COEP Technological University Pune

Department of Mathematics

(MA- 20001) - Ordinary Differential Equations and Multivariate Calculus

S.Y. B.Tech. Semester III (All Branches)

Tutorial 3 (AY: 2023-24)

---

1. Define Laplace Transform (LT) and Inverse Laplace Transform (ILT) and make a list of all standard results in two columns - one for LT and second for the corresponding ILT.
2. Why the limits of the integration in the definition of Laplace Transform is from 0 to ∞? Give the logical justification.
3. Prove the linearity property of LT and ILT.
4. State and prove the first shifting theorem of LT.
5. When do we say that a function is of **exponential order** ?
6. Which of the following functions are of exponential order and why?
   (a) $\sin(e^{t^2})$
   (b) e^{t^π}
7. Give an example of a function which of exponential order but its derivative is not of exponential order.
8. Give an example of a function whose Laplace transform exists, such that f is continuous but is not of exponential order.
9. Can you have two *distinct* functions having the same LT? Explain.
10. State the sufficient conditions for LT to exist. Are these necessary? Explain.
11. Let f be a piecewise continuous function of exponential order and F be a Laplace transform of f then prove that: $\lim_{s\to\infty} F(s) = 0$.
12. Is it possible to find piecewise functions of exponential order whose Laplace transforms are:
    (a) $F(s) = s, \quad s \in \mathbb{R}$
    (b) $F(s) = \dfrac{s-1}{s+1}, \quad s > -1$
13. Give two examples of functions that do not have LT.
14. Give an example of a function whose Laplace transform exists, such that f is not piecewise continuous but has exponential order.
15. Find Laplace transform of the first and second derivatives of a function f(t) stating clearly the necessary conditions on the function and its derivatives.
16. Find the Laplace transform of $\int_0^\infty f(\tau)\,d\tau$ stating clearly the necessary conditions under which it exists.

17. What are the steps of solving an ODE by the Laplace transform?

18. Can a discontinuous function have a Laplace transform? Give reason.

19. When and how do you use the unit step function and Dirac's delta?

20. Define Heaviside Function function and find its LT.

21. State and prove the second shifting theorem of LT.

22. Define Dirac Delta function and find its LT.

23. Is it possible to find functions (you may think of generalized functions such as Dirac delta function) whose Laplace transforms are:

    (a) $F(s) = \dfrac{s^2}{s^2+1}, \quad s \in \mathbb{R}$

    (b) $F(s) = \dfrac{s^2}{s^2-1}, \quad s > 1$

24. Is $L\{f(t)\,g(t)\} = L\{f(t)\}\ L\{g(t)\}$? Justify your answer!

25. Define convolution of two functions. Prove the commutative, associative and distributive properties of convolution of two functions.

26. Find Laplace transform of the convolution of $f(t)$ and $g(t)$ where $f(t) = \cos \omega t$ and $g(t) = e^{-at}$.

27. State and prove the convolution theorem for Laplace transforms.

28. Find the Laplace transform of a periodic function and hence find the Laplace transform of half wave rectification of $\sin \omega t$.

29. Find the LTs of the following functions indicating the formula/ theorem used clearly at each step :

    (a) $(5e^{2t}-3)^2$ Ans. $\dfrac{25}{s-4} - \dfrac{30}{s-2} + \dfrac{9}{s}$

    (b) $\sin 3t - 2\cos 5t$ Ans. $\dfrac{3}{s^2+9} - 2\dfrac{s}{s^2+25}$

    (c) $\cosh at - \cos at$ Ans. $\dfrac{2a^2 s}{s^4-a^4}$

    (d) $e^t(1+t)^2$ Ans. $\dfrac{s^2+1}{(s-1)^3}$

    (e) $f(t) = \begin{cases} t, & 0 < t < 1 \\ e^{1-t}, & t > 1. \end{cases}$ Ans. $\dfrac{1}{s^2}[1 - e^{-s}(\dfrac{2s+1}{s+1})]$

    (f) $t^{7/2}e^{3t}$ Ans. $\dfrac{105\sqrt{\pi}}{16(s-3)^{9/2}}$

    (g) $f(t) = t\cos at$ Ans. (Use $\mathcal{L}\{tf(t)\}$). $\dfrac{s^2-a^2}{(s^2+a^2)^2}$

(h) $\sin^2 t$ Ans. (Use $\mathcal{L}\{f'\}$). $\dfrac{2}{s(s^2+4)}$

(i) $\dfrac{e^{-at}-e^{-bt}}{t}$ Ans. (Use $\mathcal{L}\{f(t)/t\} = \int\limits_{s}^{\infty} F(u)du$). $\ln\dfrac{s+b}{s+a}$

(j) $\dfrac{1}{2}t^2\cos\dfrac{\pi}{2}t$ Ans. $16\dfrac{s(4s^2-3\pi^2)}{(4s^2+\pi^2)^3}$

(k) $e^{-t}\sinh 4t$ Ans. $\dfrac{4}{s^2+2s-15}$

(l) $\dfrac{\cos at - \cos bt}{t}$ Ans. $\dfrac{1}{2}\ln(\dfrac{s^2+b^2}{s^2+a^2})$

(m) $\dfrac{\sin^2 t}{t}$ Ans. $\dfrac{1}{4}\ln\dfrac{s^2+4}{s^2}$

(n) $\dfrac{e^t\,\delta(t-2)}{t}$ Ans. $\dfrac{e^{-2(s-1)}}{2}$

(o) $\delta(t-3)\,U(t-3)$ Ans. e^{-3s}

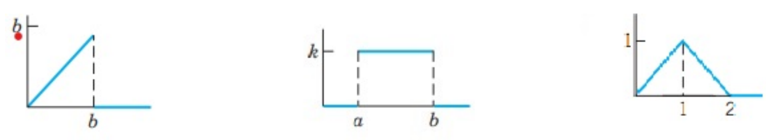
(p) $t^2\sin 2t$ Ans. (Use $\mathcal{L}\{t^2 f(t)\} = F''(s)$). $\dfrac{-4(4-3s^2)}{(s^2+4)^3}$

(q) $\int\limits_0^t \dfrac{1-e^{-u}}{u}du$ Ans. (Use $\mathcal{L}\{\int\limits_o^t f(u)du\} = \dfrac{\mathcal{L}\{f\}}{s}$). $\frac{1}{s}\ln(1+\frac{1}{s})$

(r) First sketch and express in terms of unit step: $e^{-\pi t/2}$; $1<t<3$; 0 outside.
Ans. $2\dfrac{e^{-s-\pi/2}-e^{-3s-3\pi/2}}{2s+\pi}$

(s) $4t * e^{-2t}$, $*$ denotes the convolution. Ans. $\dfrac{8}{s^3(s+2)}$

30. Find the LTs of the following functions indicating the formula/ theorem used clearly at each step :



(a) (b) (c)

$(d) f(t) = \sin^4 t$ $(e) f(t) = e^{-t}\sinh 4t$ $(f) f(t) = t\,e^{-t}\cos t$ (g) $\displaystyle\int_0^\infty e^{-\sqrt{3}t}\,\frac{\sin t}{t}\,dt$

$(h) f(t) = t^2\sin 3t$ $(i)\ f(t) = 16\,t^2\,U(t-1/4)$ $(j)\ f(t) = 1*\sin\omega t$

$(k)\ f(t) = e^{-2t}(\cos 2t - 4\sin 2t)$ $(l) \displaystyle\int_0^\infty e^{-3t}\,\{t\cos t\}\,dt$

31. Find the ILT of the following functions indicating the formula/ theorem used clearly at each step, assuming the interval of validity for the values of s :

(a) $\dfrac{0.1s+0.9}{s^2+3.24}$ Ans. $0.1\cos 1.8t + 0.5\sin 1.8t$

(b) $\dfrac{-s-10}{s^2-s-2}$ Ans. $3e^{-t}-4e^{2t}$

(c) $\dfrac{1}{(s-1)(s^2+4)}+\dfrac{4}{s^5}$ Ans. $\dfrac{e^t}{5}-\dfrac{\cos 2t}{5}-\dfrac{\sin 2t}{10}+\dfrac{t^4}{6}$

(d) $\dfrac{3s+1}{s^2+6s+13}$ Ans. $e^{-3t}(3\cos 2t-4\sin 2t)$

(e) $\dfrac{s^2}{(s-1)^4}$ Ans. $e^t(t+t^2+\frac{t^3}{6})$

(f) $\dfrac{e^{-\pi s}}{s^2+9}$ Ans. $\frac{1}{3}\sin 3(t-\pi)U(t-\pi)$

(g) $\dfrac{1-e^{-s}}{s^2}$ Ans. t, if $t<1$ and 1 if $t>1$.

(h) $\cot^{-1}\dfrac{s}{\omega}$ Ans. (Let $f(t)=\mathcal{L}^{-1}\{F(s)\}$. Use $\mathcal{L}^{-1}\{F'(s)\}=-tf(t)$). $(\sin \omega t)/t$.

(i) $\frac{1}{2}\ln(\dfrac{s^2-a^2}{s^2})$ Ans. $\dfrac{1-\cosh at}{t}$

(j) $\ln\sqrt{\dfrac{s^2+b^2}{s^2+a^2}}$ Ans. $\dfrac{\cos at-\cos bt}{t}$

(k) $\dfrac{e^{-2s}}{s^6}$. Also sketch $f(t)$. Ans. $\dfrac{1}{120}(t-2)^5U(t-2)$

(l) $\dfrac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}$ Ans. $e^t(t\sin t+\cos t)$

(m) $s\ln(\dfrac{s}{\sqrt{s^2+1}})$ Ans.(Use $\mathcal{L}^{-1}F''(s)=t^2f(t)$).

(n) $\dfrac{e^{-s}}{s}\tan^{-1}(\dfrac{s-1}{4})$

Ans. Let $F(s)=e^{-s}/s$, $G(s)=\tan^{-1}(\dfrac{s-1}{4})$. Then $\mathcal{L}^{-1}\{F(s)\}=U(t-1)$ and $\mathcal{L}^{-1}\{G(s)\}=\dfrac{-e^t\sin 4t}{t}$. By convolution thm, the required ans is $\mathcal{L}^{-1}\{F(s)G(s)\}=U(t-1)*\dfrac{-e^t\sin 4t}{t}$.

(o) $\dfrac{18s}{(s^2+36)^2}$ Ans. $3(t\sin 6t)/2$

32. Find the ILT of the following functions indicating the formula/ theorem used clearly at each step, assuming the interval of validity for the values of s :

(1) $F(s)=\dfrac{5s+1}{s^2-25}$ (2) $G(s)=\dfrac{4s+32}{s^2-16}$ (3) $H(s)=\dfrac{a_0}{s+1}+\dfrac{a_1}{(s+1)^2}+\dfrac{a_2}{(s+1)^3}$

(4) $F(s)=(\dfrac{s-1}{s^2})e^{-s}$ (5) $G(s)=\dfrac{3s}{s^2-2s+2}$ (6) $F(s)=\dfrac{1}{(s+\sqrt{2})(s-\sqrt{3})}$

(6) $G(s)=\dfrac{1}{(s+1)^3}$ (7) $F(s)=\dfrac{6s+7}{2s^2+4s+10}$ (8) $G(s)=\dfrac{a(s+k)+b\pi}{(s+k)^2+\pi^2}$

(9) $F(s) = \dfrac{20}{s^3 - 2\pi s^2}$ (10) $G(s) = \dfrac{1}{s^4 - s^2}$ (11) $H(s) = \dfrac{3s+4}{s^4 + k^2 s^2}$

(12) $F(s) = \dfrac{4}{s}(e^{-2s} - 2e^{-5s})$ (13) $H(s) = (1 + e^{-2\pi(s+1)})\dfrac{s+1}{(s+1)^2+1}$

(14) $G(s) = \dfrac{2}{s^2-4}(e^{-s} - e^{-3s})$ Also sketch the functions in the t domain.

(15) $G(s) = \dfrac{s^2+s-6}{s^2+s+1}$ (16) $\displaystyle\sum_{k=1}^{4} \frac{(k+1)^2}{s+k^2}$ (17) $\tan^{-1}\left(\dfrac{2}{s^2}\right)$

33. Solve using Laplace transform:

 (a) $y'' + y = r(t)$, $r(t) = t$ if $1 < t < 2$, 0 otherwise. $y(0) = y'(0) = 0$
 Ans. $y = [t - \cos(t-1) - \sin(t-1)]U(t-1) + [-t + 2\cos(t-2) + \sin(t-2)]U(t-2)$
 (b) $y'' + y = e^{-2t}\sin t, y(0) = y'(0) = 0.$
 Ans. $y = \frac{1}{8}[\sin t - \cos t + e^{-2t}(\sin t + \cos t)]$
 (c) $y'' + 2y' + 5y = 50t - 150, y(3) = -4, y'(3) = 14.$
 Ans. $y = 10(t-3) - 4 + 2e^{-(t-3)}\sin 2(t-3)$
 (d) $y'' + 2y' + 5y = e^{-t}\sin t,\ y(0) = 0, y'(0) = 1$
 Ans. $y = e^{-t}(\sin t + \sin 2t)/3$

34. Solve the following IVPs using LTs showing clearly all the details:

 (a) $y'' + 9y = e^{-t}\ ; y(0) = 0\ , y'(0) = 0$
 (b) $y'' - 6y' + 5y = 29\cos 2t\ ; y(0) = 3.2\ , y'(0) = 6.2$
 (c) $y'' + 0.4y = 0.02t^2\ ; y(0) = -25\ , y'(0) = 0$
 (d) $y'' + 2y' + 5y = 50t - 100\ , y(2) = -4\ , y'(2) = 14$

35. Find and graph/sketch the solution of IVP (you may use Geogebra or some similar graphing tool to get an idea of the solution):

 (a) $y'' + 16y = 4\delta(t - 3\pi)\ , y(0) = 2\ , y'(0) = 0$
 (b) $y'' + 4y' + 5y = \delta(t-1)\ ; y(0) = 0\ , y'(0) = 3$

36. Solve the following IVP and express the solution as a piece-wise defined function:
 $y'' + 5y' + 6y = \delta(t - \pi/2) + U(t-\pi)\cos t\ ; y(0) = 0 = y'(0)$

37. Solve the following linear integral equations:

 (a) $y(t) = \sin 2t + \int_0^t y(\tau)\sin 2(t-\tau)d\tau.$ Ans. $\sqrt{2}\sin\sqrt{2}\,t$

 (b) $y(t) = 1 - \sinh t + \int_0^t (1+\tau)\,y(t-\tau)d\tau.$ Ans. $\cosh t$

 (c) $y(t) + 4\int_0^t y(\tau)(t-\tau)d\tau = 2t$

 (d) $y(t) - \int_0^t y(\tau)\sin(2t - 2\tau)d\tau = \sin 2t$

(e) $\int_0^t y(\tau)(t-\tau)^2 d\tau - y(t) = \frac{t^2}{2} - 2$

38. Find the current $i(t)$ in an LC circuit assuming $L = 1$henry, $C = 1$ farad, zero initial current and charge on the capacitor and

$$v(t) = \begin{cases} 1 - e^{-t}, & 0 < t < \pi \\ 0, & otherwise. \end{cases}$$

Ans. $\frac{1}{2}(e^{-t} - \cos t + \sin t)$, if $0 < t < \pi$ and $\frac{1}{2}[-(1+e^{-\pi})\cos t + (3 - e^{-\pi})\sin t]$, if $t > \pi$.

39. Find the current in an RLC circuit if $R = 4\Omega, L = 1H, C = 0.05F$ and the applied voltage is

$$v(t) = \begin{cases} 34\, e^{-t}\, V, & 0 < t < 4 \\ 0, & t > 4 \end{cases}.$$

Assume that current and charge are 0 initially. Solve using Laplace transform method showing all the details.

40. Write a summary on Laplace transforms in your own words not exceeding 500 words.

41. Note that any problem similar to the problems in CO3 in a new or unknown situation can be treated as a problem of CO4 or CO5. Hence you should try to solve all problems in the exercises from the text book.

Please report any mistakes in the problems and/or answers given here.