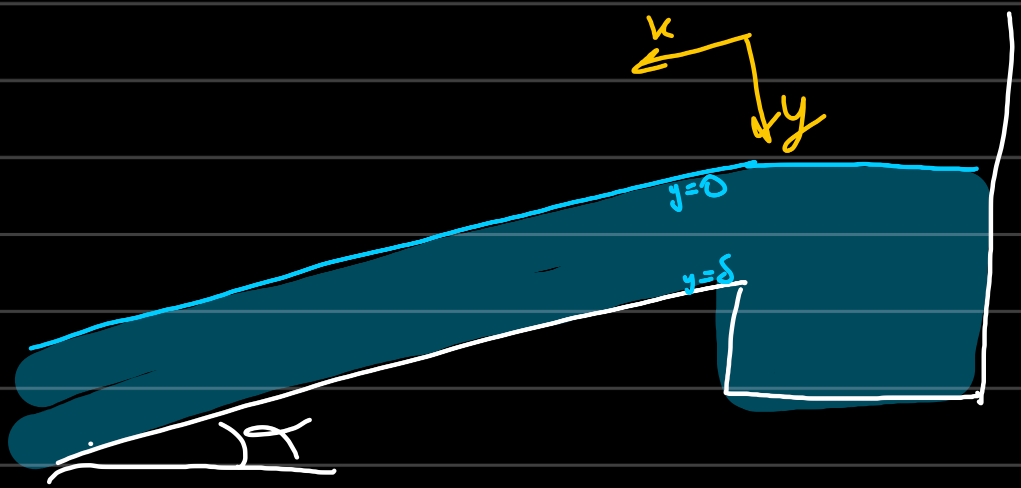
Tutorial - 9

1) Cauchy Equations:

$$\rho\left(\frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{v}\cdot\underset{\sim}{\nabla}\underset{\sim}{v}\right) = -\underset{\sim}{\nabla}\cdot\underset{\approx}{\tau} + \rho\underset{\sim}{g}$$



→ Considering fluid inertia negligible and steady state.

$$\rho\cancel{\left(\frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{v}\cdot\underset{\sim}{\nabla}\underset{\sim}{v}\right)}^{\,0} = -\underset{\sim}{\nabla}\cdot\underset{\approx}{\tau} + \rho\underset{\sim}{g}$$

$$0 = -\underset{\sim}{\nabla}\cdot\underset{\approx}{\tau} + \rho\underset{\sim}{g}$$

→ x - momentum balance:

$$\underset{\sim}{\nabla}\cdot\underset{\approx}{\tau} = \frac{\partial}{\partial x_i}\underset{\sim}{\delta_i}\cdot\tau_{pq}\underset{\sim}{\delta_p}\underset{\sim}{\delta_q} = \frac{\partial}{\partial x_i}\tau_{iq}\underset{\sim}{\delta_q}$$

$$0 = \frac{\partial}{\partial x_i}\tau_{ix} + \rho g_x$$

also from equation of continuity $\nabla \cdot \underset{\sim}{v} = 0$

& $v_y, v_z = 0 \Rightarrow$ $\underline{v_x = f(y)}$

$$0 = -\frac{\partial}{\partial x}\cancel{\tau_{xx}}^{0} + -\frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\cancel{\tau_{zx}}^{0} + \rho g \sin\theta$$

$$0 = \frac{-\partial}{\partial y}\tau_{yx} + \rho g \sin\theta$$

$$\tau_{yx} = \rho g y \sin\theta + \cancel{C_1}$$

$$\tau_{yx}\big|_{y=0} = C_1 = 0 \quad \{\text{air-liq interface}\}$$

$$\therefore \boxed{\tau_{yx} = \rho g y \sin\theta}$$

For a Power Law fluid: $\tau = -\mu(\dot{\gamma})\frac{dv_x}{dy}$

where $\mu(\dot{\gamma}) = m\left|\frac{dv_x}{dy}\right|^{n-1}$

$$\mu(\dot{\gamma}) = m\left(-\frac{dv_x}{dy}\right)^{n-1}$$

$$-m\left(-\frac{\partial v_x}{\partial y}\right)^{n-1} \times \frac{\partial v_x}{\partial y} = \rho g y \sin\alpha$$

$$m\left(-\frac{\partial v_x}{\partial y}\right)^{n} = \rho g y \sin\alpha$$

$$-\frac{\partial v_x}{\partial y} = \left(\frac{\rho g y \sin\alpha}{m}\right)^{1/n}$$

$$v_x = -\frac{y^{\frac{1}{n}+1}}{\frac{1}{n}+1}\left(\frac{\rho g \sin\alpha}{m}\right)^{1/n} + C_2$$

$$v_x = \left(\frac{-n}{n+1}\right) y^{\frac{n+1}{n}} \left(\frac{\rho g \sin\alpha}{m}\right)^{1/n} + C_2$$

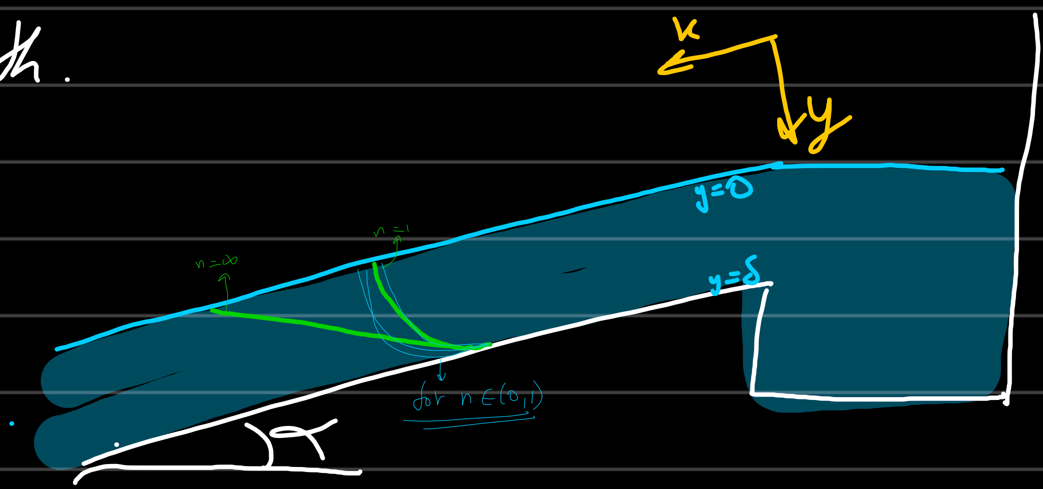
$$v_x(y=\delta) = 0 \Rightarrow C_2 = \left(\frac{n}{n+1}\right)\delta^{\frac{n+1}{n}}\left(\frac{\rho g \sin\alpha}{m}\right)^{1/n}$$

$$\boxed{v_x = \left(\frac{n}{n+1}\right)\left(\frac{\rho g \sin\alpha}{m}\right)^{1/n}\left[\delta^{\frac{n+1}{n}} - y^{\frac{n+1}{y}}\right]}$$

Volumetric flow rate:

$$\dot{Q} = \int_0^{\delta} v_x \, dy \, dz$$

volumetric flow rate per unit depth.



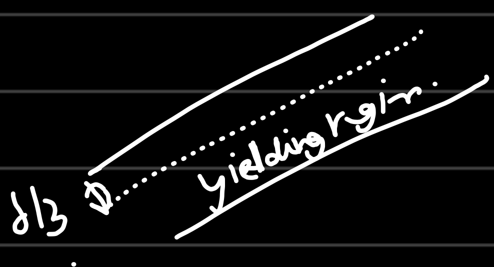
2> $\tau = -\mu(\dot{\gamma}) \dfrac{dv_x}{dy}$ $\quad \mu(\dot{\gamma}) = \infty \left\{ \dfrac{dv_x}{dy} = 0 \right\}$

for $\tau \leq \tau_0$

$$\mu(\dot{\gamma}) = \mu_0 + \frac{\tau_0}{\left|\frac{dv_x}{dy}\right|} \quad \text{for } \tau \geq \tau_0$$

$$\tau_{yx} = \rho g y \sin\alpha$$

b> yielding character at $H = \dfrac{2\delta}{3}$ $\Rightarrow y = \dfrac{\delta}{3}$



$$\mu = \mu_0 - \frac{\tau_0}{\frac{du_x}{dy}}$$

$$\tau_{yx} = -\mu_0 \frac{dv_x}{dy} + \tau_0 = \rho g y \sin\theta$$

$$-\mu_0 v_x + \tau_0 y = \frac{\rho g y^2 \sin\theta}{2} + c_1$$

$$v_x = -\frac{\rho g y^2 \sin\theta}{2\mu_0} + \frac{\tau_0}{\mu_0} y + \frac{c_1}{\mu_0}$$

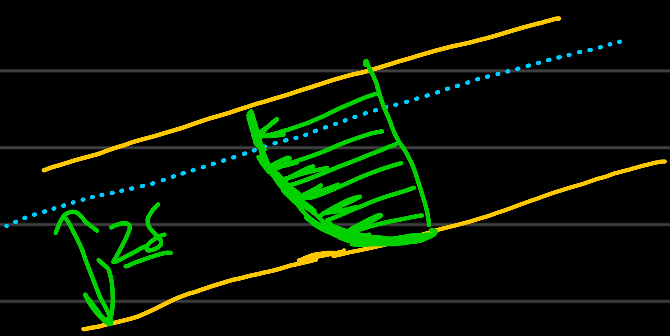
$v_x(y=\delta) = 0$

$$c_1 = \mu_0 \left(\frac{\rho g \delta^2 \sin\theta}{2\mu_0} - \frac{\tau_0 \delta}{\mu_0} \right)$$

$$c_1 = \frac{\rho g \sin\theta\, \delta^2}{2} - \tau_0 \delta$$

$$v_x = \frac{\rho g \sin\theta}{2\mu_0} \left[\delta^2 - y^2 \right] + \frac{\tau_0}{\mu_0} (y - \delta)$$

$$v_x = \frac{\rho g \sin\theta\, \delta^2}{2\mu_0} \left[1 - \frac{y^2}{\delta^2} \right] - \frac{\tau_0}{\mu_0} \delta \left(1 - \frac{\delta}{y} \right)$$



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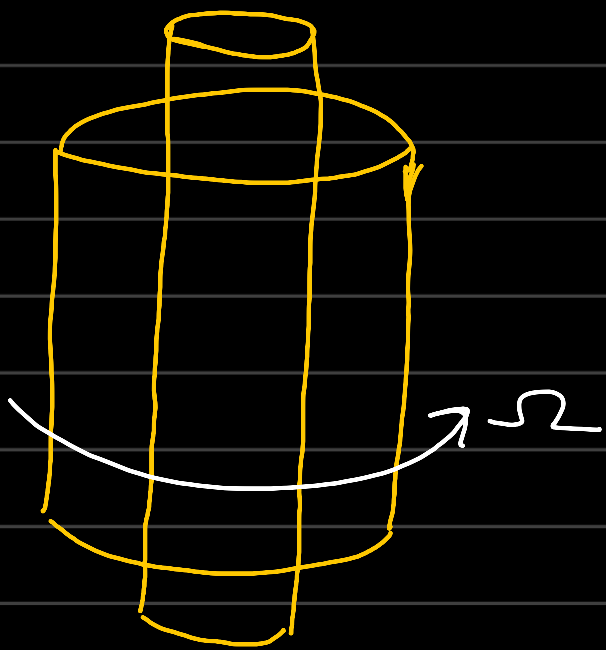

Cylindrical coordinates (r, θ, z):

$$\rho\left(\cancelto{0}{\frac{\partial v_r}{\partial t}} + \cancelto{0}{v_r \frac{\partial v_r}{\partial r}} + \cancelto{0}{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}} + \cancelto{0}{v_z \frac{\partial v_r}{\partial z}} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(\cancelto{0}{r v_r})\right) + \frac{1}{r^2}\cancelto{0}{\frac{\partial^2 v_r}{\partial \theta^2}} + \cancelto{0}{\frac{\partial^2 v_r}{\partial z^2}} - \frac{2}{r^2}\cancelto{0}{\frac{\partial v_\theta}{\partial \theta}}\right] + \rho g_r \qquad \text{(B.6-4)}$$

$$\rho\left(\cancelto{0}{\frac{\partial v_\theta}{\partial t}} + \cancelto{0}{v_r}\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\cancelto{0}{\frac{\partial v_\theta}{\partial \theta}} + \cancelto{0}{v_z}\frac{\partial v_\theta}{\partial z} + \cancelto{0}{\frac{v_r v_\theta}{r}}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\cancelto{0}{\frac{\partial^2 v_\theta}{\partial \theta^2}} + \cancelto{0}{\frac{\partial^2 v_\theta}{\partial z^2}} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right] + \rho g_\theta \qquad \text{(B.6-5)}$$

$$\rho\left(\cancelto{0}{\frac{\partial v_z}{\partial t}} + \cancelto{0}{v_r}\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\cancelto{0}{\frac{\partial v_z}{\partial \theta}} + \cancelto{0}{v_z}\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z \qquad \text{(B.6-6)}$$

$$-\underset{\sim}{\nabla} \cdot \underset{\approx}{\tau} \Rightarrow -\nabla_i \cdot \tau_{ij} \Rightarrow \underline{\underline{-\nabla_i \cdot \tau_{i\theta}}}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \tau_{r\theta}\right)$$