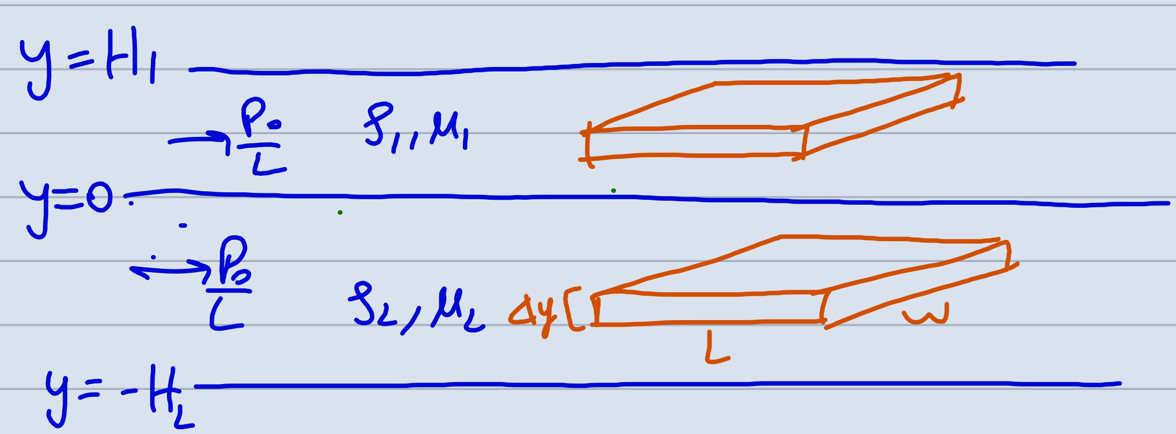
Tutorial - 7

1)



Momentum shell balance in Fluid 2:

$$\Phi_{ix} = \Delta y W \left[\phi_{xx}|_x - \phi_{xx}|_{x+L} \right]$$

$$+ LW \left[\phi_{yx}|_x - \phi_{yx}|_{x+L} \right]$$

$$+ L\Delta x \left[\phi_{zx}|_x - \phi_{zx}|_{x+L} \right]$$

$$\underline{\underline{\Phi}} = -p\underline{\underline{\delta}} + \mu \nabla \underline{v} + \rho \underline{v}\,\underline{v} \qquad v_y = 0 \quad v_z = 0$$

$$\phi_{xx} = -P + \cancel{\rho v_x^2}^{\,0} \qquad \phi_{yx} = \mu \frac{dv_x}{dy} \qquad \phi_{zx} = 0$$

as $\nabla \cdot \underline{v} = 0$ $\quad v_x \neq f(x)$

$$\Delta x W \left[-P|_x + P|_{x+L} \right] + LW \left[\mu \frac{dv_x}{dy}\Big|_x - \mu \frac{dv_x}{dy}\Big|_{x+L} \right]$$

$$+ \rho g_x L W \Delta y = 0$$

$$\frac{P_0}{L} - \frac{d}{dy}\left[\mu \frac{dv_x}{dy} \right] = 0$$

$$\mu \frac{dv_x}{dy} = \frac{P_0 y}{L} + C_1$$

$$v_x^{(2)} = \frac{P_0 y^2}{2L\mu} + \frac{C_1 y}{\mu} + C_2$$

Similarly

$$v_x^{(1)} = \frac{P_0 y^2}{2L\mu} + \frac{C_3 y}{\mu} + C_4$$

Boundary conditions:

$$v_x^{(1)}(y=H_1) = \frac{P_0 H_1^2}{2L\mu_1} + \frac{C_3 H_1}{\mu_1} + C_4 = 0$$

$$v_x^{(2)}(y=-H_2) = \frac{P_0 H_2^2}{2\mu_2} - \frac{C_1 H_2}{\mu_2} + C_2 = 0$$

$$v_x^{(1)}(y=0) = v_x^{(2)}(y=0)$$

$$\underline{\underline{C_4 = C_2}}$$

$$\mu_1 \frac{\partial v_x^{(1)}}{\partial y}(y=0) = \mu_2 \frac{\partial v_x^{(2)}}{\partial y}(y=0)$$

$$\underline{\underline{C_1 = C_3}}$$

$$\frac{P_0 H_1^2}{2L\mu} + \frac{C_1 H_1}{\mu_1} + \cancel{C_2} = \frac{P_0 H_2^2}{2L\mu} - \frac{C_3 H_2}{\mu_2} \cancel{+C_4}$$

$$C_1 = \frac{\frac{P_0}{2L}\left[\frac{H_2^2}{\mu_1} - \frac{H_1^2}{\mu_2}\right]}{\left[\frac{H_1}{\mu_1} + \frac{H_2}{\mu_2}\right]} = C_3$$

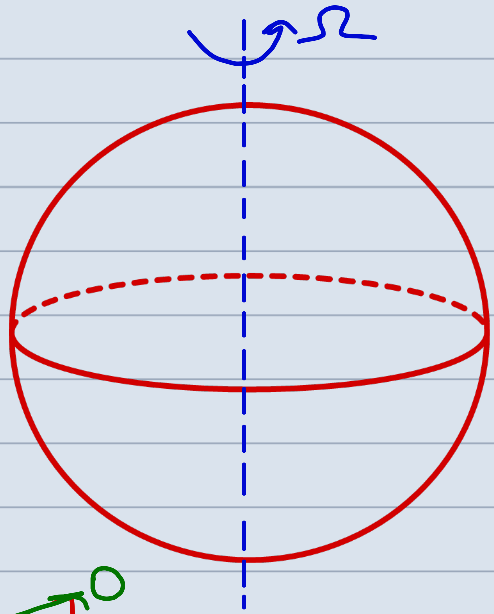
Volumetric flow rate $= \dfrac{\iint v_x \, dy\, dz}{\iint dy\, dz}$

$$V_{(1)} = \frac{\int_0^{H_1} v_x \, dy}{\int_0^{H_1} dy} = \frac{\frac{P_0 H_1^3}{6L\mu_1} + \frac{c_1 H_1^2}{2\mu_1} + c_2 H_1}{H_1}$$

$$V_{(1)} = \frac{P_0 H_1^2}{6L\mu_1} + \frac{c_1 H_1}{2\mu_1} + c_2$$

$$V_{(2)} = \frac{P_0 H_2^2}{6L\mu_2} - \frac{c_3 H_2}{2\mu_1} - c_4$$

2)



$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$v_\phi = v_\phi(r,\theta)$

$p(r,\theta)$

$$\rho\left[\cancel{\frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{v}\cdot\nabla\underset{\sim}{v}}\right]^{0} = -\underset{\sim}{\nabla} p + \mu\nabla^2\underset{\sim}{v} + \rho\underset{\sim}{g}$$

Stokes Eqns: $\underset{\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim}{-\underset{\sim}{\nabla} p + \mu\nabla^2\underset{\sim}{v} + \rho\underset{\sim}{g} = 0}$

In spherical coordinates:

r-direction:

Spherical coordinates (r, θ, ϕ):

$$\rho\left(\frac{\partial \cancel{v_r}}{\partial t} + v_r \frac{\partial \cancel{v_r}}{\partial r} + \frac{\cancel{v_\theta}}{r}\frac{\partial \cancel{v_r}}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial \cancel{v_r}}{\partial \phi} - \frac{\cancel{v_\theta^2} + v_\phi^2}{r}\right) = -\frac{\partial p}{\partial r}$$
$$+ \mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(\cancel{r^2 v_r}) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \cancel{v_r}}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \cancel{v_r}}{\partial \phi^2}\right] + \rho g_r \quad \text{(B.6-7)}^a$$

$$\rho\left(\frac{\partial \cancel{v_\theta}}{\partial t} + v_r \frac{\partial \cancel{v_\theta}}{\partial r} + \frac{v_\theta}{r}\frac{\partial \cancel{v_\theta}}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial \cancel{v_\theta}}{\partial \phi} + \frac{\cancel{v_r v_\theta} - v_\phi^2 \cot\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta}$$
$$+ \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\theta \sin\theta)\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 v_\theta}{\partial \phi^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{2 \cot\theta}{r^2 \sin\theta}\frac{\partial v_\phi}{\partial \phi}\right] + \rho g_\theta \quad \text{(B.6-8)}$$

$$\rho\left(\cancel{\frac{\partial v_\phi}{\partial t}}^{0} + \cancel{v_r} \frac{\partial v_\phi}{\partial r} + \frac{\cancel{v_\theta}}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\cancel{\frac{\partial v_\phi}{\partial \phi}}^{0} + \frac{v_\phi \cancel{v_r} + \cancel{v_\theta} v_\phi \cot\theta}{r}\right) = -\frac{1}{r \sin\theta}\frac{\partial p}{\partial \phi}$$
$$+ \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\phi \sin\theta)\right) + \frac{1}{r^2 \sin^2\theta}\cancel{\frac{\partial^2 v_\phi}{\partial \phi^2}}^{0} + \frac{2}{r^2 \sin\theta}\frac{\partial \cancel{v_r}}{\partial \phi} + \frac{2 \cot\theta}{r^2 \sin\theta}\frac{\partial \cancel{v_\theta}}{\partial \phi}\right] + \cancel{\rho g_\phi}^{0} \quad \text{(B.6-9)}$$

θ - momentum balance:

$$0 = \frac{-1}{r\sin\theta}\cancel{\frac{\partial p}{\partial \phi}}^{0} + \mu\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\phi \sin\theta)\right)\right]$$

$p(r, \theta)$

Consider $v_\phi = f(r, \theta)$

Using seperation variable. $= f(r)\underbrace{g(\theta)}_{\to \sin\theta}$

Boundary conditions: $v_\phi(R, \theta) = \Omega R \sin\theta$

$v_\phi(r \to \infty)$

$$\frac{1}{r^2}\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{d}{d\theta}\left(f(r)\sin\theta \times \sin\theta\right)\right)$$

$$\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\cancel{\sin\theta}}\times f(r)\times 2\cancel{\sin\theta}\cos\theta\right) = \frac{-2f(r)\sin\theta}{r^2}$$

$$\frac{1}{r^2}\left[\frac{\partial}{\partial r}(r^2 f(r)\sin\theta)\right] = \frac{\sin\theta}{r^2}\left[2rf'(r) + r^2 f''(r)\right]$$

$$\frac{1}{r^2}\left[2rf'(r)+r^2f''(r)\right] - \frac{2f(r)}{r^2} = 0$$

Let $f(r) = r^m$

$$\frac{1}{r^2}\left[2r\times m r^{m-1} + r^2\times m(m-1)r^{m-2}\right] - 2r^{m-2} = 0$$

$$2m + m(m-1) - 2 = 0$$

$$m^2 + m - 2 = 0 \qquad m = -2$$

$$m^2 + 2m - m - 2 = 0 \qquad m = 1$$

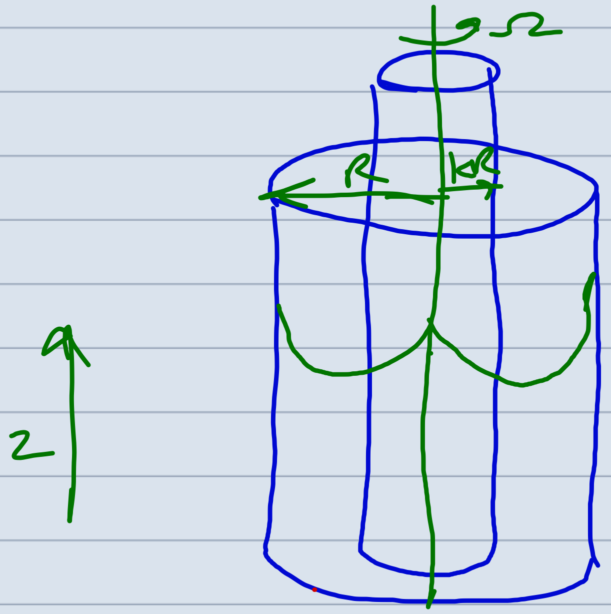
$$v_\phi = \left(\frac{c_1}{r^2} + c_2 r\right)\sin\theta$$

$$v_\theta(r=R) = \left(\frac{c_1}{R^2} + c_2 R\right)\sin\theta = \Omega R\sin\theta$$

$$v_\theta(r\to\infty) = c_2 R\sin\theta = 0 \Rightarrow c_2 = 0$$

$$\boxed{v_\phi = \frac{\Omega R^3}{r^2}\sin\theta}$$

3)



$v_r = 0 \,;\, v_z = 0$

$v_\theta = f(r)$ $\quad \dfrac{\partial v_\theta}{\partial \theta} = 0 \Downarrow \underline{\nabla \cdot v = 0}$

$\dfrac{\partial v_\theta}{\partial z} = 0$

↓ only long cylinder

Equation of Motion in θ direction:

ylindrical coordinates (r, θ, z):

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + \rho g_r \quad \text{(B.6-4)}$$

$$\rho\left(\cancel{\frac{\partial v_\theta}{\partial t}}^{0} + \cancel{v_r}\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\cancel{\frac{\partial v_\theta}{\partial \theta}}^{0} + \cancel{v_z}\frac{\partial v_\theta}{\partial z} + \cancel{\frac{v_r v_\theta}{r}}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_\theta}{\partial \theta^2}}^{0} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}}^{0} + \frac{2}{r^2}\cancel{\frac{\partial v_r}{\partial \theta}}^{0}\right] + \cancel{\rho g_\theta}^{0} \quad \text{(B.6-5)}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z \quad \text{(B.6-6)}$$

$$0 = \cancel{-\frac{1}{r}\frac{\partial p}{\partial \theta}}^{0} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)\right]$$

axi-symmetry of fluid flow.

$$\frac{c_1}{\mu} r = \frac{\partial}{\partial r}(rv_\theta) \qquad \Rightarrow v_\theta = \underline{\underline{\frac{c_1 r}{2\mu} + \frac{c_2}{r}}}$$

$$rv_\theta = \frac{c_1 r^2}{2\mu} + c_2$$

Boundary conditions: $v_\theta(r = kR) = \Omega kR$ {No slip condition}

$v_\theta(r = R) = 0$ {No slip}

$$-\Omega kR = \frac{C_1 kR}{2\mu} + \frac{C_2}{kR} \quad ; \quad 0 = \frac{C_1 R}{2\mu} + \frac{C_2}{R}$$

$$C_2 = -\frac{C_1 R^2}{2\mu}$$

$$\Omega kR = \frac{C_1 kR}{2\mu} - \frac{C_1 R}{2\mu k}$$

$$\frac{\Omega kR \times 2\mu k}{k^2 R - R} = C_1 \Rightarrow C_1 = \frac{2\mu\Omega k^2}{k^2-1}$$

$$C_2 = -\frac{\Omega k^2 R^2}{k^2-1}$$

$$V_\theta = \frac{\Omega k^2 r}{k^2-1} - \frac{\Omega k^2}{(k^2-1)}\frac{R^2}{r}$$

$$V_\theta = \frac{\Omega k^2}{k^2-1}\left[r - \frac{R^2}{r}\right]$$

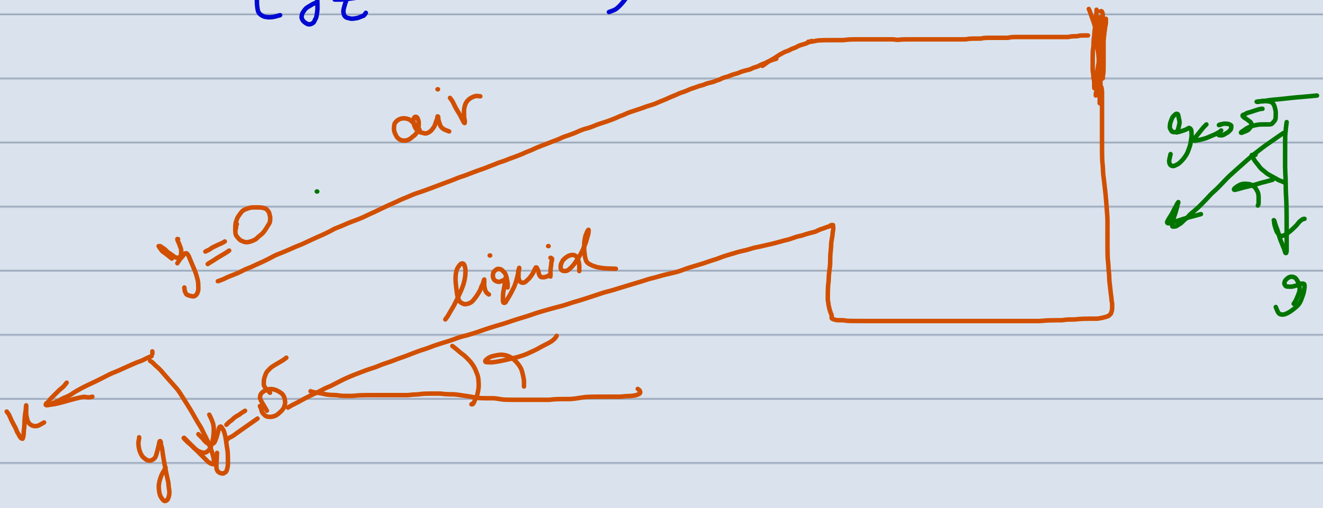
r-direction momentum balance:

$$\frac{\rho V_\theta^2}{r} = -\frac{\partial P}{\partial r} \qquad \int \partial P = -\int \frac{\rho}{r}\left[\left(\frac{\Omega k^2}{k^2-1}\right)^2\left[r^2 + \frac{R^4}{r^2} - 2R^2\right]\right]$$

Tutorial-9

1) Cauchy Equations:

$$\rho\left(\frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{v}\cdot\underset{\sim}{\nabla}\underset{\sim}{v}\right) = -\underset{\sim}{\nabla}\cdot\underset{\sim}{\tau} + \rho\,\underset{\sim}{g}$$



x - balance :

$$0 = -\underset{\sim}{\nabla}\cdot\underset{\sim}{\tau} + \rho\underset{\sim}{g}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underset{\sim}{v}) = 0$$

$$0 = -\frac{\partial}{\partial x}\left[\tau_{xx}\right] - \frac{\partial}{\partial y}\left(\tau_{yx}\right) - \frac{\partial}{\partial z}\left(\tau_{zx}\right) + \rho\underset{\sim}{g}$$

$$0 = -\frac{\partial}{\partial x}\left[-\mu\left[\cancelto{0}{\frac{\partial v_x}{\partial x}} + \frac{\partial v_x}{\partial x}\right]\right] - \frac{\partial}{\partial y}\left[-\mu\left[\frac{\partial v_x}{\partial y} + \cancelto{0}{\frac{\partial v_y}{\partial x}}\right]\right]$$

$$-\frac{\partial}{\partial z}\left[-\mu\left[\cancelto{0}{\frac{\partial v_x}{\partial z}} + \cancelto{0}{\frac{\partial v_z}{\partial x}}\right]\right] + \rho g$$

$\mu(\dot{\gamma})$

$$0 = \mu\frac{d^2 v_x}{dy^2} + \rho g\cos\alpha$$

$$\underline{\mu(\dot{\gamma}) = m\left|\frac{dv_x}{dy}\right|^{n-1}} = m\left(-\frac{dv_x}{dy}\right)^{n-1}$$

$$0 = m\left(\frac{-\partial v_x}{\partial y}\right)^{n-1}\left(\frac{\partial^2 v_x}{\partial y^2}\right) + \rho g \cos\theta$$

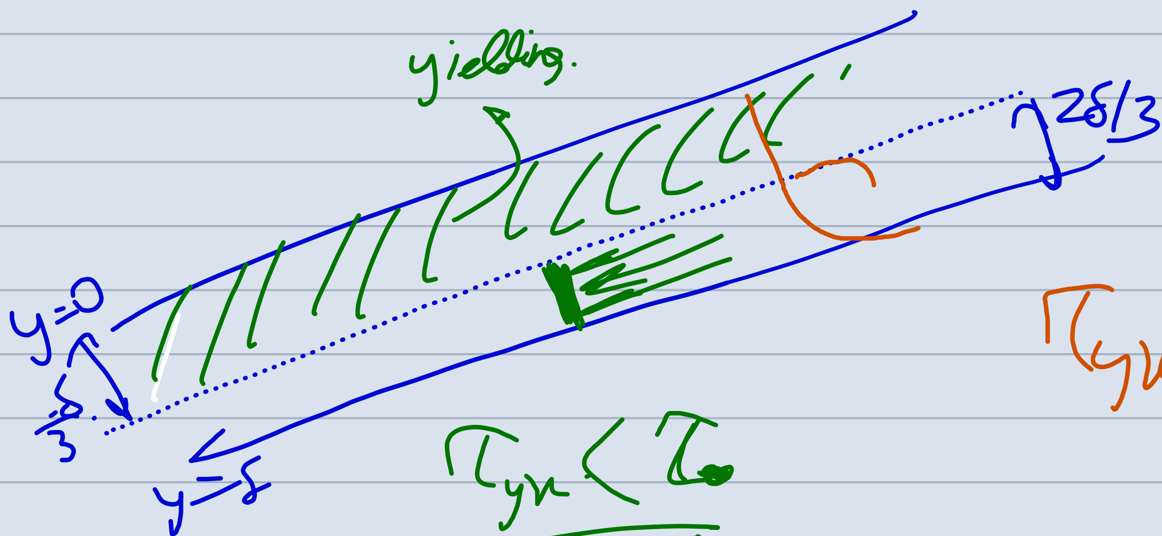
$$\rho g \cos\theta\, y = m\left(\frac{-\partial v_x}{\partial y}\right)^{n}$$

$$\frac{-\partial v_x}{\partial y} = \left(\frac{\rho g \cos\theta y}{m}\right)^{1/n}$$

$$v_x = -\frac{y^{\frac{1}{n}+1}}{\left(\frac{1}{n}+1\right)}\left(\frac{\rho g \cos\theta}{m}\right)^{1/n} + C_1$$

$$\boxed{v_x = \left(\frac{n}{n+1}\right)\left(\frac{\rho g \sin\theta}{m}\right)^{1/n}\left[\delta^{\frac{n+1}{n}} - y^{\frac{n+1}{y}}\right]}$$

Q)



$\tau_{yx} < \tau_0$

$\underline{\underline{\tau_{yx} < \tau_0}}$

$$\frac{\partial}{\partial y}(\tau_{yx}) = \rho g \sin\theta$$

$$\tau_{yx} = \rho g y \sin\theta + C$$

$$-\left(\mu_0 + \frac{\tau_0}{-\frac{\partial v_x}{\partial y}}\right)\frac{\partial v_x}{\partial y} = \rho g y \sin\theta + C$$

$$-\mu_0 \frac{\partial v_x}{\partial y} + \tau_0 = \rho g y \sin\theta + C$$

$$v_x =$$

Cylindrical coordinates (r, θ, z):[b]

$$\rho\left(\cancel{\frac{\partial v_r}{\partial t}} + v_r \cancel{\frac{\partial v_r}{\partial r}} + \frac{v_\theta}{r}\cancel{\frac{\partial v_r}{\partial \theta}} + v_z \cancel{\frac{\partial v_r}{\partial z}} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} - \left[\frac{1}{r}\frac{\partial}{\partial r}(\cancelto{0}{r\tau_{rr}}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta r} + \frac{\partial}{\partial z}\tau_{zr} - \frac{\tau_{\theta\theta}}{r}\right] + \rho g_r \quad \text{(B.5-4)}$$

$$\rho\left(\cancelto{0}{\frac{\partial v_\theta}{\partial t}} + \cancel{v_r}\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\cancelto{0}{\frac{\partial v_\theta}{\partial \theta}} + v_z \cancelto{0}{\frac{\partial v_\theta}{\partial z}} + \cancel{\frac{v_r v_\theta}{r}}\right) = -\frac{1}{r}\cancelto{0}{\frac{\partial p}{\partial \theta}} - \left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta\theta} + \frac{\partial}{\partial z}\tau_{z\theta} + \frac{\tau_{\theta r} - \tau_{r\theta}}{r}\right] + \rho g_\theta \quad \text{(B.5-5)}$$

$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}} + \cancelto{0}{v_r}\cancel{\frac{\partial v_z}{\partial r}} + \frac{v_\theta}{r}\cancel{\frac{\partial v_z}{\partial \theta}} + \cancel{v_z}\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} - \left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta z} + \frac{\partial}{\partial z}\tau_{zz}\right] + \rho g_z \quad \text{(B.5-6)}$$

$\tau = \mu \underset{\sim}{\nabla}\underset{\sim}{v}$

Q3)

Cylindrical coordinates (r, θ, z):

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + \rho g_r \quad \text{(B.6-4)}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right] + \rho g_\theta \quad \text{(B.6-5)}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z \quad \text{(B.6-6)}$$

Cylindrical coordinates (r, θ, z):

$$\tau_{rr} = -\mu\left[2\cancelto{0}{\frac{\partial v_r}{\partial r}}\right] + \left(\tfrac{2}{3}\mu - \kappa\right)\cancelto{0}{(\nabla \cdot \mathbf{v})}$$

$$\tau_{\theta\theta} = -\mu\left[2\left(\frac{1}{r}\cancelto{0}{\frac{\partial v_\theta}{\partial \theta}} + \cancelto{0}{\frac{v_r}{r}}\right)\right] + \left(\tfrac{2}{3}\mu - \kappa\right)\cancelto{0}{(\nabla \cdot \mathbf{v})}$$

$$\tau_{zz} = -\mu\left[2\cancelto{0}{\frac{\partial v_z}{\partial z}}\right] + \left(\tfrac{2}{3}\mu - \kappa\right)\cancelto{0}{(\nabla \cdot \mathbf{v})}$$

$$\tau_{r\theta} = \tau_{\theta r} = -\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\cancelto{0}{\frac{\partial v_r}{\partial \theta}}\right]$$

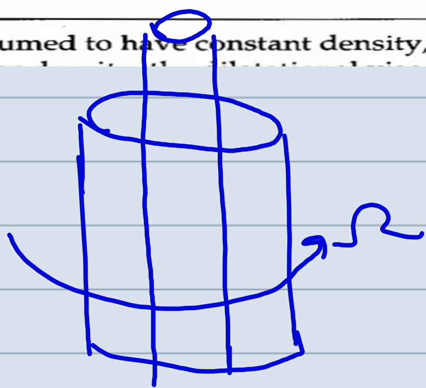
$$\tau_{\theta z} = \tau_{z\theta} = -\mu\left[\frac{1}{r}\cancel{\frac{\partial v_z}{\partial \theta}} + \cancelto{0}{\frac{\partial v_\theta}{\partial z}}\right]$$

$$\tau_{zr} = \tau_{rz} = -\mu\left[\cancelto{0}{\frac{\partial v_r}{\partial z}} + \cancelto{0}{\frac{\partial v_z}{\partial r}}\right]$$

in which

$$(\nabla \cdot \mathbf{v}) = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

[a] When the fluid is assumed to have constant density, the term containing $(\nabla \cdot \mathbf{v})$ may b



$V_\theta = f(r)$

$V_r = V_z = 0$

$V_\theta(r=R) = \Omega R$

$V_\theta(r=\text{large}) = 0$

Momentum balance in θ-direction:

$$-\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta\theta} + \frac{\partial}{\partial z}\tau_{z\theta} + \frac{\tau_{\theta r} - \tau_{r\theta}}{r}\right] + \rho g_\theta$$

$$\tau_{r\theta} = -\mu r\left[\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\right]$$

$$\tau_{\theta\theta} = 0$$

$$\tau_{z\theta} = 0$$

$$0 = -\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\tau_{r\theta}\right)\right]$$

Cauchy Eqn.

$$\tau_{r\theta} = \frac{c_1}{r^2}$$

$$-\mu r\left[\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\right] = \frac{c_1}{r^2}$$

$$\mu = m\left|\frac{\partial v_\theta}{\partial r}\right|^{n-1} = m\left(\frac{\partial v_\theta}{\partial r}\right)^{n-1}$$

$$m\left[\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right]^{n-1} \times r\left[\frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2}\right] = \frac{c_1}{r^2}$$

$$m\left[\frac{\partial v_\theta}{r} - \frac{v_\theta}{r}\right]^{n-1} \times \left[\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right] = \frac{c_1}{r^2}$$

$$m\left[\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right]^n=\frac{c_1}{r^2}$$

$$\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}=\frac{c_1'}{m\,r^{2/n}}$$

$$e^{\int -\frac{1}{r}dr}=e^{-\ln r}=\frac{1}{r}$$

$$\frac{v_\theta}{r}dr=\int \frac{1}{r}\times\frac{c_1'}{m\,r^{2/n}}\,dr$$

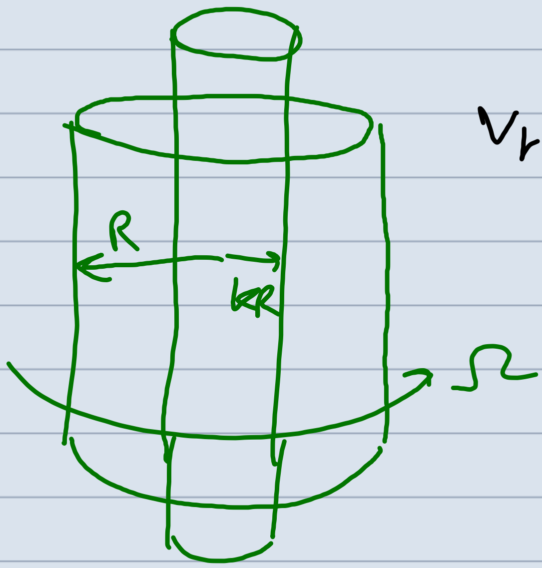
$$-\left(1+\frac{2}{n}\right)$$

$$\frac{v_\theta}{r}=\frac{c_1'}{m}\times\frac{-1}{\left(\frac{2}{n}\right)}r^{-2/n}$$

$$\boxed{v_\theta=-\frac{c_1'}{m}r^{1-\frac{2}{n}}}$$

Tutorial-8

1)



$v_r=v_z=0$

$$\rho\left[\frac{\partial \underset{\sim}{v}}{\partial t}+\underset{\sim}{v}\cdot\nabla\underset{\sim}{v}\right]=\underset{\sim}{\nabla}p+\mu\nabla^2\underset{\sim}{v}+\rho\underset{\sim}{g}$$

Navier-Stokes Equation.

$$\frac{\partial v_\theta}{\partial\theta}=0$$

$v_\phi=f(r,z)$ $\quad v_\phi\neq f(\theta)$ as $\underset{\sim}{\nabla}\cdot\underset{\sim}{v}=0$

conservation of mass.

$\frac{\partial v_\phi}{\partial z} = 0$ as $L >> R$ {infinite dimensionality in z direction}

$v_\phi = f(r)$

$$\rho\left(\cancel{\frac{\partial v_\theta}{\partial t}}^{0} + \cancel{v_r}\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\cancel{\frac{\partial v_\theta}{\partial \theta}}^{0} + \cancel{v_z}\frac{\partial v_\theta}{\partial z} + \frac{\cancel{v_r} v_\theta}{r}\right) = -\frac{1}{r}\cancel{\frac{\partial p}{\partial \theta}}^{0} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_\theta}{\partial \theta^2}}^{0} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}}^{0} + \frac{2}{r^2}\cancel{\frac{\partial v_r}{\partial \theta}}^{0}\right] + \rho \cancel{g_\theta}^{0}$$

$$0 = \mu \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right)$$

$$\frac{c_1 r}{\mu} = \frac{\partial}{\partial r}(r v_\theta) \qquad r v_\theta = \frac{c_1 r^2}{2\mu} + c_2$$

$$v_\theta = \frac{c_1 r}{2\mu} + \frac{c_2}{r}$$

$$\tau_{r\theta} = \tau_{r\theta} = \tau_{\theta r} = -\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\cancel{\frac{\partial v_r}{\partial \theta}}^{0}\right]$$

$$= -\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\cancel{\frac{\partial v_r}{\partial \theta}}^{0}\right]$$

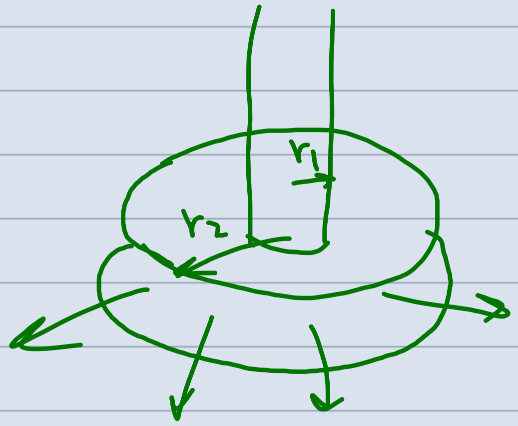
$$= -\mu\left[r\frac{\partial}{\partial r}\left(\frac{c_1}{2\mu} + \frac{c_2}{r}\right)\right] = -\mu\left[r\left(-\frac{2c_2}{r^3}\right)\right] = \frac{2\mu c_2}{r^2}$$

$$\tau_{r\theta} = \frac{2\mu c_2}{r^2}$$

$$T_z = \left(\frac{2\mu c_2 \times 2\pi(\cancel{kR})L}{\cancel{r^2}}\right)\cancel{\times kR}$$

$$= \frac{4\pi\mu c_2 \cancel{kR^2} L}{\cancel{r^2}} = 4\pi\mu c_2 L$$

2>



THE EQUATION OF CONTINUITY[a]

$$[\partial\rho/\partial t + (\nabla\cdot\rho\mathbf{v}) = 0]$$

Cartesian coordinates (x, y, z):

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0 \qquad \text{(B.4-1)}$$

Cylindrical coordinates (r, θ, z):

$$\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0 \qquad \text{(B.4-2)}$$

Spherical coordinates (r, θ, ϕ):

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho v_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\rho v_\phi) = 0 \qquad \text{(B.4-3)}$$

[a] When the fluid is assumed to have constant mass density ρ, the equation simplifies to $(\nabla\cdot\mathbf{v}) = 0$.

$$\cancel{\frac{\partial\rho}{\partial t}}^{0} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(\cancel{\rho v_\theta})^{0} + \frac{\partial}{\partial z}(\rho v_z) = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) = 0$$

$$v_r + r\frac{\partial v_r}{\partial r} = 0$$

$$\boxed{\frac{\partial v_r}{\partial r} = -\frac{v_r}{r}}$$

θ-component balance in Navier-Stokes:

$$\rho\left(\cancel{\frac{\partial v_\theta}{\partial t}}^{0} + v_r\cancel{\frac{\partial v_\theta}{\partial r}} + \cancel{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}} + v_z\cancel{\frac{\partial v_\theta}{\partial z}} + \cancel{\frac{v_r v_\theta}{r}}\right) = -\cancel{\frac{1}{r}\frac{\partial p}{\partial\theta}}^{0} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\cancel{v_\theta})^{0}\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_\theta}{\partial\theta^2}}^{0} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}}^{0} + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta}\right] + \rho g_\theta$$

$$0 = \frac{2\mu}{r^2}\frac{\partial v_r}{\partial \theta}$$

v_r

Cylindrical coordinates (r, θ, z):

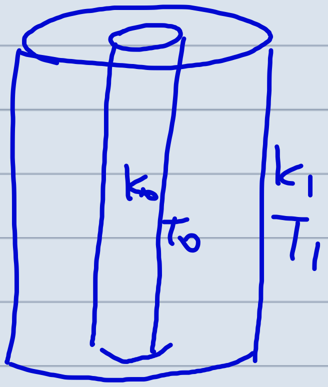
$$\rho\left(\cancel{\frac{\partial v_r}{\partial t}} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \cancel{\frac{v_\theta^2}{r}}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}} + \frac{\partial^2 v_r}{\partial z^2} - \cancel{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}\right] + \rho g_r \quad \text{(B.6-4)}$$

$$\rho\left(\cancel{\frac{\partial v_\theta}{\partial t}} + v_r\cancel{\frac{\partial v_\theta}{\partial r}} + \frac{v_\theta}{r}\cancel{\frac{\partial v_\theta}{\partial \theta}} + v_z\cancel{\frac{\partial v_\theta}{\partial z}} + \cancel{\frac{v_r v_\theta}{r}}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right] + \rho g_\theta \quad \text{(B.6-5)}$$

$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}} + v_r\cancel{\frac{\partial v_z}{\partial r}} + \frac{v_\theta}{r}\cancel{\frac{\partial v_z}{\partial \theta}} + v_z\cancel{\frac{\partial v_z}{\partial z}}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\cancel{\frac{\partial v_z}{\partial r}}\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_z}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}\right] + \rho g_z \quad \text{(B.6-6)}$$

Tutorial-10

1)



$$k = aT + b$$

$$k_0 = aT_0 + b$$

$$k_1 = aT_1 + b$$

$$k_0 - k_1 = a(T_0 - T_1)$$

$$a = \frac{k_0 - k_1}{T_0 - T_1}$$

$$b = k_0 - \left(\frac{k_0 - k_1}{T_0 - T_1}\right)T_0$$

$$b = \frac{\cancel{k_0T_0} - T_1k_0 - \cancel{k_0T_0} + k_1T_0}{T_0 - T_1}$$

$$\boxed{k = \left(\frac{k_0 - k_1}{T_0 - T_1}\right)T + \frac{k_1T_0 - k_0T_1}{T_0 - T_1}}$$

r - direction energy shell balance:

$$2\pi r\left[q_r\big|_r - q_r\big|_{r+\Delta r} \right] = 0$$

$$\frac{1}{r}\frac{d}{dr}(r q_r) = 0$$

$$q_r = \frac{c}{r}$$

$$-k\frac{dT}{dr} = \frac{c}{r}$$

$$-\left[\left(\frac{k_0-k_1}{T_0-T_1}\right)T + \frac{k_1T_0-k_0T_1}{T_0-T_1}\right]dT = \frac{c}{r}dr$$

$$-\left[\left(\frac{k_0-k_1}{T_0-T_1}\right)(T-T_1) + \frac{\cancel{T_1k_0}-T_1k_1+k_1T_0-\cancel{k_0T_1}}{T_0-T_1}\right]dT = \frac{c}{r}dr$$

$$\Theta = \frac{T-T_1}{T_0-T_1} \qquad \bar{r} = \frac{r}{R}$$

$$-\left[(k_0-k_1)\Theta + k_1\right]d\Theta = \frac{c}{\bar{r}}d\bar{r}$$

$$-(k_0-k_1)\frac{\Theta^2}{2} - k_1\Theta = c_1\ln\bar{r} + c_2$$

$$c_1\ln k + c_2 = \frac{(-k_0+k_1)}{2} - k_1 = \frac{-k_0}{}$$

$$c_2 = 0$$

$$c_1 = \frac{-(k_0+k_1)}{\ln k \; 2}$$

$$\frac{(k_0-k_1)\,(1\cdot 1)^2}{2} + k_1\,\Theta = \frac{\ln \bar{r}}{\ln k}$$

2) $$\frac{4\pi r^2\left[q_r|_r - q_r|_{r+\Delta r}\right]}{4\pi r^2 \Delta r} = 0$$

$$\frac{1}{r^2}\frac{d}{dr}(r^2 q_r) = 0$$

$$q_r = \frac{c}{r^2}$$

$$-k\frac{dT}{dr} = \frac{c}{r^2}$$

$$-\left[(k_0-k_1)\Theta + k_1\right] d\Theta = \frac{c}{k\bar{r}^2}\,d\bar{r} \qquad \Theta = \frac{T-T_1}{T_0-T_1}$$

$$-\frac{(k_0-k_1)\Theta^2}{2} - k_1\Theta = \frac{-c}{k\bar{r}} + c_2$$

$$0 = \frac{-c}{k} + c_2$$

$$-\frac{(k_0-k_1)}{2} - k_1 = \frac{-c}{kk} + c_2$$

$$-\left(\frac{k_0+k_1}{2}\right) = \frac{-c}{kR} + \frac{c}{R}$$

$$-\left(\frac{k_0+k_1}{2}\right) = \frac{c}{R}\left[1 - \frac{1}{k}\right]$$

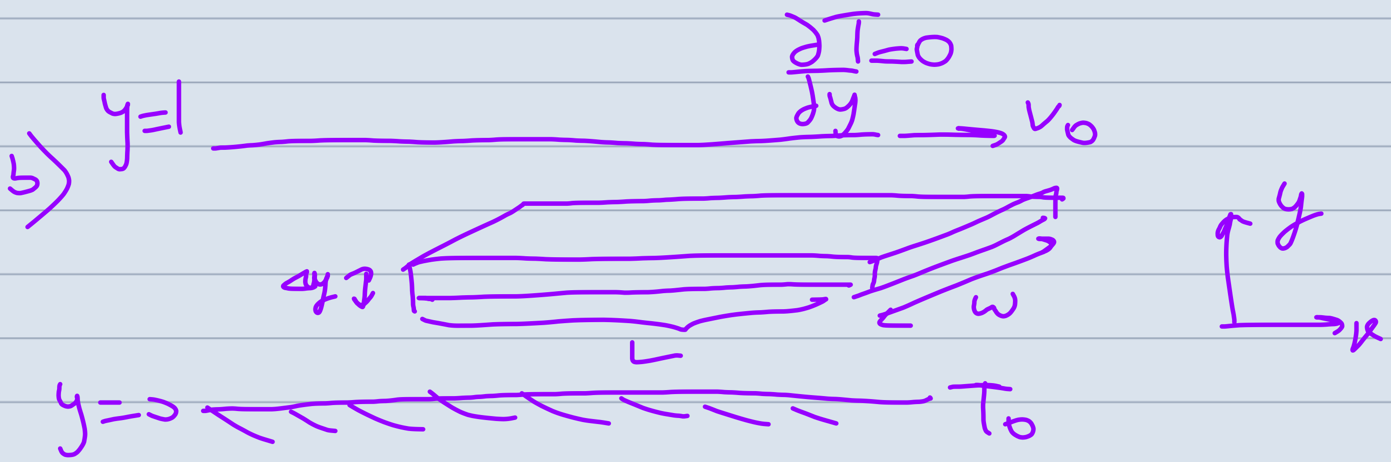
$$c = \frac{-Rk}{(k-1)}\left[\frac{k_0+k_1}{2}\right] = \frac{Rk}{1-k}\left[\frac{k_0+k_1}{2}\right]$$

$$c_L = \frac{k}{1-k}\left[\frac{k_0+k_1}{2}\right]$$

$$\frac{(k_0-k_1)\,\Theta^2}{2} + k_1\Theta = \frac{kR}{\gamma(1-k)}\left[\frac{k_0+k_1}{2}\right] - \frac{k}{1-k}\left(\frac{k_0+k_1}{2}\right.$$

$$\frac{(k_0-k_1)\,\Theta^2}{2} + k_1\Theta = \frac{k}{1-k}\left[\frac{k_0+k_1}{2}\right]\left[\frac{R}{\gamma} - 1\right]$$

b)



$$\underset{\sim}{e} = \left(\frac{1}{2}\rho v^2 + \rho\hat{H}\right)\underset{\sim}{v} + \underset{\approx}{\tau}\cdot\underset{\sim}{v} + \underset{\sim}{\nabla}q$$

Energy shell balance:

$$W\Delta y\left[e_x|_x - e_x|_{x+L}\right] + LW\left[e_y|_y - e_y|_{y+\Delta y}\right]$$

$$+\Delta y L\left[e_z|_z - e_z|_{z+W}\right] + LW\Delta y(\text{source}) = 0$$

$$\cancel{\frac{1}{L}\left[e_x|_x - e_x|_{x+L}\right]}^{0} + \frac{1}{\Delta y}\left[e_y|_y - e_y|_{y+\Delta y}\right] + \cancel{\frac{1}{W}\left[e_z|_z - e_z|_{z+W}\right]}^{0} = 0$$

$$\frac{de_y}{dy} = 0$$

$$\underline{e_y = \text{const}}$$

$$\left(\frac{1}{2}\rho v^2 + \rho\hat{H}\right)\cancel{v_y}^{0} + \tau_{yi}v_i + q_y = \text{Const}$$

$$\tau_{yx}v_x \quad \left(-k\frac{dT}{dy}\right) = C$$

$$-\mu\left(\frac{dv_x}{dy}\right)v_x - k\frac{dT}{dy} = C$$

From Navier-Stokes eqn:

$$\rho\left(\frac{d\underset{\sim}{v}}{dt} + \underset{\sim}{v}\cdot\nabla\underset{\sim}{v}\right) = \underset{\sim}{\nabla}p + \mu\nabla^2\underset{\sim}{v} + \rho\underset{\sim}{g}$$

$$\underline{v_x = \frac{v_0 y}{H}}$$

$$-\frac{\mu V_0}{H}\frac{V_0 y}{H} - K\frac{dT}{dy} = C$$

$$K\frac{dT}{dy} = -\frac{\mu V_0^2}{H^2}y + C_1$$

$$T = -\frac{\mu V_0^2}{2H^2K}y^2 + \frac{C_1}{K}y + C_2$$

Boundary conditions: $y=0 \; : \; T = T_0$

$y=H \; : \; \left.\frac{dT}{dy}\right|_{y=H} = 0$

$T_0 = C_2$

$$0 = -\frac{\mu V_0^2 \cancel{H}}{H^{\cancel{2}}\cancel{K}} + \frac{C_1}{K}$$

$$C_1 = \frac{\mu V_0^2}{H}$$

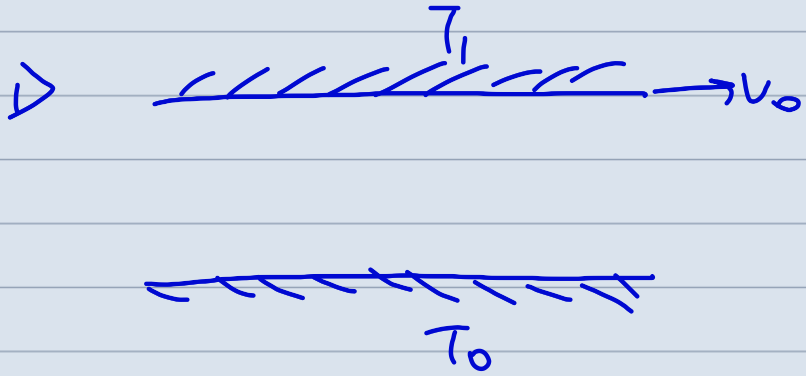
$$T = -\frac{\mu V_0^2}{2H^2K}y^2 + \frac{\mu V_0^2}{HK}y + T_0$$

$$T = T_0 + \frac{\mu V_0^2}{HK}\left[y - \frac{y^2}{2H}\right]$$

$$T = T_0 - \frac{\mu V_0^2 y^2}{2H^2K}\left[1 - \frac{2H}{y}\right]$$

Brinkmann number

Tutorial - 11



$$\underset{\sim}{\tau} = \mu(\dot\gamma)\underset{\sim}{\dot\gamma}$$

$$\mu(\dot\gamma) = m|\dot\gamma|^{n-1}$$

$$\frac{\partial e_y}{\partial y} = 0$$

$$e_y = \text{const.}$$

$$\left(\frac{1}{2}\rho v^2 + \rho \hat{h}\right)\cancelto{0}{v_y} + \tau_{yi} v_i + q_y = \text{const.}$$

$$\tau_{xy} v_x - k\frac{dT}{dy} = c_1$$

$$V_x = \frac{V_0 y}{H} \qquad \frac{dv_x}{dy} = \frac{V_0}{H} > 0$$

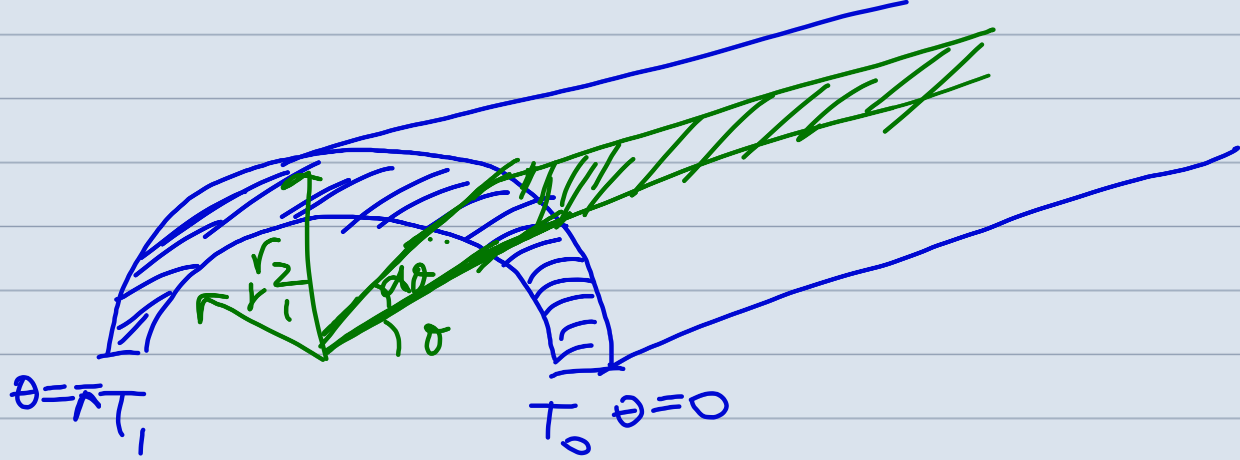
$$\mu(\dot\gamma)\frac{\partial v_x}{\partial y} v_x - k\frac{dT}{dy} = c_1$$

$$m\left(\frac{\partial v_x}{\partial y}\right)^n V_x - k\frac{dT}{dy} = c_1$$

$$m\left(\frac{V_0}{H}\right)^{n+1} - k\frac{dT}{dy} = c_1$$

$$\boxed{T = \frac{m}{k}\left(\frac{V_0}{H}\right)^{n+1} y + C_2}$$

2>



Energy balance in θ direction: $\frac{1}{r}\frac{\partial e_\theta}{\partial \theta} = 0$

$e_\theta = \text{const.}$

$$-k\frac{1}{r}\frac{\partial T}{\partial \theta} = c_1$$

$$k = k_0 + (k_1 - k_0)\Theta \qquad \Theta = \frac{T - T_0}{T_1 - T_0}$$

$$\left(k_0 + (k_1 - k_0)\Theta\right)\frac{1}{r}\frac{\partial \Theta}{\partial \theta} = c_1$$

$$\frac{k_0}{r}\Theta + \frac{(k_1 - k_0)}{r}\frac{\Theta^2}{2} = c_1\theta + \cancel{c_2}^{\,0}$$

$$\frac{k_0}{r} + \frac{k_1 - k_0}{2r} = c_1\pi$$

$$c_1 = \frac{k_1 + k_0}{2r\pi}$$

$$\underline{\underline{\frac{k_0\Theta}{r} + \frac{k_1 - k_0}{r}\frac{\Theta^2}{2} = \frac{(k_1 + k_0)\theta}{2r\pi}}}$$

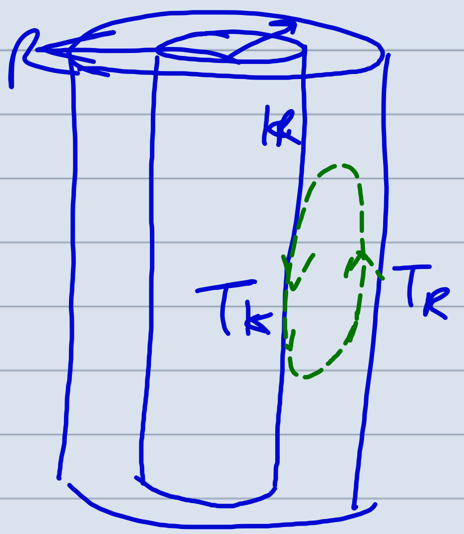
$$Q = \int_{z=0}^{L} \int_{r=r_1}^{r=r_2} (-k\nabla T)\,dr\,dz \Big|_{\theta=\pi}$$

$$= (T_1 - T_0) \int_{r_1}^{r_2} -\frac{k}{r}\frac{\partial \Theta}{\partial \theta}\Big|_{\theta=\pi} dr$$

$$= (T_1 - T_0) \int_{r_1}^{r_2} -\frac{k}{r}\left(\frac{k_0 + k_1}{2\pi k_1}\right) dr\,dz$$

$$\boxed{Q = -(T_1 - T_0)\ln\left(\frac{r_2}{r_1}\right)\left(\frac{k_0 + k_1}{2\pi}\right)}$$

3)



→ Free convection: velocity profile influenced by temperature

$$2\pi r L\left(q_r|_r - q_r(r+\Delta r)\right) = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r q_r) = 0$$

$$q_r = \frac{c}{r}$$

$$\bar{r} = \frac{r}{R}$$

$$-k\frac{dT}{dr} = \frac{c}{r}$$

$$\Theta = \frac{T - T_k}{T_R - T_k}$$

$$-k\frac{d\Theta}{d\bar{r}} = \frac{c}{\bar{r}}$$

$$\Theta = -c\ln\bar{r} + c_2$$

$$1 = -c\ln(R/R) + c_2 = c_2$$

$$0 = -c\ln k + c_2$$

$$c = \frac{1}{\ln k} \qquad c_2 = 1$$

$$\Theta = 1 - \frac{\ln\bar{r}}{\ln 1}$$

$$\Theta - 1 = -\frac{\ln\bar{r}}{\ln k} \qquad \boxed{\Theta' = \frac{\ln\bar{r}}{\ln k}}$$

$$\frac{T-T_k}{T_R-T_K} - 1 = -\frac{\ln\bar{r}}{\ln k} \qquad \frac{T-T_R}{T_K-T_R} = \frac{\ln\bar{r}}{\ln k}$$

Now $\rho(T) \Rightarrow$

In navier Stokes equation:

$$\rho\left(\frac{\partial \bar{v}}{\partial t} + \underset{\sim}{v}\cdot\nabla\underset{\sim}{v}\right) = \nabla p + \mu\nabla^2 v + \rho g$$

z - component:

$$-\frac{\partial p}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right) + \rho g_z = 0$$

$$\rho(T) = \rho(\bar{T}) + \left.\frac{\partial \rho}{\partial T}\right|_{T=\bar{T}}(T-\bar{T}) + O(\bar{T})^2$$

$$= \rho\bar{T} - \beta\rho(\bar{T})(T-\bar{T})$$

velocity profile can be integrated.