Tutorial - 2

1) $\underset{\sim}{a} \cdot (\underset{\sim}{b} \wedge \underset{\approx}{A}) = (\underset{\sim}{a} \wedge \underset{\sim}{b}) \cdot \underset{\approx}{A}$

$$a_i \underset{\sim}{\delta_i} \cdot \left(b_j \underset{\sim}{\delta_j} \wedge A_{pq} \underset{\sim}{\delta_p} \underset{\sim}{\delta_q} \right)$$

$$= a_i \underset{\sim}{\delta_i} \cdot \left(b_j A_{pq} \varepsilon_{kjp} \underset{\sim}{\delta_k} \underset{\sim}{\delta_q} \right)$$

$$= a_i b_j A_{pq} \varepsilon_{kjp} \underset{\sim}{\delta_q} \left(\underset{\sim}{\delta_i} \cdot \underset{\sim}{\delta_k} \right)$$

$$= a_i b_j A_{pq} \varepsilon_{kjp} \underset{\sim}{\delta_q} \delta_{ik} = a_i b_j A_{pq} \varepsilon_{ijp} \underset{\sim}{\delta_q}$$

$$(\underset{\sim}{a} \wedge \underset{\sim}{b}) \cdot \underset{\approx}{A} = \left(a_i \underset{\sim}{\delta_i} \wedge b_j \underset{\sim}{\delta_j} \right) \cdot A_{pq} \underset{\sim}{\delta_p} \underset{\sim}{\delta_q}$$

$$= a_i b_j \varepsilon_{ijk} \underset{\sim}{\delta_k} \cdot A_{pq} \underset{\sim}{\delta_p} \underset{\sim}{\delta_q}$$

$$= a_i b_j \varepsilon_{ijk} A_{pq} \underset{\sim}{\delta_q} \left(\underset{\sim}{\delta_k} \cdot \underset{\sim}{\delta_p} \right)$$

$$= a_i b_j \varepsilon_{ijk} A_{pq} \delta_{kp} \underset{\sim}{\delta_q}$$

$$= a_i b_j \varepsilon_{ijk} A_{kq} \underset{\sim}{\delta_q}$$

$$2) \quad (a \wedge b) \cdot (a \wedge b) + (a \cdot b)^2$$

$$(a_i \underset{\sim}{\delta_i} \wedge b_j \underset{\sim}{\delta_j}) \cdot (a_i \underset{\sim}{\delta_i} \wedge b_j \underset{\sim}{\delta_j}) + (a_i \underset{\sim}{\delta_i} \cdot b_j \underset{\sim}{\delta_j})^2$$

$$= (a_i b_j \varepsilon_{ijk} \underset{\sim}{\delta_k}) \cdot (a_i b_j \varepsilon_{ijk} \underset{\sim}{\delta_k}) + (a_i b_j \delta_{ij})^2$$

$$= (a_i b_j)^2 \varepsilon_{ijk} \varepsilon_{kij} (\underset{\sim}{\delta_k} \cdot \underset{\sim}{\delta_k}) + (a_i b_i)^2$$

$$= (a_i b_j)^2 [\delta_{ii} \delta_{jj} - \delta_{ij} \delta_{ji}] + (a_i b_i)^2$$

$$= (a_i b_j)^2 - \cancel{(a_i b_i)^2} + \cancel{(a_i b_i)^2}$$

$$= (a_i b_j)^2$$

$$(a \wedge b)^2 = (\varepsilon_{ijk} a_j b_k)^2$$

$$= \varepsilon_{ijk} a_j b_k \varepsilon_{imn} a_m b_n$$

$$= a_i^2 b_j^2 .$$

$$(a \wedge b) \wedge (a \wedge b) = \varepsilon_{ijk} a_i b_j \underset{\sim}{\delta}_k \wedge \varepsilon_{pqr} a_p b_q \underset{\sim}{\delta}_r$$

$$= \varepsilon_{ijk} a_i b_j \varepsilon_{pqr} a_p b_q \varepsilon_{lkr} \underset{\sim}{\delta}_l$$

$$= \varepsilon_{ijk} a_i b_j a_p b_q \varepsilon_{pqr} \varepsilon_{rlk} \underset{\sim}{\delta}_l$$

$$= \varepsilon_{ijk} a_i b_j a_p b_q \{ \delta_{pl} \delta_{qk} - \delta_{pk} \delta_{ql} \} \underset{\sim}{\delta}_l$$

$$= \varepsilon_{ijq} a_i b_j a_l b_q \underset{\sim}{\delta}_l - \varepsilon_{ijp} a_i b_j a_p b_l \underset{\sim}{\delta}_l$$

$$= \underset{\sim}{a} [\underset{\sim}{b} \cdot (\underset{\sim}{a} \wedge \underset{\sim}{b})] - \underset{\sim}{b} [\underset{\sim}{a} \cdot (\underset{\sim}{a} \wedge \underset{\sim}{b})]$$

$$= \underset{\sim}{a} [\underset{\sim}{a} \cdot \cancel{(\underset{\sim}{b} \wedge \underset{\sim}{b})}] - \underset{\sim}{b} [\underset{\sim}{b} \cdot \cancel{(\underset{\sim}{a} \wedge \underset{\sim}{a})}]$$

$$= 0$$

4) $A : B = A_{ij}\, \underset{\sim}{\delta_i}\, \underset{\sim}{\delta_j} : B_{pq}\, \underset{\sim}{\delta_p}\, \underset{\sim}{\delta_q}$

$$= A_{ij} B_{pq} (\underset{\sim}{\delta_j} \cdot \underset{\sim}{\delta_p}) (\underset{\sim}{\delta_i} \cdot \underset{\sim}{\delta_q})$$

$$= A_{ij} B_{ji}$$

$$\underset{\approx}{\delta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\approx}{\delta} : \underset{\approx}{\delta} = \delta_{ij}\, \underset{\sim}{\delta_i}\, \underset{\sim}{\delta_j} : \delta_{pq}\, \underset{\sim}{\delta_p}\, \underset{\sim}{\delta_q}$$

$$= \delta_{ij}\, \delta_{pq} (\delta_j \cdot \delta_p)(\underset{\sim}{\delta_i} \cdot \underset{\sim}{\delta_q})$$

$$= \delta_{ij} \cdot \delta_{ji} = \underline{\underline{\delta_{ii}}} = \underline{\underline{1+1+1=3}}$$

(Keeping $i=j$ using one of the δ_{ij})

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{v} = \frac{\partial}{\partial x_i} \underset{\sim}{\delta_i} \cdot (\omega_k r_p \varepsilon_{jkp} \underset{\sim}{\delta_j})$$

$$= \frac{\partial}{\partial x_i} (\omega_k r_p \varepsilon_{ikp}) = \omega_k \varepsilon_{ikp} \frac{\partial r_p}{\partial x_i}$$

$$= \omega_k \varepsilon_{iki} = 0$$