COEP Technological University Pune
Department of Mathematics
(MA- 20001) - Ordinary Differential Equations and Multivariate Calculus
S.Y. B.Tech. Semester III (All Branches)
Tutorial 2 (AY: 2023-24)

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1. Let the electric equipotential lines (curves of constant potential) between two concentric cylinders with the z-axis in space be given by $u(x, y) = x^2 + y^2 = c$. Find their orthogonal trajectories (the curves of electric force).

2. The lines of electric force of two opposite charges of the same strength at (-1,0) and (1,0) are the circles through (-1,0) and (1,0) (dashed in Fig.1). Show that these circles are given by $x^2 + (y - c)^2 = 1 + c^2$. Show that the equipotential lines (which are orthogonal trajectories of those circles) are the circles given by $(x + c^*)^2 + \tilde{y}^2 = c^{*^2} - 1$.

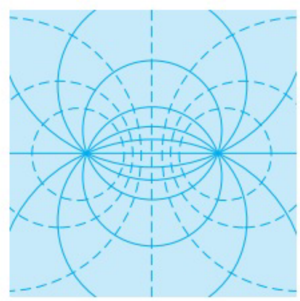

Figure:1

3. Let the isotherms (curves of constant temperature) in a body in the upper half-plane $y > 0$ be given by $4x^2 + 9y^2 = c$. Find the orthogonal trajectories (the curves along which heat will flow in regions filled with heat-conducting material and free of heat sources or heat sinks).

4. Write a note on unifying power of mathematics.

5. What is the Wronskian? What is it used for?

6. What does an initial value problem of a second-order ODE look like?

7. What form does an initial value problem for an n^{th}-order linear ODE have?

8. Apply the given operator to the given function (show all the steps in detail)

   (a) $8D^2 + 2D - I;$ $\cosh \frac{x}{2},\ \sinh \frac{x}{2},\ e^{\frac{x}{2}}$

   (b) $(D + 5I)(D - I);$ $e^{-3x} \sin x,\ e^{3x},\ x^2$

   (c) $(D - 4I)(D + 3I);$ $x^3 - x^2,\ \sin 4x,\ e^{-3x}$

9. Define Lnear independence and dependence of function.

10. Check whether the following functions linearly independent or dependent on the given interval ?

    $(a)\ x^2, \frac{1}{x^2}, 0; \quad (x \geq 0) \qquad (b)\ e^{-x}, \cos\left(\frac{x}{2}\right), 0; \quad (-1 \leq x \leq 1)$

(c) x^2, $\ln x^2$; $(x > 1)$ (d) $\cosh 2x$, $\sinh 2x$, e^{2x}; $(x \geq 0)$

(e) x^2, $x|x|$; $[-1, 1]$ (f) x^2, $x|x|$; $(0, \infty)$

11. Find linear ODE for which the following functions are linearly independent solutions:

    (1) $1,\ e^{-2x}$ (2) $\cos 5x,\ \sin 5x$ (3) $x^2,\ x^3$ (4) $e^x,\ xe^x$ (5) $x,\ x\ln x$

    (6) $1,\ x,\ \cos 2x,\ \sin 2x$ (7) $e^x,\ xe^x,\ \cos x,\ \sin x,\ x\cos x,\ x\sin x$

    (8) $e^{-(s+it)x},\ e^{-(s-it)x}$ (9) $e^x,\ e^{-x},\ e^{2x}$ (10) $e^{-kx}\cos \pi x,\ e^{-kx}\sin \pi x$

12. What is the superposition or linearity principle? For what n^{th}-order ODEs does it hold?

13. List some other basic theorems that extend from second-order to n^{th}-order ODEs.

14. Reduce to first order and solve, showing each step in detail.

    (a) $y'' + (y')^3 \sin y = 0$
    (b) $2xy'' = 3y'$

15. Find the curve through the origin in the $xy-$ plane which satisfies $y'' = 2y'$ and whose tangent at the origin has slope 1.

16. If the roots of the auxillary equation of second order homogeneous linear ODE $y'' + by' + cy = 0$ are real and equal then find the first solution, and the second solution using the method of reduction of order, and hence write the basis.

17. Using reduction of order, find a second linearly independent solution $y_2(x)$ of the $xy'' + 2y' + xy = 0$ by , if $y_1(x) = \dfrac{\cos x}{x}$ is one solution.

18. Show that x and $x\ln x$ are linearly independent solutions of $x^2y'' - xy' + y = 0$.
    Hence solve the IVP $y(1) = 1;\ y'(1) = 2$

19. Verify that $e^{-4x},\ xe^{-4x},\ x^2e^{-4x}$ are linearly independent solutions of
    $y''' + 12y'' + 48y' + 64y = 0$.

20. Solve the following :

    (a) $y'' + \pi y = 0; \quad y(0) = 3, \quad y'(0) = -\pi$
    (b) $(D^4 + k^4)\,y = 0$
    (c) $y'' + 4y' + (\pi^2 + 4)y = 0$
    (d) $4y'' - 4y' - 3y = 0$
    (e) $y'' + 2k^2y' + k^4y = 0$
    (f) $y'' - 2y' - 3y = 0; \quad y(-1) = e, \quad y'(-1) = -e/4$
    (g) $(D^3 - D^2 - D + I)y = 0$
    (h) $y^{(4)} - 9y^{(2)} - 400y = 0; \quad y(0) = 3.4, \quad y'(0) = 0, \quad y''(0) = 2.5, \quad y'''(0) = 3.5$
    (i) $(D^2 + 3D + 2.5I)y = 0$
    (j) $9y'' - 30y' + 25y = 0; \quad y(0) = 3.3, \quad y'(0) = 10$

(k) $(x^2D^2 - xD + 5I)y = 0$

(l) $(9x^2D^2 + 3xD + I)y = 0$

(m) $(D + 2I)^2 y = 0$

(n) $(D^3 - 3D^2 + 9D - 27I)y = 0$

(o) $x^2y'' - xy' + 2y = 0$

(p) $x^2y'' + 3xy' + y = 0$

(q) $(10\,x^2\,D^2 - 20\,x\,D + 22.4I)y = 0$

(r) $\left(\dfrac{d}{dx} + \dfrac{1}{x}\right)^2 y = 0$

21. Using the method of undetermined coefficients, obtain a real general solution of following non-homogeneous linear differential equations (LDEs):

(a) $y'' - y' - 2y = 3\,e^x$

(b) $y'' - 4y' = 8\,\cos\pi x$

(c) $y'' + 6y' + 9y = 50\,e^{-x}\,\cos x$

(d) $y'' + 6y' + 9y = e^{-x}\cos 2x; \quad y(0) = 1, \quad y'(0) = -1$

(e) $(2D^2 - 3D - 2I)y = 13 - 2x^2$

(f) $(D^2 - I)y = \sinh x$

(g) $y'' - 4y' + 3y = 10\,\cos x; \quad y(0) = 1, \quad y'(0) = -1.$

(h) $y'' - 9y' = x^3 + e^{2x} - \sin 3x$

(i) $y''' + y' = 3x^2 + 4\sin x - 2\cos x$

(j) $y'' - 2y' = 12e^{2x} - 8e^{-2x}$

22. Solve the following non-homogeneous LDEs using method of variation of parameters.

(a) $y'' + 4y = \cos 2x$

(b) $(D^3 + 4D)y = \sin x$

(c) $(D^2 + 2D + 2I)y = 4e^{-x}\sec^3 x$

(d) $y'' - 4y' + 4y = \dfrac{e^{2x}}{x}$

(e) $(D^2 + 6D + 9I)y = \dfrac{16\,e^{-3x}}{x^2 + 1}$

(f) $y'' + 9y = \sec 3x$

(g) $x^2y'' - xy' + y = x\ln x$

(h) $(D^2 + I)y = \cot x$

(i) $(D^3 + D)y = cosecx$

(j) $(x^2D^2 - 2xD + 2I)y = x^3\cos x$

(k) $y'' - 4y' + 5y = e^{2x}cosecx$

(l) $y'' - y' = (3 + x)\,x^2\,e^x$

23. For the following non-homogeneous equation, a solution y_1 of the corresponding homogeneous equation is given. Find a second solution y_2 of the corresponding homogeneous equation and the general solution of the non-homogeneous equation using the method of variation of parameters.

$$(1+x^2)y'' - 2xy' + 2y = x^3 + x, \qquad y_1(x) = x$$

24. A capacitor $C = 0.2\ farads$ in series with a resistor $R = 20\ ohms$ is charged from a source $E_0 = 24\ volts$. Find the voltage $v(t)$ on the capacitor, assuming that at $t = 0$ the capacitor is completely uncharged.

25. Consider the RC circuit equation $R\dfrac{dQ}{dt} + \dfrac{Q}{C} = E(t)$. Determine the charge and current at time $t > 0$ if $R = 10\ ohms$, $C = 2 \times 10^{-4}\ farads$, and $E(t) = 100\ volts$. Given that $Q(0) = 0$.

26. The charge Q on the plate of a condenser of capacity C charged through a resistance R by a steady voltage V satisfies the differential equation $R\dfrac{dQ}{dt} + \dfrac{Q}{C} = V$. If $Q = 0$ at $t = 0$, show that $Q = CV\left(1 - e^{-\dfrac{t}{RC}}\right)$. Find the current flowing into the plate at any time t. $\left(Ans : i(t) = \frac{V}{R}\ e^{-\dfrac{t}{RC}}\right)$

27. A decaying $e.m.f. E = 200\ e^{-5t}$ is connected in series with a 20 ohm resistor and 0.01 $farad$ capacitor. Find the charge and current at any time assuming $Q = 0$ at $t = 0$. Show that the charge reaches a maximum, calculate it and find the time when it is reached. $\left(Ans : t = \dfrac{1}{5},\ \max.of\ Q = 0.74\right)$

28. In a circuit containing inductance L, resistance R and voltage E, the current I is given by $E = RI + L\dfrac{dI}{dt}$. Given $L = 640H$, $R = 250\ ohm$ and $E = 500\ volts$. I being zero when $t = 0$. Find the time that elapses, before it reaches 90% of its maximum value. $\left(Ans : t = \dfrac{64}{25}\ln 10\right)$

29. Show that the current in RL circuit when a constant e.m.f. E_0 is applied reaches 63% of its final value in $\dfrac{L}{R}$ seconds. Further if $L = 10$ henries, determine the value of R so that the current will reach 99% of its final value at $t = 1$ seconds ? $(Ans : R = 46.06)$

30. Find the current $I(t)$ in the RC circuit with $E = 100\ volts$, $C = 0.25\ farads$, R is variable according to

$$R = \begin{cases} (200-t)\ ohms, & 0 \le t \le 200\ \text{sec} \\ 0, & t > 200\ \text{sec} \end{cases}$$

and $I(0) = 1\ amp.$ Ans $(I = (200)^{-3}(200-t)^3$ and 0)

31. Find the time when the capacitor in an RC circuit with no external e.m.f. has lost 99% of its initial charge of Q_0 Coulomb. (Ans: t = 4.605 RC)

32. Find the steady state solution for $Q(t)$ in an RC circuit when R = 50 ohm, C = 0.04 farad, and $E(t) = 100\ \cos 2t + 25\ \sin 2t + 200\ \cos 4t + 25\ \sin 4t$.

33. Find the steady state and transient state motion of the mass-spring system with mass 4 kg, damping constant $c = 8kg/\sec$, spring constant $k = 3kg/\sec^2$, and driving force $r(t) = 425\ \sin 2t$ newton, where $y(0) = 16$ and $y'(0) = 26$.

34. Find the steady state and transient state motion of the mass-spring system with mass $m = 4kg$, damping constant $c = 4kg/\sec$, spring constant $k = 17kg/\sec^2$, and the driving force $r(t) = 202\ \cos 3t$ newton.

35. A large tank is initially filled with 100 L of brine in which 1 kg of salt is dissolved. Brine containing 0.5 kg of salt per L is pumped into the tank at a rate of 6 L/min. The well-mixed brine is pumped out of the tank at a slower rate of 4 L/min. Assuming that the tank does not overflow, find the amount of salt in the tank after t minutes . Give your answer to the nearest gram. $Ans : x(t) = \left[50 + t - \dfrac{122500}{(50+t)^2}\right] \times 1000\, gm$

36. A force of 400N stretches a spring 2 meters. A mass of 50 kg is attached to the end of the spring and is initially released from the equilibrium position with an upward velocity of 10m/s. Find the equation of motion. Also detrmine the natural frequency, period and amplitude.

37. State some applications that can be modeled by systems of ODEs.

38. How can you transform an ODE into a system of ODEs?

39. What are eigenvalues? What role did they play in the system of ODEs ?

40. Find the general solution of the given ODE by **first converting it to a system of equations**.

    (a) $y'' - 4y = 0$
    (b) $y'' + 2y' - 24y = 0$
    (c) $y" + 15y' + 50y = 0$
    (d) $y'' + 4y' + 3y = 0$

41. Find a real general solution of the following systems.

    (a) $y_1' = 6y_1 + 9y_2$; $y_2' = y_1 + 6y_2$
    (b) $y_1' = y_2$; $y_2' = -y_1 + y_3$; $y_3' = -y_2$
    (c) $y_1' = y_1 + 2y_2$; $y_2' = y_2$
    (d) $y_1' = 2y_1 + 2y_2$; $y_2' = 5y_1 - y_2$; $y_1(0) = 0$; $y_2(0) = 7$
    (e) $y_1' = y_2$; $y_2' = y_1$; $y_1(0) = 0$; $y_2(0) = 0$
    (f) $y_1' = 7y_1 + y_2$; $y_2' = -4y_1 + 3y_2$; $y_1(0) = 2$; $y_2(0) = -5$
    (g) $y_1' = 2y_1$; $y_2' = 2y_2$
    (h) $y_1' = -y_1 + y_2$; $y_2' = -y_1 - y_2$
    (i) $y_1' = y_2 + e^{3t}$; $y_2' = y_1 - 3e^{3t}$
    (j) $y_1' = y_1 + y_2 + 10\cos t$; $y_2' = 3y_1 - y_2 - 10\sin t$

(k) $y_1' = y_1 + 2y_2 + e^{2t} - 2t$; $\quad y_2' = -y_2 + 1 + t$; $\quad y_1(0) = 1$; $\quad y_2(0) = -4$

(l) $y_1' = 4y_2$; $\quad y_2' = 4y_1 - 16t^2 + 2$

42. Solve the following system of ODEs by the method of variation of parameters:

$$y_1' = -3y_1 + y_2 - 6e^{-2t}, \qquad y_2' = y_1 - 3y_2 + 2e^{-2t}$$

43. Tank T_1 in Fig. 2 initially contains 200 gal of water in which 160 lb of salt are dissolved. Tank T_2 initially contains 100 gal of pure water. Liquid is pumped through the system as indicated, and the mixtures are kept uniform by stirring. Find the amounts of salt and in T_1 and T_2 , respectively.

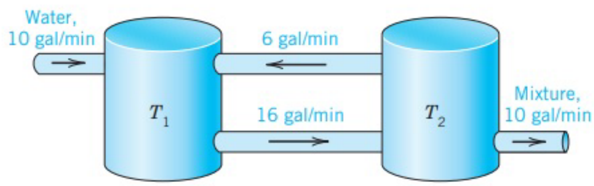

Figure:2

44. Show that a model for the currents I_1 and I_2 in Fig.3 is

$$\frac{1}{C}\int I_1\,dt + R\,(I_1 - I_2) = 0, \qquad L\,I_2' + R\,(I_2 - I_1) = 0$$

Find a general solution, assuming that $R = 3\ \Omega$, $\quad L = 4\,H$, $\quad C = \frac{1}{12}\,F$.

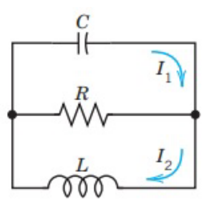

Figure:3

45. Find the currents in Fig. 4 when $R = 2.5\,\Omega$, $\quad L = 1\,H$, $\quad C = 0.04F$, $\quad E(t) = 169\sin t\ V$, $I_1(0) = 0$, $\quad I_2(0) = 0$.

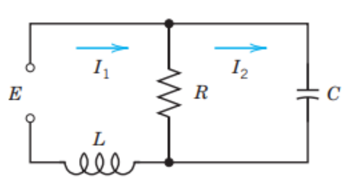

Figure: 4

Please report any mistakes in the problems and/or answers given here.