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Integration of Laplace Transform :

Let $f(t)$ be a function of t such that

(i) $f(t)$ is piecewise continuous on every finite interval in the range $t \geq 0$.

(ii) there exists constants $k > 0$, and $M > 0$ such that $|f(t)| \leq M e^{kt}$, $t \geq 0$

(iii) $\lim_{t \to 0^+} \left\{ \frac{f(t)}{t} \right\}$ exists finitely,

then

$$L\left\{ \frac{f(t)}{t} \right\} = \int_s^\infty F(u)\, du \text{. Hence } L^{-1}\left\{ \int_s^\infty F(u)\,du \right\} = \frac{f(t)}{t}$$

Proof:-

By definition

$$F(s) = L\{F(s)\} = \int_0^\infty e^{-st} f(t)\, dt \quad -(1)$$

Integrating (1) both side from s to ∞ we get.

$$\int_s^\infty F(u)du = \int_s^\infty \left(\int_0^\infty e^{-ut} f(t)\, dt \right) du.$$

$$= \int_0^\infty \left(\int_s^\infty e^{-ut} f(t)\, du \right) dt$$

---- [By change of order of integration]

$$= \int_0^\infty f(t) \left[\int_s^\infty e^{-ut}\, du \right] dt$$

$$= \int_0^\infty f(t) \left(\frac{e^{-ut}}{-u} \right)_s^\infty dt.$$

$$\int_s^\infty F(u)\,du = \int_0^\infty e^{-st}\left[\frac{f(t)}{t}\right]dt \quad , \quad s > k$$

$$= L\left\{\frac{f(t)}{t}\right\}, \text{ by definition L.T.}$$

Hence

$$\boxed{\int_s^\infty F(u)\,du = L\left\{\frac{f(t)}{t}\right\}}$$

Thus, we have

$$\boxed{L^{-1}\left\{\int_s^\infty F(u)\,du\right\} = \frac{f(t)}{t} = \frac{1}{t}\,L^{-1}\{F(s)\}}$$

Corollary If $L\{f(t)\} = F(s)$, then $\int_0^\infty \frac{f(t)}{t}\,dt = \int_0^\infty F(s)\,ds$ provided that the integral converges.

Proof:- From above th^m,

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du$$

$$\Rightarrow \int_0^\infty e^{-st}\frac{f(t)}{t}\,dt = \int_s^\infty F(u)\,du.$$

Taking limit of both side as $s \to 0^+$ and assuming that the integral converges, we get

$$\int_0^\infty \frac{f(t)}{t}\,dt = \int_0^\infty F(u)\,du.$$

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Example (1) show that $\int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}$.

Sol^n Use above corollary to prove that

Let $f(t) = \sin t$ so that $F(s) = L\{\sin t\} = \frac{1}{s^2+1}$.

But by corollary

$$\int_0^\infty \frac{f(t)}{t}\, dt = \int_0^\infty F(u)\, du.$$

$$\therefore \int_0^\infty \frac{\sin t}{t}\, dt = \int_0^\infty \frac{1}{u^2+1}\, du$$

$$= \left[\tan^{-1} u\right]_0^\infty$$

$$= \frac{\pi}{2}.$$

Example (2) Prove that

$$L\left\{\frac{\cos at - \cos bt}{t}\right\} = \frac{1}{2} \log \frac{s^2+b^2}{s^2+a^2}.$$

Sol^n :- $L\{\cos at - \cos bt\} = L\{\cos at\} - L\{\cos bt\}$

$$= \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2} \rightarrow F(s) \text{ say. } (*)$$

By Integration of transform th^m,

$$\int_s^\infty F(u)\, du = L\left\{\frac{f(t)}{t}\right\}$$

$$\therefore L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty F(u)\,du$$

$$= \int_s^\infty \left(\frac{u}{u^2+a^2} - \frac{u}{u^2+b^2}\right) du.$$

-- by (*)

$$= \frac{1}{2}\int_s^\infty \left(\frac{2u}{u^2+a^2} - \frac{2u}{u^2+b^2}\right) du$$

$$= \frac{1}{2}\left[\log(u^2+a^2) - \log(u^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{u^2+a^2}{u^2+b^2}\right)\right]_{u=s}^{\infty}$$

$$= \frac{1}{2}\left\{\lim_{u\to\infty}\left[\log\left(\frac{u^2+a^2}{u^2+b^2}\right)\right] - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}$$

$$= \frac{1}{2}\left\{\lim_{u\to\infty}\left[\log\left(\frac{1+a^2/u^2}{1+b^2/u^2}\right)\right] + \log\left(\frac{s^2+b^2}{s^2+a^2}\right)\right\}$$

$$= 0 + \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

$$= \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right).$$

$= R.H.S.$

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Example (3) $L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = ?$

Sol^n :-

$$L\{e^{-at}-e^{-bt}\} = L\{e^{-at}\} - L\{e^{-bt}\}$$

$$= \frac{1}{s+a} - \frac{1}{s+b} \rightarrow F(s) \text{ (say) } (A)$$

By Integration of L.T th^m we get

$$\therefore L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du$$

$$L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \int_s^\infty \left[\frac{1}{u+a} - \frac{1}{u+b}\right] du$$

$$= \left[\log(u+a) - \log(u+b)\right]_{u=s}^{\infty}$$

$$= \left[\log\left(\frac{u+a}{u+b}\right)\right]_{u=s}^{\infty}$$

$$= \lim_{u\to\infty}\left[\log\left(\frac{u+a}{u+b}\right)\right] - \log\left(\frac{s+a}{s+b}\right)$$

$$= \lim_{u\to\infty}\left[\log\left(\frac{1+a/u}{1+b/u}\right)\right] + \log\left(\frac{s+b}{s+a}\right)$$

$$= 0 + \log\left(\frac{s+b}{s+a}\right) = \log\left(\frac{s+b}{s+a}\right)$$

Example (4) Prove that $\bar{L}^{1}\left\{\int_{s}^{\infty}\frac{1}{u(u+1)}du\right\} = \frac{1-\bar{e}^{t}}{t}$.

Sol^n

As you know

$$L\left\{\frac{f(t)}{t}\right\} = \int_{s}^{\infty} F(u)\, du, \text{ hence}$$

$$\bar{L}^{1}\left\{\int_{s}^{\infty} F(u)\,du\right\} = \frac{1}{t} f(t) = \frac{1}{t}\bar{L}^{1}\{F(s)\} \quad ---(1)$$

Here

$$F(s) = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

$$\bar{L}^{1}\{F(s)\} = \bar{L}^{1}\left\{\frac{1}{s}\right\} - \bar{L}^{1}\left\{\frac{1}{s+1}\right\}$$

$$= 1 - \bar{e}^{t} \quad ----(2)$$

Thus

$$\bar{L}^{1}\left\{\int_{s}^{\infty}\frac{1}{u(u+1)}du\right\} = \frac{1}{t}(1-\bar{e}^{t}) \quad ---[\text{by } (1) \text{ and } (2)].$$

Homework

(1) $\bar{L}^{1}\left\{\frac{s}{(s^2+16)^2}\right\}$ (2) $L\left\{\frac{\sin 2t}{t}\right\}$

(3) Prove that $L\left\{\frac{\sin^2 t}{t}\right\} = \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$

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Unit Step Function :-

It is also called as Heaviside function. It is defined as

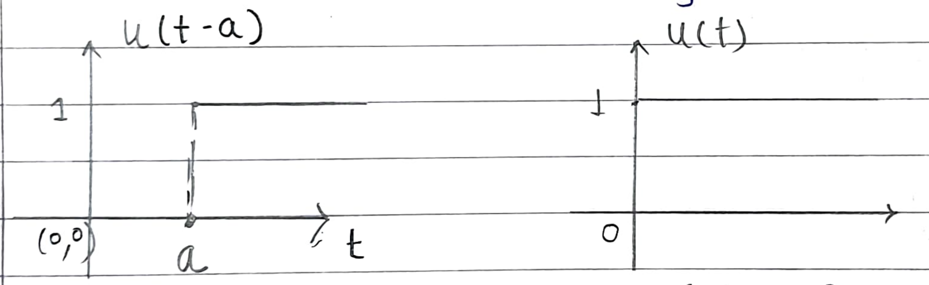
$$u(t-a)=\begin{cases}0, & \text{if } t<a \\ 1, & \text{if } t>a\end{cases} \quad (a\geq 0)$$

(see fig A)

(shifted a unit to the right hand side).

- Here $u(t-a)=1$, at $t=a$ (where we leave it undefined).
- The special case $u(t)$, which has it's jump at zero.

i.e $u(t)=\begin{cases}0 & \text{if } t<0 \\ 1 & \text{if } t>0\end{cases}$ (see fig B)



(fig A) (fig B)

- The unit step function is a typical "engineering function" made to measure for engineering aplications, which often involve function (mecanical or electrical driving forces) that are either 'off' or 'on'. Multiplying $f(t)$ with $u(t-a)$ we can produce all sorts of effects.
- The unit step function are very helpfull when solving ODEs with right sides function i.e $r(t)$ is complicated function.

- If you have following type of function

$$f(t) = \begin{cases} f_1(t), & 0 < t < a_1 \\ f_2(t), & a_1 < t < a_2 \\ f_3(t), & t > a_2. \end{cases}$$

Then you can write into unit step function as follow.

$$f(t) = f_1(t)\left[u(t) - u(t-a_1)\right] + f_2(t)\left[u(t-a_1) - u(t-a_2)\right] + f_3(t)\left[u(t-a_2)\right].$$

So using unit step function, you can write piecewise continuous function into single line.

- If

$$f(t) = \begin{cases} f_1(t), & 0 < t < a_1 \\ f_2(t), & a_1 < t < a_2 \\ \vdots & \\ f_{n-1}(t), & a_{n-2} < t < a_{n-1} \\ f_n(t), & t > a_{n-1} \end{cases}$$

then using unit step funct it express as.

$$f(t) = f_1(t)\left[u(t) - u(t-a_1)\right] + f_2(t)\left[u(t-a_1) - u(t-a_2)\right] + \cdots + f_n(t)\left[u(t-a_{n-1})\right]$$

That is in single expression.

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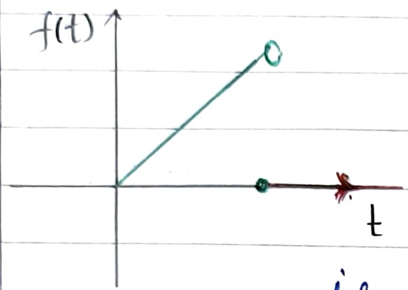
Examples (1) If $f(t) = \begin{cases} t, & \text{for } 0 \le t < 4 \\ 0 & \text{otherwise} \end{cases}$

then f in terms of Heaviside function.

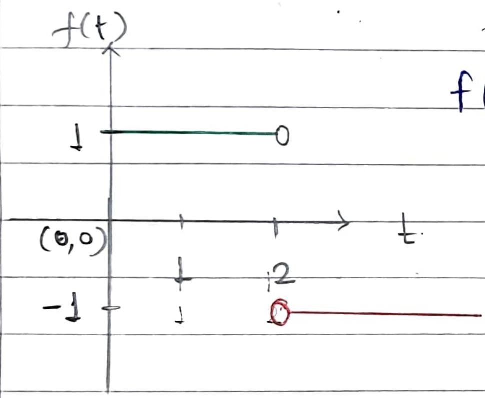


At $t = 4$, $f(t) = 0$, and

$$f(t) = t\left[u(t) - u(t-4)\right], \quad t \neq 4$$

ie $$f(t) = t \cdot u(t) - t\, u(t-4).$$

(2) $f(t) = \begin{cases} 1 & t < 2 \\ -1 & t > 2. \end{cases}$



$$f(t) = 1 \cdot \left[u(t) - u(t-2)\right] + (-1)\left[u(t-2)\right]$$

$$= u(t) - 2\,u(t-2)$$

$$= 1 - 2\,u(t-2)$$

$$\left(u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} \right).$$

(3) $f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t+1 & t > 1. \end{cases}$

$$f(t) = 0\left[u(t) - u(t-1)\right] + (t+1)\,u(t+1).$$

$$= (t+1)\,u(t+1).$$

Homework: Express in terms of Heaviside's fu^n

(a) $f(t) = \begin{cases} \cos t & 0 < t < \pi \\ \sin t & t > \pi \end{cases}$

sketch it

(b) $f(t) = \begin{cases} \cos t & 0 < t < \pi \\ \cos 2t & \pi < t < 2\pi \\ \cos 3t & t > 2\pi. \end{cases}$

Laplace Transform of unit step function $u(t-a)$

By definition of Laplace transform,

$$L\{u(t-a)\} = \int_0^\infty e^{-st}\, u(t-a)\, dt$$

$$= \int_0^a e^{-st} u(t-a)\, dt + \int_a^\infty e^{-st} u(t-a)\, dt$$

$$= \int_0^a e^{-st}\, 0\, dt + \int_0^\infty e^{-st}(1)\, dt \quad \left(\because u(t-a) = \begin{cases} 0 & t<a \\ 1, & t>a \end{cases}\right)$$

$$= 0 + \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= 0 + \lim_{s\to\infty}\left[\frac{e^{-st}}{-s}\right] + \frac{e^{-as}}{s}$$

$$= \frac{e^{-as}}{s}, \quad s>0.$$

$$\boxed{L\{u(t-a)\} = \frac{e^{-as}}{s}, \quad s>0}$$

In particular

$$L\{u(t)\} = \frac{e^{-(0)s}}{s} = \frac{1}{s}, \quad s>0$$

$$\boxed{L\{u(t)\} = \frac{1}{s}, \quad s>0}$$

[use $u(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$ and defn of LT find out $L\{u(t)\}$].