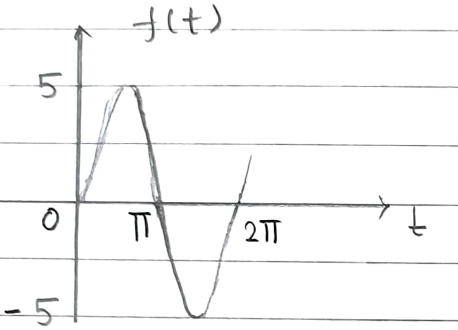
Name: Class & Div. : Page No. :

Subject : Topic : Date :

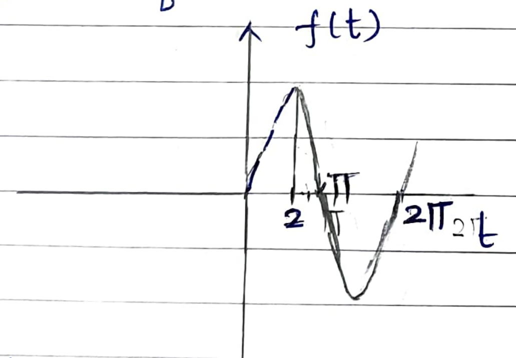
Unit Step function :- (Via Example & it's graph)

(I) Let $f(t) = 5\sin t$, given function



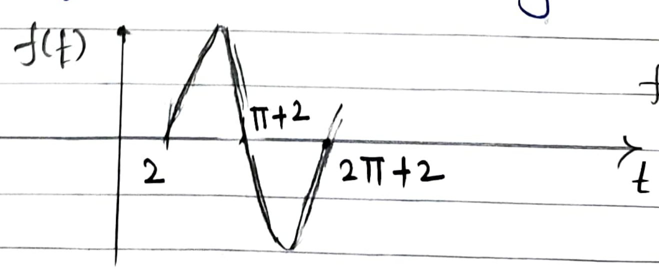
$f(t) = 5\sin t$

(II) When the function $f(t)$ is multiplied by unit step function $u(t-2)$ the resultant function $f(t)\,u(t-2)$ ie $(5\sin t \cdot u(t-2))$ will be part of the function $f(t)$ on the right of $t=2$ and the part of the function $f(t)$ on the left of $t=2$ is switched off (cut off)



$$f(t)\,u(t-2) = \begin{cases} 0 & t < 2 \\ f(t) & t > 2 \end{cases}$$

(III) When this displaced function $f(t-2)$ is multiplied by $u(t-2)$, the resultant function will shifted to the right by 2 units.



$$f(t-2)\,u(t-2) = \begin{cases} 0 & t < 2 \\ f(t-2) & t > 2 \end{cases}$$

Time Shifting (t-Shifing); Second Shifting Th^m

If $L\{f(t)\} = F(s)$, then $L\{f(t-a)u(t-a)\} = e^{-as} L\{f(t)\}$.

Here
$$f(t-a)u(t-a) = \begin{cases} 0 & \text{if } t<a \\ f(t-a) & \text{if } t>a \end{cases}, \text{ Shifted function } \quad \text{---- (1)}$$

Hence $L^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$.

Proof :-

$L\{f(t-a)u(t-a)\}$

$$= \int_0^\infty e^{-st}[f(t-a)u(t-a)]\,dt$$

$$= \int_0^a e^{-st} f(t-a)\cdot(0)\,dt + \int_a^\infty e^{-st} f(t-a)\,dt \quad \text{--- by (1)}$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

let $t-a=q \Rightarrow dq = dt$.

t	a	∞
q	0	∞

$$\therefore L\{f(t-a)u(t-a)\} = \int_0^\infty e^{-s(q+a)} f(q)\,dq$$

$$= \int_0^\infty e^{-sq} e^{-as} f(q)\,dq$$

$$= e^{-as}\int_0^\infty e^{-sq} f(q)\,dq = e^{-as} L\{F(t)\}$$

$$= e^{-as} F(s).$$

$$\boxed{\begin{aligned} L\{f(t-a)\,u(t-a)\} &= e^{-as}\,L\{f(t)\} = e^{-as}F(s) \quad (*) \\ L^{-1}\{e^{-as}F(s)\} &= f(t-a)\,u(t-a) \quad (\#) \end{aligned}}$$

Corollary :- If $a=0$, then from above result $(*)$ become

$$L\{f(t)\,u(t)\} = L\{f(t)\} = F(s).$$

- Laplace Transform of $f(t)\,u(t-a)$.

By def^n of Laplace transform

$$L\{f(t)\,u(t-a)\} = \int_0^\infty e^{-st} f(t)\,u(t-a)\,dt.$$

$$= \int_0^a e^{-st}(0)\,dt + \int_a^\infty e^{-st} f(t)\,dt$$

$$\left(\because f(t)\,u(t-a) = \begin{cases} 0 & t<a \\ f(t) & t>a \end{cases}\right)$$

$$= \int_a^\infty e^{-st} f(t)\,dt.$$

Putting $t-a = x$ $\therefore dx = dt$ and

t	a	∞
u	0	∞

$$= \int_0^\infty e^{-s(x+a)} f(x+a)\,dx$$

$$= e^{-as}\int_0^\infty e^{-sx} f(x+a)\,dx$$

$$= e^{-as}\,L\{f(x+a)\}.$$

Name: ______ Class & Div. : ______ Page No. : ______

Subject : ______ Topic : ______ Date : ______

Express the following function in terms of Heaviside's function and hence find their Laplace transform.

(1)
$$f(t) = \begin{cases} t^2 & 0 < t < 1 \\ 4t & t > 1. \end{cases}$$

→

$$f(t) = t^2 [u(t) - u(t-1)] + 4t\, u(t-1)$$

$$\Rightarrow f(t) = t^2 u(t) + (4t - t^2)\, u(t-1).$$

$$\therefore L\{f(t)\} = L\{t^2 u(t)\} + L\{(4t - t^2)\, u(t-1)\}$$

--- (∵ LT is linear)

$$= L\{t^2\} + L\{\underbrace{(4t - t^2)}_{f}\, u(t-1)\}$$

$(L\{f(t)\, u(t)\} = L\{f(t)\})$

$$= L\{t^2\} + e^{-as} L\{4(t+1) - (t+1)^2\}$$

--- $(L\{f(t)\, u(t-a)\} = e^{-as} L\{f(t+a)\})$ --- by SST.

$$= \frac{2}{s^3} + e^{-as} L\{3 + 2t - t^2\}$$

$$= \frac{2}{s^2} + e^{-as}\left(\frac{3}{s} + \frac{2}{s^2} - \frac{2}{s^3}\right).$$

(2) $L\{ (t-4)\, u(t-4)\} = ?$

Here $f(t-4) = (t-4)$ so that $f(t) = t$

$\therefore$ By SST, $L\{f(t-a)\, u(t-a)\} = e^{-as} L\{f(t)\}$.

Thus
$$L\{(t-4)\, u(t-4)\} = e^{-4s} L\{t\}$$
$$= \frac{e^{-4s}}{s^2}$$

(3) $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \cos 2t, & \pi < t < 2\pi \\ \cos 3t, & t > 2\pi . \end{cases}$

$\rightarrow$ $f(t) = \cos t\,[u(t) - u(t-\pi)] + \cos 2t\,[u(t-\pi) - u(t-2\pi)]$
$+ \cos 3t\; u(t-2\pi)$.

$= \cos t\; u(t) - (\cos 2t - \cos t)\, u(t-\pi)$
$+ (\cos 3t - \cos 2t)\; u(t-2\pi)$.

$\therefore L\{f(t)\} = L\{\cos t\; u(t)\} - L\{(\cos 2t - \cos t)\; u(t-\pi)\}$
$- L\{(\cos 3t - \cos 2t)\, u(t-2\pi)\}$
$= L\{\cos t\} - e^{-\pi s} L\{\cos 2(t+\pi) - \cos(t+\pi)\}$
$+ e^{-\pi s} L\{\cos 3(t+2\pi) - \cos 2(t+2\pi)\}$.

---- by SST.

$= L\{\cos t\} - e^{-\pi s} L\{\cos(2t+2\pi) - \cos(t+\pi)\}$
$+ e^{-2\pi s} L\{\cos(3t+6\pi) - \cos(2t+2\pi)\}$.

Name: Class & Div. : Page No. :

Subject : Topic : Date :

$= L\{\cos t\} - e^{-\pi s} L\{\cos 2t + \cos t\} + e^{-2\pi s} L\{\cos 3t - \cos 2t\}$

$(\because \cos(\pi+\theta) = -\cos\theta$
$\cos(2\pi+\theta) = +\cos\theta)$

$$= \frac{s}{s^2+1} - e^{-\pi s}\left(\frac{s}{s^2+4} + \frac{s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{s}{s^2+9} - \frac{s}{s^2+4}\right)$$

(4) $f(t) = \begin{cases} \sin\left(t-\frac{\pi}{3}\right), & t > \frac{\pi}{3} \\ 0, & t < \frac{\pi}{3}. \end{cases}$

$f(t) = \sin\left(t-\frac{\pi}{3}\right) \cdot u\left(t-\frac{\pi}{3}\right).$

Here $f(t) = \sin t$, $a = \frac{\pi}{3}$.

By SST, $L\{f(t-a)\,u(t-a)\} = e^{-as} L\{f(t)\}$.

$\therefore L\{f(t)\} = L\left\{\sin\left(t-\frac{\pi}{3}\right) u\left(t-\frac{\pi}{3}\right)\right\}$

$= e^{-\frac{\pi}{3}s} L\{\sin t\}$

$= e^{-\pi/3\, s} \frac{1}{s^2+1}, \quad s > 0.$

(5) Find I.L.T for :

(a) $L^{-1}\left\{\frac{e^{-4s}}{(s-3)^4}\right\}$

Recall

$L\{f(t-a)\,u(t-a)\} = e^{-as}F(s)$

$L^{-1}\{e^{-as}F(s)\} = f(t-a)\,u(t-a)$.

Here $F(s) = \frac{1}{(s-3)^4} = L\{f(t)\}$

$\therefore f(t) = L^{-1}\left\{\frac{1}{(s-3)^4}\right\} = e^{3t}\,L^{-1}\left\{\frac{1}{s^4}\right\}$ --- by using FST. (t shift)

$= e^{3t}\frac{t^3}{3!}$

By SST, (s-shift).

$L^{-1}\left\{\frac{e^{-4s}}{(s-3)^4}\right\} = e^{3(t-4)}\frac{(t-4)^3}{6}\,u(t-4)$.

$$f(t-4) = \begin{cases} e^{3(t-4)}\dfrac{(t-4)^3}{6} & t>4 \\ 0 & t<4. \end{cases}$$

Hence $f(t) = \begin{cases} e^{3t}\frac{t^3}{6} & t>4 \\ 0 & t<4 \end{cases}$

(b) $L^{-1}\left\{\frac{s+1}{(s^2+s+1)}e^{-\pi s}\right\}$.

$\rightarrow$ Here $F(s) = \frac{s+1}{s^2+s+1} = L\{f(t)\}$

$\therefore f(t) = L^{-1}\left\{\frac{s+1}{s^2+s+1}\right\}$

$= L^{-1}\left\{\frac{(s+\frac{1}{2})+\frac{1}{2}}{(s+1/2)^2+3/4}\right\}$

Name: Class & Div. : Page No. :

Subject : Topic : Date :

$$= e^{-\frac{t}{2}} L^{-1}\left\{\frac{s+1/2}{s^2+(3/4)}\right\}, \text{ by first shifting Th}^m.$$

$$= e^{-t/2}\left[L^{-1}\left\{\frac{s}{s^2+(\sqrt{3}/2)^2}\right\} + \frac{1}{2} L^{-1}\left\{\frac{1}{s^2+(\sqrt{3}/2)^2}\right\}\right]$$

$$= e^{-\frac{t}{2}}\left[\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1}{2}\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\sin\left(\frac{\sqrt{3}}{2}t\right)\right]$$

$$= \frac{1}{\sqrt{3}} e^{-\frac{t}{2}}\left[\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}t\right) + \sin\left(\frac{\sqrt{3}}{2}t\right)\right].$$

By second shifting th^m. $L^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$

$$L^{-1}\left\{\frac{s+1}{(s^2+s+1)} e^{-\pi s}\right\}$$

$$= \frac{1}{\sqrt{3}} e^{-(t-\pi)/2}\left[\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}(t-\pi)\right) + \sin\left(\frac{\sqrt{3}}{2}(t-\pi)\right)\right] \cdot u(t-\pi).$$

$$= \begin{cases} \frac{1}{\sqrt{3}} e^{\frac{-(t-\pi)}{2}}\left[\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}(t-\pi)\right) + \sin\left(\frac{\sqrt{3}}{2}(t-\pi)\right)\right], & t > \pi \\ 0 & t < \pi. \end{cases}$$

$$\text{Hence } f(t) = \begin{cases} \frac{1}{\sqrt{3}} e^{-t/2}\left[\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}t\right) + \sin\left(\frac{\sqrt{3}}{2}t\right)\right], & t > \pi \\ 0, & 0 < t < \pi \end{cases}$$

(c) $\dfrac{e^{-\pi s}}{(s^2+1)s}$

Here $F(s) = \dfrac{1}{(s^2+1)s} = L\{f(t)\}$.

$L^{-1}\left\{\dfrac{F(s)}{s}\right\} = \int_0^t f(u)\,du$

$\therefore f(t) = L^{-1}\left\{\dfrac{1}{(s^2+1)}\right\}$

$= \int_0^t L^{-1}\left\{\dfrac{1}{(s^2+1)}\right\} du$ --- (by L.T of integ. Thm).

$= \int_0^t \sin u\, du = -\cos u \Big|_{u=0}^{t}$

$= 1 - \cos t$.

By SST,

$L^{-1}\left\{\dfrac{e^{-\pi s}}{s(s^2+1)}\right\} = [1 - \cos(t-\pi)] \cdot u(t-\pi)$

$= \begin{cases} 1 - \cos(t-\pi), & t > \pi \\ 0 & 0 < t < \pi. \end{cases}$

$(\because L^{-1}\{e^{-as}F(s)\} = f(t-a)\,u(t-a))$

Hence

$f(t) = \begin{cases} 1 - \cos t, & t > \pi \\ 0 & 0 < t < \pi. \end{cases}$

Name: ______ Class & Div. : ______ Page No. : ______

Subject : ______ Topic : ______ Date : ______

(d) $\bar{L}^1\left\{\frac{e^{-8\pi s}+e^{-2\pi s}}{s^2+4}\right\}$

$\Rightarrow \bar{L}^1\left\{\frac{e^{-8\pi s}}{s^2+4}\right\}+\bar{L}^1\left\{\frac{e^{-2\pi s}}{s^2+4}\right\}.$

Here $F(s)=\frac{1}{s^2+4}=L\{f(t)\}$

$\therefore f(t)=\bar{L}^1\left\{\frac{1}{s^2+4}\right\}=\frac{1}{2}\bar{L}^1\left\{\frac{2}{s^2+4}\right\}=\frac{\sin 2t}{2}.$

By SST

$L\{e^{-as}F(s)\}=f(t-a)\,u(t-a).$

$\therefore \bar{L}^1\left\{\frac{e^{-8\pi s}}{s^2+4}\right\}+\bar{L}^1\left\{\frac{e^{-2\pi s}}{s^2+4}\right\}$

$=\frac{\sin 2(t-8\pi)}{2}\,u(t-8\pi)+\frac{\sin 2(t-2)}{2}\,u(t-2\pi)$

(I) Find $L\{(t-\pi)\sin t\,u(t-\pi)\}.$

$L\{\underline{(t-\pi)\sin t}\,u(t-\pi)\}$ $\quad L\{f(t)u(t-a)\}=e^{-as}f(t+a)$

$=e^{-\pi s}L\{f(t+\pi)\}$ --- by SST

$=e^{-\pi s}L\{(t+\cancel{\pi}-\cancel{\pi})\sin(t+\pi)\}$

$=e^{-\pi s}L\{t\sin(t+\pi)\}.$

$$= e^{-\pi s} L\left\{t\left(\sin t\cdot \cancel{\cos(\pi)}^{(-1)} + \cos t\, \cancel{\sin(\pi)}^{(0)}\right)\right\}$$

$$= e^{-\pi s} L\{-t \sin t\}$$

$$= e^{-\pi s}\left[(-1)\frac{d}{ds}\left[L\{\sin t\}\right]\right] \text{ --- by } L\{t f(t)\} = (-1)F'(s)$$

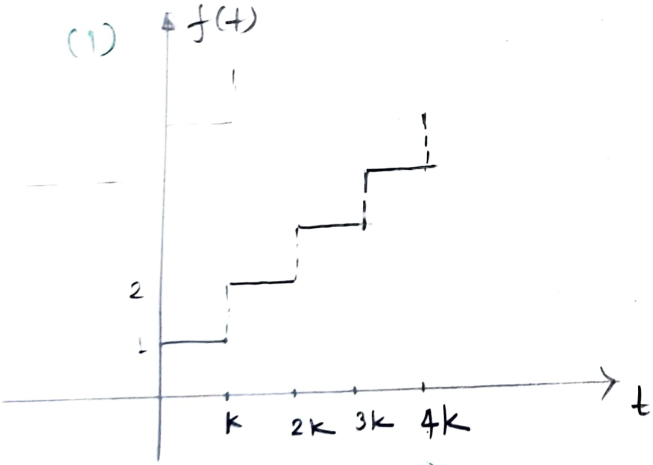
$$= e^{-\pi s}\left[-\frac{d}{ds}\left[\frac{1}{s^2+1}\right]\right]$$

$$= e^{-\pi s}\left[\frac{-2s}{(s^2+1)^2}\right]$$

Home work

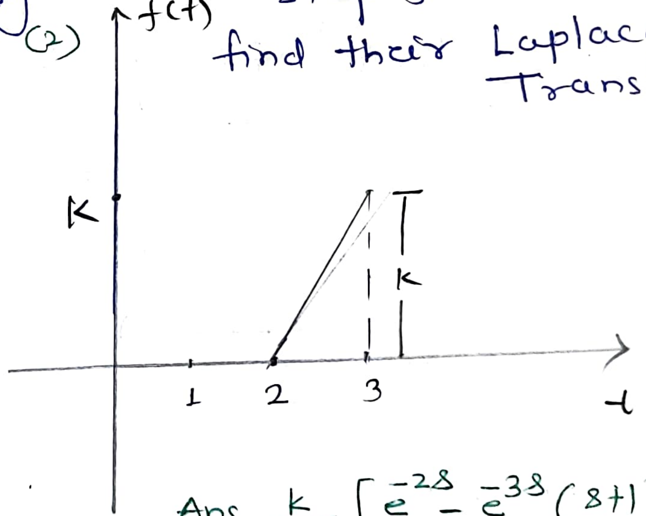
Represent the function for following figures in terms of unit step fuⁿ and hence find their Laplace Transform.

(1)



Ans $\frac{1}{s}\left(\frac{1}{1-e^{-ks}}\right)$

(2)



Ans $\frac{k}{s^2}\left[e^{-2s} - e^{-3s}(s+1)\right]$

(3) If $f(t) = \begin{cases} t+1, & 0 \le t < 2 \\ 3 & t > 2 \end{cases}$, find $L\{f(t)\}$ and $L\{f'(t)\}$.

(4) $L^{-1}\left\{\frac{e^{-3s}}{s^2+8s+25}\right\}$ (5) $L^{-1}\left\{\frac{e^{-2s}}{\sqrt{s+5}}\right\}$

Name: ______ Class & Div. : ______ Page No. : ______

Subject : ______ Topic : ______ Date : ______

Convolution :-

As we know, If $L\{f\} = F(s)$, $L\{g\} = G(s)$ then

$$L\{f+g\} = L\{f\} + L\{g\}$$

$$L\{\alpha f\} = \alpha L\{f\} \quad , \quad \alpha \in \mathbb{R} \text{ (scalar)}.$$

Also $L\{f-g\} = L\{f\} - L\{g\}$.

But

$$\boxed{L\{f \cdot g\} \neq L\{f\} \cdot L\{g\}}$$

For example

Take $f(t) = e^t$, $g(t) = 1$.

Then

$$f \cdot g = e^t.$$

$$\Rightarrow L\{f \cdot g\} = L\{e^t\} = \frac{1}{s-1} \quad \text{— (i)}$$

But

$$L\{f\}L\{g\} = \left(\frac{1}{s}\right)\left(\frac{1}{s-1}\right) = \frac{1}{s(s-1)} \quad \text{— (ii)}$$

From (i) $\neq$ (ii)

Convolution (Def^n) :-

Let f and g defined on $[0, \infty)$. Then the convolution of f and g denoted by standard notation $f*g$ and defined by the integral

$$(f*g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$