Working Rule (useful for problem solving)

Step 1 - Convert the given equation to the standard form of linear differential equation

$$\frac{dy}{dx} + p(x)\, y = r(x)$$

Step 2 - Find the integrating factor

$$I.F = e^{\int p(x)dx}$$

Step 3 - Then the solⁿ is

$$y\,(I.F) = \int r(x)(I.F)\,dx + C.$$

Examples ① Solve $(x+1)\frac{dy}{dx} - y = e^{x}(x+1)^2$

$\rightarrow \quad \frac{dy}{dx} - \frac{y}{x+1} = e^{x}(x+1)$

Here $p(x) = -\frac{1}{x+1}$ and $Q(x) = e^{x}(x+1)$

$$I.F = e^{\int p(x)dx} = e^{-\int\frac{1}{x+1}} = e^{\log(x+1)^{-1}} = \frac{1}{x+1}$$

The solⁿ is

$$y\frac{1}{x+1} = \int e^{x}\cancel{(x+1)}\cdot\frac{1}{\cancel{x+1}}\,dx + C$$

$$= \int e^{x}dx + C$$

$$\frac{y}{x+1} = e^{x} + C.$$

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② solve for $x>0$

$$y' + \frac{y}{x} = x$$

subject to $y(1)=0$

$\rightarrow$ Here $p(x) = \frac{1}{x}$, $x>0$, $r(x) = x$

Now,

$$I.F = e^{\int p(x)dx} = e^{\int \frac{1}{x}dx} = x \quad (x>0)$$

sol^n is

$$y(IF) = \int r(x)\, IF\, dx + C$$

$$yx = \int x \cdot x\, dx + C$$

$$= \int x^2 dx + C$$

$$yx = \frac{x^3}{3} + C$$

$$y(x) = \frac{x^2}{3} + \frac{C}{x} \quad \text{This is a G.S.}$$

Our IC gives

$$0 = y(1) = \frac{1}{3} 1^2 + C \cdot 1$$

$$\Rightarrow C = -\frac{1}{3}$$

Thus our Particular sol^n is

$$y(x) = \frac{1}{3}x^2 - \frac{1}{3x}$$

$$= \frac{1}{3}\left(x^2 - \frac{1}{x}\right)$$

[Here $-\frac{1}{3x}$ is the response to the initial data and $\frac{1}{3}x^2$ is response to the input x.]

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(3) Solve $ye^y dx = (y^3 + 2xe^y)dy$

sol^n

$$ye^y dx = (y^3+2xe^y)dy$$

$$\Rightarrow \frac{dx}{dy} = \frac{y^3+2xe^y}{ye^y}$$

$$\Rightarrow \frac{dx}{dy} = \frac{y^2}{e^y} + \frac{2x}{y}$$

$$\Rightarrow \frac{dx}{dy} - \frac{2}{y}x = \frac{y^2}{e^y}$$

which is linear in x

$$p = -\frac{2}{y}, \quad r(y) = \frac{y^2}{e^y}, \quad IF = \frac{1}{y^2}$$

sol^n is

$$x \cdot IF = \int r(y) \cdot I(F)\, dy + c$$

$$x \cdot \frac{1}{y^2} = \int \frac{\cancel{y^2}}{e^y} \cdot \frac{1}{\cancel{y^2}}\, dy + c$$

$$\frac{x}{y^2} = \int e^{-y}\, dy + c$$

$$\frac{x}{y^2} = -e^{-y} + c$$

$$\frac{x}{y^2} + e^{-y} = C$$ which geral sol^n

Equation Reducible to the linear form (Bernoulli Equation)

If $n=0$, what happens?

An equation of the form

$$\frac{dy}{dx} + Py = Qy^n \quad , \quad n \in \mathbb{R} \qquad — (1)$$

where P and Q are continuous functions of x only or constants. is known as Bernoulli's equation. Though not linear, it can be made linear.

Dividing both sides of (1) by y^n we have

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$$

Putting $z(x) = y^{1-n}$ then

$$\Rightarrow \frac{dz}{dx} = (1-n)\, y^{-n}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^n}{(1-n)}\frac{dz}{dx} \qquad — (2)$$

So, we have

$$y^{-n} \cdot \frac{y^n}{1-n}\frac{dz}{dx} + Pz = q$$

i.e $\frac{1}{(1-n)}\frac{dz}{dx} + p(x)\, z = q(x)$

i.e $\frac{dz}{dx} + (1-n)\,p(x)\, z = (1-n)\, q(x)$

This is a linear ODE and you know How solve it.

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Example.

① $x^2 dy + y(x+y)\,dx = 0, \quad x \neq 0 \neq y$

→ we have

$$x^2 dy + y(x+y)\,dx = 0$$

$$\frac{dy}{dx} = -\frac{(yx + y^2)}{x^2}$$

$$\frac{dy}{dx} = -\left[\frac{y}{x} + \frac{y^2}{x^2}\right]$$

$$\frac{dy}{dx} + \frac{y}{x} = -\frac{y^2}{x^2}$$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = -\frac{1}{x^2}$$

put $-\frac{1}{y} = z$ so that

$$\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$$

The given equation reduce to a linear differential eqn in z

$$\frac{dz}{dx} - \frac{z}{x} = -\frac{1}{x^2}$$

$$I.F = e^{-\int \frac{1}{x} dx} = e^{-\log x} = \frac{1}{x}.$$

Hence the soln is

$$z \cdot \frac{1}{x} = \int -\frac{1}{x^2} \cdot \frac{1}{x}\, dx + C$$

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$\Rightarrow \frac{z}{x} = \int -x^{-3} dx + c$

$\Rightarrow -\frac{1}{xy} = \frac{-x^{-2}}{-2} + c$

$\Rightarrow \frac{1}{xy} = -\frac{1}{2x^2} - c$

(2) $3\frac{dy}{dx} + 3\frac{y}{x} = 2x^4 y^4$

$\rightarrow 3\frac{dy}{dx} + 3\frac{y}{x} = 2x^4 y^4$

$\div$ by y^4

$\frac{3}{y^4}\frac{dy}{dx} + \frac{3}{xy^3} = 2x^4$

put $\frac{1}{y^3} = z$

$y^{-3} = z$

$-3y^{-4}\frac{dy}{dx} = \frac{dz}{dx}$

$3y^{-4}\frac{dy}{dx} = -\frac{dz}{dx}$

$-\frac{dz}{dx} + \frac{3}{x}z = 2x^4$

$P(x) = -\frac{3}{x}$, $Q(x) = -2x^4$

$I.F = e^{\int P(x)dx} = e^{\int -3/x \, dx} = e^{-3\log x} = \frac{1}{x^3}$

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$$z \cdot \frac{1}{x^3} = \int 2\frac{x^4}{x^3}\, dx + c$$

$$\frac{z}{x^3} = \int 2x\, dx + c$$

$$\frac{z}{x^3} = x^2 + c$$

$$\frac{z}{x^3} = x^2 + c$$

$$z = x^5 + cx^3$$

$$\frac{1}{y^3} = x^5 + cx^3.$$

Home work

(I) State the order and degree of the ODE. verify that the given function is sol^n (a, b, c are arbitary constent)

① $y'' + \pi^2 y = 0$, $y = a\cos \pi x + b\sin \pi x$

② $y''' = \cos x$, $y = -\sin x + ax^2 + bx + c$

(II) Find the 1st order differential equation of all circles touching the axis of y at the origin and center on the axis of x.

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(III) Does the ODE $(y')^2 = -1$ have a real solⁿ (Justify your ans)

(IV) Does the ODE $|y'| + |y| = 0$ have a general solⁿ?

(V) Solve the following ODE

1. $y' = x e^{x^2/2}$
2. $y' = y - x$
3. $yy' = (x-1) e^{-y^2}$, $y(0) = 1$
4. $-2xy \sin(x^2) + \cos(x^2)\, dy = 0$
5. $-\sin xy\,(y\,dx + x\,dy) = 0$, $y(1) = \pi$
6. $y' + y = y^2$, $y(0) = -1$