Name: Class & Div. : Page No. :

Subject : Topic : Date :

Convolution :-

As we know, If $L\{f\} = F(s)$, $L\{g\} = G(s)$
then

$$L\{f+g\} = L\{f\} + L\{g\}$$

$$L\{\alpha f\} = \alpha L\{f\} \quad , \quad \alpha \in \mathbb{R} \text{ (scalar)}.$$

Also $L\{f-g\} = L\{f\} - L\{g\}$.

But

$$\boxed{L\{f\cdot g\} \neq L\{f\}\cdot L\{g\}}$$

For example

Take $f(t) = e^{t}$, $g(t) = 1$.

Then

$$f\cdot g = e^{t}.$$

$$\Rightarrow L\{f\cdot g\} = L\{e^{t}\} = \frac{1}{s-1} \quad \text{--- (i)}.$$

But

$$L\{f\}L\{g\} = \left(\frac{1}{s}\right)\left(\frac{1}{s-1}\right) = \frac{1}{s(s-1)} \quad \text{--- (ii)}$$

From (i) $\neq$ (ii)

Convolution (Def^n) :-

Let f and g defined on $[0,\infty)$.
Then the convolution of f and g denoted by standard notation $f*g$ and defined by the integral

$$(f*g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

Examples

① If $f(t)=t^2$, $g(t)=t$ find $(f*g)(t)$.

$$(f*g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$= \int_0^t \tau^2 (t-\tau)\, d\tau$$

$$= \int_0^t (t\tau^2 - \tau^3)\, d\tau = \frac{t^4}{12}$$

② $f(t) = 1$, $g(t) = t^2$ find $(f*g)(t)$

$$(f*g)(t) = \int_0^t 1\,(t-\tau)^2\, d\tau$$

$$= \int_0^t (t^2 - 2t\tau + \tau^2)\, d\tau$$

$$= t^3 - t^3 + \frac{t^3}{3} = \frac{t^3}{3}.$$

③ Homework : ① $f(t) = t$ $g(t) = \sin t$

② $e^t * e^t$ ③ $\cos wt * \cos wt$.

Properties of Convolution :-

(1) $f*g = g*f$ (Commutative)

(2) $f*(g_1+g_2) = f*g_1 + f*g_2$ (Distrib

(3) $f*(g*h) = (f*g)*h$

(4) $f*0 = 0*f = 0$

But $f*1 \neq f$ for all f.

For example

$$t*1 = \int_0^t \tau\, 1\, d\tau = \int_0^t \tau\, d\tau = \frac{t^2}{2}$$

Name: Class & Div. : Page No. :

Subject : Topic : Date :

Show that $\underbrace{1*1*\cdots*1}_{n+1} = t^n/n!$ $t*1 \neq t$

What is 1 convolution 1.? $(1*1)$

$$1*1 = \int_0^t 1\,1\,d\tau = t = \frac{t^1}{1!}$$

$$1*1*1 = (1*1)*1 = t*1 = \frac{t^2}{2} = \frac{t^2}{2!}$$

$$1*1*1*1 = (1*1*1)*1 = \frac{t^2}{2}*1 = \frac{t^3}{6}$$

$$\underbrace{1*1*1*\cdots\cdots*1}_{n+1\text{ - times}} = \frac{t^n}{n!}$$

Convolution Theorem :- (No proof)

If two functions f and g satisfy the assumptions in the existence theorem (Conts $[0,\infty)$, expo order) so that their Laplace transform F and G exist. Then

$$L\{(f*g)(t)\} = L\{f(t)\}\cdot L\{g(t)\} = F(s)\cdot G(s). \quad \text{--- (1)}$$

Hence

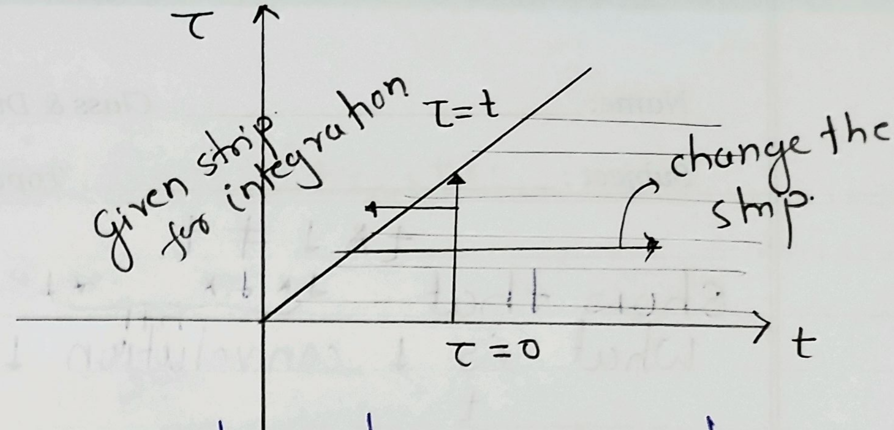
$$L^{-1}\{F(s)\cdot G(s)\} = (f*g)(t).$$

Proof :→

$$L\{(f*g)(t)\} = L\left\{\int_0^t f(\tau)\,g(t-\tau)\,d\tau\right\}$$

$$= \int_{t=0}^{\infty} e^{-st}\left[\int_{\tau=0}^{t} f(\tau)\,g(t-\tau)\,d\tau\right]dt$$

--- by def^n of LT.



Change the order of integration, we get

$$L\{(f*g)(t)\} = \int_{\tau=0}^{\infty}\left[\int_{t=\tau}^{\infty} e^{-st} f(\tau)\, g(t-\tau)\, dt\right] d\tau$$

$$= \int_{\tau=0}^{\infty} f(\tau)\left[\int_{t=\tau}^{\infty} e^{-st} g(t-\tau)\, dt\right] d\tau$$

$$= \int_{\tau=0}^{\infty} f(\tau)\left[\int_{p=0}^{\infty} e^{-s(\tau+p)} g(p)\, dp\right] d\tau$$

(On putting $t-\tau = p \therefore d\tau = dp$ and

t	τ	∞
p	0	∞

)

$$= \int_{0}^{\infty} e^{-s\tau} f(\tau)\, d\tau \int_{0}^{\infty} e^{-sp} g(p)\, dp$$

$$= \int_{0}^{\infty} e^{-st} f(t)\, dt \int_{0}^{\infty} e^{-st} g(t)\, dt$$

$$= F(s)\, G(s)$$

Name: Class & Div. : Page No. :

Subject : Topic : Date :

Hence

$$\boxed{L\{(f*g)(t)\} = L\left[\int_0^t f(\tau)\, g(t-\tau)\, d\tau\right] = F(s)\, G(s) = L\{f(t)\}\cdot L\{g(t)\}}$$

In words, this theorem states that Laplace transform of convolution of two functions is equal to product of their Laplace transforms.

Examples

Use the convolution theorem to evaluate

(i) $L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\}$

By Convolution th^m

$$L\{f(t)*g(t)\} = L\left[\int_0^t f(\tau)\, g(t-\tau)\, d\tau\right] = F(s)\, G(s)$$

Hence

$$L^{-1}\{F(s)\, G(s)\} = \int_0^t f(\tau)\, g(t-\tau)\, d\tau \quad \text{---} (*)$$

Let

$$F(s) = \frac{1}{s+a}, \quad G(s) = \frac{1}{s+b}$$

$$\therefore f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$$

$$g(t) = L^{-1}\{G(s)\} = L^{-1}\left\{\frac{1}{s+b}\right\} = e^{-bt}$$

By (*)

$$\bar{L}^1\left\{\frac{1}{(s+a)(s+b)}\right\} = \int_0^t e^{-a\tau} e^{-b(t-\tau)} d\tau$$

$$= e^{-bt} \int_0^t e^{(b-a)\tau} d\tau$$

$$= e^{-bt} \left[\frac{e^{(b-a)\tau}}{b-a}\right]_0^t$$

$$= \frac{e^{-bt}}{b-a}\left[e^{(b-a)t} - 1\right]$$

$$= \frac{1}{b-a}\left[e^{-at} - e^{-bt}\right].$$

(2) $\bar{L}^1\left\{\frac{1}{s(s+1)^3}\right\}$

Let $F(s) = \frac{1}{s}$ and $G(s) = \frac{1}{(s+1)^3}$

Then

$$\bar{L}^1\{F(s)\} = \bar{L}^1\left\{\frac{1}{s}\right\} = 1, \text{ and} \quad \text{---- (1)}$$

$$\bar{L}^1\{G(s)\} = \bar{L}^1\left\{\frac{1}{(s+1)^3}\right\} = e^{-t}\,\bar{L}^1\left\{\frac{1}{s^3}\right\} = e^{-t}\frac{t^2}{2!} \quad \text{--- (2)}$$

---- by first shifting th^m.

Now, by Convolution theorem,

$$\bar{L}^1\{F(s)G(s)\} = \int_0^t g(\tau) f(t-\tau) d\tau$$

$$\bar{L}^1\left\{\frac{1}{s(s+1)^3}\right\} = \int_0^t \left(\frac{1}{2}\tau^2 e^{-\tau}\right)\cdot 1\, d\tau \quad \text{--- by (1) and (2)}$$

$$= 1 - e^{-t}\left(1 + t + \frac{t^2}{2}\right)$$

Name: Class & Div. : Page No. :

Subject : Topic : Date :

Convolution also helps in solving certain integral equations, that is equation in which the unknown function $y(t)$ appears in an integral.

solve the integral equations

(1) $y(t) + \int_0^t y(s)\,ds = 1 - e^{-t}$

Taking Laplace transform of both sides.

$L\{y(t)\} + L\left\{\int_0^t y(s)\,ds\right\} = L\{1\} - L\{e^{-t}\}$.

... L is linear.

$L\{y(t)\} + \frac{1}{s} L\{y(t)\} = \frac{1}{s} - \frac{1}{s+1}$ .... (by LT of integral thm)

$L\{y(t)\}\left(1 + \frac{1}{s}\right) = \frac{1}{s} - \frac{1}{s+1}$

$L\{y(t)\} \frac{s+1}{s} = \frac{s+1-\cancel{s}}{s(s+1)}$

$L\{y(t)\} = \frac{1}{\cancel{s}(s+1)} \cdot \frac{\cancel{s}}{(s+1)} = \frac{1}{(s+1)^2}$

Taking inverse Laplace transform we get

$y(t) = L^{-1}\left\{\frac{1}{(s+1)^2}\right\}$

$= e^{-t} L^{-1}\left\{\frac{1}{s^2}\right\}$, by 1st shifting thm

$= t e^{-t}$.

Home work

$$\frac{dy}{dt} + 3y + 2\int_0^t y(s)\,ds = t \quad \text{with } y(0)=0.$$

(Ans $\frac{1}{2} - e^{-t} + \frac{1}{2}\cdot e^{-2t}$).

(2) $y(t) = 1 + \int_0^t y(\tau)\sin(t-\tau)\,d\tau$.

→ Given $y(t) = 1 + \int_0^t y(\tau)\sin(t-\tau)\,d\tau$ ——— (1)

Using the definition of convolution, (1) can be written as.

$$y(t) = 1 + y(t) * \sin t \quad \text{---} \quad (2)$$

$\left(\because f(t)*g(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau\right)$.

Applying Laplace transform of (2) we get

$$L\{y(t)\} = L\{1\} + L\{y(t)*\sin t\}$$

$$= \frac{1}{s} + L\{y(t)\}\,L\{\sin t\}$$

---- (by the Convolution thm $L\{(f*g)(t)\} = L\{f(t)\}\cdot L\{g(t)\}$)

$$= \frac{1}{s} + L\{y(t)\}\frac{1}{s^2+1}$$

$$\left(1 - \frac{1}{s^2+1}\right)L\{y(t)\} = \frac{1}{s}$$

$$\left(\frac{s^2}{s^2+1}\right)L\{y(t)\} = \frac{1}{s}$$

Name: Class & Div. : Page No. :

Subject : Topic : Date :

$$L\{y(t)\} = \frac{s^2+1}{s^3} = \frac{1}{s} + \frac{1}{s^3}.$$

Taking Laplace inverse on both side.

$$y(t) = L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= 1 + \frac{t^2}{2}$$

(3) $y(t) - \int_0^t y(\tau)\, d\tau = 1.$

Taking L.T on both side.

$$L\{y(t)\} - L\left\{\int_0^t y(\tau)\, d\tau\right\} = L\{1\}$$

$$L\{y(t)\} - L\{y(t) * 1\} = \frac{1}{s}$$

$$\left(\because f(t) * g(t) = \int_0^t f(\tau) g(t-\tau)\, d\tau\right)$$

$$L\{y(t)\} - L\{y(t)\} L\{1\} = \frac{1}{s}$$

$$\left(\because L\{f(t) * g(t)\} = L\{f(t)\} \cdot L\{g(t)\}\right)$$

$$L\{y(t)\}\left(1 - \frac{1}{s}\right) = \frac{1}{s}$$

$$L\{y(t)\}\left(\frac{s-1}{s}\right) = \frac{1}{s}$$

$$L\{y(t)\} = \frac{1}{s-1}$$

$$y(t) = L^{-1}\left\{\frac{1}{s-1}\right\} = e^t.$$

Homework $y(t) - \int_0^t y(\tau)(t-\tau)\,d\tau = 1$.

(4) $y(t) = t^2 + \int_0^t y(\tau)\sin(t-\tau)\,d\tau$

→ given $y(t) = t^2 + \int_0^t y(\tau)\sin(t-\tau)\,d\tau$

$L\{y(t)\} = L\{t^2\} + L\{\int_0^t y(\tau)\sin(t-\tau)\,d\tau\}$

$L\{y(t)\} = L\{t^2\} + L\{y(t) * \sin t\}$ --- by def^n of Convolution.

$= L\{t^2\} + L\{y(t)\}\cdot L\{\sin t\}$ --- by convolution th^m

$L\{y(t)\} = \frac{2}{s^3} + L\{y(t)\}\left(\frac{1}{s^2+1}\right)$

$\left(1 - \frac{1}{s^2+1}\right) L\{y(t)\} = \frac{2}{s^3}$

$\left(\frac{s^2}{s^2+1}\right) L\{y(t)\} = \frac{2}{s^3}$

$L\{y(t)\} = \frac{2(1+s^2)}{s^5}$

$= \frac{2}{s^5} + \frac{2}{s^3}$

$y(t) = L^{-1}\left\{\frac{2}{s^5}\right\} + L^{-1}\left\{\frac{2}{s^3}\right\}$

$= 2\frac{t^4}{4!} + 2\frac{t^2}{2!}$

$= t^2 + \frac{t^4}{12}$.

Name: Class & Div. : Page No. :

Subject : Topic : Date :

Short Impulses. Dirac's Delta Function

Phenomena of an impulsive nature, such as the action of forces or voltages over short intervals of time, arise in various applications, for examples:

- if a mechanical system is hit by a hammerblow
- an airplane make a "hard" landing
- a ship is hit by a single high wave
- we hit tennisball by a racket, and so on.

Our goal is to show how such problems are modeled by "Dirac's delta function" and can be solved very efficiently by the Laplace transform.

To model situations of that types, we consider the function.

$$f_k(t-a) = \begin{cases} \frac{1}{k}, & \text{if } a \le t \le a+k \\ 0 & \text{otherwise.} \end{cases} \quad (k > 0 \text{ and small}).$$

where k is positive and small.

This function represent, for instance, a force of magnitude $\frac{1}{k}$ acting from $t = a$ to $t = a+k$, ($k > 0$ and small.)

Impulse (Def^n, Mechanics point of view)

The integral of a force acting over a time interval $a \le t \le a+k$. (k is +ve and small) is called the impulse of the force.

What is the impulse of $f_k = \begin{cases} \frac{1}{k} & a \le t \le a+k \\ 0 & \text{otherwise} \end{cases}$ --- (1)

($k > 0$ and small).

$$\underset{\downarrow}{I_k} = \int_0^\infty f_k(t-a)\, dt$$

(By def^n of impulse, it is a integral of $f_k(t-a)$ from limit $t=0$ to ∞ (t - time parameter).

(Denote that impulse by I_k)

$$= \int_0^a f_k(t-a)\, dt + \int_a^{a+k} f_k(t-a)\, dt + \int_{a+k}^\infty f_k(t-a)\, dt$$

$$= \int_a^{a+k} f_k(t-a)\, dt \quad \text{--- (by def}^n \text{ of } f_k)$$

$$= \int_a^{a+k} \frac{1}{k}\, dt \quad \text{--- (by def}^n \text{ of } f_k)$$

$$= \frac{1}{k}\left[t\right]_a^{a+k}$$

$$= \frac{1}{k}\left[\cancel{a}+k-\cancel{a}\right]$$

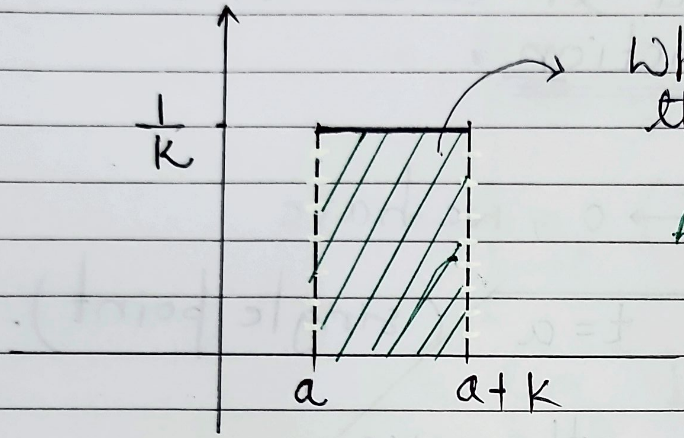
$$= 1.$$

Name: ______ Class & Div. : ______ Page No. : ______

Subject : ______ Topic : ______ Date : ______

What is the graph of $f_k(t-a) = \begin{cases} \frac{1}{k}, & a \le t \le a+k \\ 0 \end{cases}$

($k > 0$ and small).



What is the area of the rectangle?

Area = base × hight
$= k \times \frac{1}{k} = 1.$

Note that,

Impulse of f_k = Area of the rectangle = 1
(Area under the curve)

In some case, we need to analyze the behaviour of $f_k(t-a)$, when k is very very very very small.

Thus to find out what will happen if k becomes smaller and smaller, we take the limit of $f_k(t-a)$ as $k \to 0$ ($k > 0$)

This limit is denoted by $\delta(t-a)$, that is

$$\delta(t-a) = \lim_{k \to 0} f_k(t-a)$$

$$= \lim_{k \to 0} \left(f_k(t-a) = \begin{cases} \frac{1}{k}, & a \le t \le a+k \\ 0, & \text{otherwise} \end{cases} \right). \quad \text{—(1)}$$

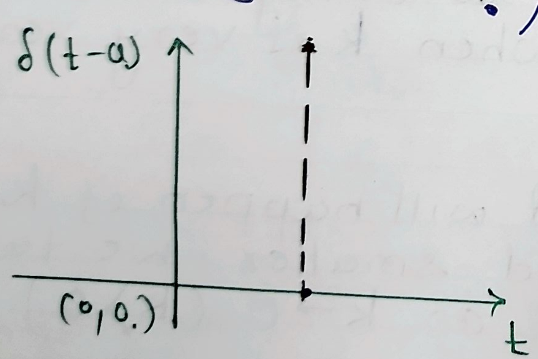
$\delta(t-a)$ is called the <u>Dirac</u> <u>delta function</u> or <u>unit</u> <u>impulse</u> <u>function</u>.

$\delta(t-a)$ is Not a function in the ordinary sence as we used in calculus., but a so-called <u>generalizef</u> <u>function</u>.

From (1) as $k \to 0$, we have

$$\boxed{\delta(t-a) = \begin{cases} \infty & t = a \leftarrow (\text{single point}). \\ 0 & \text{otherwise} \end{cases}}$$

Note that in calculus function out put is always finite value.. (So that sense it is not a function!)



Graph of $\delta(t-a)$

From, the graph what will be the area under the curve?

→ zero (If it is function in calculuse sense).

BUT is NOT, let me tell you why?

Name: ______ Class & Div. : ______ Page No. : ______

Subject : ______ Topic : ______ Date : ______

$$\int_0^\infty \delta(t-a)\,dt = \int_0^\infty \left(\lim_{k\to 0} f_k(t-a)\right) dt$$

$$= \lim_{k\to 0} \int_0^\infty f_k(t-a)\,dt$$

$$= \lim_{k\to 0} (\text{Impulse of } f_k)$$

$$= \lim_{k\to 0} 1$$

$$= 1$$

$$\therefore \int_0^\infty \delta(t-a)\,dt = 1$$

Thus $\delta(t-a)$ is NOT a function in calculus sense.

But this is generalized function, and the are are very useful.

Imp Sifting property of $\delta(t-a)$:-

$$\int_0^\infty g(t)\,\delta(t-a)\,dt = g(a)$$

where g is continuous function.

... delta function

[[Proof: As g is a continuous function.

$$\int_0^\infty g(t)\,\delta(t-a)\,dt.$$

$$= \int_a^{a+k} \frac{1}{k}\, g(t)\,dt$$

$$= k\,\frac{1}{k}\, g(c), \quad \text{where } a \le c \le a+k$$

(by Mean value theorem for integrals.

$$\int_a^b f(t)\,dt = (b-a)f(c).)$$

As $k \to 0$, $c \to a$ it means k become very very small at time c is go to a

Thus

$$\int_0^\infty g(t)\,\delta(t-a)\,dt = g(a)]].$$

Laplace transform of Dirac Delta function :-

Recall

$$\delta(t-a) = \lim_{k\to 0} f_k(t-a)$$

where
$$f_k(t-a) = \begin{cases} \frac{1}{k}, & a \le t \le a+k \\ 0 & \text{otherwise.} \end{cases}$$

($k>0$ and small).

Name: ______ Class & Div. : ______ Page No. : ______

Subject : ______ Topic : ______ Date : ______

We can write

$$f_k(t-a) = \begin{cases} 0, & t<a \\ \frac{1}{k} & a \le t \le a+k \\ 0 & t > a+k \end{cases}$$

This function, we can write into unit step function.

$$f_k(t-a) = \frac{1}{k}\left[u(t-a) - u(t-(a+k))\right].$$

$$\therefore L\{f_k(t-a)\} = \frac{1}{k}L\{u(t-a)\} - \frac{1}{k}L\{u(t-(a+k))\}$$

$$= \frac{1}{k}\frac{e^{-as}}{s} - \frac{1}{k}\frac{e^{-(a+k)s}}{s}$$

$$= \frac{e^{-as}}{ks}\left[1 - e^{-ks}\right].$$

Thus

$$L\{\delta(t-a)\} = \lim_{k\to 0} L\{f_k(t-a)\}$$

$$= \lim_{k\to 0} \frac{e^{-as}}{ks}\left[1 - e^{-ks}\right]$$

$$= \frac{e^{-as}}{s} \lim_{k\to 0} \frac{1-e^{-ks}}{k} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$L\{\delta(t-a)\} = \frac{e^{-as}}{s} \lim_{k\to 0} \frac{s e^{-ks}}{1} \quad \text{(by L Hopital rule)}$$

$$= \frac{e^{-as}}{s} \cdot s.$$

$$= e^{-as}.$$

$$\boxed{L\{\delta(t-a)\} = e^{-as}.}$$

In particular $a=0$ $L\{\delta(t)\} = e^{-0s} = 1.$

$$\Rightarrow L^{-1}\{1\} = \delta(t).$$

(Here $\delta(t-a)$ is generalized function!).

Is it possible to find generalized function whose Laplace transform is

$$F(s) = \frac{s^2}{s^2+1}.$$

→ Note that

$$\lim_{s\to\infty} F(s) = \lim_{s\to\infty} \frac{\cancel{s^2}}{\cancel{s^2}(1+1/s^2)} = 1 \neq 0$$

Hence there does not exist piecewise continuous function of exponential order

Name: ______ Class & Div. : ______ Page No. : ______

Subject : ______ Topic : ______ Date : ______

whose Laplace transform is $\frac{s^2}{1+s^2}$

But, there exists a generalized function whose Laplace transform is $\frac{s^2}{s^2+1}$.

$$F(s) = \frac{s^2}{1+s^2}$$

$$= \frac{1+s^2-1}{1+s^2} \quad \frac{}{1+s^2}$$

$$= 1 - \frac{1}{s^2+1}$$

To find generalized function, we look at

$$L^{-1}\{F(s)\} = L^{-1}\{1\} - L^{-1}\left\{\frac{1}{s^2+1}\right\}.$$

$$= \delta(t) - \sin t.$$

Homework!

Find the generalized function whose Laplace transform is

$$F(s) = \frac{s}{s+1}.$$

Examples

Find the Laplace transform of each of the following functions:-

(a) $\sin 2t\ \delta(t-2)$

As we know

$$L\{\delta(t-a)\} = e^{-as}$$

By shifting property.

$$\int_0^\infty g(t)\,\delta(t-a)\,dt = g(a). \quad \text{------} \quad (1)$$

$$L\{f(t)\,\delta(t-a)\} = \int_0^\infty e^{-st} f(t)\,\delta(t-a)\,dt$$

$$= \int_0^\infty g(t)\cdot\delta(t-a)\,dt$$

--- where $g(t) = e^{-st} f(t)$

$$= g(a) \quad \text{--- by } (1)$$

$$= e^{-sa} f(a).$$

Thus

$$L\{\sin 2t\ \delta(t-2)\} = e^{-2s} f(2)$$

$$= e^{-2s} \sin 2(2)$$

$$= e^{-2s} \sin(4).$$

continuous function of exponential order

Name: ______ Class & Div. : ______ Page No. : ______

Subject : ______ Topic : ______ Date : ______

Solve the differential equations

(1) $y'' + 4y = \begin{cases} 1, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$, $y(0) = 0$, $y'(0) = 1$.

Here $r(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & t > 1. \end{cases}$

Expressed in terms of Heaviside's functions.

i.e $r(t) = [u(t) - u(t-1)]$ ... Thus we have.

$$y'' + 4y = u(t) - u(t-1) \quad -(1)$$

Taking Laplace transform on both side.

$$L\{y''\} + 4L\{y\} = L\{u(t)\} - L\{u(t-1)\}$$

$$s^2 L\{y(t)\} - s\cancel{y(0)}^{0} - y'(0) + 4L\{y(t)\} = \frac{1}{s} - \frac{e^{-s}}{s}$$

$$(s^2+4) L\{y(t)\} - 1 = \frac{1}{s} - \frac{e^{-s}}{s}$$

$$L\{y(t)\} = \frac{1}{s^2+4} + \frac{1}{s(s^2+4)} - \frac{e^{-s}}{s(s^2+4)}$$

$$= \frac{1}{s^2+4} + \frac{1}{4}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] - \frac{e^{-s}}{4}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

Taking inverse lapace on both side.

$$y(t) = L^{-1}\left\{\frac{1}{s^2+4}\right\} + \frac{1}{4} L^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+4}\right\} + L^{-1}\left\{\frac{e^{-s}}{4}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)\right\}$$

$$y(t) = \frac{1}{2}\sin 2t + \frac{1}{4}(1-\cos 2t) - \frac{1}{4}(1-\cos 2(t-1))\, u(t-1).$$

....(by second shifting thm)

(2) $y'' + 4y = u(t-2)$, $y(0) = 0$, $y'(0) = 1$

→ Taking Laplace transform on both side.

$$L\{y''\} + 4L\{y\} = L\{u(t-2)\}$$

$$s^2L\{y(t)\} - \cancel{s\,y(0)}^{\,0} - y'(0) + 4L\{y(t)\} = \frac{e^{-2s}}{s}$$

$$L\{y(t)\} = \frac{1}{s^2+4} + \frac{e^{-2s}}{s(s^2+4)}$$

$$= \frac{1}{s^2+4} + \frac{e^{-2s}}{4}\left[\frac{1}{s} - \frac{s}{s^2+4}\right].$$

Taking inverse Laplace transform, we get

$$y(t) = \frac{1}{2}\sin 2t + \frac{1}{4}L^{-1}\left\{e^{-2s}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)\right\}$$

$$= \frac{1}{2}\sin 2t + \frac{1}{4}\left[1 - \cos 2(t-2)\, u(t-2)\right].$$

... by second shifting thm)

(3). $y'' + 4y = \delta(t-2)$ $y(0) = 0$, $y'(0) = 1$.

→ Taking Laplace transform on both side.

$$L\{y''\} + 4L\{y\} = L\{\delta(t-2)\}$$

$$s^2L\{y(t)\} - \cancel{s\,y(0)}^{\,0} - y'(0) + 4L\{y(t)\} = e^{-2s}$$

Name: ____________ Class & Div. : ____________ Page No. : ____________

Subject : ____________ Topic : ____________ Date : ____________

$$L\{y(t)\}(s^2+4) = 1 + e^{-2s}$$

$$L\{y(t)\} = \frac{1}{s^2+4} + \frac{e^{-2s}}{s^2+4}$$

Taking inverse Laplace transform on both side

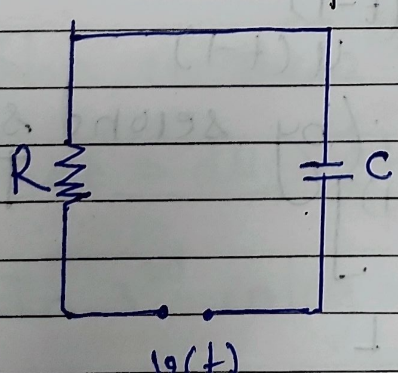
$$y(t) = L^{-1}\left\{\frac{1}{s^2+4}\right\} + L^{-1}\left\{\frac{e^{-2s}}{s^2+4}\right\}$$

$$= \frac{1}{2}\sin 2t + \frac{1}{2}\sin 2(t-2)\,u(t-2)$$

---- (by Second shifting Th^m).

(H.W). Find and graph the current $i(t)$ in the RC-circuit in fig, where $R = 100\,\Omega$, $C = 10^{-3}$ F, $v(t) = 100t$ if $0<t<2$,

$v(t) = 200$, if $t>2$ and the initial charge on the capacitor is 0.



Find the responce of the system under a unit impulse at time $t = 1$.

$$y'' + 3y' + 2y = \delta(t-1), \quad y(0) = 0, \quad y'(0) = 0.$$

$\rightarrow$ $y'' + 3y' + 2y = \delta(t-1)$

Taking Laplace transform.

$$s^2 L\{y(t)\} - \cancel{s\,y(0)} - \cancel{y'(0)} + 3s L\{y(t)\} - \cancel{3y(0)} + 2L\{y(t)\}$$
$$= L\{\delta(t-1)\}$$

$$L\{y(t)\}(s^2 + 3s + 2) = e^{-s}$$

$$L\{y(t)\} = \frac{e^{-s}}{(s^2+3s+2)} = \frac{e^{-s}}{(s+1)(s+2)}$$

Taking inverse Laplace on both side.

$$y(t) = L^{-1}\left\{\frac{e^{-s}}{(s+1)(s+2)}\right\}$$

$$= L^{-1}\left\{\frac{e^{-s}}{s+1} - \frac{e^{-s}}{s+2}\right\}$$

$$= e^{-(t-1)}u(t-1) \quad e^{-2(t-1)}u(t-1)$$

---- (by second shifting Thm).

$$= u(t-1)\left[e^{-(t-1)} - e^{-2(t-1)}\right]$$

$$= \begin{cases} 0 & \text{if } 0 < t < 1 \\ e^{-(t-1)} - e^{-2(t-1)} & \text{if } t > 1 \end{cases}$$

Name: Class & Div. : Page No. :

Subject : Topic : Date :

(self study)

Laplace Transform of Periodic Function :-

Def^n :- A function $f(t)$ defined for $t>0$ is said to be periodic with period $T(>0)$ if

$$f(t+T) = f(t), \ \forall \ t>0.$$

Theorem :- The Laplace transform of a piecewise continuous function $f(t)$ with period T is

$$L\{f(t)\} = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)\, dt \quad (s>0).$$

Proof :- By def^n of Laplace transform.

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} e^{-st} f(t)\,dt + \cdots$$

$\cdots$ (f is piecewise continuous fu^n with period p).

Now, in the 2^{nd} integral, let $t = u+T$; then

$dt = du$,

t	T	$2T$
u	0	T

In 3^{rd} integral, let $t = u+2T$, then

$dt = du$ and

t	$2T$	$3T$
u	0	T

$\cdots$ and so on

$$L\{f(t)\} = \int_0^T e^{-su} f(u)\,du + \int_0^T e^{-su} e^{-sT} f(u)\,du + \int_0^T e^{-2sT} e^{-su} f(u)\,du + \cdots$$

$$\left[\because f(u+T) = f(u+2T) = \cdots = f(u)\right]$$

$$= \left(1 + e^{-sT} + e^{-2sT} + \cdots\right) \int_0^T e^{-su} f(u)\,du$$

$$= \frac{1}{1-e^{-sT}} \int_0^T e^{-su} f(u)\,du$$

[infinite sum of GP $a + ar^2 + ar^3 + \cdots$ is $\frac{a}{1-r}$ if $|r| < 1$]

$$= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt, \quad s > 0.$$

Hence

$$\boxed{L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}}$$

Name: ______ Class & Div. : ______ Page No. : ______

Subject : ______ Topic : ______ Date : ______

① If $f(t) = t^2$, $0 < t < 2$ and $f(t+2) = f(t)$ find $L\{f(t)\}$.

sol^n:-

$f(t+2) = f(t) \Rightarrow f(t)$ is periodic with period $T = 2$.

$$\therefore L\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

$$= \frac{1}{1 - e^{-2s}} \int_0^2 e^{-st} t^2\, dt$$

$$= \frac{1}{1 - e^{-2s}} \left[(t^2)\left(\frac{e^{-st}}{-s}\right) - (2t)\left(\frac{e^{-st}}{+s^2}\right) + (2)\left(-\frac{1}{s^3} e^{-st}\right) \right]_0^2$$

$$= \frac{2 - (4s^2 + 4s + 2) e^{-2s}}{s^3 (1 - e^{-2s})}.$$

② If $f(t) = t$, $0 < t < 1$ with period 1., find $L\{f(t)\}$.