COEP Technological University Pune, Department of Mathematics

(MA-20001) - Ordinary Differential Equations and Multivariate Calculus

S.Y. B. Tech. Semester III (All Branches)

Tutorial 4 (AY: 2023-24)

1. Describe the function's domain, find the function's range and also describe the function's level curves.

   a) $f(x,y) = y - x$
   b) $f(x,y) = \sqrt{y-x}$
   c) $f(x,y) = 4x^2 + 9y^2$
   d) $f(x,y) = x^2 - y^2$
   e) $f(x,y) = xy$
   f) $f(x,y) = y/x^2$
   g) $f(x,y) = \dfrac{1}{\sqrt{16 - x^2 - y^2}}$
   h) $f(x,y) = \sqrt{9 - x^2 - y^2}$
   i) $f(x,y) = \ln(x^2 + y^2)$

2. Sketch the surface $z = f(x,y)$.

   a) $f(x,y) = y^2$
   b) $f(x,y) = x^2 + y^2$
   c) $f(x,y) = \sqrt{x^2 + y^2}$
   d) $f(x,y) = -(x^2 + y^2)$
   e) $f(x,y) = 4 - x^2 - y^2$
   f) $f(x,y) = 1 - |x| - |y|$

3. Sketch typical level surface of the given functions.

   a) $f(x,y,z) = x^2 + y^2 + z^2$
   b) $f(x,y,z) = \ln(x^2 + y^2 + z^2)$
   c) $f(x,y,z) = x + z$
   d) $f(x,y,z) = z$
   e) $f(x,y,z) = z - x^2 - y^2$
   f) $f(x,y,z) = (x^2/25) + (y^2/16) + (z^2/9)$

4. Find the following limits.

   a) $\displaystyle\lim_{(x,y) \to (0,\pi/4)} \sec x \tan y$
   b) $\displaystyle\lim_{(x,y) \to (1,1)} \ln|1 + x^2 y^2|$
   c) $\displaystyle\lim_{(x,y) \to (0,0)} \frac{e^y \sin x}{x}$
   d) $\displaystyle\lim_{(x,y) \to (\pi/2,0)} \frac{\cos y + 1}{y - \sin x}$
   e) $\displaystyle\lim_{(x,y) \to (1,1)} \frac{x^3 - y^3}{x - y}$
   f) $\displaystyle\lim_{(x,y,z) \to (\pi,0,3)} z e^{-2y} \cos 2x$
   g) $\displaystyle\lim_{(x,y,z) \to (1,-1,-1)} \frac{2xy + yz}{x^2 + z^2}$
   h) $\displaystyle\lim_{(x,y) \to (0,0)} \frac{x - y + 2\sqrt{x} - 2\sqrt{y}}{\sqrt{x} - \sqrt{y}}$

5. At what points of the domain are the following functions continuous?

a) $f(x,y) = \ln(x^2 + y^2)$

b) $f(x,y) = \dfrac{x+y}{x-y}$

c) $f(x,y) = \sin\left(\dfrac{1}{xy}\right)$

d) $f(x,y) = \dfrac{x+y}{2+\cos x}$

e) $f(x,y) = \dfrac{1}{x^2 - y}$

f) $f(x,y,z) = \ln xyz$

g) $f(x,y,z) = e^{x+y}\cos z$

h) $g(x,y,z) = \dfrac{1}{|xy| + |z|}$

6. Find the limit of the following functions as $(x,y) \to (0,0)$ or show that the limit does not exist.

a) $f(x,y) = \dfrac{-x}{\sqrt{x^2+y^2}}$

b) $f(x,y) = \dfrac{x^4}{x^4 + y^2}$

c) $f(x,y) = \dfrac{x^3 - xy^2}{x^2 + y^2}$

d) $f(x,y) = \dfrac{x^4 - y^2}{x^4 + y^2}$

7. Find the first order partial derivatives with respect to each variable.

a) $f(x,y) = (xy-1)^2$

b) $f(x,y) = \tan^{-1}(y/x)$

c) $f(x,y) = e^{-x}\sin(x+y)$

d) $f(x,y) = \ln(x+y)$

e) $f(x,y) = e^{xy}\ln y$

f) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$

g) $f(x,y,z) = \sin^{-1}(xyz)$

h) $f(x,y) = e^{-(x^2+y^2+z^2)}$

i) $f(x,y,z) = e^{-(xyz)}$

j) $g(u,v) = v^2 e^{2u/v}$

k) $h(\rho,\phi,\theta) = \rho\sin\phi\cos\theta$

l) $f(t,\alpha) = \cos(2\pi t - \alpha)$

m) $g(r,\theta,z) = r(1-\cos\theta) - z$

8. Find the second order partial derivatives of the following functions.

a) $f(x,y) = x + y + xy$

b) $f(x,y) = \sin(xy)$

c) $f(x,y) = xe^y + y + 1$

d) $h(x,y) = \tan^{-1}(y/x)$

e) $r(x,y) = \ln(x+y)$

9. Verify that $f_{xy} = f_{yx}$.

a) $f(x,y) = e^x + x\ln y + y\ln x$

b) $f(x,y) = xy^2 + x^2y^3 + x^3y^4$

10) Prove that $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z}$, where $F(x,y,z) = 0$ defines z implicitly as a function of the two independent variables x and y.

11) Equation $f(x,y) = 0$ defines y as a function of x implicitly. What do you understand by this statement? Does it mean you can express y in terms of x ?

12) Find the value of $\dfrac{\partial z}{\partial x}$ at the point $(1,1,1)$ if the equation $xy + z^3x - 2yz = 0$ defines z implicitly as a function of two independent variables x and y.

13) Find the value of $\dfrac{\partial x}{\partial z}$ at the point $(1,-1,-3)$ if the equation $xz + y\ln x - x^2 + 4 = 0$ defines x implicitly as a function of two independent variables y and z.

14) In the following exercises find the derivatives $\dfrac{dw}{dt}$ by using the Chain Rule and evaluate the derivative at the given point.

a) $w = x^2 + y^2$ $\quad x = \cos t\ ,\ y = \sin t\ ,\ t = \pi$

b) $w = x^2 + y^2$ $\quad x = \cos t + \sin t\ ,\ x = \cos t - \sin t\ ,\ t = 0$

c) $w = \dfrac{x}{z} + \dfrac{y}{z}$ $\quad x = \cos^2 t\ ,\ y = \sin^2 t\ ,\ z = 1/t\ ,\ t = 3$

d) $w = 2ye^x - \ln z$ $\quad x = \ln(t^2+1)\ ,\ y = \tan^{-1} t\ ,\ z = e^t\ ,\ t = 1$

15) For the following functions find the partial derivatives $\dfrac{\partial}{\partial u}$ and $\dfrac{\partial}{\partial v}$ as functions of u and v by using the Chain Rule and also evaluate the partial derivatives at the given point.

a) $z = 4e^x \ln y$ $\quad x = \ln(u\cos v)\ ,\ y = u\sin v\ ,\ (u,v) = (2,\pi/4)$

b) $z = \tan^{-1}(x/y)`$ $\quad x = u\cos v\ ,\ y = u\sin v\ ,\ (u,v) = (1.3,\pi/6)$

c) $w = \ln(x^2+y^2+z^2)$ $\quad x = ue^v\sin u\ ,\ y = ue^v\cos u\ ,\ z = ue^v\ ,\ (u,v) = (-2,0)$

16) Express the partial derivatives $\dfrac{\partial u}{\partial x}\ ,\ \dfrac{\partial u}{\partial y}\ ,\ \dfrac{\partial u}{\partial z}$ as functions of x , y and z by using the Chain Rule and also evaluate the partial derivatives at given point.

a) $u) = \dfrac{p-q}{q-r}$ $\quad p = x+y+z\ ,\ q = x-y+z\ ,\ r = x+y-z,\ (x,y,z) = (\sqrt{3},2,1)$

b) $u = e^{qr}\sin^{-1} p$ $\quad p = \sin x\ ,\ q = z^2\ln y\ ,\ r = 1/z\ ,\ (x,y,z) = (\pi/4, 1/2, -1/2)$

17) In the following exercises write a Chain Rule Formula for each derivative .

a) w_u and w_v for $\quad w = h(x,y,z)\ ,\ x = f(u,v)\ ,\ y = g(u,v)\ ,\ z = k(u,v)$

b) w_u and w_v for $\quad w = g(x,y)\ ,\ x = h(u,v)\ ,\ y = k(u,v)$

c) w_x and w_y for $\quad w = g(u,v)\ ,\ u = h(x,y)\ ,\ v = k(x,y)$

d) y_r for $\quad y = f(u)\ ,\ u = g(r,s)$

e) w_p for $\quad w = f(x,y,z,v)\ ,\ x = g(p,q)\ ,\ y = h(p,q)\ ,\ z = k(p,q)\ ,\ v = j(p,q)$

f) w_r and w_s for $\quad w = f(x,y)\ ,\ x = h(r)\ ,\ y = k(s)$

g) w_s for $\quad w = f(x,y)\ ,\ x = g(r,s,t)\ ,\ y = h(r,s,t)$

18) Suppose $w = f(u, v)$ stisfies the Laplace equation $f_{uu} + f_{vv} = 0$. Prove that :

a) If $u = \dfrac{(x^2 - y^2)}{2}$ and $v = xy$, then w satisfies the Laplace equation $w_{xx} + w_{yy} = 0$.

b) If $u = e^x \cos y$ and $v = e^x \sin y$, then w satisfies the Laplace equation $w_{xx} + w_{yy} = 0$.

In both the parts, let $z = x + iy$ be a complex number. Express $u + iv$ as a complex valued function of z.

19) Let $w = x^2 e^{2y} \cos 3z$. Find the value of dw/dt at the point $(1, \ln 2, 0)$ on the curve $x = \cos t$, $y = \ln(t + 2)$, $z = t$.

20) Find the gradient of the function at the given point. Then sketch the gradient together with the level curve that passes through the point.

a) $f(x, y) = y - x$, $(2, 1)$

b) $f(x, y) = \ln(x^2 + y^2)$, $(1, 1)$

c) $f(x, y) = y - x^2$, $(-1, 0)$

d) $f(x, y) = \frac{x^2}{2} - \frac{y^2}{2}$, $(\sqrt{2}, 1)$

21) Find the directions in which the functions increase and decrease most rapidly at P_0 and also find the directional derivative of the functions in these directions. Which quantity do such directional derivatives give?

a) $f(x, y) = x^2 + xy + y^2$, $P_0(-1, 1)$

b) $f(x, y) = x^2 y + e^{xy} \sin y$, $P_0(1, 0)$

c) $f(x, y, z) = xe^y + z^2$, $P_0(1, \ln 2, 1/2)$

d) $f(x, y, z) = \frac{x}{y} - yz$, $P_0(4, 1, 1)$

22) Sketch the curve $f(x, y) = c$ together with ∇f and the tangent line at the given point. Then write the equation for the tangent line.

a) $x^2 + y^2 = 4$, $(\sqrt{2}, \sqrt{2})$

b) $x^2 - y = 1$, $(\sqrt{2}, 1)$

c) $xy = -4$, $(2, -2)$

d) $x^2 - xy + y^2 = 7$, $(-1, 2)$

23) In what direction is the directional derivativeof $f(x, y) = xy + y^2$ at $P(3, 2)$ equal to zero?

24) Is there a direction $\overline{u}$ in which the rate of change of the temperature function $T(x, y, z) = 2xy - yz$ (temperature in degrees Celcius, distance in feet) at $P(1, -1, 1)$ is $-3^0 C/ft$? Give reason for your answer.

25) The lengths a, b and c of the edges of a closed rectangular box are changing with time. At the instant in question $a = 1m$, $b = 2m$, $c = 3m$, $\dfrac{da}{dt} = 1m/sec$, $\dfrac{db}{dt} = 1m/sec$, $\dfrac{dc}{dt} = -3m/sec$. At what rates are the box's volume V and the surface area S changing at that instant? Are the box's interior diagonals increasing or decreasing in length?

26) Find all the local maxima, local minima and saddle points of the functions given:

a) $f(x,y) = x^2 + xy + y^2 + 3x - 3y + 4$

b) $f(x,y) = y \sin x$

c) $f(x,y) = x^3 + 3xy + y^3$

d) $f(x,y) = e^{2x} \cos y$

e) $f(x,y,z) = x^2 - xy + y^2 + yz + z^2 - 2z$

f) $f(x,y,z) = x^2 + y^2 + z^2 - w^2 + xy + zw$

27) Find the absolute maxima and minima of the function $f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x = 0$, $y = 2$, $y = 2x$ in the first quadrant.

28) Let $f(x,y)$ be a function such that the first partial derivatives exist at (a,b). State true or false and justify your answers.

a) If $f_x(a,b) = f_y(a,b) = 0$ then $f(x,y)$ has local extreme value at (a,b).

b) If $f(x,y)$ has local maximum or minimum at (a,b) then $f_x(a,b) = f_y(a,b) = 0$

29) Consider the flat circular disc given by $x^2 + y^2 \leq 1$. The disc including the boundary where $x^2 + y^2 = 1$, is heated so that the temperature at the point (x,y) is $T(x,y) = x^2 + 2y^2 - x$, then

a) Draw level curves of $T(x,y)$. And state what do they signify?

b) Find the temperatures at the hottest and coldest points on the disc.

30) Maximize the function $f(x,y,z) = xyz$ subject to the constraints

$$x + y + z = 40 \quad \text{and} \quad x + y = z$$

31) Minimize the function $f(x,y,z) = x^2 + y^2 + z^2$ subject to the constraints

$$x + 2y + 3z = 6 \quad \text{and} \quad x + 3y + 4z = 9$$

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