Non-Exact ODEs

Are exists! when we say that ODE is an non-exact ODE? Think OH!

Recall

① A first order ODE $M(x,y)dx + N(x,y)dy = 0$ — ①
is called an exact ODE if $\exists$ differentiable function $u(x,y)$ such that

$$du = M(x,y)\,dx + N(x,y)\,dy. \quad — (2)$$

Further from (1) and (2)

$$du = 0$$

On integration

$$u(x,y) = C \longrightarrow \text{implict sol}^n \text{ of } (1).$$

② Let $M(x,y)$ and $N(x,y)$ be the continuous functions in x and y.
The necessary and sufficient condition for the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

to be exact is $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

i.e $\underset{\text{exact}}{M(x,y)\,dx + N(x,y)\,dy = 0} \Leftrightarrow \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

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Example $-y\,dx + x\,dy = 0$ (1) (check it exact or Not)

→ Here $M = -y$ and $N = x$

$\Rightarrow \frac{\partial M}{\partial y} = M_y = -1$ & $\frac{\partial N}{\partial x} = N_x = 1$

Thus we have

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

i.e. $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \neq 0$

Thus the given ODE is a non-exact ODE.

How to solve it? Think on!

But if you multiply (1) by $\frac{1}{x^2}$ then we get

$$\frac{-y\,dx + x\,dy}{x^2} = 0$$

Reduce to exact ODE, check it

$\Rightarrow d\left(\frac{y}{x}\right) = 0$

on integration

$\frac{y}{x} = c$

What is a integrating factor (IF)?

Note that $\frac{1}{x^2}$ is integrating factor.

Definition [Integration Factor (I.F)]

It is a function $F(x,y)$ and when we multiply this function to non-exact ODE, it convert to an exact ODE.

Such a function $F(x,y)$ is called an <u>integration factor</u>.

Alwalys exists?

How to find IF?

If it is exists, then is it unique? Think ON!

$$\boxed{\text{Non Exact ODE}} \times \boxed{I.F} \longrightarrow \boxed{\text{Exact ODE}}$$

$\hookrightarrow$ say $F(x,y)$

<u>Theorem 1</u>:- If $\dfrac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}{N}$ $\left(\text{i.e } \dfrac{M_y - N_x}{N}\right)$

is a function of x alone say $\phi(x)$, then integrating factor for a non-exact ODE

$$M(x,y)\,dx + N(x,y)\,dy = 0 \text{ is}$$

$$e^{\int \phi(x)\,dx}.$$

Note that $\phi(x) = \dfrac{1}{N}(M_y - N_x)$

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Proof

Let $F(x)$ be some integrating factor of a non-exact ODE

$$M(x,y)\,dx + N(x,y)\,dy = 0.$$

By defn of integrating factor, this ODE become an exact ODE

i.e $F(x)M(x,y)\,dx + F(x)N(x,y)\,dy = 0$

is an exact ODE..

So we have

$$\frac{\partial}{\partial y}\left[F(x)M(x,y)\right] = \frac{\partial}{\partial x}\left[F(x)N(x,y)\right]$$

$$\Rightarrow F(x)\frac{\partial M}{\partial y} + M(x,y)\frac{d}{dy}\left[F(x)\right] = F(x)\frac{\partial N}{\partial x} + N(x,y)\frac{dF}{dx}$$

--- by product rule

$$\Rightarrow F(x)\frac{\partial M}{\partial y} = F(x)\frac{\partial N}{\partial x} + N(x,y)\frac{dF}{dx}$$

$\left(\because \frac{d}{dy}\left[F(x)\right] = 0\right.$, because F is only fun of $\left.x\right)$

$$\Rightarrow N(x,y)\frac{dF}{dx} = F(x)\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

$$\Rightarrow \frac{1}{F(x)}\,dF = \frac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx}{N(x,y)}$$

when this ODE is solvable?

$$\frac{dF}{F} = \left(\frac{M_y - N_x}{N}\right) dx$$

$$= \phi(x)\, dx \quad \left(\because \text{ given } \frac{M_y - N_x}{N} \text{ is only fu}^n \text{ of } x \text{ say } \phi(x)\right)$$

Integrating both sides

$$\ln F(x) = \int \phi(x)\, dx + c$$

$$F(x) = e^{\int \phi(x)\, dx} \quad \text{assuming } c = 1.$$

$$\boxed{I.F = F(x) = e^{\int \phi(x) dx}}$$

Theorem 2 :-

If $\dfrac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}{M}$ is a function of y alone say $\phi(y)$ then integrating factor for a non-exact ODE

$M(x,y)\,dx + N(x,y)\,dy = 0$ is

$e^{\int \phi(y) dy}$

Note that $\phi(y) = \frac{1}{M}(N_x - M_y)$.

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Conclusion of Th^m (1) and (2).

When we solve non-exact ODE, a integrating factor plays vital role.
There are two ways for finding a integrating factor:

1st way :-
First calculate $\frac{1}{N}(M_y - N_x)$
and if it is a only function of x and constant say $\phi(x)$
i.e $\phi(x) = \frac{1}{N}(M_y - N_x)$
Then integrating factor is

$$F(x) = e^{\int \phi(x) dx}.$$

2nd way
First calculate $\frac{1}{M}(N_x - M_y)$
and if it is a only function of y and constant say $\phi(y)$
i.e $\phi(y) = \frac{1}{M}(N_x - M_y)$
Then integrating factor is

$$F(y) = e^{\int \phi(y) dy}$$

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Imp Remarks on Integrating factors:-

① Integrating factor for non-exact ODE (1^{st} order) can never be a zero and constant.

→ As we have non-exact ODE, then

$$M_y \neq N_x$$

$$\Rightarrow M_y - N_x \neq 0.$$

There are 2 ways of finding integrating factor.

If $\frac{1}{N}(M_y - N_x) = \phi(x)$

then $\underline{IF = F(x) = e^{\int \phi(x)\,dx}}$

and

If $\frac{1}{M}(N_x - M_y) = \phi(y)$ then

$$\underline{IF = F(y) = e^{\int \phi(y)\,dy}}$$

So it is never be a constant because exponential fu^n is there. (In both)

Note that $\phi(x)$ and $\phi(y)$ never zero.

{Why?}

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Suppose if you assume $\phi(x) = 0$
then we have

$$\frac{1}{N}(M_y - N_x) = 0$$

$$\Rightarrow M_y - N_x = 0$$

$$\Rightarrow M_y = N_x$$

which is a contradiction since our ODE is a non-exact.

Similarly we can prove for $\phi(y)$.

(2) Integrating factor for an exact ODE is a constant
(In that case it is 1)

→ As we have an exact ODE
i.e $M_y = N_x$

$$\Rightarrow M_y - N_x = 0$$

$$\therefore IF = e^{\int \phi(x)\,dx} = e^{\int 0\,dx} = e^0 = 1$$

and

$$IF = e^{\int \phi(y)\,dy} = e^{\int 0\,dy} = e^0 = 1.$$

(3) Integrating factor need not be be a unique! (Give example).

for eg. $-y\,dx + x\,dy = 0$, $I.F = \frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{xy}, \frac{1}{(x^2+y^2)}$

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Example (1)

Solve $(2x\log x - xy)dy + 2y\,dx = 0$ — (1)

$\rightarrow$ Here

$M = 2y \quad , \quad N = 2x\log x - xy$

$\Rightarrow \frac{\partial M}{\partial y} = 2 \quad , \quad \frac{\partial N}{\partial x} = 2\log x + 2\frac{\cancel{x}}{\cancel{x}} - y$

$= 2(1+\log x) - y$

So we have $M_y \neq N_x$, thus given ODE is non-exact ODE.

Here

$\frac{1}{N}(M_y - N_x) = \frac{1}{2x\log x - xy}\left[\cancel{2} - \cancel{2} - 2\log x + y\right]$

$= \frac{1}{x[\cancel{2\log x - y}]} - [\cancel{2\log x - y}]$

$= \frac{-1}{x}$ (only fun of x) say $\phi(x)$

$\therefore IF = F(x) = e^{\int \phi(x)dx} = e^{\int -\frac{1}{x}dx} = \frac{1}{x}$

Now multiplying $\frac{1}{x}$ to given (1) we get

$\frac{2y}{x}dx + (2\log x - y)dy = 0$ — This is an exact ODE

$\frac{2y}{x}dx + 2\log x\,dy - y\,dy = 0$

$d(2y\log x) - y\,dy = 0$ — By observation

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On integrating

$2y\log x - \frac{1}{2}y^2 = c$

This your General sol^n of given non-exact ODE.

(Check it $u(x,y) = 2y\log x - \frac{1}{2}y^2$

$du = \frac{2y}{x}dx + (2\log x - y)\,dy$

$\rightarrow$ This your QEE)

(2) $e^x \cos y\, dx - e^x \sin y\, dx = 0$

$\rightarrow$ Here $M = e^x \cos y$, $N = -e^x \sin y$

$M_y = -e^x \sin y$ $\quad N_x = -e^x \sin y$

$\therefore$ It is an exact ODE

$d(e^x \cos y) = 0$

on integration

$e^x \cos y = c$

$\rightarrow$ {Note. $\frac{\partial}{\partial x}(e^x \cos y)dx + \frac{\partial}{\partial y}(e^x \cos y)dy$

$= e^x \cos y\, dx - e^x \sin y\, dy$

our ANS is correct}

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Example

$$\left(e^{x+y}+ye^{y}\right)dx+\left(xe^{y}-1\right)dy=0,\ y(0)=-1 \quad -(1)$$

→ Here

$$M=e^{x+y}+ye^{y} \Rightarrow M_y=e^{x+y}+e^{y}+ye^{y}$$

$$N=xe^{y}-1 \Rightarrow N_x=e^{y}$$

We have $M_y \neq N_x \Rightarrow$ Non exact ODE

→ Here.

$$\frac{1}{M}(N_x-M_y)=\frac{1}{e^{x+y}+ye^{y}}\left(\cancel{e^{y}}-e^{x+y}-\cancel{e^{y}}-ye^{y}\right)$$

$$=\frac{(-1)}{\cancel{e^{x+y}+ye^{y}}}\cancel{\left(e^{x+y}+ye^{y}\right)}$$

$$=-1$$

→ $IF=F(y)=e^{\int(-1)dy}=e^{-y}$

Multiplying e^{-y} to (1)

$$(e^{x}+y)dx+(x-e^{-y})dy=0$$

$$\Rightarrow e^{x}dx+ydx+xdy-e^{-y}dy=0$$

$$\Rightarrow e^{x}dx+d(xy)-e^{-y}dy=0$$

$$\Rightarrow \int e^{x}dx+d(xy)-e^{-y}dy)=0$$

on integration

$u(x,y) = e^x + xy + e^{-y} = c$

This is a general sol^n → {correct or Not check it}

Now IC ⇒ $y(0) = -1$

$u(0,-1) = 1 + 0 + e$

$= 1 + e$

So our $c = 1 + e$

$e^x + xy + e^{-y} = 1 + e$

This is a particular sol^n of given IVP.

Note

DE	IF	Exact
1) $x\,dy + y\,dx = 0$ (exact)	1	$d(xy) = 0$ on int you ANS
	$\frac{1}{xy}$	$\frac{1}{y}dy + \frac{1}{x}dx = d(\ln xy)$ —11—
2) $x\,dx + y\,dy = 0$ (exact)	$\frac{1}{2}$	$d(x^2+y^2) = 0$
	$\frac{1}{x^2+y^2}$	$\frac{1}{2}\,d\left[\ln(x^2+y^2)\right]$
	$\frac{1}{(x^2+y^2)^n}$, $n \neq 1$	$\frac{d\left[(x^2+y^2)^{1-n}\right]}{2(1-n)}$
3) $-y\,dx + x\,dy$ (nonexact)	$\frac{1}{x^2}$	$d\left(\frac{y}{x}\right)$
	$\frac{1}{y^2}$	$-\,d\left(\frac{x}{y}\right)$

$\frac{1}{xy}$ $\qquad d\left[\ln\left(\frac{y}{x}\right)\right]$

$\frac{1}{x^2+y^2}$ $\qquad d\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$

Linear Differential Equations :-

A differential equation is said to be linear if

(i) degree of $\frac{dy}{dx}$ is one

(ii) there should not be product of dependent variable and it's derivative

(iii) there should not be dependent variable attached with square, trigo, exponetial, square root.

The general form of a linear differential equation of the first order is

$$\frac{dy}{dx} + p(x)\,y = r(x) \quad , \quad x \in I \subseteq \mathbb{R} \qquad \text{--- (i)}$$

Where $p(x)$ and $r(x)$ are only functions of x or may be constant. Moreover $p(x)$ and $r(x)$ are continuous function of x on some interval I

- This DE is called linear in y.
- Interchange the role of depent. vari. (y) and indept var (x) we have

$$\frac{dx}{dy} + p(y)\,x = r(y) \quad , \quad y \in I \subseteq \mathbb{R}$$

This DE is called linear in x.

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For examples:-

1. $y' + 2xy = \cos x$ — linear ODE
2. $x' + \sqrt{x} = 100$ — not a linear ODE
3. $y' + x\sin y = 1$ — not a linear ODE
4. $y' - \frac{1}{x^2}y = 3x$ — linear ODE
5. $x' - \frac{1}{y}x = e^{x}$ — not a linear ODE

Observation 1 (See eq^n (1))

If $r(x) = 0 \ \forall\ x \in I$ then (1) become

$$y' + p(x)y = 0 \quad \text{— (2)}$$

This ODE is called Homogeneous linear 1st order ODE.

How to solve (2) ?

By using variable seperable method

$$\frac{dy}{y} = -p(x)dx$$

$$\ln|y| = -\int p(x)dx + \ln c$$

$$\boxed{y(x) = c\, e^{-\int p(x)dx}}$$

This is a general sol^n of (2).

If we choose $c = 0 \Rightarrow y(x) = 0 \ \forall x \in I$, is called the Trivial sol^n of (2)

$\boxed{\text{Observation 2}} \leftarrow$ (See eq^n (1))

If $r(x) \neq 0$ for some $x \in I$ then we have

$$y' + p(x)y = r(x) \quad \text{—— (3)}$$

This ODE is called non-homogeneous linear 1^{st} order ODE

How to solve (3) ?

From (3)

$$\frac{dy}{dx} = r - py$$

$$\Rightarrow (r - py)dx - dy = 0 \quad \text{—— (4)}$$

Here $M = r - py \Rightarrow M_y = -p$

$N = -1 \Rightarrow N_x = 0$

$\therefore M_y \neq N_x$

$\therefore$ It is a non-exact ODE

Here $\phi(x) = \frac{1}{N}(M_y - N_x)$

$= \frac{1}{-1}(-p - 0) = p(x)$

$$\boxed{I.F = F = e^{\int p(x)dx}} \longrightarrow \text{Need to remember}$$

Multiply $e^{\int p(x)dx}$ to eq^n (4) we get

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$$e^{\int p(x)dx}(\gamma - py)dx - e^{\int p(x)dx}dy = 0$$

$$\gamma e^{\int p(x)dx}dx - py\, e^{\int p(x)dx}dx - e^{\int p(x)dx}dy = 0$$

$$\gamma e^{\int p(x)dx}dx - \left(e^{\int p(x)dx}\cdot y\right)' = 0$$

$$\gamma e^{\int p(x)dx}dx = \left(e^{\int p(x)dx}\cdot y\right)'$$

On integration

$$y e^{\int p(x)dx} = \int e^{\int p(x)dx}\gamma(x)\,dx + c$$

$$\boxed{y = e^{-\int p(x)dx}\left[\int e^{\int p(x)dx}\gamma(x)\,dx + c\right]}$$

This is general solⁿ of (4).

$$\boxed{y = \frac{1}{I.F}\left[\int I.F\ \gamma(x)\,dx + c\right]}$$

Need to remember.

$$\boxed{y\cdot(IF) = \int IF\ \gamma(x)\,dx + c}$$

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Note:

Linear ODEs that can be transformed to linear form are models of various phenomenaa, for instance in physcis, biology, population and ecology.

A first-order ODE is said to be a linear (in y) if it can be written

$$y' + p(x)\, y = r(x), \quad x \in I = (a,b) : a, b \in \mathbb{R} \qquad —(1)$$

where p and r are any given functions of x. (continuous functions)

If in an application the independent variable is time, we write t instead of x.

For example

$$y' \cos x + y \sin x = x$$ is a linear

ODE and it's standard form is

$$y' + \tan x\, y = x \sec x.$$

The function $r(x)$ on the right may be a force, and the sol^n $y(x)$ a displacement in a motion or an electrical current or some other physical quantity.

In engineering, $r(x)$ is frequently

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called the input, and $y(x)$ is called the output or the response to the input (and is given to the initial condition)

Non-homogeneous linear 1st order ODE:

It's form is

$$y' + p(x)y = r(x),$$

where $r(x) \neq 0$ for some x in I.

It's general solⁿ is

$$y(x) = e^{-\int p(x)dx}\left[\int e^{\int p(x)dx} \cdot r(x)\, dx + C\right]$$

The quantity depending on a give $\underline{\underline{IC}}$ is C.

i.e

$$y(x) = e^{-\int p(x)dx}\int e^{\int p(x)dx} \cdot r(x)\, dx + C e^{-\int p(x)dx}.$$

Meaning of this solⁿ

Total Output = Response to the input $r(x)$ + Response to the initial Data

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Working Rule (Useful for problem solving)

Step 1 - Convert the given equation to the standard form of linear differential equation.

$$\frac{dy}{dx} + p(x)\,y = r(x)$$

Step 2 - Find the integrating factor

$$I \cdot F = e^{\int p(x)dx}$$

Step 3 - Then the sol^n is

$$y\,(I \cdot F) = \int r(x)(I \cdot F)\,dx + C.$$