General points from MLL212:

Tensor: an object that is invariant under a change of coordinates and has components that changes in a special, predictable way under a change of coordinates

Vector is invariant but vector components are not invariant.

Tensor Notation:

$$\underset{\sim}{v} = v_1 \underset{\sim}{\delta_1} + v_2 \underset{\sim}{\delta_2} + v_3 \underset{\sim}{\delta_3}$$

$$\underset{\sim}{v} = \sum_{i=1}^{3} v_i \underset{\sim}{\delta_i} \Rightarrow \boxed{\underset{\sim}{v} = v_i}$$

contracted form.

i) $a_i x_i \Rightarrow a_1 x_1 + a_2 x_2 + a_3 x_3$

{2 free
1 dummy}

ii) $a_{ij} b_{jk} \Rightarrow a_{i1} b_{1k} + a_{i2} b_{2k} + a_{i3} b_{3k}$

iii) $a_{ij} b_{jk} c_k \Rightarrow \sum_{j=1}^{3} \sum_{k=1}^{3} a_{ij} b_{jk} c_k$ {2 dummy indices 1 free index}

$\underset{\sim}{D} = D_{ij} \underset{\sim}{\delta_i} \underset{\sim}{\delta_j}$ 2nd order tensor

$$a_{ij} b_j = \left(\sum_{i}^{3} \sum_{j}^{3} a_{ij} \underset{\sim}{\delta_i} \underset{\sim}{\delta_j} \right) \left(\sum_{j}^{3} b_j \underset{\sim}{\delta_j} \right)$$

Operations in Tensors:

1) Scalar Multiplication:

$$g\underset{\sim}{v} = g(v_1 \underset{\sim}{\delta_1} + v_2 \underset{\sim}{\delta_2} + v_3 \underset{\sim}{\delta_3})$$

$$= g(v_i \underset{\sim}{\delta_i}) = g v_i$$

2) Addition & Subtraction:

$$\underset{\sim}{a} + \underset{\sim}{b} = a_i \underset{\sim}{\delta_i} + b_j \underset{\sim}{\delta_j}$$

$$= (a_i + b_j) = (a_i + b_j)$$

3) Dot Product:

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \sum_{i}^{3} a_i \underset{\sim}{\delta_i} \cdot \sum_{j}^{3} b_j \underset{\sim}{\delta_j}$$

$$= \sum_{i}^{3} \sum_{j}^{3} a_i b_j (\underset{\sim}{\delta_i} \cdot \underset{\sim}{\delta_j})$$

$$= \sum_{i}^{3} \sum_{j}^{3} a_i b_j \delta_{ij} \quad \underset{\sim}{\delta_i} \underset{\sim}{\delta_j} = \sum_{i}^{3} \sum_{j}^{3} a_i b_j \underset{\approx}{\delta}$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} \equiv a_i b_i$$

Cross Prdt:

$$\underset{\sim}{a} \wedge \underset{\sim}{b} = a_i \underset{\sim}{\delta_i} \wedge b_j \underset{\sim}{\delta_j} = a_i b_j (\underset{\sim}{\delta_i} \wedge \underset{\sim}{\delta_j})$$

$$= a_i b_j \varepsilon_{ijk} \underset{\sim}{\delta_k} = a_i b_j \left(\underset{\approx}{\underset{\approx}{\varepsilon}} \underset{\sim}{\delta_i} \underset{\sim}{\delta_j} \underset{\sim}{\delta_k} \right) \underset{\sim}{\delta_k}$$

$$= \sum_i^3 \sum_j^3 a_i b_j \varepsilon_{ijk} \underset{\sim}{\delta_k}$$

i, j = dummy $\quad k$ = free