Financial Engineering & Risk Management

Introduction to Term Structure Lattice Models

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Fixed Income Markets

Fixed income markets are enormous and in fact bigger than equity markets.

According to *SIFMA*, in Q3 2012 the total outstanding amount of US bonds was $\$35.3$ trillion:

Government	\$10.7	30.4%
Municipal	\$3.7	10.5%
Mortgage	\$8.2	23.3%
Corporate	\$8.6	24.3%
Agency	\$2.4	6.7%
Asset-backed	\$1.7	4.8%
Total	\$35.3 tr	100%

– in comparison, size of US equity markets is approx $\$26$ trillion.

Fixed income derivatives markets are also enormous

– includes interest-rate and bond derivatives, credit derivatives, MBS and ABS

– will focus here on interest-rate and bond derivatives

– using binomial lattice models.

(The slides and Excel spreadsheet should be sufficient, but Luenberger (1st edition) Chapter 14, Interest Rate Derivatives, (note: this chapter number may differ in different editions of the book), is an excellent reference for the material in this section.)

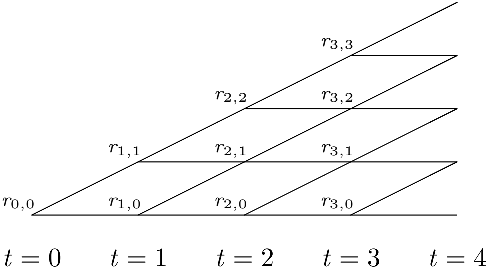
Binomial Models for the Short Rate

- Will use binomial lattice models as our vehicle for introducing:
  1. the mechanics of the most important fixed income derivative securities
     - bond futures (and forwards)
     - caplets and caps, floorlets and floors
     - swaps and swaptions
  2. the "philosophy" behind fixed income derivatives pricing
     – more on this soon.
- Fixed-income models are inherently more complex than security models
  - need to model evolution of entire **term-structure of interest rates**.
- The short-rate, r_t, is the variable of interest in many fixed income models
  – including binomial lattice models
  – r_t is the risk-free rate that applies between periods t and $t+1$
  – it is a **random process** but r_t is known by time t.

The “Philosophy” of Fixed Income Derivatives Pricing

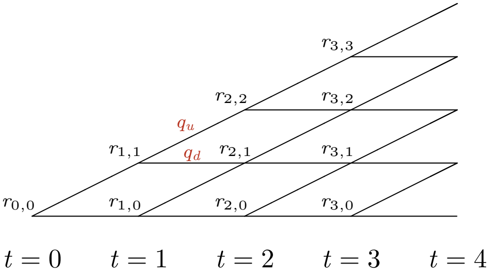
- We will simply specify risk-neutral probabilities for the short-rate, r_t
  - without any reference to the true probabilities of the short-rate
- This is in contrast to the binomial model for stocks where we specified p and $1-p$
  - and then used replication arguments to obtain q and $1-q$.
- We will price securities in such a way that **guarantees no-arbitrage**
- Will match market prices of liquid securities via a **calibration** procedure
  - often the most challenging part.
- Will see that derivatives pricing in practice is really about **extrapolating** from liquid security prices to illiquid security prices.

Binomial Models for the Short-Rate



- We will take zero-coupon bond (zcb) prices to be our basic securities
  - will use $Z_{i,j}^k$ for time i, state j price of a zcb that matures at time k
- Would like to specify binomial model by specifying all $Z_{i,j}^k$'s at all nodes
  - possible but awkward if we want to ensure no-arbitrage.
- Instead will specify the **short-rate**, $r_{i,j}$ at each node $N_{i,j}$
  - the risk-free rate that applies to the next period.

Binomial Models for the Short-Rate



- Let $Z_{i,j}$ be the date i, state j price of a non-coupon paying security.
- Will use risk-neutral pricing to price every security so that:

$$Z_{i,j} = \frac{1}{1 + r_{i,j}} \left[q_u \times Z_{i+1,j+1} + q_d \times Z_{i+1,j} \right] \tag{1}$$

  – where q_u and q_d are the risk-neutral probabilities of an up- and down-move
  – so $q_d + q_u = 1$ and must have $q_d > 0$ and $q_u > 0$.

- There can be **no arbitrage** when we price using (3). Why?

Binomial Models for the Short Rate

- If the security pays a "coupon", $C_{i+1,j}$, at date $i+1$ and state j then

$$Z_{i,j} = \frac{1}{1+r_{i,j}} \left[q_u \left(Z_{i+1,j+1} + C_{i+1,j+1} \right) + q_d \left(Z_{i+1,j} + C_{i+1,j} \right) \right] \qquad (2)$$

  – where $Z_{i+1,\cdot}$ is now the **ex-coupon** price at date $i+1$.

- If we use (3) or (2) to price securities in the lattice model then arbitrage is not possible
  - Regardless of what probabilities we use! Why is this?

- In fact it is very common to simply set $q_u = q_d = 1/2$
  - and to calibrate other parameters to market prices.
- We will assume $q_u = q_d = 1/2$ in our examples.

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The Cash Account and Pricing Zero-Coupon Bonds

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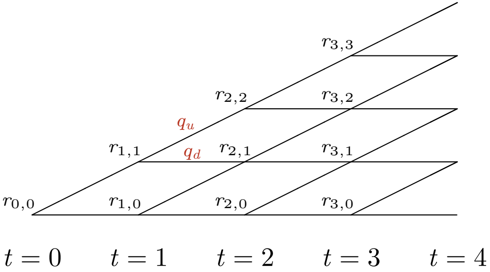
Binomial Models for the Short-Rate



- We use **risk-neutral** pricing to price every non-coupon paying security:

$$Z_{i,j} \;=\; \frac{1}{1 + r_{i,j}} \left[q_u \times Z_{i+1,j+1} \;+\; q_d \times Z_{i+1,j} \right] \tag{3}$$

  - $q_u > 0$ and $q_d > 0$ are the risk-neutral probabilities of an up- and down-move, respectively, of the short-rate.

- There can be **no arbitrage** when we price using (3). Why?

The Cash-Account

- The cash-account is a particular security that in each period earns interest at the short-rate
  - we use B_t to denote its value at time t and assume that $B_0 = 1$.
- The cash-account is **not** risk-free since B_{t+s} is not known at time t for any $s > 1$
  - it is **locally** risk-free since B_{t+1} is known at time t.
- Note that B_t satisfies $B_t \; = \; (1 + r_{0,0})(1 + r_1) \ldots (1 + r_{t-1})$
  - so that $B_t / B_{t+1} = 1/(1 + r_t)$.
- Risk-neutral pricing for a "non-coupon" paying security then takes the form:

$$
\begin{aligned}
Z_{t,j} &= \frac{1}{1 + r_{t,j}} \left[q_u \times Z_{t+1,j+1} \; + \; q_d \times Z_{t+1,j} \right] \\
&= \mathsf{E}_t^{\mathbb{Q}} \left[\frac{Z_{t+1}}{1 + r_{t,j}} \right] \\
&= \mathsf{E}_t^{\mathbb{Q}} \left[\frac{B_t}{B_{t+1}} \, Z_{t+1} \right] \qquad (4)
\end{aligned}
$$

Risk-Neutral Pricing with the Cash-Account

- Therefore for a non-coupon paying security, (4) is equivalent to

$$\frac{Z_t}{B_t} = \mathsf{E}_t^{\mathbb{Q}}\left[\frac{Z_{t+1}}{B_{t+1}}\right] \tag{5}$$

- We can iterate (5) to obtain

$$\frac{Z_t}{B_t} = \mathsf{E}_t^{\mathbb{Q}}\left[\frac{Z_{t+s}}{B_{t+s}}\right] \tag{6}$$

for any non-coupon paying security and any $s > 0$.

Risk-Neutral Pricing with the Cash-Account

- **Risk-neutral** pricing for a "coupon" paying security takes the form:

$$\begin{aligned} Z_{t,j} &= \frac{1}{1+r_{t,j}} \left[q_u \left(Z_{t+1,j+1} + C_{t+1,j+1} \right) \; + \; q_d \left(Z_{t+1,j} + C_{t+1,j} \right) \right] \\ &= \mathsf{E}_t^{\mathbb{Q}} \left[\frac{Z_{t+1} + C_{t+1}}{1 + r_{t,j}} \right] \end{aligned} \tag{7}$$

- We can rewrite (7) as

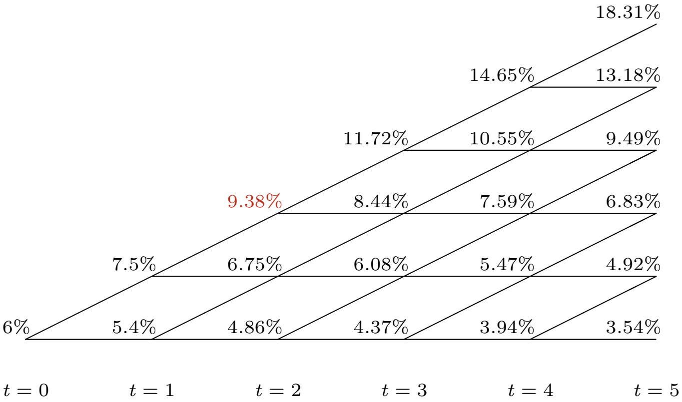
$$\frac{Z_t}{B_t} = \mathsf{E}_t^{\mathbb{Q}} \left[\frac{C_{t+1}}{B_{t+1}} + \frac{Z_{t+1}}{B_{t+1}} \right] \tag{8}$$

- More generally, we can iterate (8) we obtain

$$\frac{Z_t}{B_t} = \mathsf{E}_t^{\mathbb{Q}} \left[\sum_{j=t+1}^{t+s} \frac{C_j}{B_j} + \frac{Z_{t+s}}{B_{t+s}} \right] \tag{9}$$

- Pricing using (9) ensures **no-arbitrage**
  - note that (6) is a special case of (9).

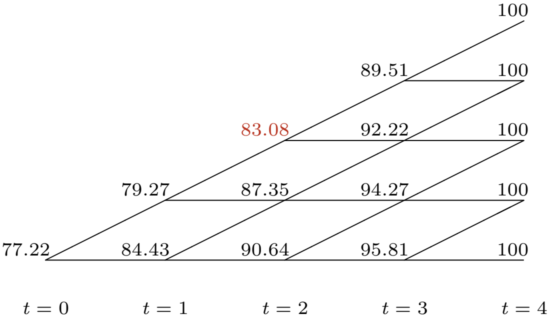
A Sample Short-Rate lattice



The short-rate, r, grows by a factor of $u = 1.25$ or $d = .9$ in each period

– not very realistic but more than sufficient for our purposes.

Pricing a ZCB that Matures at Time t=4

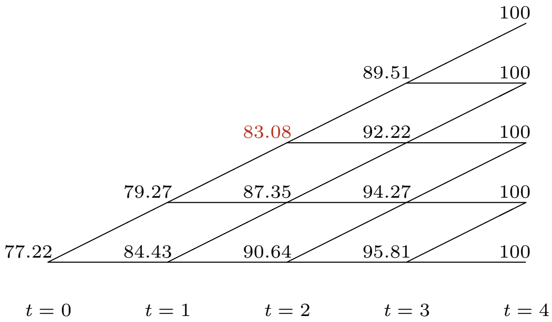


$$\text{e.g. } 83.08 = \frac{1}{1 + .0938}\left[\frac{1}{2} \times 89.51 + \frac{1}{2} \times 92.22\right].$$

Can compute the term-structure by pricing ZCB's of every maturity and then backing out the spot-rates for those maturities

- so $s_4 = 6.68\%$ assuming per-period compounding, i.e., $77.22(1 + s_4)^4 = 100$.

Pricing a ZCB that Matures at Time t=4



Therefore can compute compute Z_0^1, Z_0^2, Z_0^3 and Z_0^4

- and then compute s_1, s_2, s_3 and s_4 to obtain the **term-structure of interest rates** at time $t = 0$.

At $t = 1$ we will compute new ZCB prices and obtain a new term-structure

- model for the short-rate, r_t, therefore defines a model for the term-structure!

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Fixed Income Derivatives: Options on Bonds

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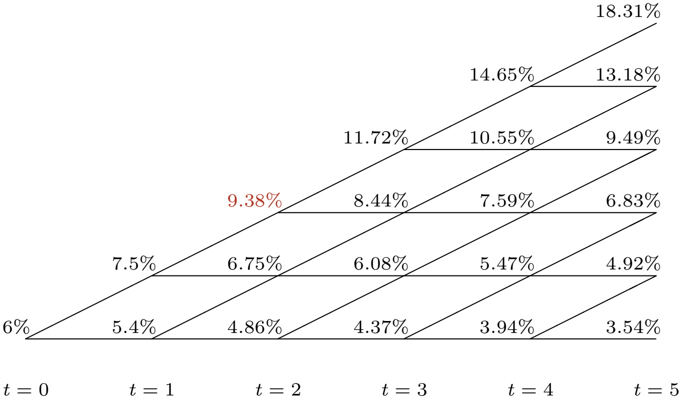
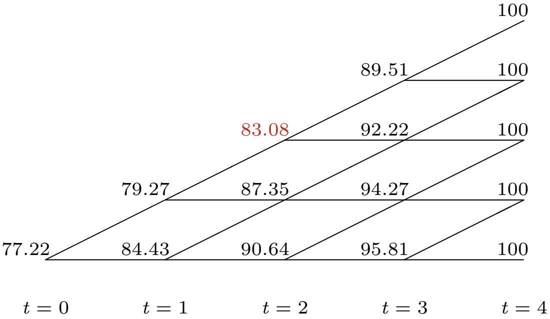
Our Sample Short-Rate lattice

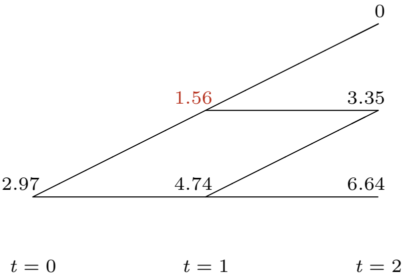


Pricing a ZCB that Matures at Time t=4



$$\text{e.g.} \quad 83.08 \;=\; \frac{1}{1+.0938}\left[\frac{1}{2}\times 89.51 \;+\; \frac{1}{2}\times 92.22\right].$$

Pricing a European Call Option on the ZCB



Strike $= \$84$

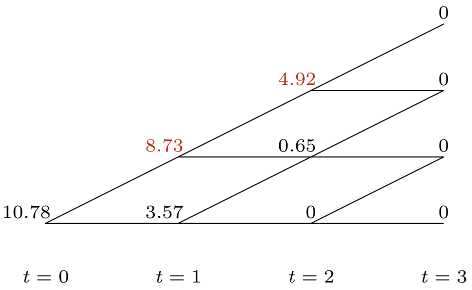
Option Expiration at $t = 2$

Option Payoff $= \max\left(0,\ Z_{2,\cdot}^{4} - 84\right)$

Underlying ZCB Matures at $t = 4$

$$\text{e.g.}\quad 1.56 \;=\; \frac{1}{1 + .075}\left[\frac{1}{2} \times 0 \;+\; \frac{1}{2} \times 3.35\right].$$

Pricing an American Put Option on a ZCB



Strike $= \$88$
Expiration at $t = 3$
Payoff at $t = 3$ is $\max(0,\ 88 - Z^4_{3,\cdot})$
Underlying ZCB Matures at $t = 4$

$$\text{e.g.}\quad 4.92 \;=\; \max\left\{88 - 83.08\,,\ \frac{1}{1 + .0938}\ \left[\frac{1}{2} \times 0 \;+\; \frac{1}{2} \times 0\right]\right\}.$$

$$\text{e.g.}\quad 8.73 \;=\; \max\left\{88 - 79.27\,,\ \frac{1}{1 + .075}\ \left[\frac{1}{2} \times 4.92 \;+\; \frac{1}{2} \times 0.65\right]\right\}.$$

Turns out it's optimal early-exercise everywhere

– not a very realistic example.

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Fixed Income Derivatives: Bond Forwards

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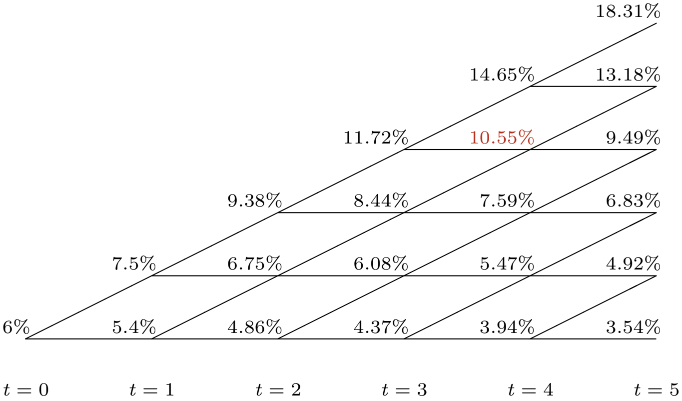
Our Sample Short-Rate lattice



Pricing a Forward on a Coupon-Bearing Bond

- Delivery at $t = 4$ of a 2-year 10% coupon-bearing bond.
- We assume delivery takes place just after a coupon has been paid.
- In the pricing lattice we use backwards induction to compute the $t = 4$ ex-coupon price of the bond.
- Let G_0 be the forward price at $t = 0$ and let Z_4^6 be the ex-coupon bond price at $t = 4$. Then risk-neutral pricing implies

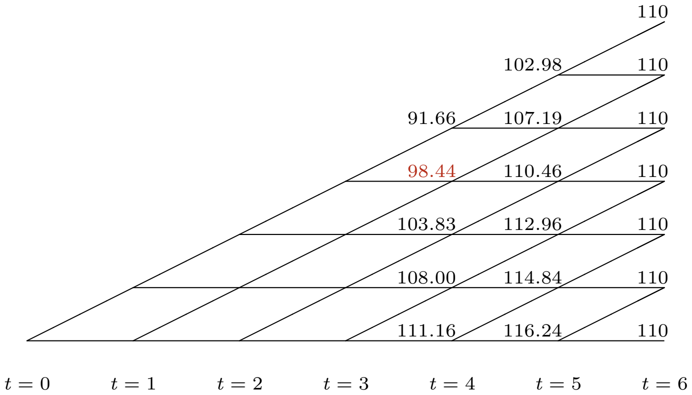
$$0 \quad = \quad \mathsf{E}_0^{\mathbb{Q}}\left[\frac{Z_4^6 - G_0}{B_4}\right]$$

  where B_4 is the value of the cash-account at $t = 4$.
- Rearranging terms and using the fact that G_0 is known at date $t = 0$ we obtain

$$G_0 \;=\; \frac{\mathsf{E}_0^{\mathbb{Q}}\left[Z_4^6/B_4\right]}{\mathsf{E}_0^{\mathbb{Q}}\left[1/B_4\right]}. \tag{10}$$

- Recall that $\mathsf{E}_0^{\mathbb{Q}}\left[1/B_4\right]$ is time $t = 0$ price of a ZCB maturing at $t = 4$ but with a face value $\$1$
  - have already calculated this to be $.7722$.

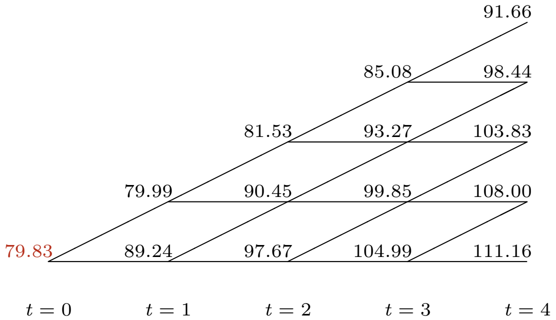
Pricing a Forward on a Coupon-Bearing Bond



First find ex-coupon price, Z_4^6, of the bond at time $t = 4$:

$$\text{e.g. } 98.44 \;=\; \frac{1}{1 + .1055}\left[\frac{1}{2} \times 107.19 \;+\; \frac{1}{2} \times 110.46\right].$$

Pricing a Forward on a Coupon-Bearing Bond



Now work backwards in lattice to compute $\mathsf{E}_0^{\mathbb{Q}}\left[Z_4^6/B_4\right] = 79.83$.

Can now use (13) to obtain

$$G_0 \;=\; \frac{79.83}{0.7722} \;=\; 103.38.$$

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Fixed Income Derivatives: Bond Futures

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Pricing Futures Contracts

- Let F_k be the date k price of a futures contract that expires after n periods.
- Let S_k denote the time k price of the security underlying the futures contract.
- Then $F_n = S_n$, i.e., at expiration the futures price and the underlying security price must coincide.
- Can compute the futures price at $t = n - 1$ by recalling that anytime we enter a futures contract, the initial value of the contract is 0.
- Therefore the futures price, F_{n-1}, at date $t = n - 1$ must satisfy (why?)

$$\frac{0}{B_{n-1}} \quad = \quad \mathsf{E}^{\mathbb{Q}}_{n-1}\left[\frac{F_n - F_{n-1}}{B_n}\right].$$

Pricing Futures Contracts

- Since B_n and F_{n-1} are both known at date $t = n-1$, we therefore have

$$F_{n-1} = \mathsf{E}^{\mathbb{Q}}_{n-1}[F_n].$$

- By the same argument, we obtain

$$F_k = \mathsf{E}^{\mathbb{Q}}_{k}[F_{k+1}] \quad \text{for } 0 \leq k < n.$$

- Can then use the law of iterated expectations to obtain

$$F_0 \ = \ \mathsf{E}^{\mathbb{Q}}_{0}\left[F_n\right].$$

- Since $F_n = S_n$ we have

$$F_0 \ = \ \mathsf{E}^{\mathbb{Q}}_{0}\left[S_n\right] \tag{11}$$

  - holds regardless of whether or not underlying security pays coupons etc.

- In contrast corresponding forward price, G_0, satisfies

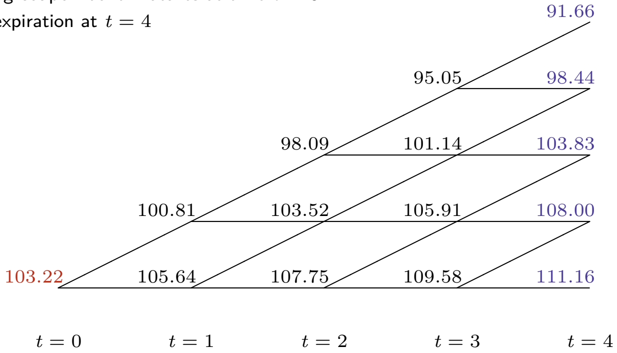
$$G_0 \ = \ \frac{\mathsf{E}^{\mathbb{Q}}_{0}\left[S_n/B_n\right]}{\mathsf{E}^{\mathbb{Q}}_{0}\left[1/B_n\right]}. \tag{12}$$

A Futures Contract on a Coupon-Bearing Bond

Futures contract written on same coupon bond as earlier forward contract

Underlying coupon bond matures at time $t = 6$

Futures expiration at $t = 4$



Note that the forward price, 103.38, and futures price, 103.22, are close

– but not equal!

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Fixed Income Derivatives: Caplets and Floorlets

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Pricing a Caplet

A caplet is similar to a European call option on the interest rate, r_t.

- Usually settled in arrears but they may also be settled in advance.
- If maturity is τ and strike is c, then payoff of a caplet (settled in arrears) at time τ is

$$(r_{\tau-1} - c)^+$$

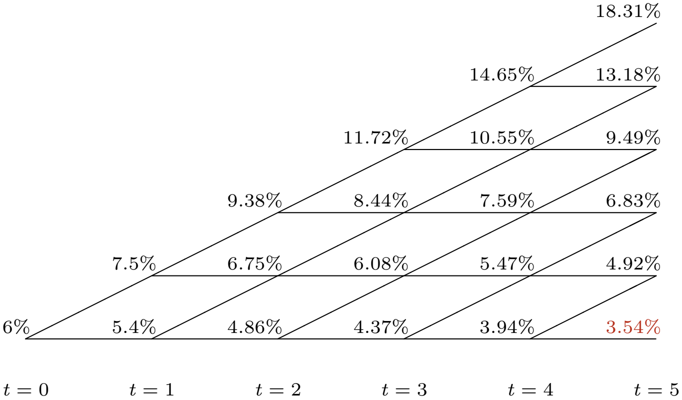
  - so the caplet is a call option on the short rate prevailing at time $\tau - 1$, settled at time τ.

A floorlet is the same as a caplet except the payoff is $(c - r_{\tau-1})^+$.

A cap consists of a sequence of caplets all of which have the same strike.

A floor consists of a sequence of floorlets all of which have the same strike.

Our Short-Rate lattice



Pricing a Caplet

Expiration at $t = 6$
Strike $= 2\%$

					.138
				.103	.099
			.080	.076	.068
		.064	.059	.053	.045
	.052	.047	.041	.035	.028
.042	.038	.032	.026	.021	.015
$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$

Note that it is easier to record the time $t = 6$ cash flows at their time 5 predecessor nodes, and then discount them appropriately:

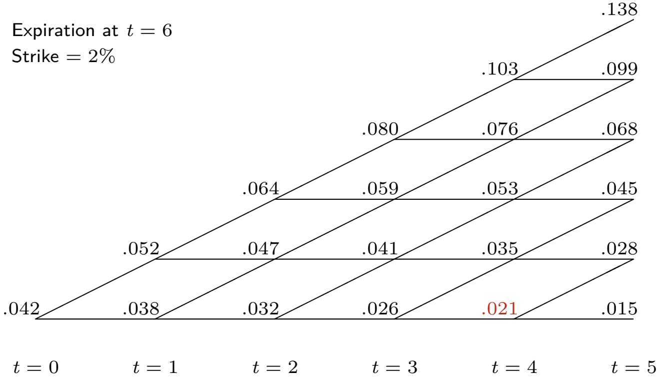
– so $(r_5 - c)^+$ at $t = 6$ is worth $(r_5 - c)^+/(1 + r_5)$ at $t = 5$.

A sample calculation:

$$0.015 \quad = \quad \frac{\max(0,\ .0354 - .02)}{1 + .0354}$$

Pricing a Caplet

Expiration at $t = 6$
Strike $= 2\%$



Now work backwards in the lattice to find the price at $t = 0$.

A sample calculation:

$$0.021 \quad = \quad \frac{1}{1.0394}\left[\frac{1}{2} \times 0.028 \; + \; \frac{1}{2} \times 0.015\right]$$

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Fixed Income Derivatives: Swaps and Swaptions

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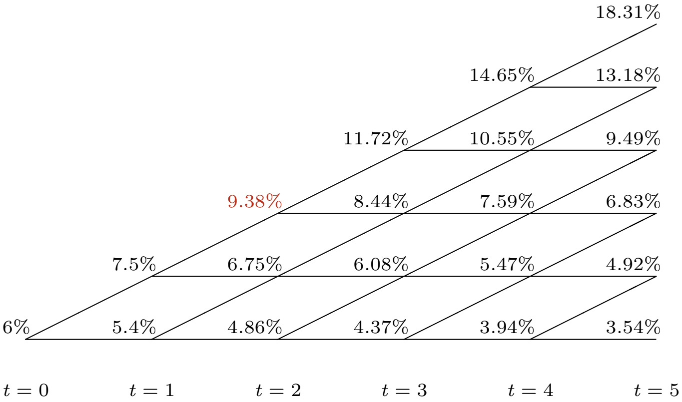
Our Short-Rate lattice



Want to price an interest-rate swap with fixed rate of 5% that expires at $t = 6$

- first payment at $t = 1$ and final payment at $t = 6$
- payment of $\pm(r_{i,j} - K)$ made at time $t = i + 1$ if in state j at time i.

Pricing Swaps

Expiration, i.e. last payment, at $t = 6$
First payment, at $t = 1$
Strike $=$ fixed rate $= 5\%$

$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
					.1125
				.1648	.0723
			.1793	.1014	.0410
		.1686	.1021	.0512	.0172
	.1403	.0829	.0400	.0122	−.0008
.0990	.0496	.0137	−.0085	−.0174	−.0141

Note that it is easier to record the time t cash flows at their time $t-1$ predecessor nodes, and then discount them appropriately:

– so $(r_{5,5} - K)$ at $t = 6$ is worth $\pm(r_{5,5} - K)/(1 + r_{5,5}) = .0723$ at $t = 5$.

A sample calculation:

$$0.1686 \quad = \quad \frac{1}{1.0938}\left[(.0938 - .05) \; + \; \frac{1}{2} \times 0.1793 \; + \; \frac{1}{2} \times 0.1021\right]$$

Pricing Swaptions

- A swaption is an option on a swap.
- Consider a swaption on the swap of the previous slide
  - will assume that the option strike is 0%
    - not to be confused with the strike, i.e. fixed rate, of underlying swap
  - and the swaption expiration is at $t = 3$.
- Swaption value at expiration is therefore $\max(0, \ S_3)$ where $S_3 \equiv$ underlying swap price at $t = 3$.
- Value at dates $0 \leq t < 3$ computed in usual manner by working backwards in the lattice
  - but underlying cash-flows of swap are not included at those times.

Pricing Swaptions

Fixed rate in Swap $= 5\%$
Underlying Swap Expiration at $t = 6$
Option Expiration at $t = 3$

$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
					.1125
				.1648	.0723
			.1793	.1014	.0410
		.1286	.1021	.0512	.0172
	.0908	.0665	.0400	.0122	−.0008
.0620	.0406	.0191	**0**	−.0174	−.0141

Swaption price is computed by determining payoff at maturity, i.e $t = 3$ and then working backwards in the lattice.

A sample calculation:

$$.0908 = \frac{1}{1 + .075}\left[\frac{1}{2} \times .1286 + \frac{1}{2} \times .0665\right]$$

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The Forward Equations

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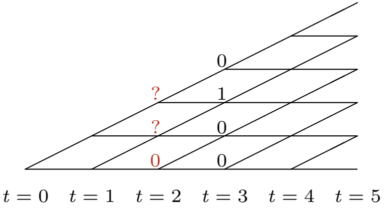
The Forward Equations

- $P^e_{i,j}$ denotes the time 0 price of a security that pays $\$1$ at time i, state j and 0 at every other time and state.
- Call such a security an elementary security and $P^e_{i,j}$ is its state price.
- Can see that elementary security prices satisfy the forward equations

$$
\begin{aligned}
P^e_{k+1,s} &= \frac{P^e_{k,s-1}}{2(1+r_{k,s-1})} + \frac{P^e_{k,s}}{2(1+r_{k,s})}, \quad 0 < s < k+1 \qquad (13) \\
P^e_{k+1,0} &= \frac{1}{2}\frac{P^e_{k,0}}{(1+r_{k,0})} \\
P^e_{k+1,k+1} &= \frac{1}{2}\frac{P^e_{k,k}}{(1+r_{k,k})}.
\end{aligned}
$$

with $P^e_{0,0} = 1$.

Deriving the Forward Equations



Consider the security that pays $\$1$ only at $t = 3$ and only in state 2

– value of this security is $P^e_{3,2}$ by definition.

But can also work backwards in lattice to price it. Its value at node $N_{2,2}$ is

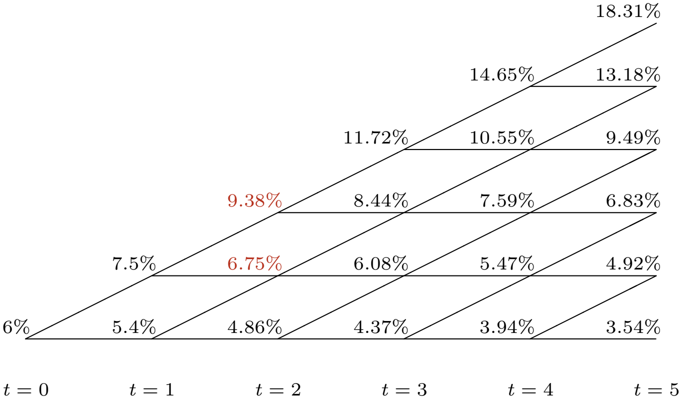
$$\frac{1}{1 + r_{2,2}} \left[\frac{1}{2} \times 0 \ + \ \frac{1}{2} \times 1 \right] \ = \ \frac{1}{2(1 + r_{2,2})}$$

its value at node $N_{2,0}$ is 0, and its value at node $N_{2,1}$ is

$$\frac{1}{1 + r_{2,1}} \left[\frac{1}{2} \times 1 \ + \ \frac{1}{2} \times 0 \right] \ = \ \frac{1}{2(1 + r_{2,1})}.$$

Therefore $P^e_{3,2} = \frac{1}{2(1+r_{2,2})} \times P^e_{2,2} \ + \ \frac{1}{2(1+r_{2,1})} \times P^e_{2,1} \ + \ 0 \times P^e_{2,0}$.

Our Short-Rate lattice



Now compute the forward prices by iterating the equations **forward** starting with $P^e_{0,0} = 1$.

… and the Corresponding Elementary Prices

Key: Value at node $N_{i,j}$ is $P^e_{i,j}$

					0.0196
				0.0449	0.1041
			0.1003	0.1868	0.2193
		.2194	0.3079	0.2901	0.2293
	.4717	.4432	0.3143	0.1992	0.1190
1	.4717	.2238	0.1067	0.0511	0.0246
$t=0$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$

Sample calculations:

$$
\begin{aligned}
.3079 &= \frac{P^e_{k,s-1}}{2(1+r_{k,s-1})} + \frac{P^e_{k,s}}{2(1+r_{k,s})} \\
&= \frac{.4432}{2(1+.0675)} + \frac{.2194}{2(1+.0938)}
\end{aligned}
$$

Derivative Prices Via Elementary Prices

Given the elementary prices the calculation of some security prices becomes very straightforward:

e.g. Can calculate Z_0^4 as

$$\begin{aligned} Z_0^4 &= 100 \times (.0449 + .1868 + .2901 + .1992 + .0511) \\ &= 77.22 \end{aligned}$$

– as calculated before.

Derivative Prices Via Elementary Prices

Consider a forward-start swap that begins at $t = 1$ and ends at $t = 3$

- notional principal is $\$1$ million
- fixed rate in the swap is 7%
- payments at $t = i$ for $i = 2, 3$ are based as usual on fixed rate minus floating rate that prevailed at $t = i - 1$

The "forward" feature of the swap is that it begins at $t = 1$

- first payment is then at $t = 2$ since payments are made in arrears.

Question: What is the value, V_0, of the forward swap today at $t = 0$?

Solution: The value is given by

$$\begin{aligned} V_0 &= \frac{(.07 - .0938)}{1.0938} \times .2194 + \frac{(.07 - .0675)}{1.0675} \times .4432 + \frac{(.07 - .0486)}{1.0486} \times .2238 \\ &\quad + \frac{(.07 - .075)}{1.075} \times .4717 + \frac{(.07 - .054)}{1.054} \times .4717 \\ &= \$5,800. \end{aligned}$$