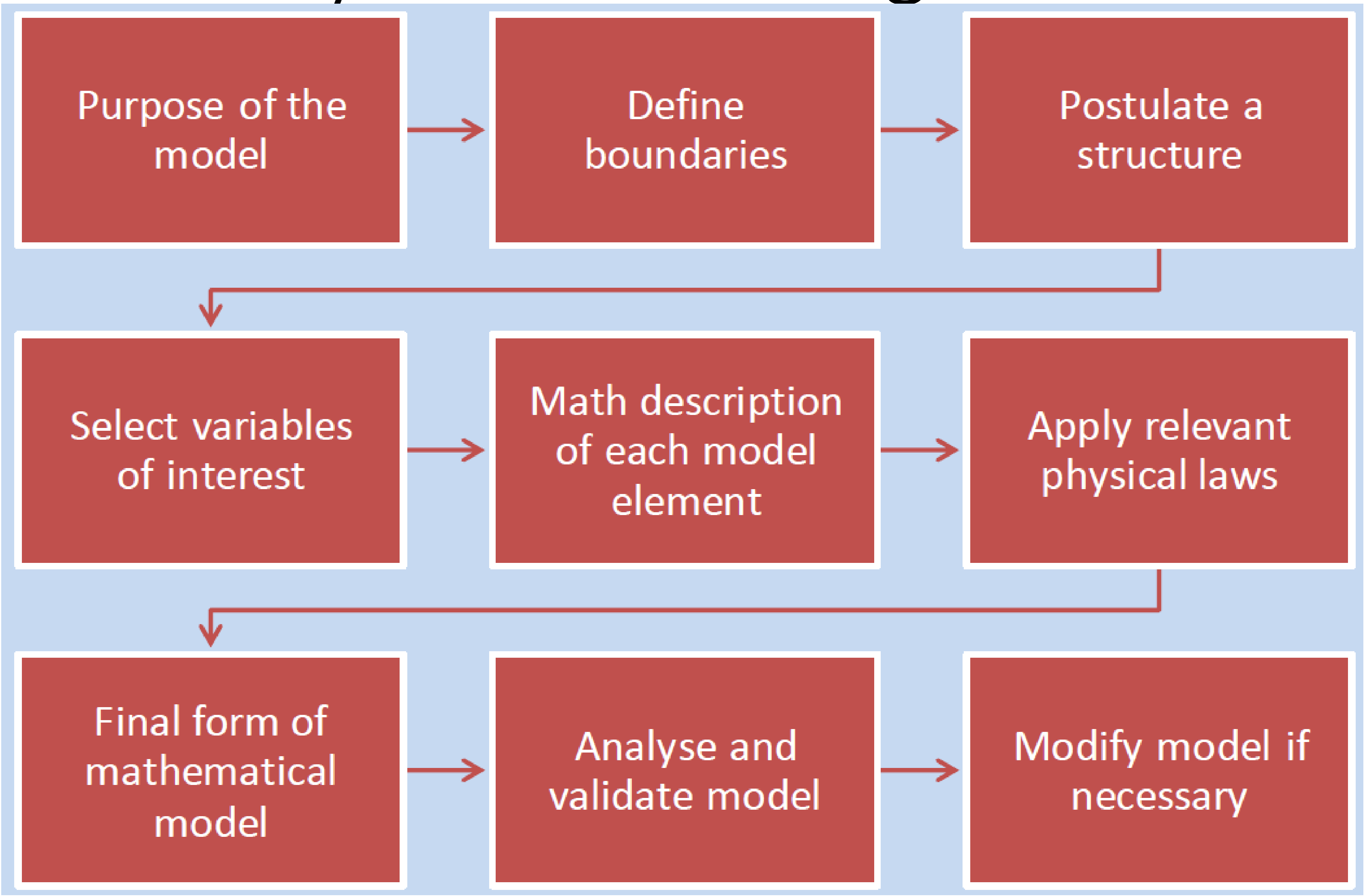
–Combination of physical modelling and mathematical modelling

Experimental modelling :

–Selection of mathematical relations which best fit the observed input-output data of a system

–Also called modelling by synthesis

Steps of Analytical Modelling



Steps of Analytical Modelling (1&2)

1.Purpose of the model:

–Model to be developed should be decided based on intended objective

–E.g. Model of a transformer to study losses & efficiency Vs Model of a transformer to study magnetic behaviour & core saturation

2.Define boundaries:

–System of interest is separated from the rest of the world (referred to as environment) by a boundary

–Boundary may be real or imaginary

–Boundaries for every system and sub-system should be defined based on purpose

Steps of Analytical Modelling (3&4)

3.Postulate a structure

–Systems store, dissipate, transfer or transform energy from one form to another
–Identify simple elements which characterize these operations on energy
–Model elements generally have two ports, sometimes more
–Represent the actual system as an interconnection of these elements
–Referred to as physical modelling

4.Select variables of interest

–First step of mathematical modelling
–Assign variables to all system attributes of interest
–E.g. current, voltage, velocity, temperature, etc.

Steps of Analytical Modelling (5&6)

5.Mathematical description of each model elements

–Identify the relations between variables at each of the model elements

–Relations may be differential or algebraic expressions

6.Apply relevant physical laws

–The most important step in getting the mathematical model

–Develop equations to describe the effects of element interconnections

–Physical laws describe these effects

–E.g. Newton’s laws of motions, Kirchhoff's voltage and current laws, Thermodynamic laws, Conservation of energy, etc.

Steps of Analytical Modelling (7,8 & 9)

7.Final form of mathematical model

–All the equations resulting from step 6 put together form the mathematical model of the system

–Can be simplified if possible

8.Analyse and validate model

–Model is never a exact representation of true system

–Verify the accuracy of the model if possible

–Comparing model results with actual results for some IO conditions

9.Modify model if necessary

–To be done if results in step 8 are not convincing

Experimental Modelling

- In most scenarios of experimental modelling, the model form is available:
  - Based on laws of physics
  - Based on reasonable assumptions
- It is the parameters (or coefficients of variables) that are either known partially or unknown
- Partially known, here, implies that either the bounds or probability distributions of parameters are known
- Parameters are determined through data fitting using regression or time series analysis

Physical Systems

- Physical systems can be classified into various types:
  - ➢**Electrical systems**
  - ➢**Mechanical systems**
  - ➢Electronic systems
  - ➢Hydraulic systems
  - ➢Thermal systems
- Each of these systems can be modelled in terms of certain basic elements
- Basic elements of all physical systems can be shown to be analogous

Electrical Systems

Based on the type of source, electrical systems can be classified as:

–Voltage sourced systems

–Current sourced systems

Voltage sourced	Current sourced
Basic System Elements	
Resistor (R)	Resistor (R)
Inductor (L)	Inductor (L)
Capacitor (C)	Capacitor (C)
Basic System Variables	
Voltage (V)	Current (I)
Charge (Q)	Flux (ϕ)

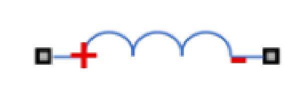
Electrical System Elements

- Resistor (R): It is an element which resists the flow of current in an electrical system

R

$$V = IR$$

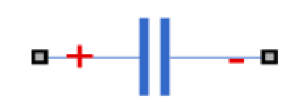
- Inductor (L): It is an element that stores electrical energy in a magnetic field



L

$$V = \frac{d\phi}{dt} = L\frac{dI}{dt}$$

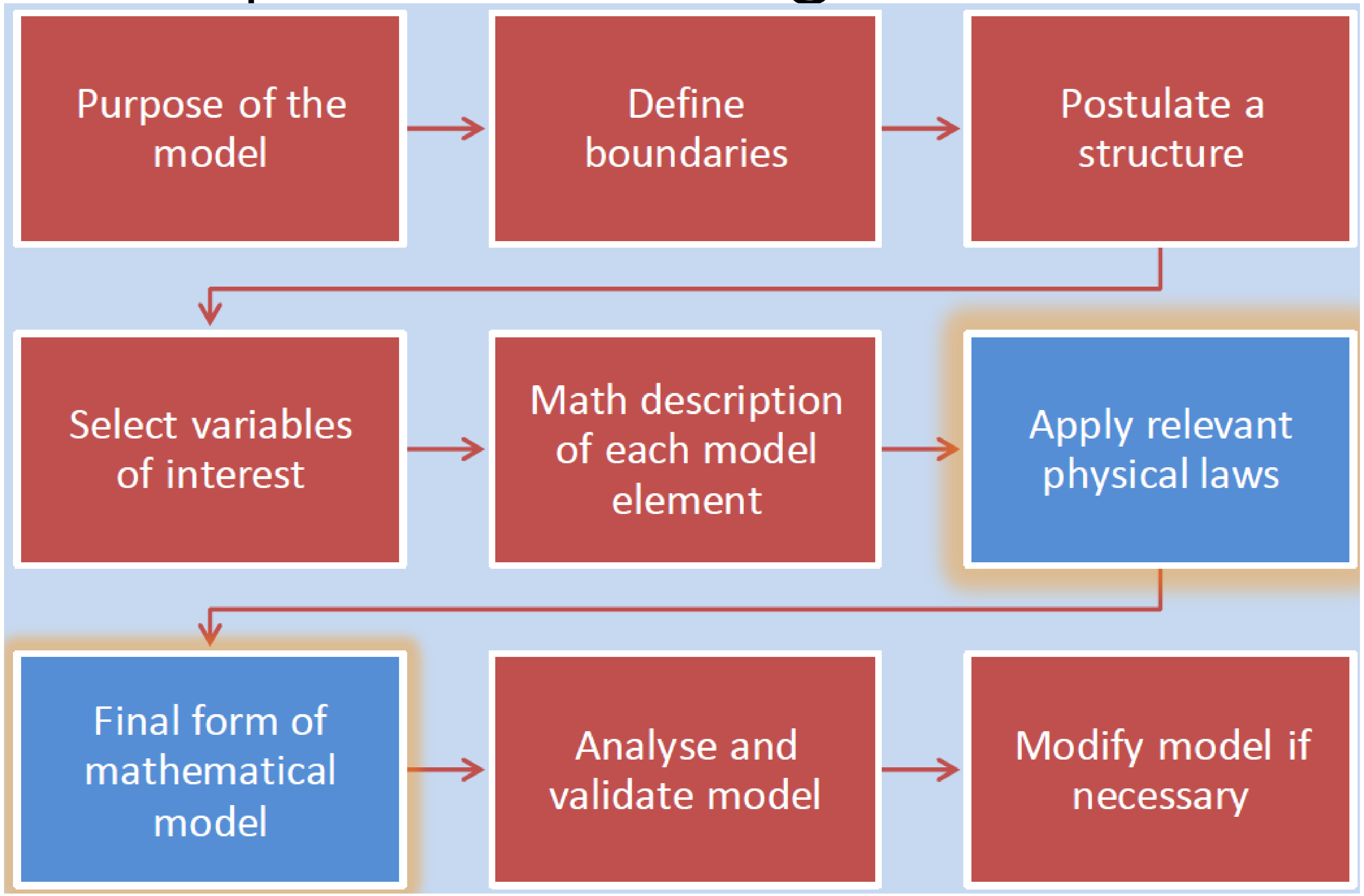
- Capacitor (C): It is an element that stores electrical energy in a electrical field



C

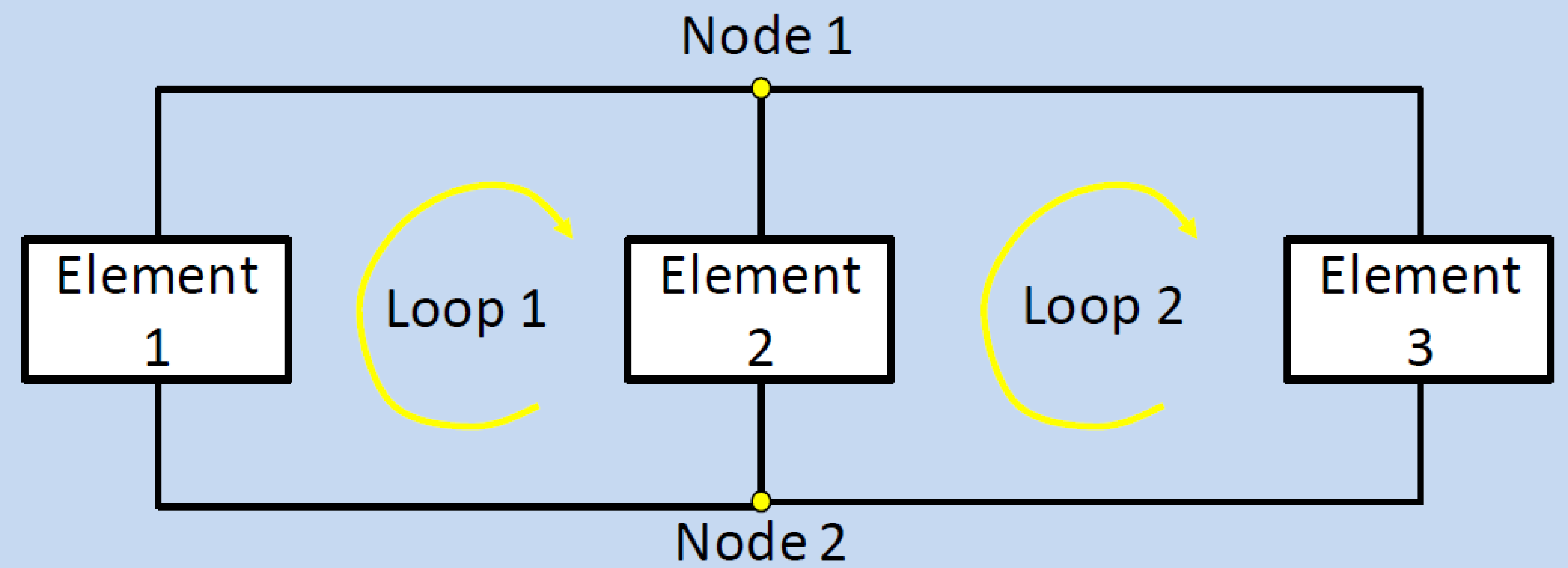
$$I = \frac{dq}{dt} = C\frac{dV}{dt}$$

Review: Steps of Modelling



Nodal and Loop Analysis

- Nodal and loop analysis form part of Step 6 in the steps of modelling
- A structured way of applying physical laws and getting model equations
- Applied after identifying the basic elements of the system and variables of interest



Analysis of Electrical Systems

1. Nodal analysis based on Kirchhoff's current law
2. Loop or Mesh analysis based on Kirchhoff’s voltage law

Kirchhoff's Current Law (KCL)

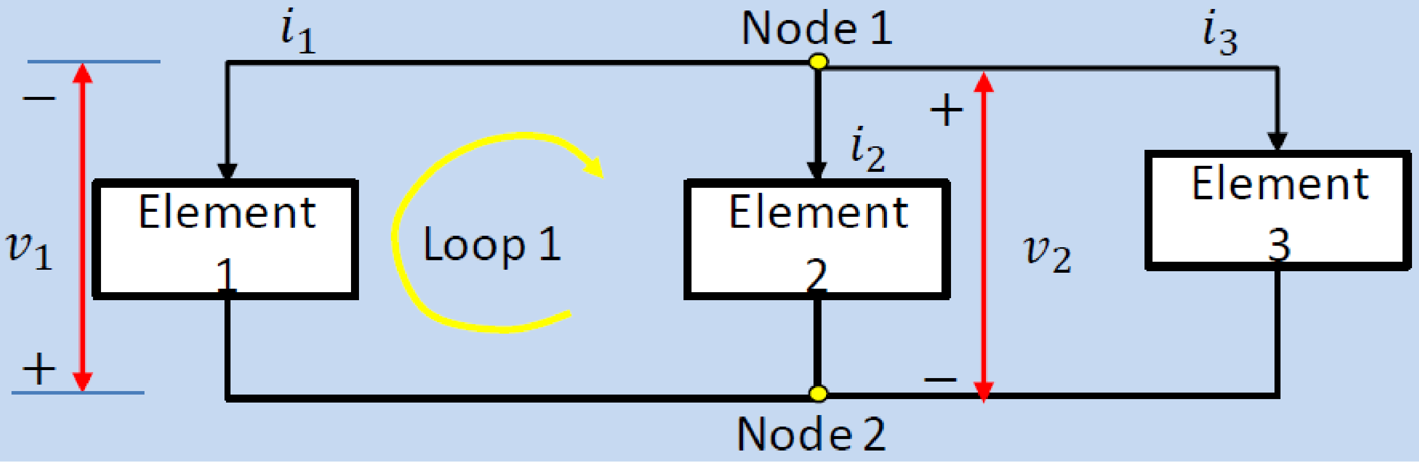
At any node in an electrical circuit, the directed sum of currents flowing out of that node is equal to zero.

Kirchhoff's Voltage Law (KVL)

In an electrical circuit, the directed sum of voltages around a closed loop is zero.

At node 1,

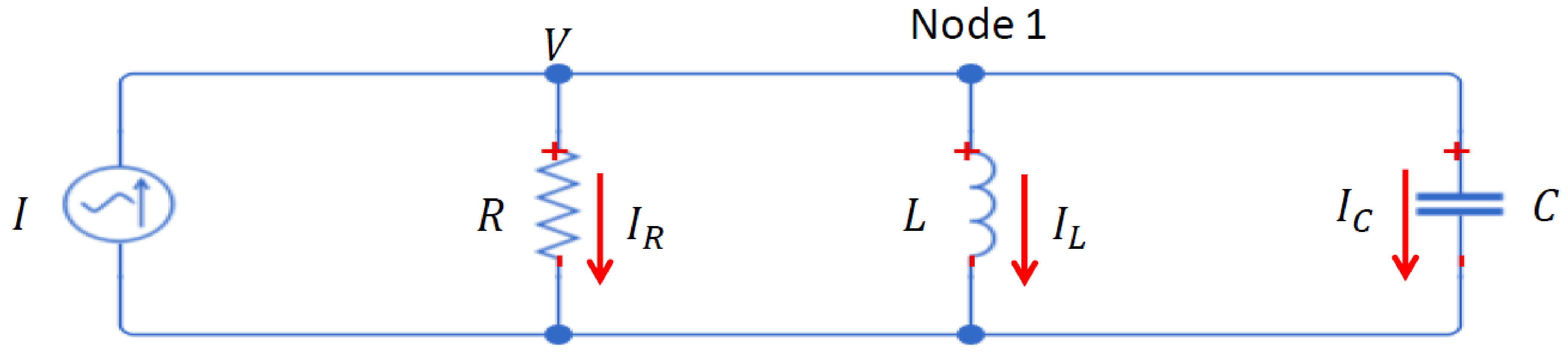
$$i_1 + i_2 + i_3 = 0$$



Around loop 1,

$$v_1 + v_2 = 0$$

Nodal Analysis : Example



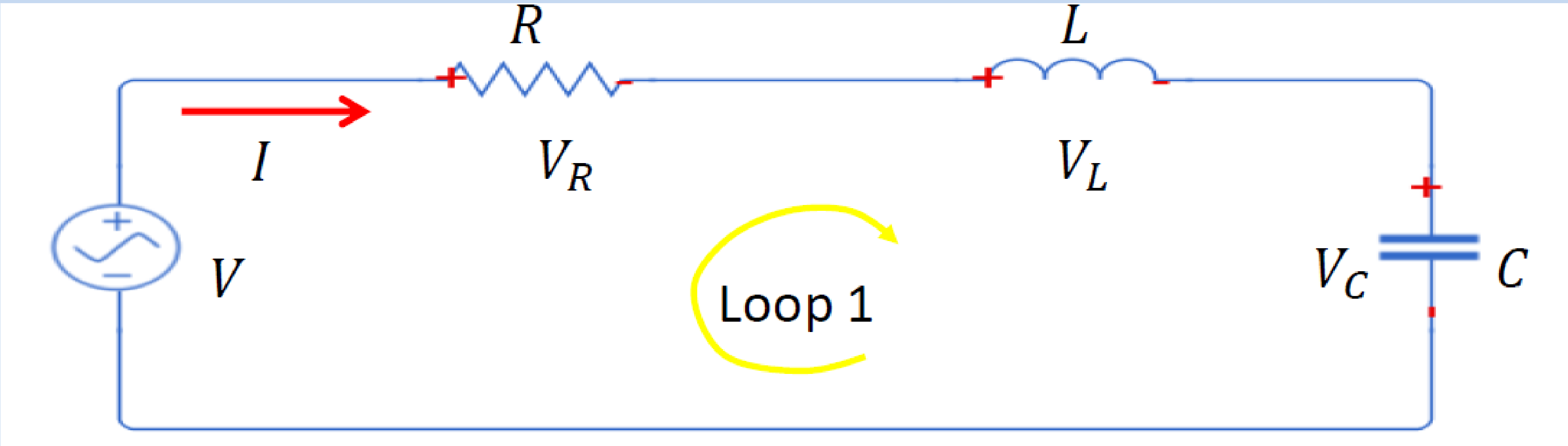
Applying KCL at node 1:

$$I = I_R + I_L + I_c$$

$$I = \frac{V}{R} + \frac{1}{L}\int V dt + C\frac{dV}{dt}$$

Substituting $V = \frac{d\phi}{dt}$:
$$I = C\frac{d^2\phi}{dt^2} + \frac{1}{R}\frac{d\phi}{dt} + \frac{\phi}{L}$$

Loop or Mesh Analysis: Example



Applying KVL around loop 1:

$$V = V_R + V_L + V_c$$

$$V = IR + L\frac{dI}{dt} + \frac{1}{C}\int Idt$$

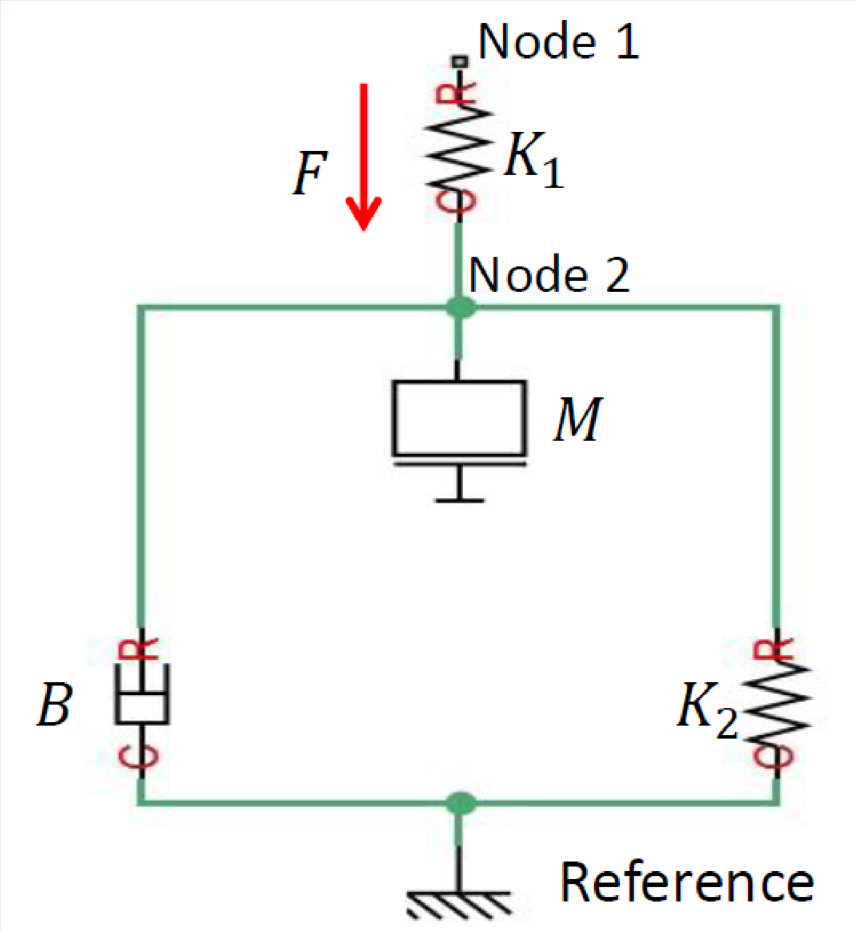
Substituting $I = \frac{dq}{dt}$: $$V = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c}$$

Nodal Analysis for Mechanical Systems

- System structure should be modified to suit nodal analysis, without loss of system characteristics

- Steps of nodal analysis are detailed by an example

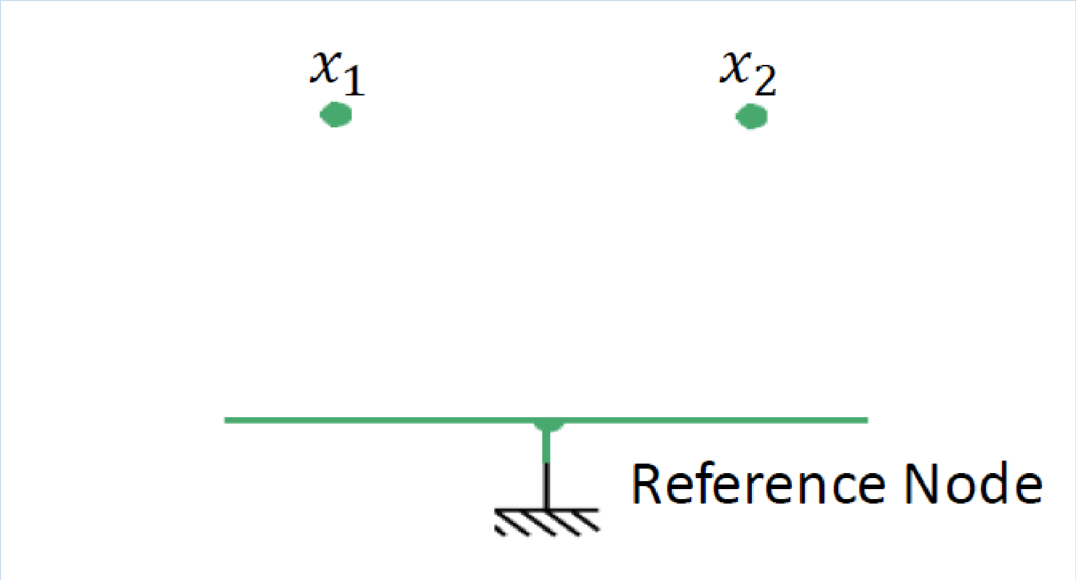
Nodal Analysis : Example



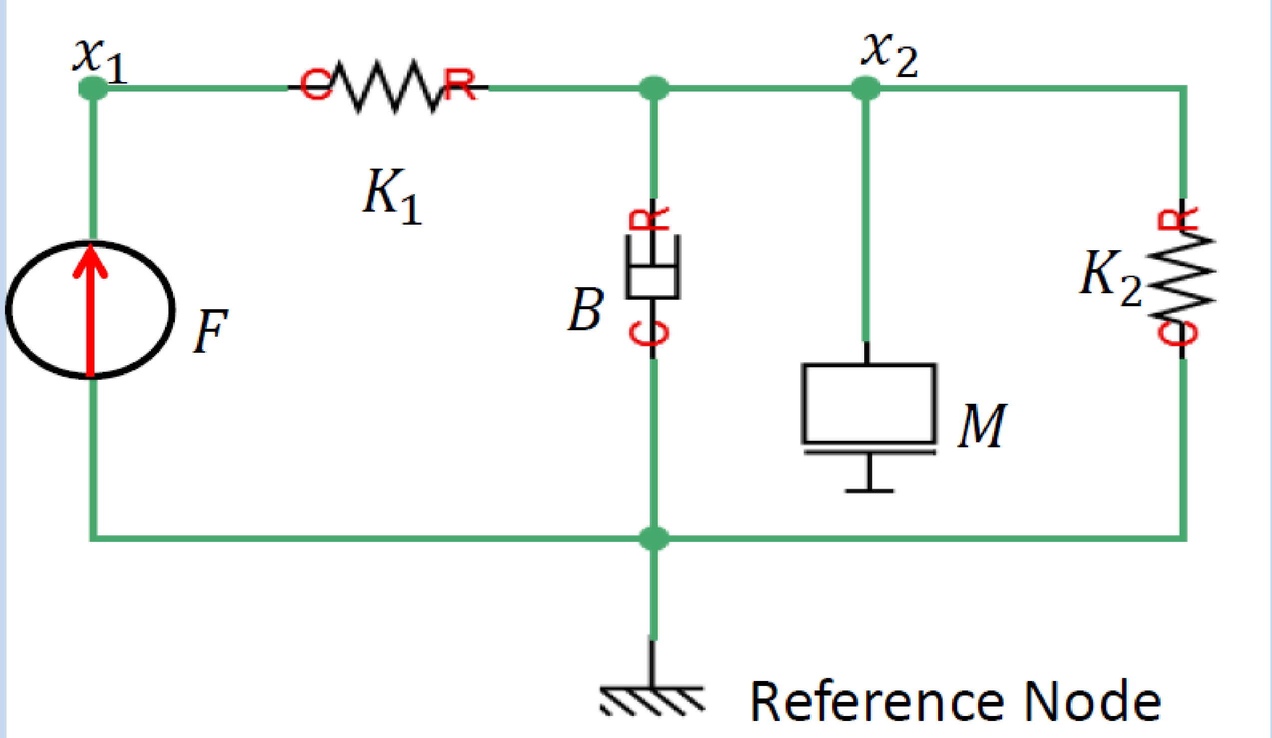
Step 1:

Number of nodes = 2

Hence, number of displacements = 2



Step 2:
Displacement and reference nodes are identified



Step 3:

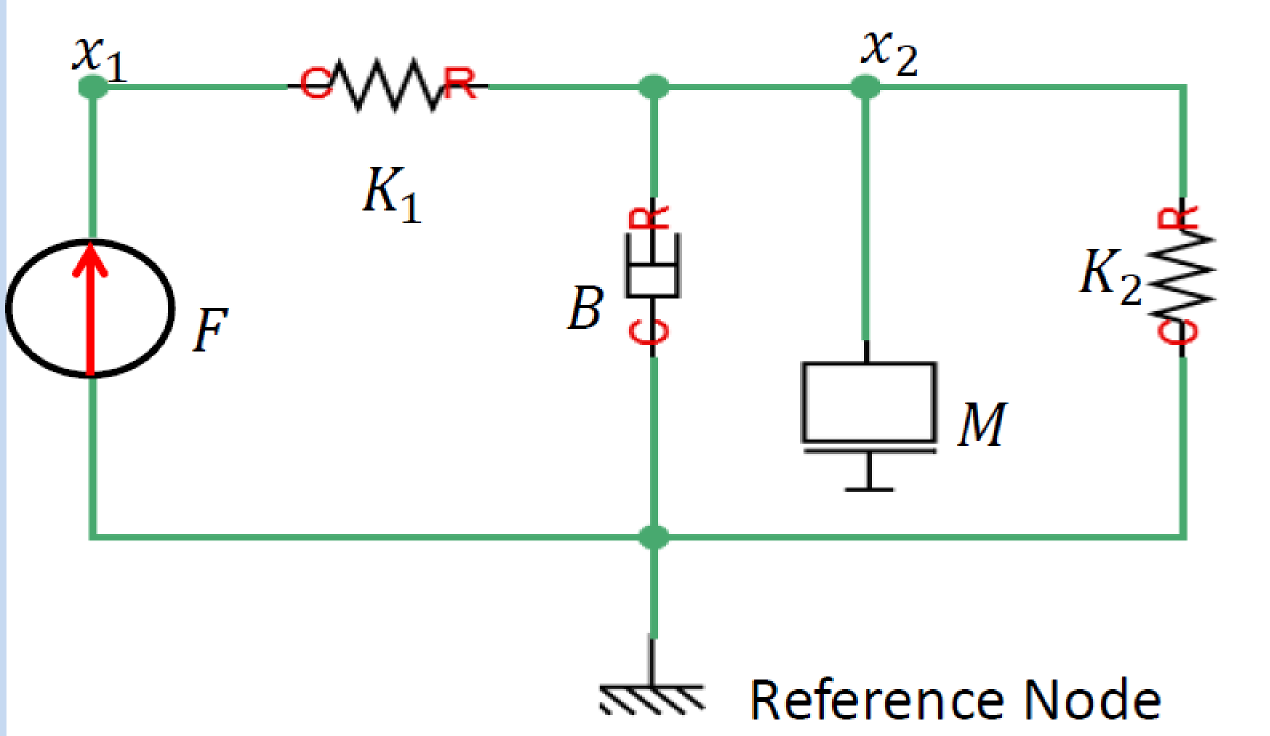
- Connect mass M between node x_2 and reference node

Step 4:

- Connect spring K_1 between nodes x_1 and x_2
- Connect spring K_2 between x_2 and reference node

Step 5:

- Connect the force F between x_1 and reference node



Step 6:

Apply Newton's 2nd law at node x_1:

$$F = K_1(x_1 - x_2) \qquad (1)$$

Apply Newton's 2nd law at node x_2:

$$0 = M\ddot{x}_2 + B\dot{x}_2 + K_1(x_2 - x_1) + K_2x_2 \qquad (2)$$

Eq.1 and Eq.2 give the mathematical model of the given mechanical system

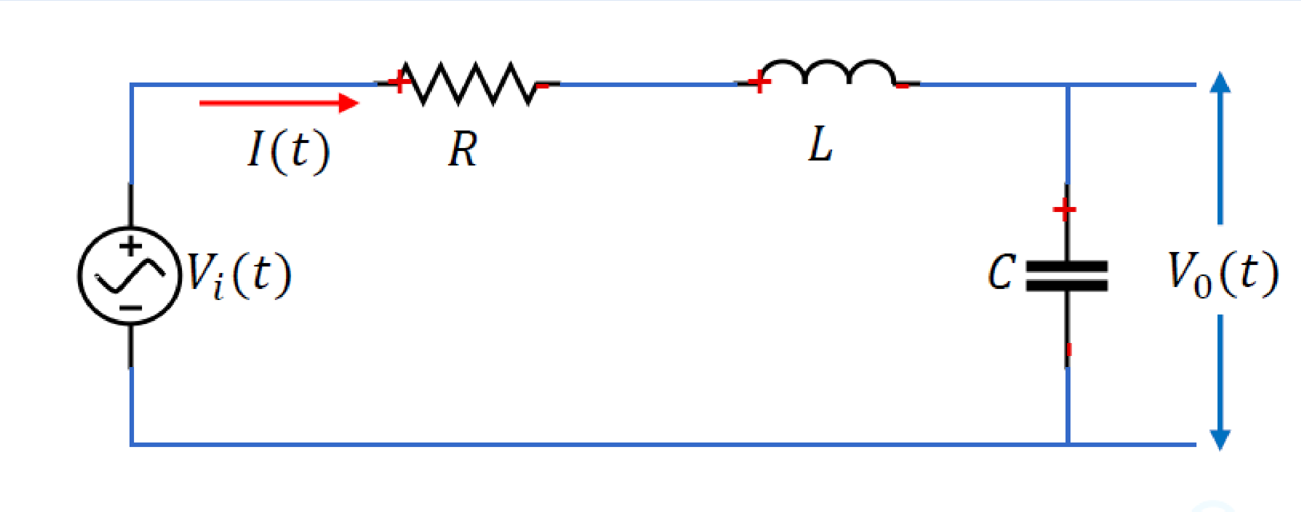
Summary: Steps of Nodal Analysis

1. Identify number of nodes which is equal to number of displacements
2. Take a reference node independent of other nodes (displacement nodes)
3. Connect all mass / inertia elements between the relevant displacement node and the reference node irrespective of its position
4. Connect spring and damper elements between relevant nodes based on their position
5. Connect force or torque between the relevant displacement node and the reference node
6. At each of the displacement nodes, apply Newton's laws of motion

Transfer function

Let consider the system:

- How to find the response of a system for an given input signal?
- E.g.



Find $V_0(t)$ for a given signal $V_i(t)$

$$V_i(t) = RI(t) + L\frac{dI}{dt} + \frac{1}{C}\int Idt$$

$$V_0(t) = \frac{1}{C}\int Idt$$

- To find the time response, we need to solve ordinary differential equations (integro-differential equations)
- When model equations are transformed to $s-$ domain, they turn out to be algebraic equations which are comparably easy to solve
- The transformed model in $s-$ domain is called transfer function model
- It is a model which is applicable for all kinds of input signals

Transfer Function

- For an LTI system, transfer function is the ratio of the Laplace transform of the output to the Laplace transform of the input with the initial conditions being zero
- Mathematically, if $U(s)$ is the Laplace transform of the input function and $Y(s)$ is the Laplace transform of the output, the transfer function $G(s)$ is given by:

$$G(s) = \frac{Y(s)}{U(s)}$$

Transfer Function as Impulse Response

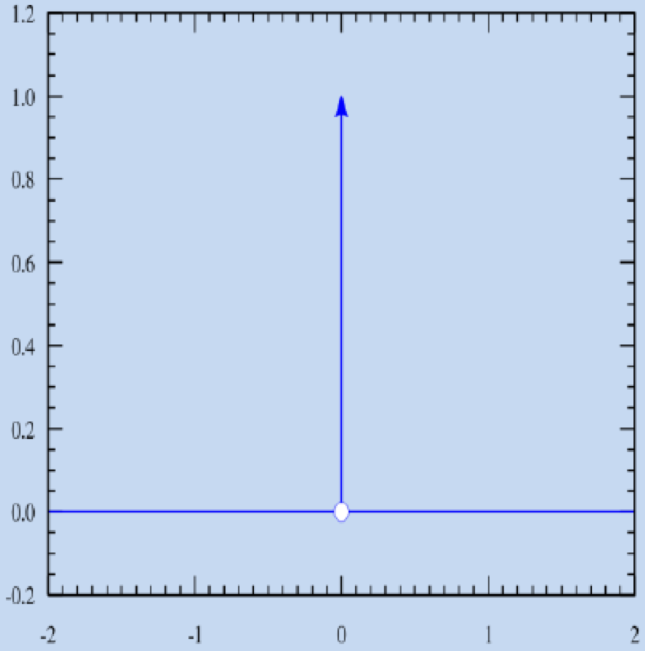
- Impulse signal $(\delta(t))$ is infinitesimally narrow and infinitely tall yet integrating to one
- It takes zero value everywhere except at $t = 0$

$$\int_{-\infty}^{\infty} \delta(t) = 1$$

- If input to the system is the unit impulse, then the output is called the impulse response i.e.,

$$u(t) = \delta(t) \Longrightarrow U(S) = 1 \Longrightarrow G(s) = Y(s)$$

- That means transfer function is the Laplace transform of the impulse response of an LTI system when the initial conditions are set to zero



Impulse function $\delta(t)$

Steps to Finding Transfer Function

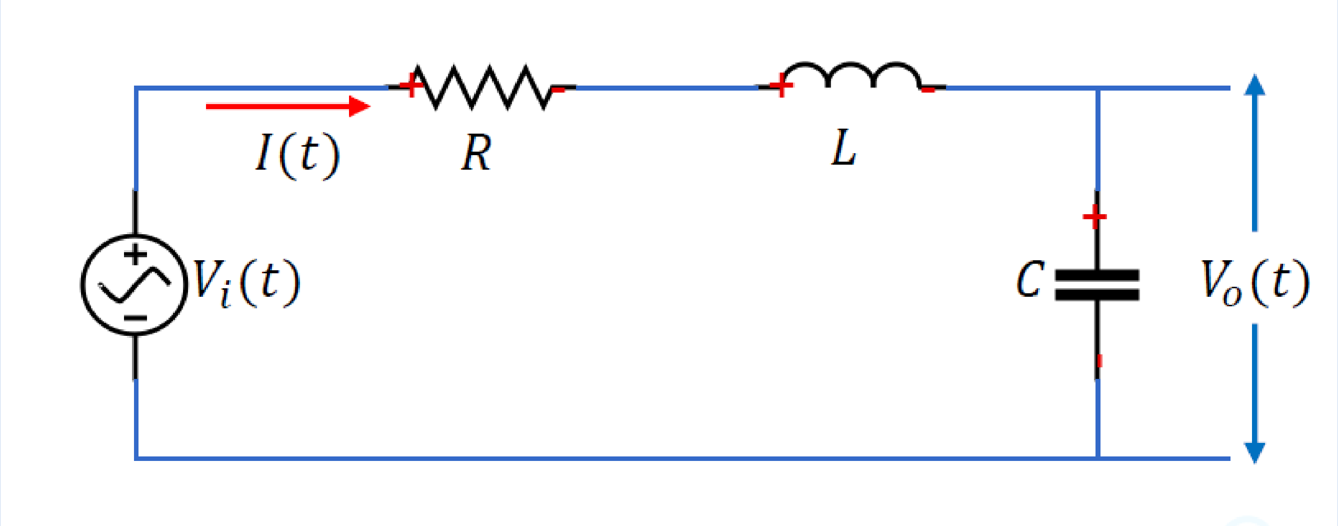
1. Find the model equations of the given system

2. Identify the system input and output variables

3. Take the Laplace transform of the model equations, assuming zero initial conditions

4. Find the ratio of the Laplace transform of the output to the Laplace transform of the input

Transfer Function : Example 1



1. Model Equations:

$$V_i(t) = RI(t) + L\frac{dI}{dt} + \frac{1}{C}\int I dt$$

$$V_0(t) = \frac{1}{C}\int I dt$$

2. **Input and Output Variables:**
   - Input: $V_i(t)$
   - Output: $V_o(t)$
3. **Laplace Transform:** (assuming initial conditions to be zero)

$$V_i(s) = RI(s) + sLI(s) + \frac{1}{sC}I(s)$$

$$V_0(s) = \frac{1}{sC}I(s)$$

4. **Transfer Function:**

$$G(s) = \frac{V_0(s)}{V_i(s)} = \frac{\frac{1}{sC}I(s)}{\left(R + sL + \frac{1}{Cs}\right)I(s)} = \frac{\frac{1}{sC}}{\left(R + sL + \frac{1}{Cs}\right)} = \frac{1}{s^2LC + sRC + 1}$$

Transfer Function : Example 2

- Find the transfer function of a system described by following equation: $\frac{d^3y}{dt^3} + 10\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + y = 10\frac{du}{dt} + u$ with zero initial conditions

➢ **Laplace Transform:**

$$Y(s)[s^3 + 10s^2 - 5s + 1] = U(s)[10s + 1]$$

➢ **Transfer Function:**

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10s + 1}{s^3 + 10s^2 - 5s + 1}$$

Properties of Transfer Function

- Transfer function of a system is independent of the magnitude and nature of input
- Using the transfer function, the response can be studied for various inputs to understand the nature of the system
- Transfer function does not provide any information concerning the physical structure of the system i.e., two different physical systems can have the same transfer function

E.g. MSD system : $G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2+Bs+K} = \frac{1}{s^2+s+1}$ $(M = B = K = 1)$

Series RLC circuit : $G(s) = \frac{V_0(s)}{V_i(s)} = \frac{1}{s^2LC+sRC+1} = \frac{1}{s^2+s+1}$ $(R = L = C = 1)$

Transfer Function : General Form

- General form of transfer function of a system:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^m + b_1 S^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

$$= \frac{K'\big((s - z_1)(s - z_2) \dots (s - z_m)\big)}{(s - p_1)(s - p_2) \dots (s - p_n)}$$

  - n: Order of the system
  - K: System gain or Gain factor – A proportional value that relates the magnitude of the input to that of the output signal at steady state
  - $z_1, z_2, \dots, z_m$: Zeros of the system
  - $p_1, p_2, \dots, p_n$: Poles of the system
  - $n \geq m$ because the system becomes non-causal and is not physically realizable if $n < m$

Poles and Zeros

- **Poles:**
  - Roots of the denominator polynomial of the transfer function
  - Values of s at which the transfer function becomes unbounded

$$\lim_{s \to p_i} G(s) = \infty$$

- **Zeros:**
  - Roots of the numerator polynomial of the transfer function
  - Values of s at which the transfer function vanishes

$$\lim_{s \to z_i} G(s) = 0$$

- Poles and zeros together with the system gain K characterise the input-output system dynamics

Gain, Poles and Zeros : Example

- Find the system gain, poles and zeros of the system with following transfer function: $\frac{6s+12}{s^3+3s^2+7s+5}$

➢ $G(s) = \frac{6s+12}{s^3+3s^2+7s+5}$

➢ System gain: $K = \frac{12}{5}$

➢ Zeros: $s - 2 = 0 \Longrightarrow s = 2 \Rightarrow z_1 = 2$

➢ Poles: $s^3 + 3s^2 + 7s + 5 = 0 \Longrightarrow s = -1, -1 + 2j, -1 - 2j$

$$\Longrightarrow p_1 = -1, p_2 = -1 + 2j, p_3 = -1 - 2j$$

Note: Poles and zeros are purely real or appear in complex conjugates $(a \mp jb)$ because all the coefficients of transfer function are real

Other definition of Transfer Function

- This function will allow separation of the input, system, and output into three separate and distinct parts, unlike the differential equation. The function will also allow us to algebraically combine mathematical representations of subsystems to yield a total system representation.
- Let us begin by writing a general nth-order, linear, time-invariant differential equation,

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t)$$

- where c(t) is the output, r(t) is the input, and the ai's, bi's, and the form of the differential equation represent the system. Taking the Laplace transform of both sides,

$$a_n s^n C(s) + a_{n-1}s^{n-1}C(s) + \cdots + a_0 C(s) + \text{initial condition terms involving } c(t)$$

$$= b_m s^m R(s) + b_{m-1}s^{m-1}R(s) + \cdots + b_0 R(s) + \text{initial condition terms involving } r(t) \tag{2.51}$$

Equation (2.51) is a purely algebraic expression. If we assume that *all initial conditions are zero*, Eq. (2.51) reduces to

$$(a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0)C(s) = (b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0)R(s) \tag{2.52}$$

Now form the ratio of the output transform, $C(s)$, divided by the input transform, $R(s)$:

$$\boxed{\frac{C(s)}{R(s)} = G(s) = \frac{(b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0)}{(a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0)}} \tag{2.53}$$

- The transfer function is defined as the ratio of Laplace transform of the output to the Laplace transform of input with all initial conditions are zeros.

Notice that Eq. (2.53) separates the output, $C(s)$, the input, $R(s)$, and the system, the ratio of polynomials in s on the right. We call this ratio, $G(s)$, the *transfer function* and evaluate it with *zero initial conditions.*

The transfer function can be represented as a block diagram, as shown in Figure 2.2, with the input on the left, the output on the right, and the system transfer function inside the block. Notice that the denominator of the transfer function is identical to the characteristic polynomial of the differential equation. Also, we can find the output, $C(s)$ by using

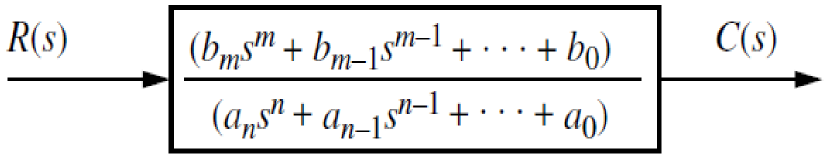


FIGURE 2.2 Block diagram of a transfer function

$$\boxed{C(s) = R(s)G(s)} \tag{2.54}$$

Let us apply the concept of a transfer function to an example and then use the result to find the response of the system.

- Characteristic Equation of a Transfer function:

The characteristic equation of a linear system can be obtained by equating the denominator polynomial of the transfer function to zero. Thus, the characteristic equation of the transfer function will be,

$$a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + ... + a_0 = 0$$

...(1.5)

1.9. POLES AND ZEROS OF A TRANSFER FUNCTION

Consider the equation 1.3, the numerator & denominator can be factored in m and n terms respectively, then the equation 1.3 can be expressed as

$$\frac{C(s)}{R(s)} = G(s) = \frac{K(s+Z_1)(s+Z_2)\left(as^2+bs+c\right)}{(s+p_1)(s+p_2)\left(As^2+Bs+C\right)} \quad \text{...(1.6)}$$

Where $K = \frac{b_m}{a_n}$ is known as the gain factor, s is the complex frequency

POLES : The poles of $G(s)$ are those values of 's' which make $G(s)$ tend to infinity. For example in equation 1.6 we have poles at $s = -p_1, s = -p_2$ and a pair of poles at

$$s = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \quad \text{...(1.7)}$$

ZEROS : The zeros of $G(s)$ are those values of 's' which make $G(s)$ tend to zero. For example in eq. 1.6 we have zeros at $s_1 = -Z_1, s_2 = -Z_2$ and a pair of zeros at

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{...(1.8)}$$

If either poles or zeros coincide, then such type of poles or zeros are called multiple poles or multiple zeros, otherwise they are known as simple poles or simple zeros. Multiple poles are due to the repetitive factor in denominator and multiple zeros are due to the repetitive factor in numerator of a transfer function.

Transfer Function for a Differential Equation

PROBLEM: Find the transfer function represented by

$$\frac{dc(t)}{dt} + 2c(t) = r(t) \tag{2.55}$$

SOLUTION: Taking the Laplace transform of both sides, assuming zero initial conditions, we have

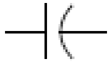
$$sC(s) + 2C(s) = R(s) \tag{2.56}$$

The transfer function, $G(s)$, is

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2} \tag{2.57}$$

Electrical Network Transfer Functions

TABLE 2.3 Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
 Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ – V (volts), $i(t)$ – A (amps), $q(t)$ – Q (coulombs), C – F (farads), R – Ω (ohms), G – Ω (mhos), L – H (henries).

Transfer Function—Single Loop via the Differential Equation

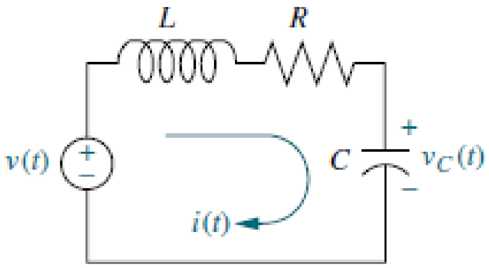


FIGURE 2.3 *RLC* network

PROBLEM: Find the transfer function relating the capacitor voltage, $V_C(s)$, to the input voltage, $V(s)$ in Figure 2.3.

SOLUTION: In any problem, the designer must first decide what the input and output should be. In this network, several variables could have been chosen to be the output—for example, the inductor voltage, the capacitor voltage, the resistor voltage, or the current. The problem statement, however, is clear in this case: We are to treat the capacitor voltage as the output and the applied voltage as the input.

Summing the voltages around the loop, assuming zero initial conditions, yields the integro-differential equation for this network as

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t) \tag{2.61}$$

Changing variables from current to charge using $i(t) = dq(t)/dt$ yields

$$L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = v(t) \tag{2.62}$$

From the voltage-charge relationship for a capacitor in Table 2.3,

$$q(t) = Cv_C(t) \tag{2.63}$$

Substituting Eq. (2.63) into Eq. (2.62) yields

$$LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = v(t) \tag{2.64}$$

Taking the Laplace transform assuming zero initial conditions, rearranging terms, and simplifying yields

$$(LCs^2 + RCs + 1)V_C(s) = V(s) \tag{2.65}$$

Solving for the transfer function, $V_C(s)/V(s)$, we obtain

$$\frac{V_C(s)}{V(s)} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \tag{2.66}$$

as shown in Figure 2.4.

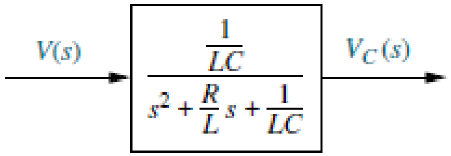


FIGURE 2.4 Block diagram of series RLC electrical network

Example 1.1. Find the transfer function of the given network

Solution : Step 1 : Apply *KVL* in mesh (1)

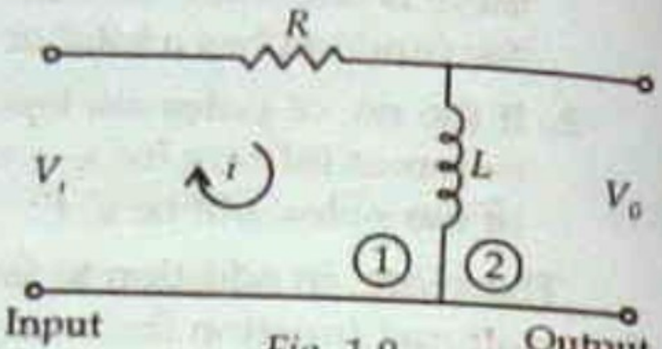


Fig. 1.9

$$V_i = Ri + L\frac{di}{dt} \qquad ...(1.13)$$

Apply *KVL* in mesh (2)

$$V_0 = L\frac{di}{dt} \qquad ...(1.14)$$

Step 2 : Take laplace transform of equation 1.13 & 1.14 with assumption that all initial conditions are zero.

$$V_i(s) = RI(s) + sLI(s) \qquad ...(1.15)$$

$$V_0(s) = sLI(s) \qquad ...(1.16)$$

Step 3 : Calculation of transfer function

$$\frac{V_0(s)}{V_i(s)} = \frac{sLI(s)}{(R+sL)I(s)}$$

$$\frac{V_0(s)}{V_i(s)} = \frac{sL}{R+sL} \qquad ...(1.17)$$

Eq. 1.17 is the required transfer function.

Example 1.2. Determine the transfer function of the electrical network shown in fig. 1.10.

Solution : Step 1 : Apply *KVL* in both meshes

$$E_i = Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt \qquad ...(1.18)$$

$$E_0 = \frac{1}{C}\int i\,dt \qquad ...(1.19)$$

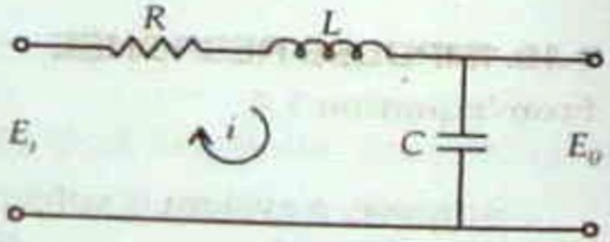


Fig. 1.10.

Step 2 : Take laplace transform of eqn 1.18 & 1.19

$$E_i(s) = RI(s) + sLI(s) + \frac{1}{Cs}I(s)$$

$$= I(s)\left[R + sL + \frac{1}{Cs}\right]$$

$$E_i(s) = I(s)\left[\frac{RCs + s^2LC + 1}{Cs}\right] \qquad ...(1.20)$$

$$E_0(s) = \frac{1}{Cs}I(s) \qquad ...(1.21)$$

Step 3 : Determination of transfer function

$$\frac{E_0(s)}{E_i(s)} = \frac{I(s)}{Cs}\cdot\frac{Cs}{I(s)\left[s^2Lc + SRC + 1\right]}$$

$$\frac{E_0(s)}{E_i(s)} = \frac{1}{S^2LC + SRC + 1} \qquad \textbf{Ans.} \qquad ...(1.22)$$

Example 1.3. Obtain the transfer function $\frac{V_2(s)}{V_1(s)}$ for fig. 1.11.

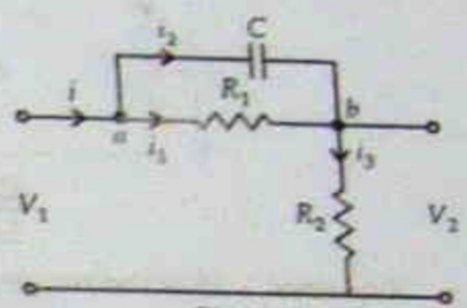


Fig. 1.11.

Solution : Step 1 : *KCL* at node '*a*'

$$i = i_1 + i_2 \qquad ...(1.23)$$

$$i_1 = \frac{V_1 - V_2}{R_1}$$

$$i_2 = C\frac{d}{dt}(V_1 - V_2)$$

$$i = i_3 = \frac{V_2}{R_2}$$

Put all these values in eq^n 1.23

$$\frac{V_2}{R_2} = \frac{V_1 - V_2}{R_1} + C\frac{d}{dt}(V_1 - V_2) \qquad ...(1.24)$$

Step 2 : Take laplace transform of *eq^n* 1.24

$$\frac{V_2(s)}{R_2} = \frac{1}{R_1}V_1(s) - \frac{1}{R_1}V_2(s) + Cs\,V_1(s) - Cs\,V_2(s)$$

$$\frac{V_2(s)}{R_2} + \frac{1}{R_1}V_2(S) + Cs\,V_2(s) = \frac{1}{R_1}V_1(s) + Cs\,V_1(s)$$

$$V_2(s)\left[\frac{1}{R_1} + \frac{1}{R_2} + Cs\right] = V_1(s)\left[\frac{1}{R_1} + Cs\right]$$

Step 3 : Determination of transfer function

$$V_2(s)\left[\frac{R_1 + R_2 + R_1R_2Cs}{R_1R_2}\right] = V_1(s)\left[\frac{1 + R_1Cs}{R_1}\right]$$

$$\frac{V_2(s)}{V_1(s)} = \frac{R_2 + R_1R_2Cs}{R_1 + R_2 + R_1R_2Cs} \qquad \textbf{Ans.} \qquad ...(1.25)$$

Example 1.4. Find the transfer function of lag network shown in fig. 1.12.

Solution : Step 1 : Apply *KVL* in both meshes

$$e_i(t) = R_1 i(t) + R_2 i(t) + \frac{1}{C}\int i(t)\,dt \qquad ...(1.26)$$

$$e_0(t) = R_2 i(t) + \frac{1}{C}\int i(t)\,dt \qquad ...(1.27)$$

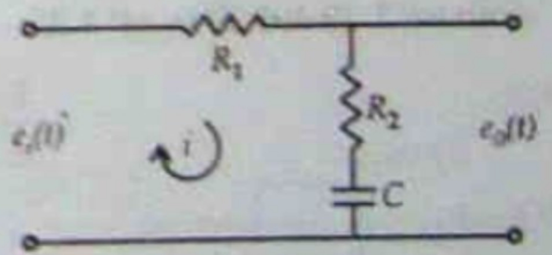


Fig. 1.12

Step 2 : Laplace transform of eq" (1.26) & (1.27)

$$E_i(s) = \left[R_1 + R_2 + \frac{1}{Cs}\right] I(s)$$

$$E_0(s) = \left[R_2 + \frac{1}{Cs}\right] I(s)$$

Step 3 : Calculation of transfer function

$$E_0(s) = \frac{\left[R_2 + \frac{1}{Cs}\right] I(s)}{\left[\frac{R_1Cs + R_2Cs + 1}{Cs}\right] I(s)}$$

$$\frac{E_0(s)}{E_i(s)} = \frac{1 + R_2Cs}{1 + R_1Cs + R_2Cs} \qquad ...(1.28)$$

Equation 1.28 is the required transfer function.

Example 1.5. Determine the transfer function of fig 1.13.

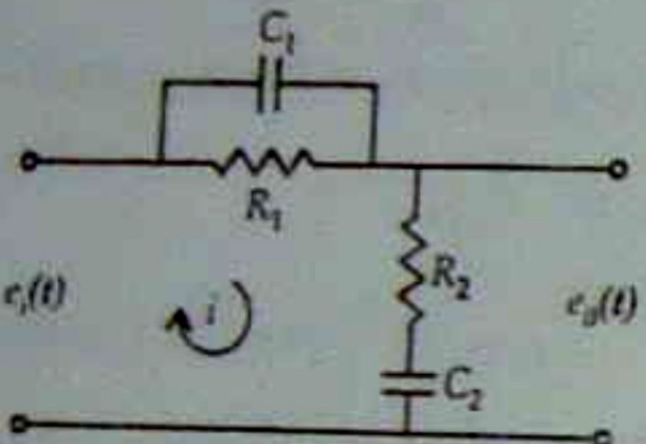


Fig. 1.13.

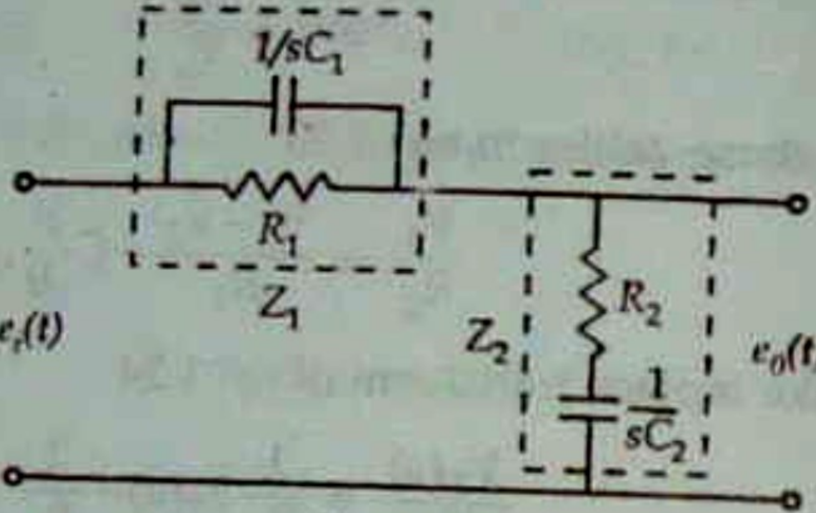


Fig. 1.14.

Fig. 1.14.

Solution : Step 1 : calculation of Z_1 :

$$Z_1 = \frac{R_1 \frac{1}{sC_1}}{R_1 + \frac{1}{sC_1}} = \frac{R_1}{R_1 C_1 s + 1} \quad ...(1.29)$$

Step 2 : Calculation of Z_2 :

$$Z_2 = R_2 + \frac{1}{sC_2} = \frac{R_2 C_2 S + 1}{sC_2} \quad ...(1.30)$$

Step 3 : Calculation of transfer function in terms of Z_1 & Z_2

$$\frac{E_0(s)}{E_i(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)} \quad ...(1.31)$$

Step 4 : Calculation of transfer function in terms of R_1, R_2, C_1 & C_2 Put the values of Z_1 (s) & $Z_2(s)$ from eq^n 1.29 & 1.30 in eq^n 1.31

$$\frac{E_0(s)}{E_i(s)} = \frac{(1 + R_2 C_2 S)/Sc_2}{\frac{R_1}{C_1 R_1 S + 1} + \frac{R_2 C_2 S + 1}{SC_2}}$$

$$\frac{E_0(s)}{E_i(s)} = \frac{(1 + R_1 C_1 S)(1 + R_2 C_2 S)}{(1 + R_1 C_1 S)(1 + R_2 C_2 S) + R_1 C_2 S} \quad ...(1.32)$$

The above eq^n is the required transfer function of the given circuit.

Operational amplifier

- An operational amplifier, pictured in Figure is an electronic amplifie used as a basic building block to implement transfer functions. It has the following characteristics:

1. Differential input, $V_2(t) - v_1(t)$
2. High input impedance, $Z_i = \infty$ (ideal)
3. Low output impedance, $Z_o = 0$ (ideal)
4. High constant gain amplification, $A = \infty$ (ideal)

The output, $v_o(t)$, is given by

$$v_o(t) = A(v_2(t) - v_1(t))$$

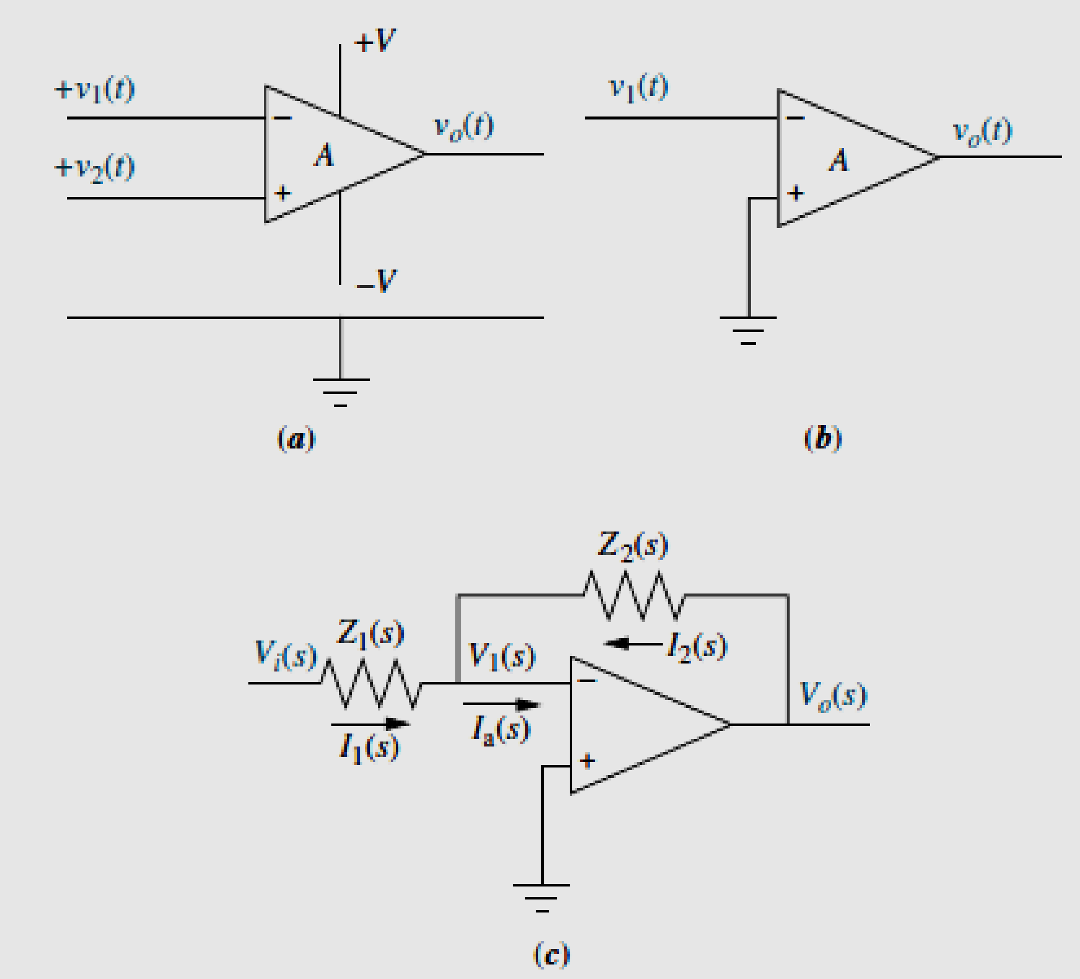


Fig. a. Operational amplifier;

Fig. b. schematic for an inverting operational amplifier;

Fig. c. inverting operational amplifier configured for transfer function realization.

Example

- Figure shows an electrical circuit involving an operational amplifier. Obtain the output eo/ei.

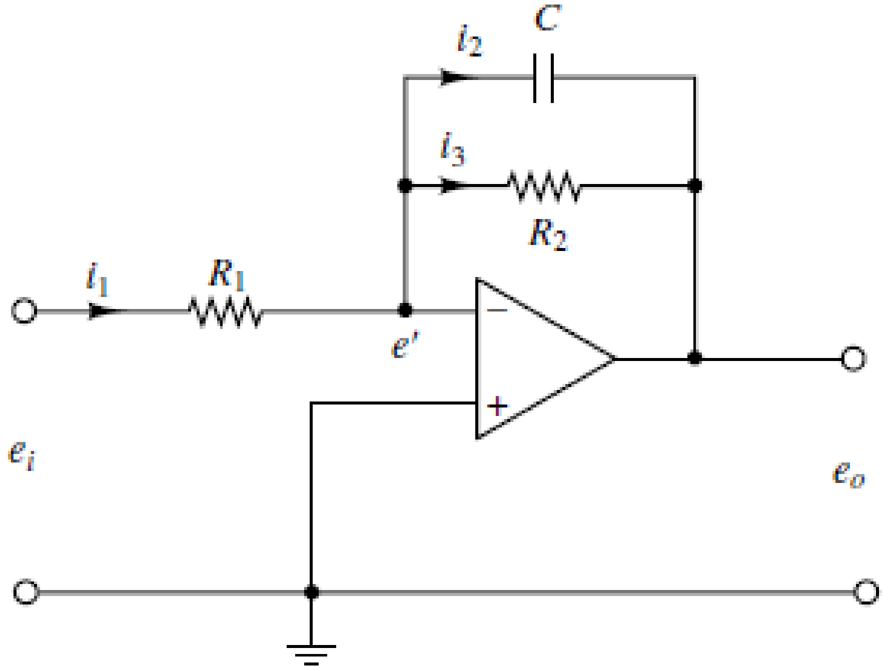


Fig. First-order lag circuit using operational amplifier.

Let us define

$$i_1 = \frac{e_i - e'}{R_1}, \qquad i_2 = C\frac{d(e' - e_o)}{dt}, \qquad i_3 = \frac{e' - e_o}{R_2}$$

Noting that the current flowing into the amplifier is negligible, we have

$$i_1 = i_2 + i_3$$

Hence

$$\frac{e_i - e'}{R_1} = C\frac{d(e' - e_o)}{dt} + \frac{e' - e_o}{R_2}$$

Since $e' \doteqdot 0$, we have

$$\frac{e_i}{R_1} = -C\frac{de_o}{dt} - \frac{e_o}{R_2}$$

Taking the Laplace transform of this last equation, assuming the zero initial condition, we have

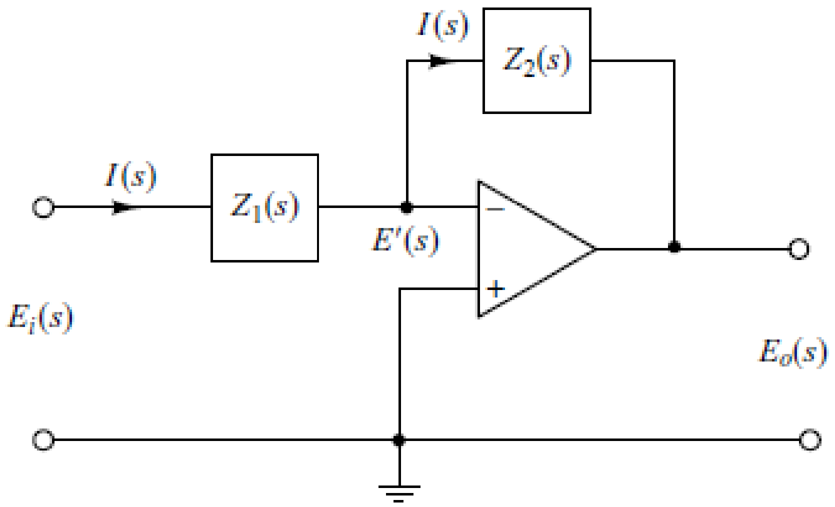
$$\frac{E_i(s)}{R_1} = -\frac{R_2Cs + 1}{R_2}E_o(s)$$

which can be written as

$$\frac{E_o(s)}{E_i(s)} = -\frac{R_2}{R_1}\frac{1}{R_2Cs + 1}$$

Impedance Approach to Obtaining Transfer Functions

- Same Example:



use of the impedance approach.

The complex impedances $Z_1(s)$ and $Z_2(s)$ for this circuit are

$$Z_1(s) = R_1 \qquad \text{and} \qquad Z_2(s) = \frac{1}{Cs + \frac{1}{R_2}} = \frac{R_2}{R_2Cs + 1}$$

The transfer function $E_o(s)/E_i(s)$ is, therefore, obtained as

$$\frac{E_o(s)}{E_i(s)} = -\frac{Z_2(s)}{Z_1(s)} = -\frac{R_2}{R_1}\frac{1}{R_2Cs + 1}$$

Mechanical Systems

Classification based on type of motion:

–**Translational systems** having linear motion

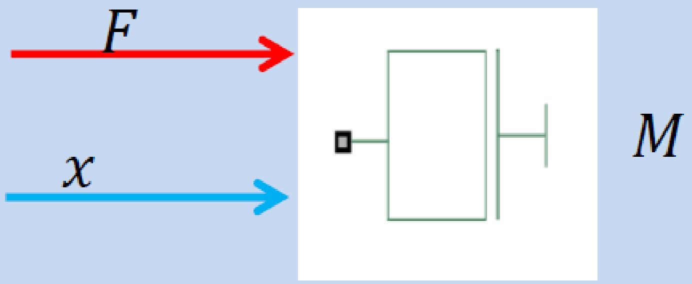
–**Rotational systems** having angular motion about a fixed axis

Translational	Rotational
Basic System Elements	
Mass (M)	Inertia (J)
Damper (B)	Damper (D)
Linear spring (K)	Torsional spring (K)
Basic System Variables	
Force (F)	Torque (T)
Displacement (x)	Angular displacement (θ)

Mass Vs Inertia

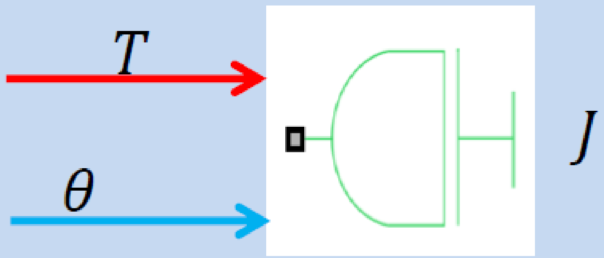
Mass

- Property of an element that stores the kinetic energy due to translational motion
- When a force is acting on a body of mass M causing displacement x, then:
- $F = \frac{dP}{dt} = M\frac{d^2x}{dt^2} = M\ddot{x}$



Inertia

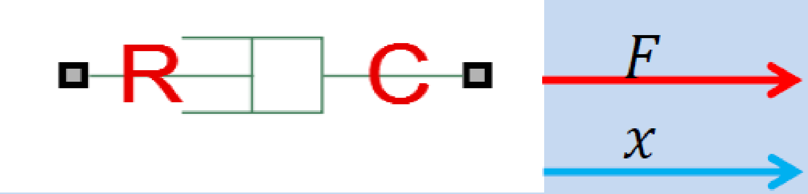
- Property of an element that stores the kinetic energy due to rotational motion
- When a torque is acting on a body of inertia J causing displacement θ, then:
- $T = J\frac{d^2\theta}{dt^2} = J\ddot{\theta}$



Damper

- Damper is an element that generates force which acts opposite to the direction of motion, translational or rotational
- Damper resists motion
- Friction or dashpot are examples of dampers

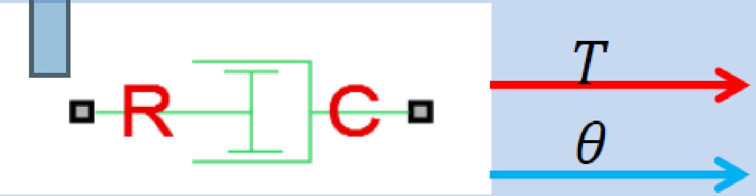
Translational



B

$$F = B\frac{dx}{dt} = B\dot{x}$$

Rotational



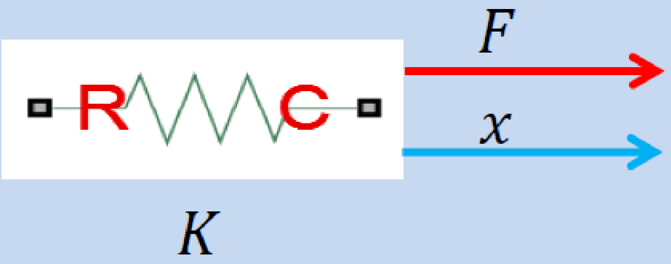
D

$$T = D\frac{d\theta}{dt} = D\dot{\theta}$$

Linear Vs Torsional Spring

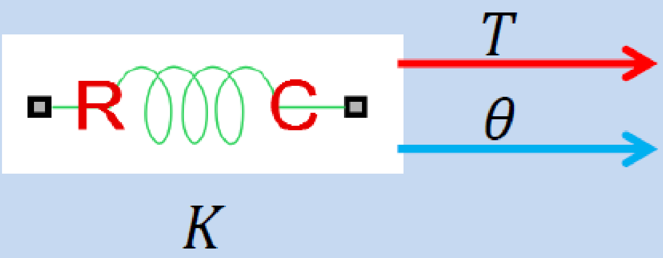
Linear Spring

- Property of an element that stores the potential energy due to translational motion
- When a spring of spring constant K is applied a force F causing an elastic displacement x, then:
- $F = Kx$



Torsional spring

- Property of an element that stores the potential energy due to rotational motion
- When a torsional spring of constant K is applied a torque T causing an angular displacement θ, then:
- $T = K\theta$

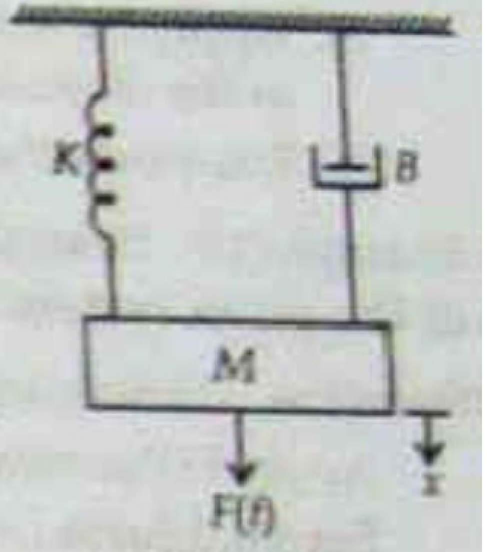


D'ALEMBERT'S PRINCIPLE

This principle states that "for any body, the algebraic sum of externally applied forces and the forces resisting motion in any given direction is zero".

D' Alembert principle is useful in writing the equation of motion of mechanical system. Consider, a system shown in fig. 1.21. consisting of a mass M, spring & dashpot.

First choose a reference direction. All the forces in the direction of reference direction considered as positive & the forces opposite to the reference direction taken as negative.

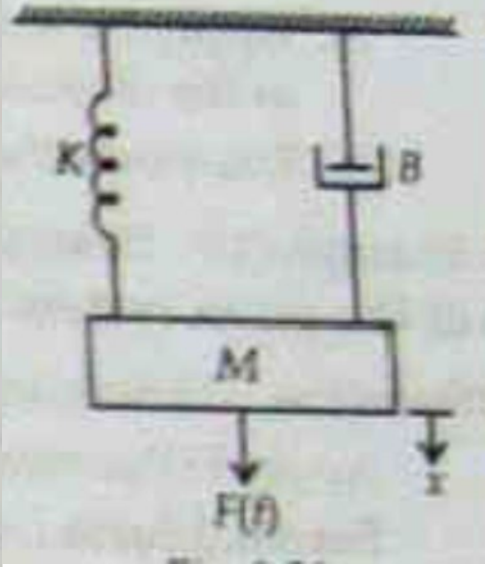


External Force : $F(t)$

Resisting Forces : *a.* Inertia Force $F_m(t) = -M\frac{d^2}{dt^2}x(t)$

b. Damping Force $F_D(t) = -B\frac{d}{dt}x(t)$

c. spring Force $F_K(t) = -Kx(t)$



According to D' Alembert's principle

$$F(t) + F_m(t) + F_D(t) + F_K(t) = 0$$

$$F(t) - M\frac{d^2}{dt^2}x(t) - B\frac{d}{dt}x(t) - Kx(t) = 0$$

or, $$F(t) = M\frac{d^2}{dt^2}x(t) + B\frac{d}{dt}x(t) + Kx(t)$$

External Torque : $T\,(t)$

Resisting Torque : *a.* Inertia Torque $T_I(t) = -J\dfrac{d\omega(t)}{dt}$

b. Damping Torque $T_D(t) = -B\dfrac{d}{dt}\theta(t)$

c. spring Torque $T_K(t) = -K\theta$

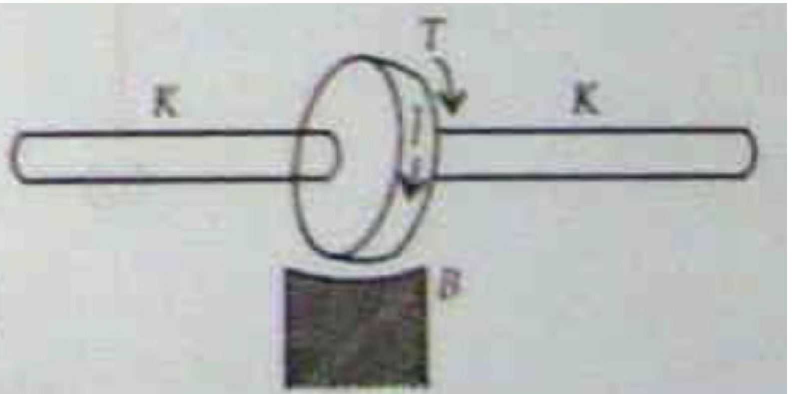


Fig. 1.22.

According to D'Alembert principle

$$T(t) + T_I + T_D + T_K = 0$$

$$T(t) - J\frac{d}{dt}\omega(t) - B\frac{d}{dt}\theta(t) - K\theta(t) = 0$$

or,
$$T(t) = J\frac{d}{dt}\omega(t) + B\frac{d}{dt}\theta(t) + K\theta(t) \qquad \text{...(1.48)}$$

So, D'Alembert principle for rotational motion is

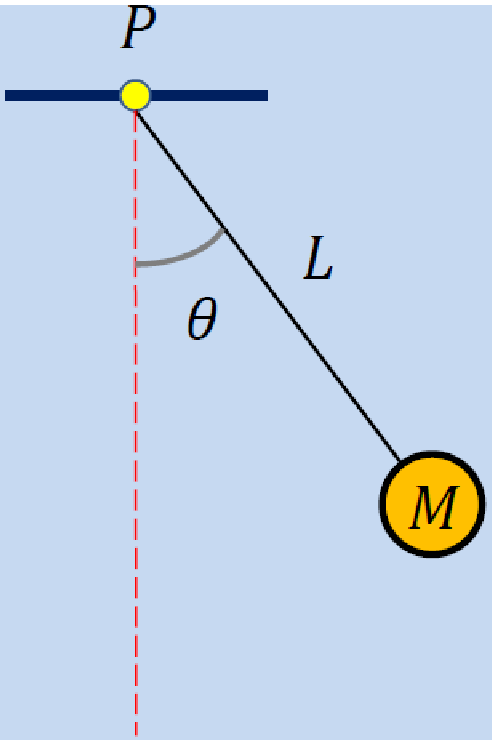
"For any body, the algebaric sum of externally applied torques and the torques resisting rotation about any axis is zero".

1.14. PROCEDURE OF WRITING THE MODELS OF MECHANICAL SYSTEM

1. Assume system is in equilibrium.
2. Assume that the system is given same arbitrary displacement if no. of distributing forces are present.
3. Draw the free body diagram of forces exerted on each mass in the system.
4. Apply Newton's law of motion to each diagram, using the convention that any force acting in the direction of assume displacement is positive.
5. Rearrange the equations in suitable form to be solved by any means.

Example: Simple Pendulum

- A simple pendulum consists of a mass M hanging from a string of length L and fixed at a pivot point P
- Pendulum oscillates back and forth when released at an angle to its equilibrium point
- We derive a mathematical model to study the motion of the pendulum



Modelling a Pendulum

1.Purpose of the model

–Model for studying the motion of a simple pendulum

2.Define boundaries

–Pivot, string and mass form the system

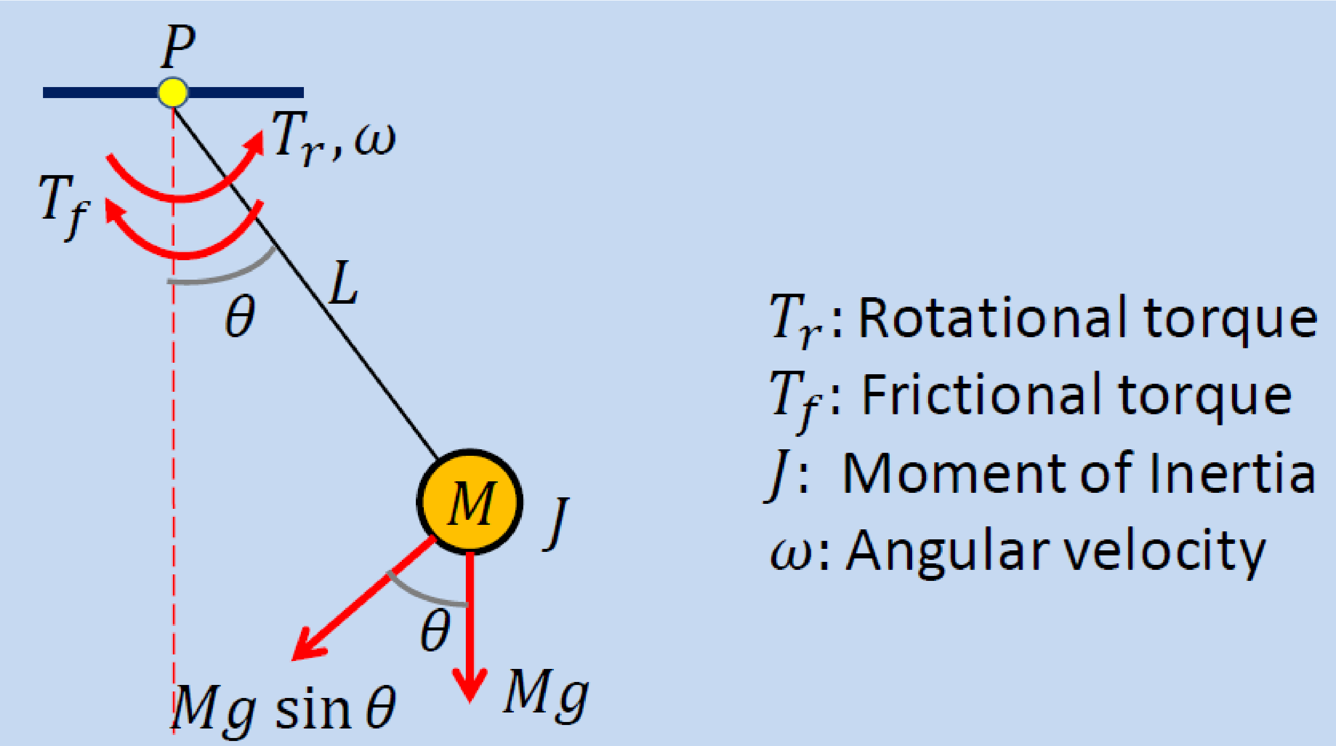
3.Postulate a structure

–In this system, energy transformation happens between potential and kinetic energies successively

–System is similar to a rotational mechanical system

–Models are derived for both cases of neglecting and considering friction

Modelling a Pendulum



4.Select variables of interest

– Angular displacement θ and angular velocity ω are variables of interest

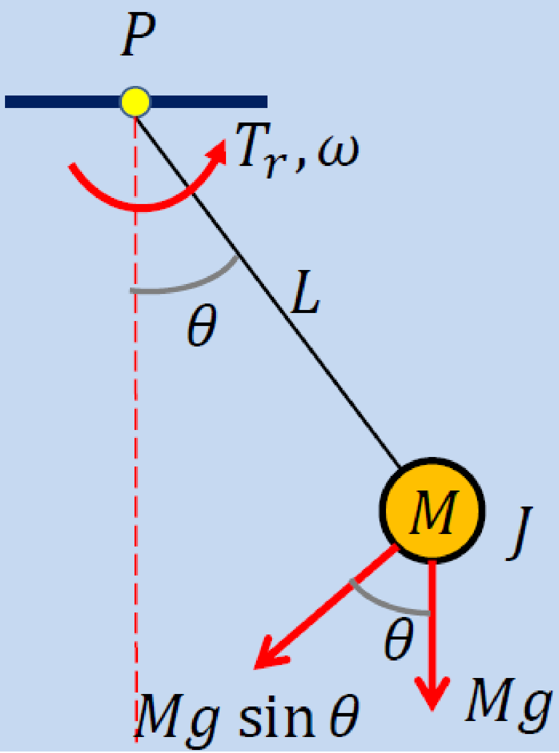
Modelling Pendulum (Lossless)

5. **Mathematical description of each model elements**

(neglecting friction)

– Rotational torque: $T_r = J\omega = J\ddot{\theta} = ML^2\ddot{\theta}$

– Torque due to gravity: $T_g = Mg\sin\theta\,L$



6. **Apply relevant physical laws**

– Both the torques act in opposite directions

$$T_r = -T_g$$

$$ML^2\ddot{\theta} = -Mg\sin\theta\,L$$

7. **Final mathematical model**

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0$$

Modelling Pendulum (Lossy)

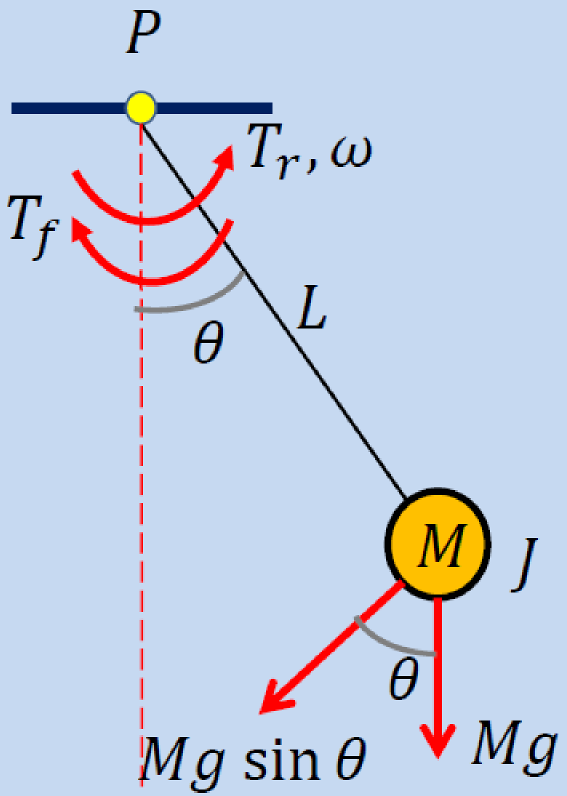
5. Mathematical description of each model elements

– Rotational torque: $T_r = J\omega = J\ddot{\theta} = ML^2\ddot{\theta}$

– Torque due to gravity: $T_g = Mg\sin\theta\, L$

– Frictional torque: $T_f = B\dot{\theta}$

(with friction)



6. Apply relevant physical laws

$$T_r = -T_g - T_f$$

$$ML^2\ddot{\theta} = -Mg\sin\theta\, L - B\dot{\theta}$$

7. Final mathematical model

$$\ddot{\theta} + \frac{B}{ML^2}\dot{\theta} + \frac{g}{L}\sin\theta = 0$$

Example 1.7. Draw the free body diagram and write the diffrential equation of the given system shown in fig. 1.23.

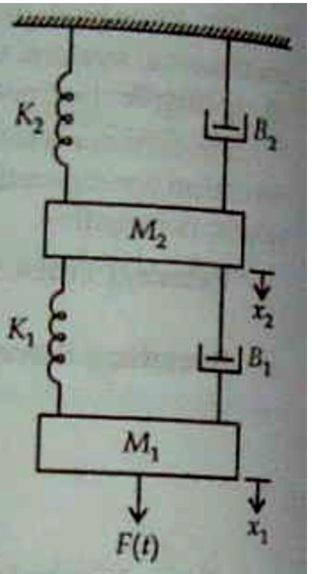


Fig. 1.23

Solution : Differential equation For M_1 :

Apply D'Alembert principle

External force : $F(t)$

Resisting forces :

1. Inertia force $\quad F_M = -M_1 \dfrac{d^2}{dt^2} x_1$

2. Damping force

$$F_D = -B_1 \frac{d}{dt}(x_1 - x_2)$$

3. Spring force $\quad F_K = K_1(x_1 - x_2)$

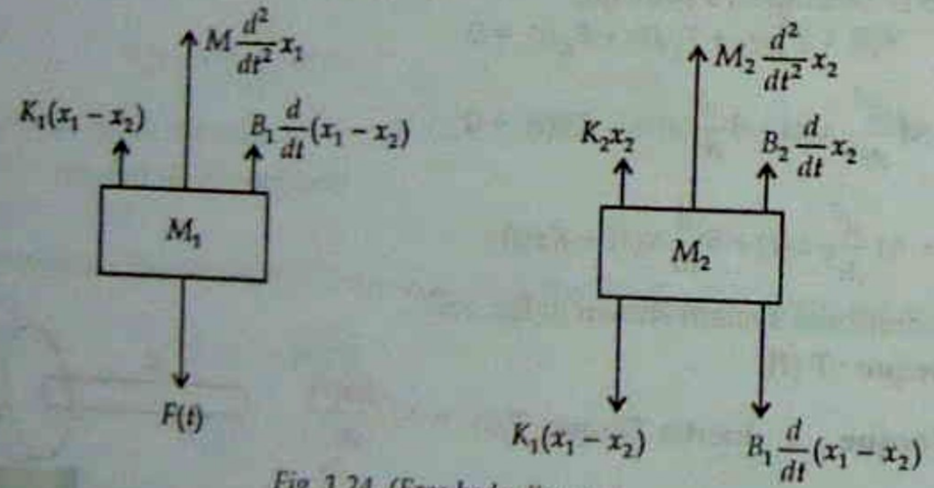


Fig. 1.24. (Free body diagram)

$\therefore$

$$F(t) = M_1 \frac{d^2}{dt^2} x_1 + B_1 \frac{d}{dt}(x_1 - x_2) + K_1 (x_1 - x_2) \qquad \text{...(1.49)}$$

Similarly for mass M_2 :

$$K(x_1 - x_2) + B_1 \frac{d}{dt}(x_1 - x_2) = M_2 \frac{d^2 x_2}{dt^2} + B_2 \frac{dx_2}{dt} + K_2 x_2 \qquad \text{...(1.50)}$$

Equation (1.49) & (1.50) are the required equations.

Example 1.8. Write the differential equations describing the dynamics of the system shown in fig. 1.25 and find the ratio $\frac{X_2(s)}{F(s)}$.

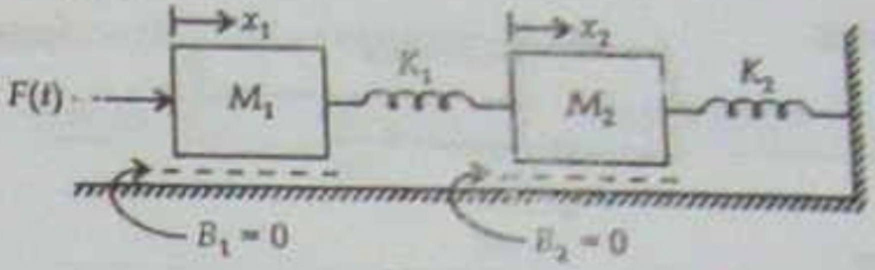


Fig. 1.25

Solution : Free body diag. For mass M_1 :

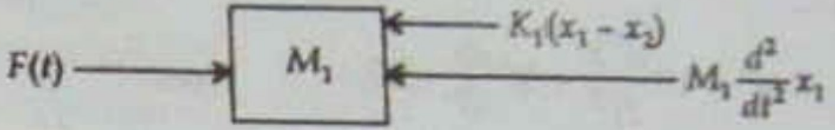


Fig. 1.26(a)

Free body diag. For mass M_2 :

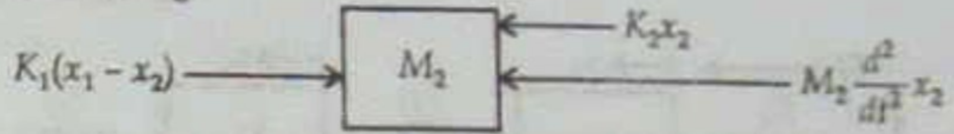


Fig. 1.26(b)

Differential equation for mass M_1

$$F(t) = M_1 \frac{d^2}{dt^2} x_1 + K_1(x_1 - x_2) \quad \text{...(1.51)}$$

$$K_1(x_1 - x_2) = K_2 x_2 + M_2 \frac{d^2}{dt^2} x_2 \quad \text{...(1.52)}$$

Take laplace transform of eq^n (1.51), assume initial conditions zero.

$$F(t) = M_1 \frac{d^2}{dt^2} x_1 + K_1 x_1 - K_1 x_2$$

$$F(s) = M_1 S^2 X_1(s) + K_1 X_1(s) - K_1 X_2(s) \qquad \text{...(1.53)}$$

laplace transform of equation (1.52)

$$K_1 X_1 - K_1 X_2 = K_2 X_2 + M_2 \frac{d^2}{dt^2} X_2$$

$$K_1 X_1(s) - K_1 X_2(s) = K_2 X_2(s) + M_2 s^2 X_2(s)$$

$$X_1(s) = \frac{X_2(s)}{K_1} [s^2 M_2 + K_1 + K_2] \qquad \text{...(1.54)}$$

Put the value of $X_1(s)$ in eq^n (1.53)

$$F(s) = \frac{X_2(s)}{K_1} [s^2 M_2 + K_1 + K_2]\,[s^2 M_1 + K_1] - K_1 X_2(s)$$

$$\frac{X_2(s)}{F(s)} = \frac{K_1}{(S^2 M_2 + K_1 + K_2)(K_1 + s^2 M_1) - K_1^2} \qquad \textbf{Ans.}$$

Drive the system equations & find the value of $X_2(s)/F(s)$ for the system

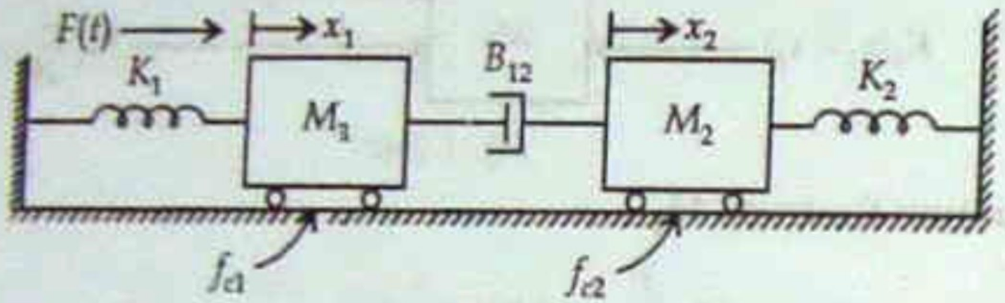
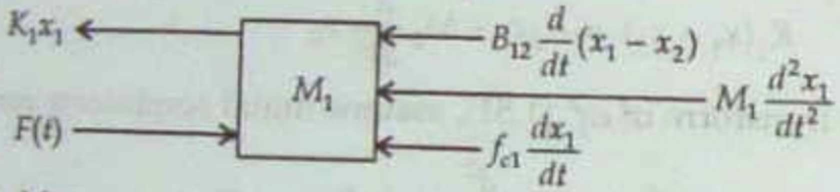
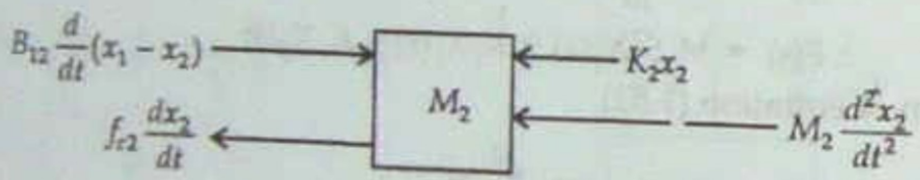


Fig. 1.29.

Solution : Free body diag. for mass M_1:



Free body diag. For mass M_2:



System equation for mass M_1:

$$F(t) = M_1\frac{d^2x_1}{dt^2} + fc_1\frac{dx_1}{dt} + B_{12}\frac{d}{dt}(x_1 - x_2) + K_1x_1 \qquad \text{.....(1.56)}$$

System equation for mass M_2:

$$B_{12}\frac{d}{dt}(x_1 - x_2) = M_2\frac{d^2x_2}{dt^2} + fc_2\frac{dx_2}{dt} + K_2x_2 \qquad \text{....(1.57)}$$

Laplace transform of equation 1.56

$$F(s) = X_1(s)[s^2M_1 + sB_{12} + sfc_1 + K_1] - B_{12}\,sX_2(s) \qquad \text{.....(1.58)}$$

Laplace transform of equation 1.57

$$B_{12}sX_1(s) = (s^2M_2 + B_{12}s + sfc_2 + K_2)\,X_2(s)$$

$$\therefore \quad X_1(s) = \frac{X_2(s)\left(s^2 M_2 + B_{12}s + sfc_2 + K_2\right)}{B_{12}s} \qquad \text{....(1.59)}$$

Put the value of $X_1(s)$ for eqn (1.59) in equation (1.58)

$$F(s) = \frac{X_2(s)\left[S^2 M_2 + B_{12}S + sfc_2 + K_2\right]\left[S^2 M_1 + SB_{12} + sfc_1 + K_1\right]}{B_{12}S} - B_{12}SX_2(s)$$

$$\therefore \quad \frac{X_2(s)}{F(s)} = \frac{SB_{12}}{(s^2 M_1 + sB_{12} + sfc_1 + K_1)\left(s^2 M_2 + sB_{12} + sfc_2 + K_2\right) - s^2 B_{12}^2} \qquad \textbf{Ans.}$$

Block Diagram : Motivation

- How to visualise a complex system wih many components?
- How to understand the flow and transformation of signals in a complex system?
- How to find the transfer function of a complex system?

Block Diagram Representation

Block Diagram of a System

- It is a short hand pictorial representation of the system which depicts
  - Each functional component or sub-system and
  - Flow of signals from one sub-system to another
- Block diagram provides a simple representation of complex systems
- Block diagram enables calculating the overall system transfer function provided the transfer functions of each of the components or sub-systems are known

Components of Block Diagram

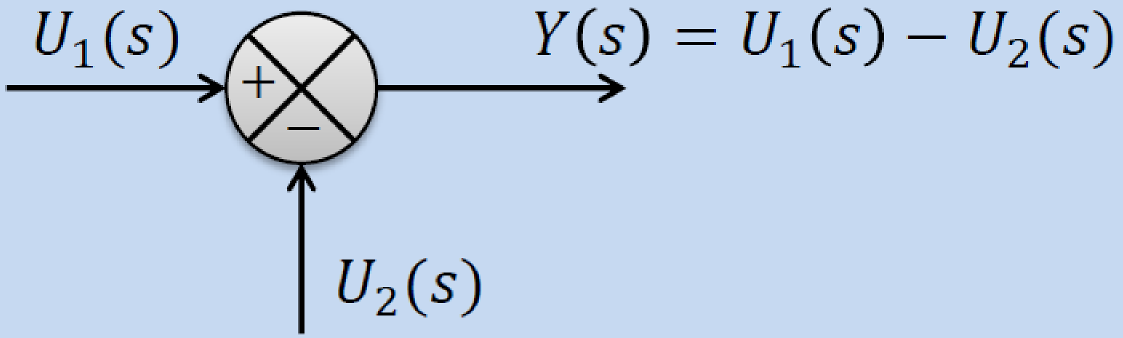
- Block diagrams have four components:
  1. **Blocks:** To represent the components or sub-systems

     $\boxed{G(s)}$ $G(s)$ is the transfer function of sub-system

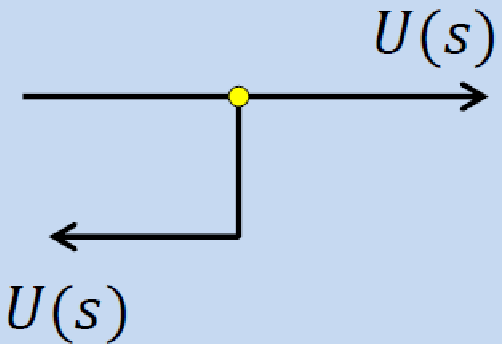
  2. **Arrows:** To represent the direction of flow of signals

     $\xrightarrow{U(s)}$ $\qquad$ $\xrightarrow{Y(s)}$

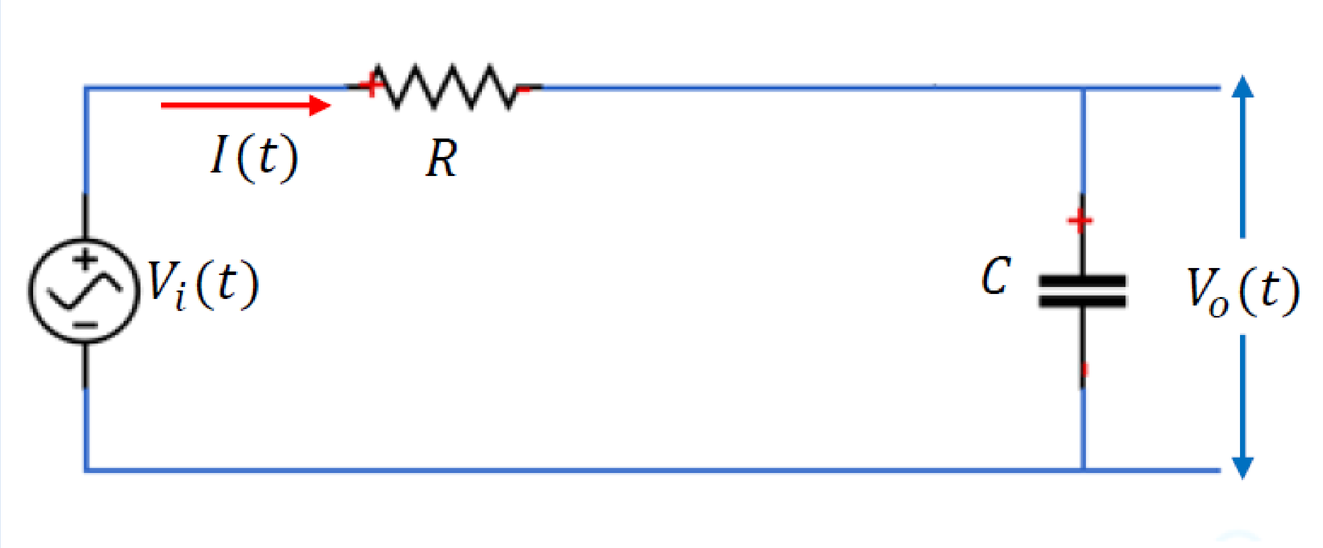
3. Summing points: To represent the summation of two or more signals



4. Take-off points: To represent the branching of a signal



Block Diagram Example



$$\xrightarrow{V_i(s)} \boxed{\frac{1}{1+sRC}} \xrightarrow{V_0(s)}$$

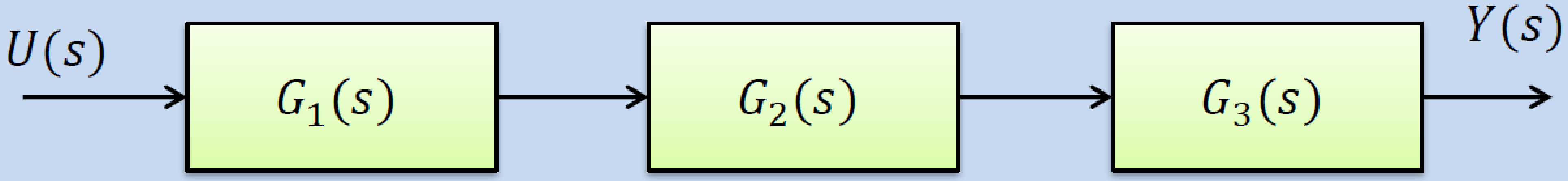
Transfer Function:

$$G(s) = \frac{V_0(s)}{V_i(s)} = \frac{1}{1+sRC}$$

Typical Block Diagram Forms

➢ Cascaded Form / Series Form:

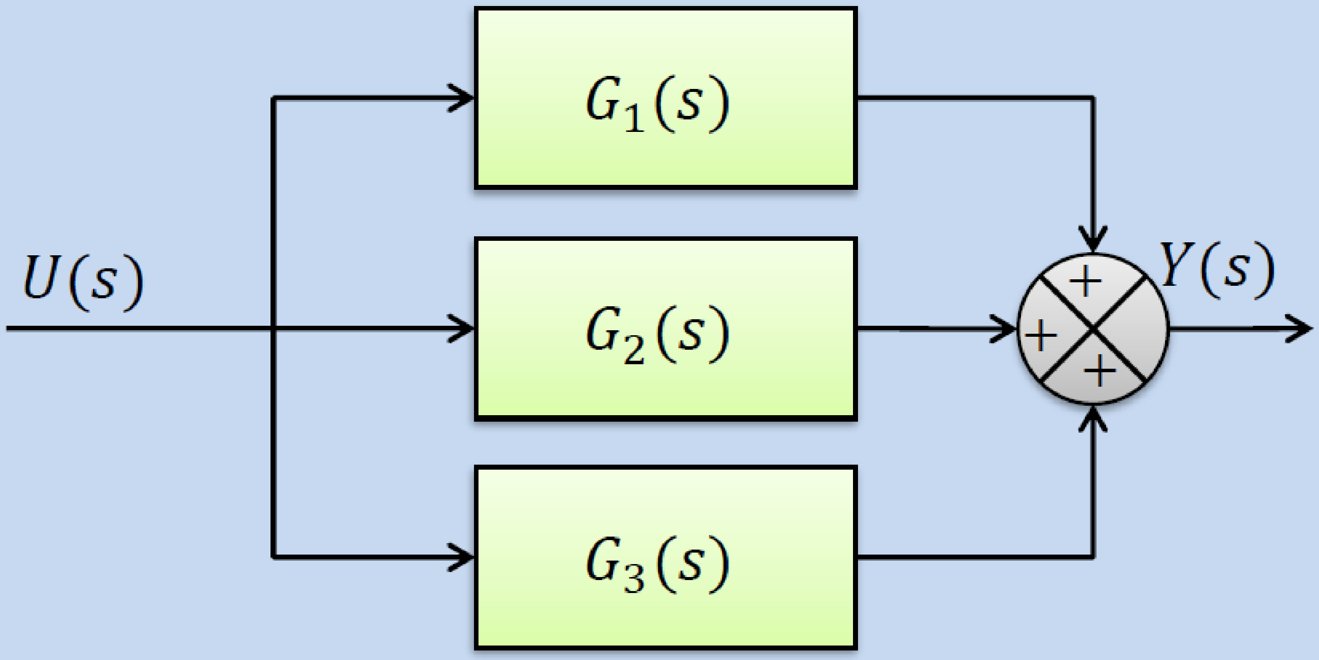
- Components or sub-systems of a system are connected in series each having its own transfer function
- Overall transfer function is product of individual transfer functions



Transfer Function: $G(s) = \frac{Y(s)}{U(s)} = G_1(s)G_2(s)G_3(s)$

Typical Block Diagram Forms

- **Parallel Form:**
  - Components or sub-systems of a system are connected in parallel
  - Overall transfer function is sum of individual transfer functions



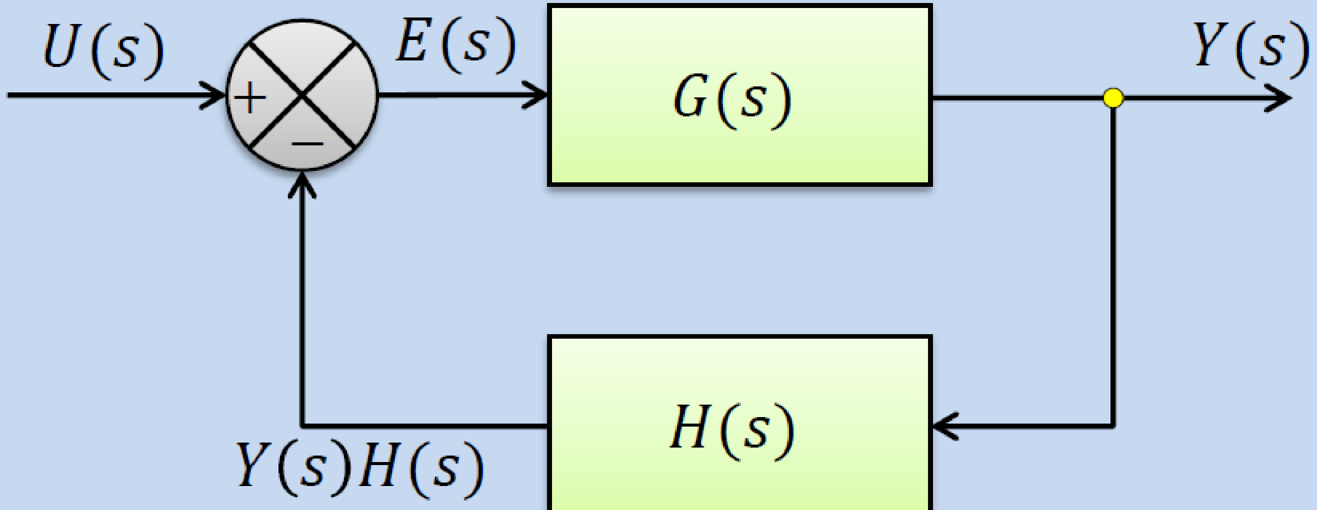
Transfer Function:

$$G(s) = \frac{Y(s)}{U(s)} = G_1(s) + G_2(s) + G_3(s)$$

Typical Block Diagram Forms

➢ **Feedback Form:**

– One component is present in the feedback loop of another component



Negative Feedback Loop

Transfer Function:

$$Y(s) = G(s)E(s)$$

$$Y(s) = G(s)[U(s) - Y(s)H(s)]$$

$$Y(s) = G(s)U(s) - G(s)H(s)Y(s)$$

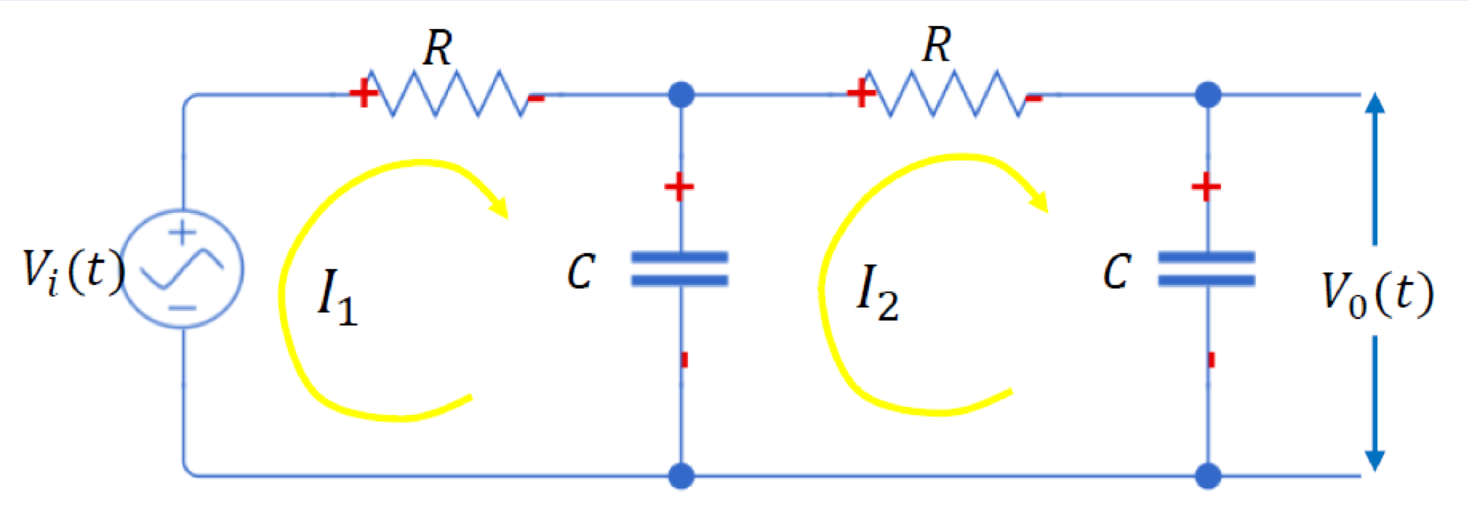
$$Y(s)[1 + G(s)H(s)] = G(s)U(s)$$

$$\frac{Y(s)}{U(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Loading Effect on Transfer Function

- **Assumption:** While deriving transfer function, there is no loading i.e., no power is drawn at the output of the system
- This assumption must be satisfied even while deriving transfer functions for each component in a block diagram
- If one component is acting as a load on another component:
  - Transfer function of each component cannot be determined separately
  - Transfer function of both components combined should be determined
  - Both components are put in the same block in the block diagram representation

Loading Effect : Example



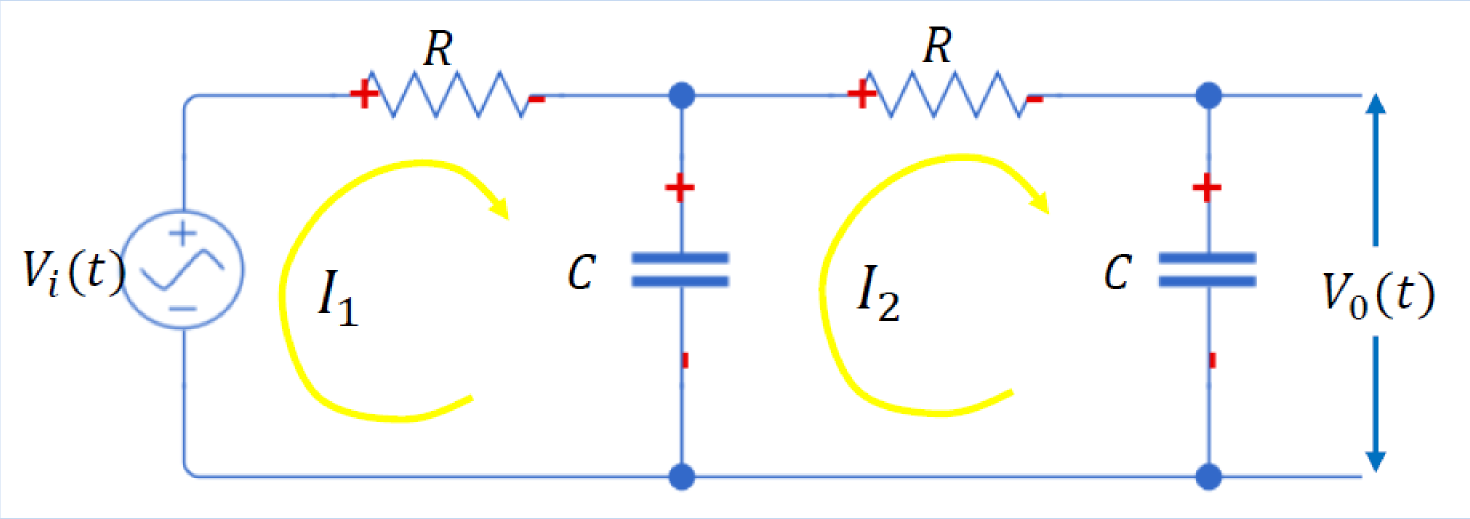
RC Circuits in cascade

- Model Equations in Laplace form:

$$V_i(s) = RI_1(s) + \frac{1}{sC}(I_1(s) - I_2(s)) \qquad (1)$$

$$V_0(s) = RI_2(s) + \frac{1}{sC}(I_2(s) - I_1(s)) \qquad (2)$$

$$V_0(s) = -\frac{1}{sC}I_2(s) \qquad (3)$$



- Transfer function is obtained by elimination $I_1(s)$ and $I_2(s)$ from Eqs.1, 2, 3

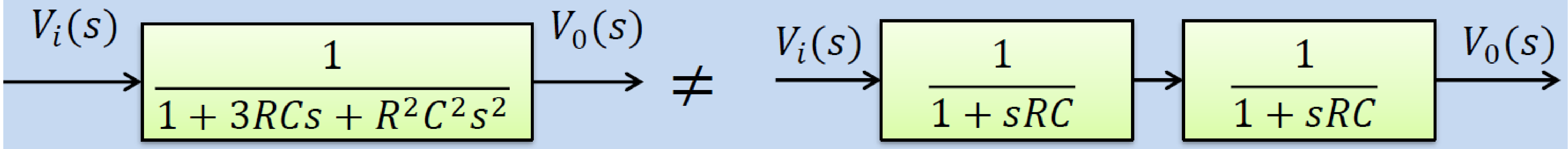
$$\frac{V_0(s)}{V_i(s)} = \frac{1}{R^2C^2s^2 + 3RCs + 1}$$

Loading Effect : Example

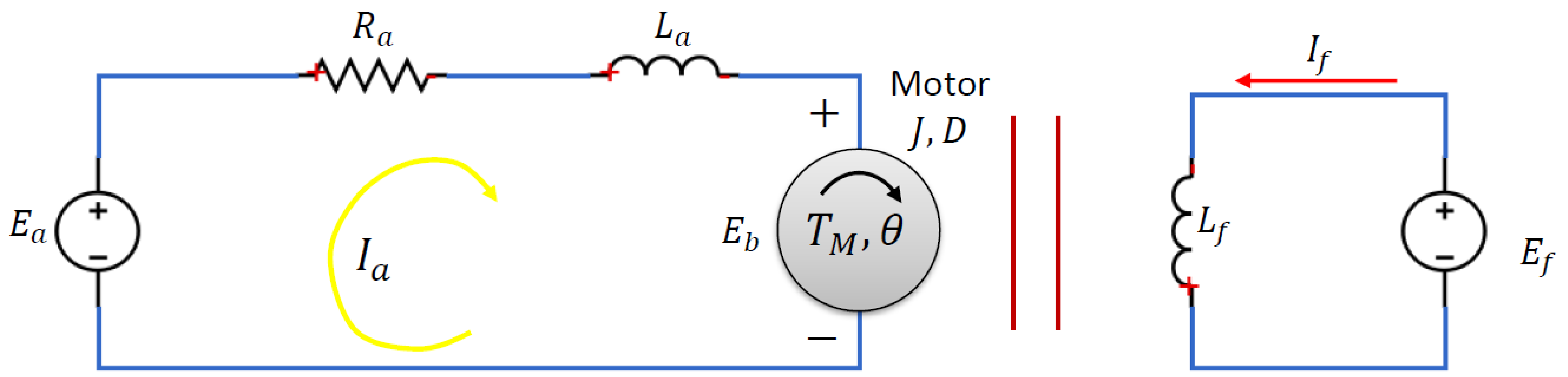
- Observe:

$$\frac{V_0(s)}{V_i(s)} = \frac{1}{R^2C^2s^2 + 3RCs + 1} \neq \frac{1}{(1 + RCs)(1 + RCs)}$$

- Overall transfer function is not equal to product of transfer functions of two RC circuits in cascade
- This is because the assumption of no loading fails when individual transfer functions are derived
- Here, the second RC circuit draws energy from the first one and hence its individual transfer function is not valid when in cascade



Block Diagram to Transfer Function: Example



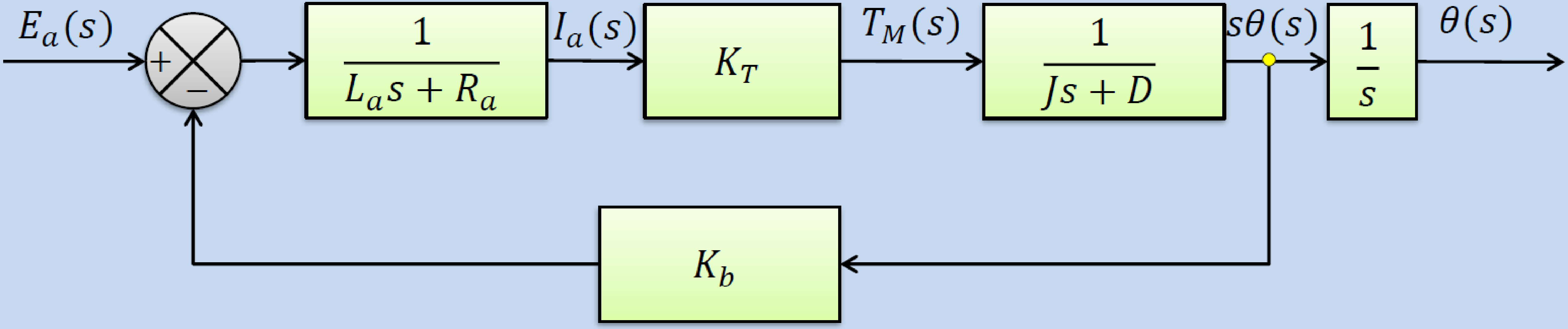
Armature controlled DC motor

➢ Variables and Constants in the model:

- R_a = resistance of armature (Ω)
- L_a = inductance of armature (H)
- I_a = armature current (A)
- I_f = field current (A)
- E_a = voltage applied to armature (V)
- E_b = back emf (V)
- T_M = torque developed by motor (Nm)
- θ = angular displace of motor shaft (rad)
- J = moment of inertia of motor and load referred to motor shaft $(kg - m^2)$
- D = friction coefficient of motor and load referred to motor shaft $\left(\frac{Nm}{rad-s}\right)$

K_T: Motor torque constant

K_b: Back emf constant



$$TF = \frac{\theta(s)}{E_a(s)} = \frac{\frac{K_T}{R_s}}{Js^2 + s(D + K_T K_b)R_s}$$

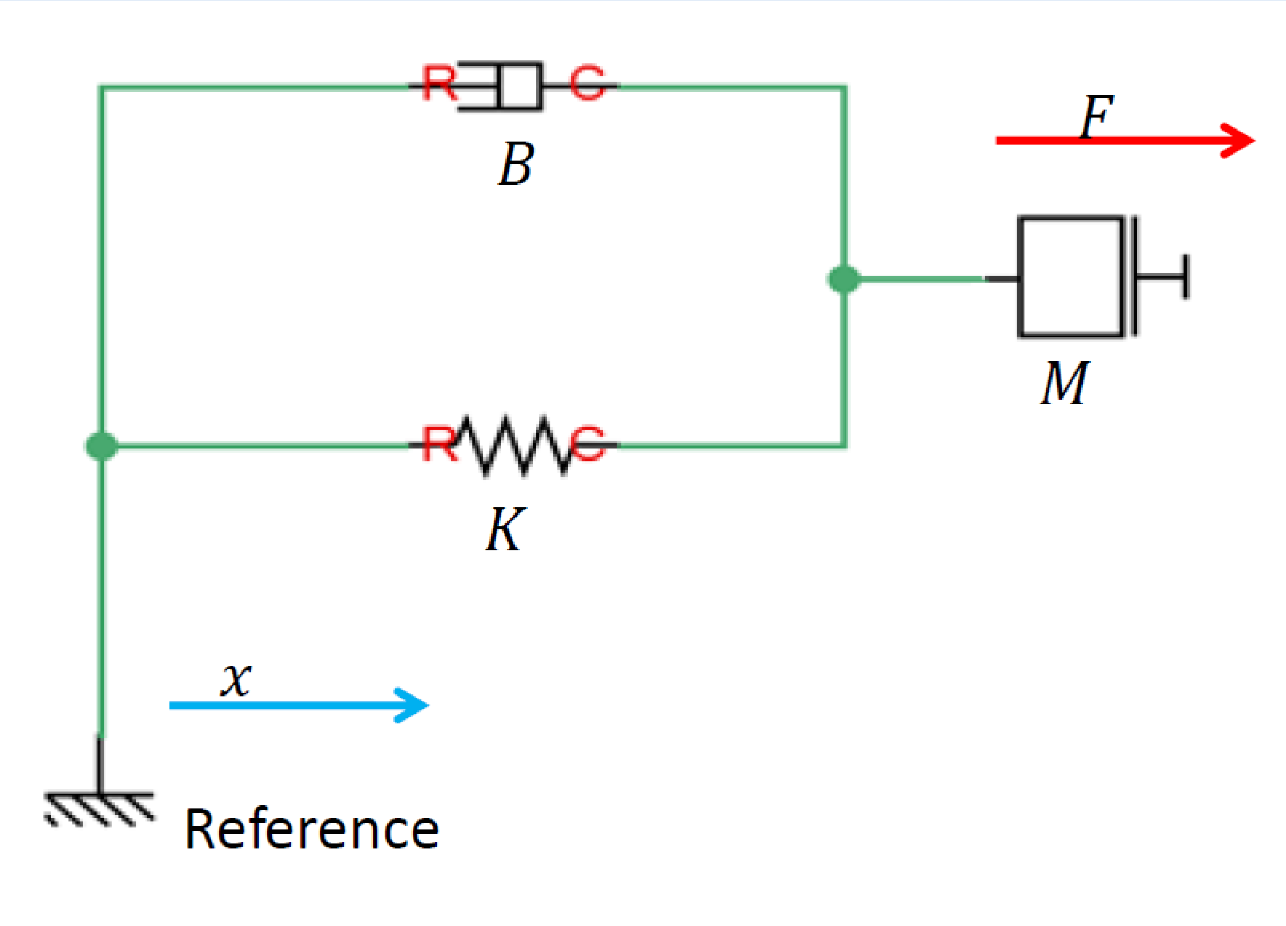
Analogous Systems

- Mechanical systems can be represented using electrical elements by the following analogies

- Two types of analogies:
  - Force (Torque) - Voltage analogy (F-V analogy)

- Force is analogous to voltage
  - Force (Torque) - Current Analogy (F-I analogy)
- Force is analogous to current

Mass-Spring-Damper (MSD) System



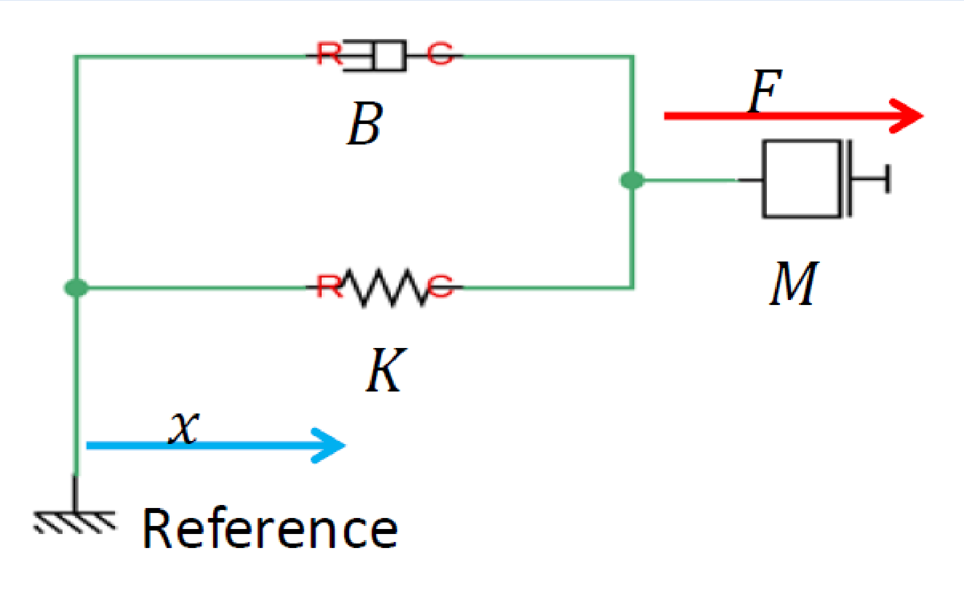
Based on Newton's 2nd law,

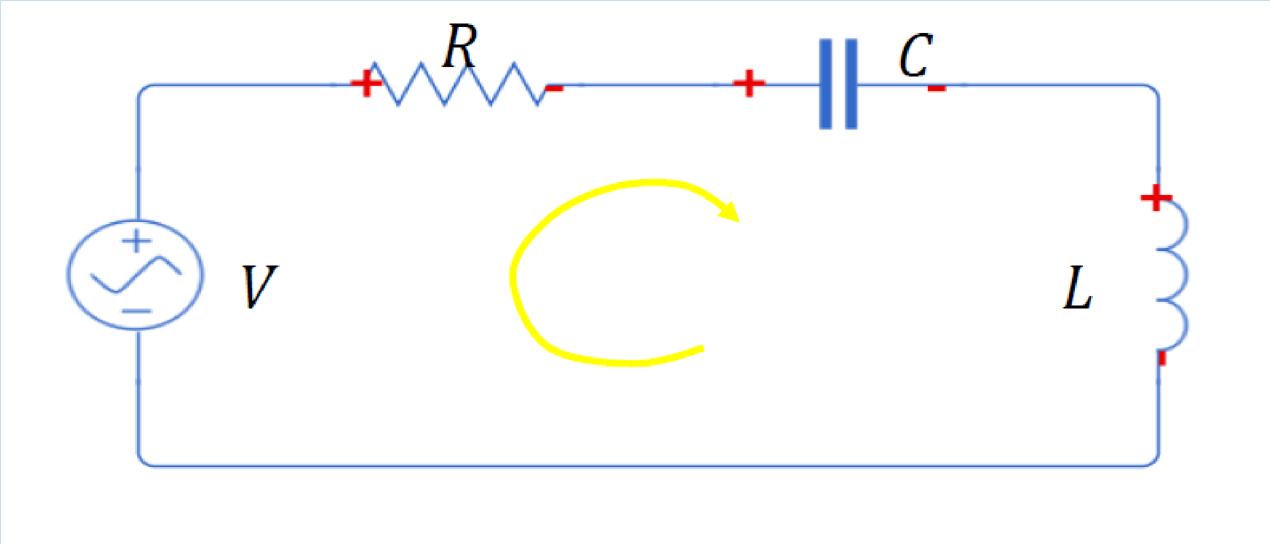
$$F = M\ddot{x} + B\dot{x} + Kx$$

Similarly for a rotational system,

$$T = J\ddot{\theta} + D\dot{\theta} + K\theta$$

F-V Analogy of MSD System





$$F \rightarrow V$$
$$M \rightarrow L$$
$$B \rightarrow R$$
$$K \rightarrow \frac{1}{C}$$
$$x \rightarrow q$$

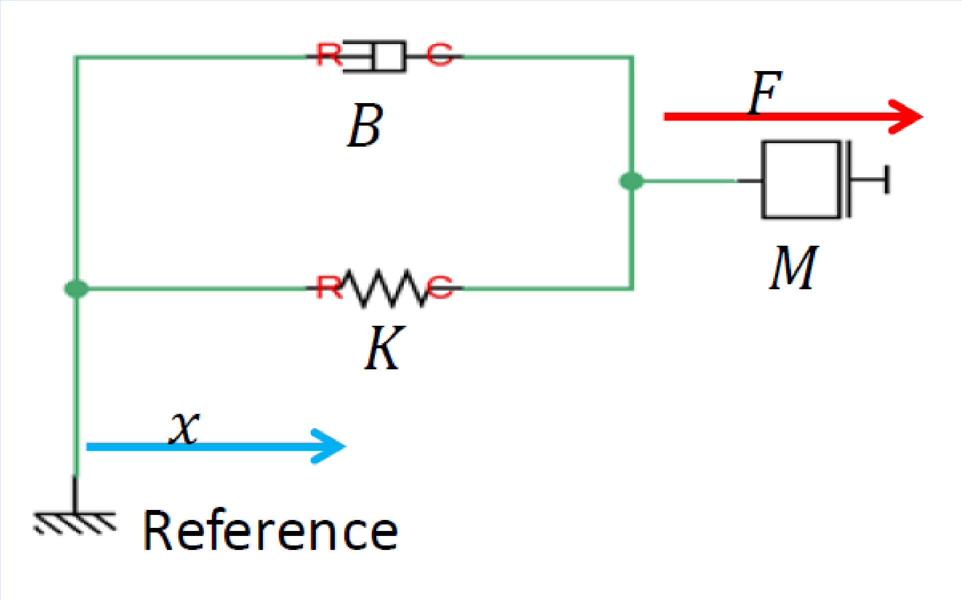
Based on Newton's 2nd law,

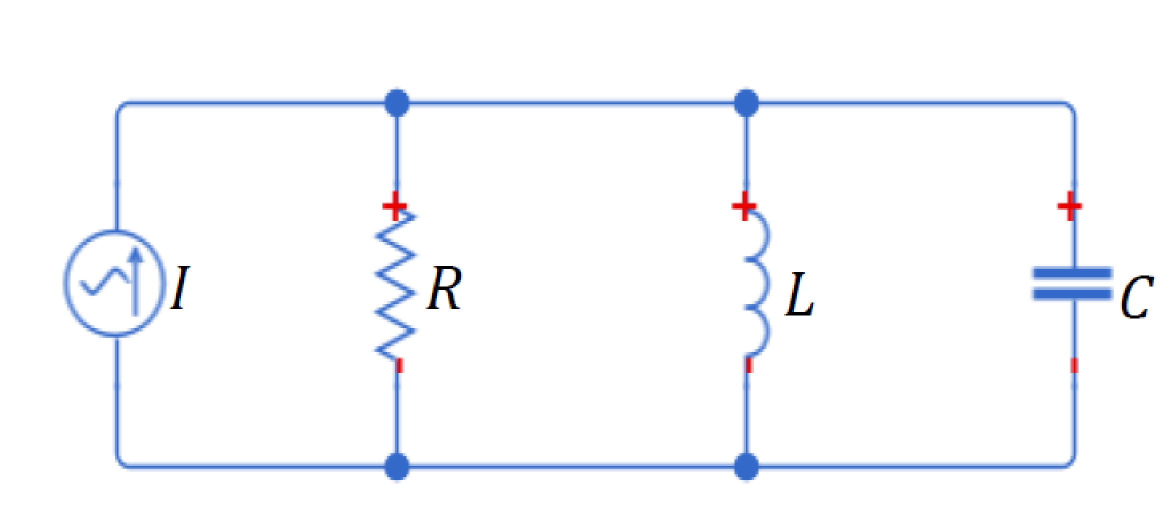
$$F = M\ddot{x} + B\dot{x} + Kx$$

Based on KVL around the loop,

$$V = L\ddot{q} + R\dot{q} + \frac{q}{C}$$

F-I Analogy of MSD System





$$F \rightarrow I$$
$$M \rightarrow C$$
$$B \rightarrow \frac{1}{R}$$
$$K \rightarrow \frac{1}{L}$$
$$x \rightarrow \phi$$

Based on Newton's 2nd law,

$$F = M\ddot{x} + B\dot{x} + Kx$$

Based on KVL around the loop,

$$I = C\ddot{\phi} + \frac{\dot{\phi}}{R} + \frac{\phi}{L}$$

Summary: Analogous Systems

- Following table shows the analogue between the elements of mechanical and electrical systems:

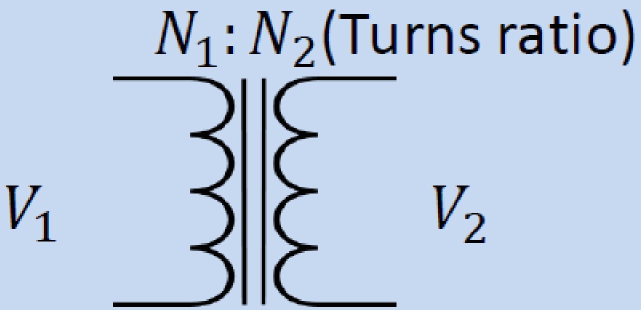
Mechanical System		Electrical System	
Translational	**Rotational**	**F-V Analogy**	**F-I Analogy**
Force (F)	Torque (T)	Voltage (V)	Current (I)
Mass (M)	Inertia (J)	Inductor (L)	Capacitor (C)
Friction (B)	Friction (D)	Resistor (R)	Conductor $(1/R)$
Linear spring (K)	Torsional spring (K)	Capacitor $(1/C)$	Inductor $(1/L)$
Displacement (x)	Displacement (θ)	Charge (q)	Flux (ϕ)

Transformer Vs Gears

Transformer

- Transmits electrical energy from one circuit to another through electromagnetic induction
- Changes voltage level

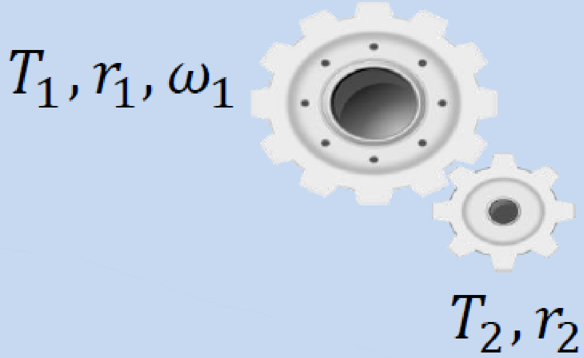
$$\frac{V_1}{V_2} = \frac{N_1}{N_2} = \frac{I_2}{I_1}$$



Gears

- A rotating machine to transmit torque
- Changes speed and direction of motion

$$\frac{T_1}{T_2} = \frac{r_1}{r_2} = \frac{\omega_2}{\omega_1}$$



T:Torque
r:Radius
ω:Angular velocity

HomeWork

1.23.1 Armature Controlled d.c. Motor

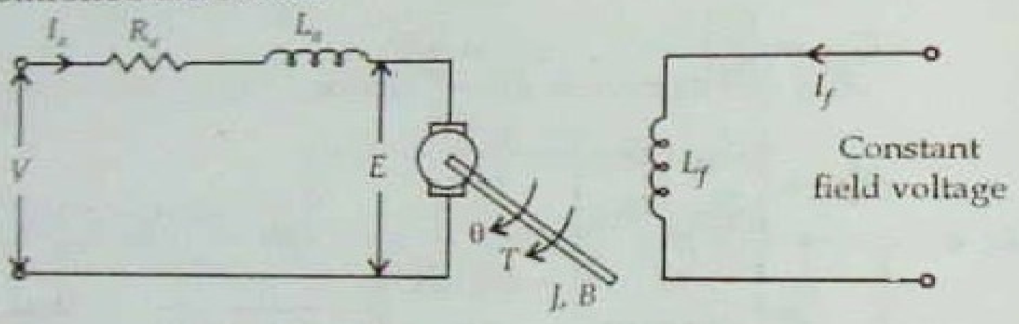


Fig. 1.97.

Consider the armature controlled d.c. motor and assume that the demagnetizing effect of armature reaction is neglected, magnetic circuit is assumed linear and field voltage is constant *i.e.* i_f = constant.

Let R_a = Armature resistance

L_a = Armature self inductance caused by armature flux

i_a = armature current

i_f = field current

E = Induced e.m.f in armature

V = Applied voltage

T = Torque developed by the motor

θ = Angular displacement of the motor shaft

J = Equivalent moment of inertia of motor shaft & load referred to the motor

B = equivalent coefficient of friction of motor and load referred to the motor

1.23.2. Field Control d.c. Motor

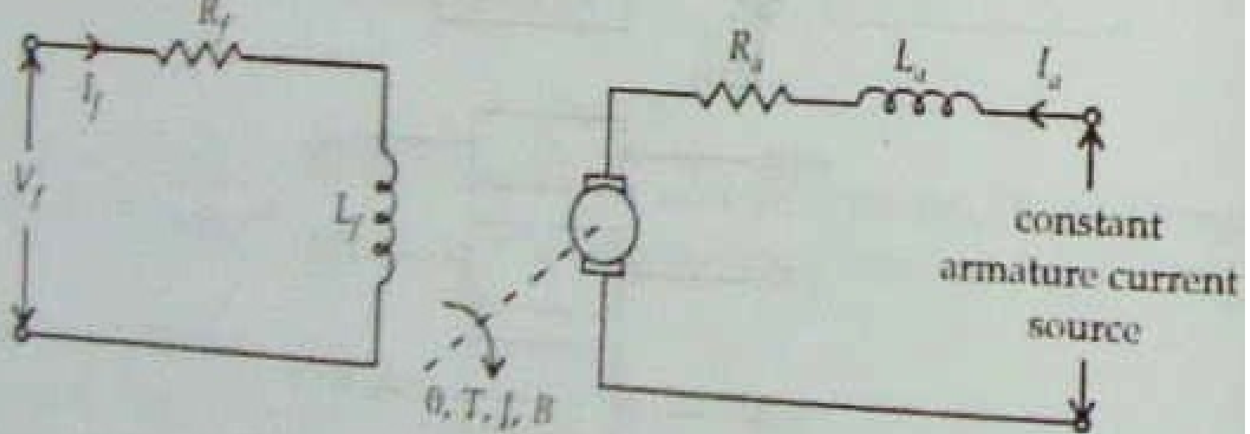


Fig. 1.99

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