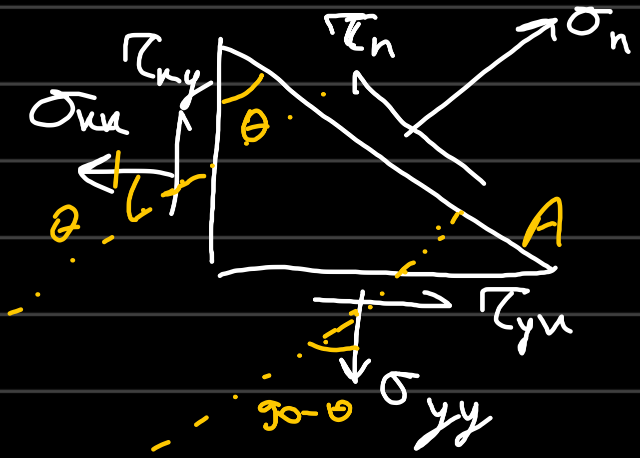
Lecture 3

Transformation of stresses in 2D.



$$\sigma_n = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta + 2\tau_{xy} \sin\theta\cos\theta$$

$$\tau_n = -\frac{\sigma_x}{2} \sin 2\theta + \frac{\sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

Force equilibrium:

$$\sigma_n(A) = (\sigma_x A\cos\theta)\cos\theta + (\sigma_y A\sin\theta)\sin\theta + (\tau_{yx} A\sin\theta)\cos\theta + (\tau_{xy} A\cos\theta)\sin\theta$$

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta\cos\theta$$

$$\tau_n(A) = (\sigma_y A\sin\theta)\cos\theta + (\tau_{yx} A\sin\theta)\sin\theta + (\sigma_x A\cos\theta)\sin\theta - (\tau_{xy} A\cos\theta)\cos\theta$$

$$\tau_{x'y'} = (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

$$\begin{bmatrix} \sigma_x' \\ \sigma_y' \\ \tau_{xy}' \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & & \\ & & \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

$$\sigma_x' = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_y' = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{xy} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{12}\cos 2\theta$$

Principal Stresses:

↳ no shear stress, maximum normal stress

↳ For any state of stress it is always possible to define

$$\tau_{x'y'} = 0$$

$$\tau_{xy}(\cos^2\theta - \sin^2\theta) + (\sigma_x - \sigma_y)\sin\theta\cos\theta = 0$$

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$\tan 2\theta = \tan(\pi + 2\theta)$: 2 angles obtained 90° apart.

↳ Hence two principal planes θ, & $90-\theta$, from original coordinates (x, y, z)

$$\left.\begin{array}{l} \sigma_{max} = \sigma_1 \\ \sigma_{min} = \sigma_2 \end{array}\right\} = \frac{\sigma_x + \sigma_y}{2} \pm \left\{\left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2\right]^{1/2}\right\}$$

To find plane for maximum shear stress:

$$\frac{d\tau_{x'y'}}{d\theta} = 0$$

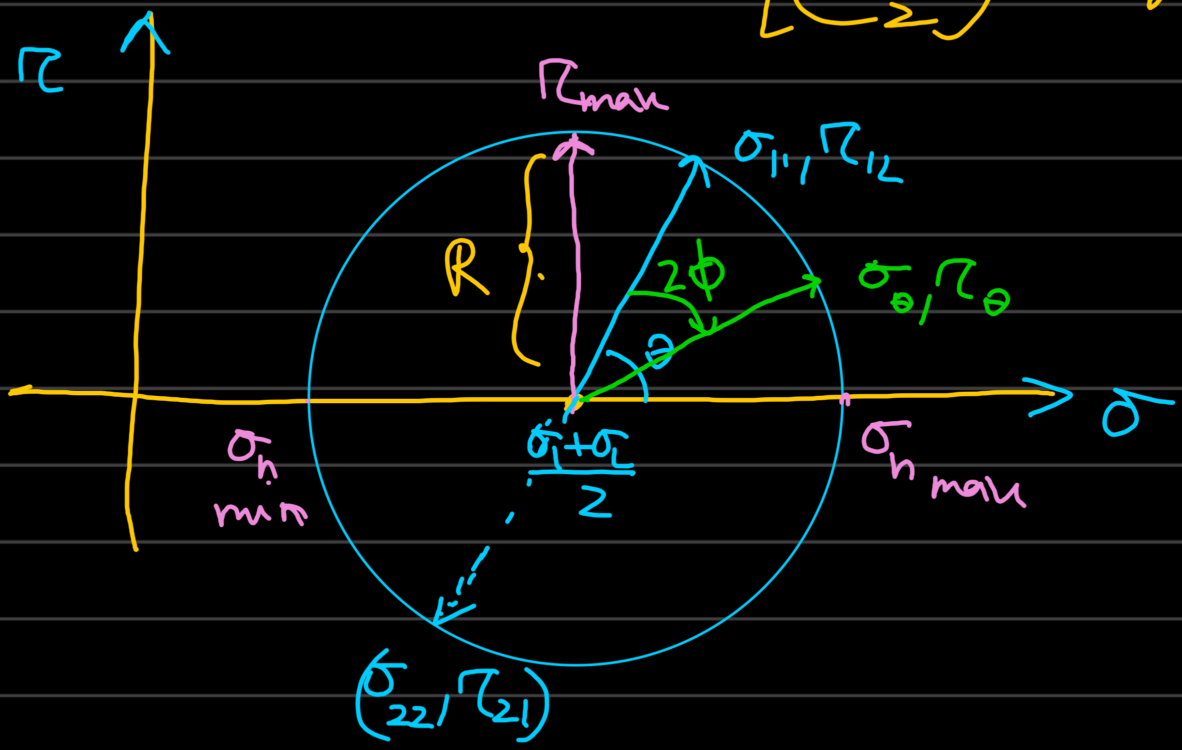
$$= (\sigma_y - \sigma_x)\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0$$

$$\tan 2\theta = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}}$$

$$\tau_{max} = \pm\left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2\right]^{1/2}$$

Mohr's Circle:

$$R = \left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2\right]$$



Principal stresses in 3D.

$$\begin{vmatrix} \sigma - \sigma_x & -\tau_{yx} & -\tau_{zx} \\ -\tau_{yx} & \sigma - \sigma_y & -\tau_{zy} \\ -\tau_{xz} & -\tau_{zx} & \sigma - \sigma_z \end{vmatrix} = 0$$

$$\sigma^3 - \sigma^2 I_1 + \sigma I_2 - I_3 = 0$$

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2$$

$$I_3 = \sigma_x \sigma_y \sigma_z + 2\tau_{xy}\tau_{yz}\tau_{xz} - \sigma_x \tau_{yz}^2 - \sigma_y \tau_{xz}^2 - \sigma_z \tau_{xy}^2$$

$$= \det(\underline{\underline{\sigma}})$$

$$= \begin{vmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{vmatrix}$$

I_1, I_2, I_3 are called the stress invariants.

Hydrostatic & Deviatoric stress

$$\sigma_{ij}' = \sigma_{ij} - \underbrace{\sigma_m \delta_{ij}}_{\text{Hydrostatic stress}} \qquad \sigma_m = \frac{1}{3}\sigma_{kk}$$

The principal values of deviatoric stress:

$$(\sigma')^3 - J_1(\sigma')^2 - J_2(\sigma') - J_3 = 0$$

J_1, J_2, J_3 : deviatoric stress invariants