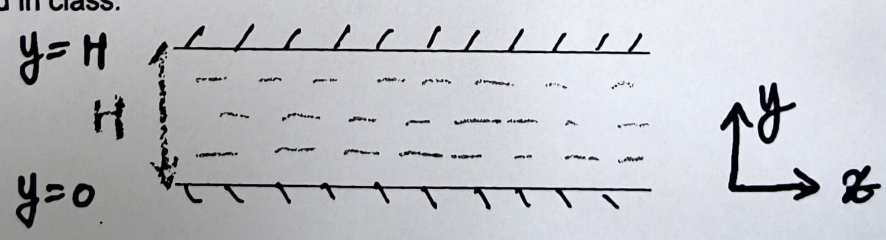
Tutorial 1- CLL110B

1. In the class we discussed the dot product. Consider the three unit vectors of the three-dimensional Cartesian coordinate system $\boldsymbol{\delta}_x$, $\boldsymbol{\delta}_y$, $\boldsymbol{\delta}_z$.
   - Write down the components of each of them.
   - What is $\boldsymbol{\delta}_x \cdot \boldsymbol{\delta}_y$, $\boldsymbol{\delta}_y \cdot \boldsymbol{\delta}_z$, $\boldsymbol{\delta}_y \cdot \boldsymbol{\delta}_y$, $\boldsymbol{\delta}_z \cdot \boldsymbol{\delta}_z$, $\boldsymbol{\delta}_z \cdot \boldsymbol{\delta}_x$?
   - If δ_{ij} form the components of a $\boldsymbol{\delta}$ matrix (a 3x3 matrix), such that i, j form the elements $\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j$ then can you write it down in matrix form?
   - For an arbitrary vector $\boldsymbol{a}$, what is $\boldsymbol{\delta} \cdot \boldsymbol{a}$? First write down the expansion of the summation convention, and then simplify the expression. This operation is referred to as "index shifting". The Kronecker Delta is very handy in shifting indices, often to get parity of dummy indices in large expressions.
   - Can you express a vector $\boldsymbol{v}$ in terms of it's magnitude by using the "dot" product operator?

2. A vector $\boldsymbol{c}$ is coplanar to two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, such that $\boldsymbol{c} = \alpha\boldsymbol{a} + \beta\boldsymbol{b}$ in a 3D Cartesian coordinate system.
   - Write down the components of $\boldsymbol{c}$ in terms of the sum of components vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.
   - You now have a set of linear equations to solve. What condition do you obtain for the coplanarity (hint: eliminate α, β)? From your course in Linear algebra in the 1st year (or from the lecture in the previous week), do you recall what have you calculated?

3. Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ have an angle θ between them. Use the cosine rule of triangle along with the expression for $\boldsymbol{a} \cdot \boldsymbol{b}$ to obtain the relation between the direction cosines of $\boldsymbol{a}$ and $\boldsymbol{b}$ and $\cos\theta$. Assume that the direction cosines of $\boldsymbol{a}$ can be represented as μ_i and that of $\boldsymbol{b}$ as γ_i for the index $i \in [1,3]$ for the 3 Cartesian coordinate directions.

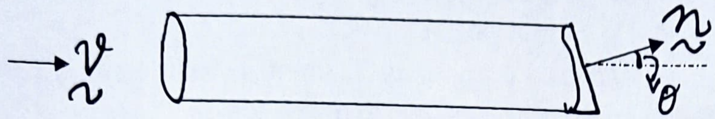
4. Consider a fluid sandwiched between two infinite parallel plates, like the one we discussed in class:



The bottom and top plates are moving with velocity $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ to the left and right, respectively.
   - Can you pictorially show how you'd expect the velocity profile to look like in the fluid medium at steady state?
   - Write down the expression for the $\boldsymbol{v}$ in terms of its x and y components at steady state. Note that this can be considered as a 2D system.
   - The Newtons law of viscosity states that the stress tensor $\boldsymbol{\tau}$ is propotional to the velocity gradient $\nabla\boldsymbol{v}$, with the constant of proportionality being the shear viscosity μ. Can you write down the expression for the stress components τ_{yx},

τ_{xx} and τ_{yy} at steady state? What direction does the first and second index of τ_{ij} signify (Hint: Look at the indices in the velocity gradient)?

5. Consider the below figure of a pipe. The normal of the exit face of the pipe, on the right has an inclination of with the direction of flow.



Fluid with velocity $\boldsymbol{v}$, enters through the inlet on the left-hand side. We would like to evaluate the mass rate of flow of the fluid through the exit. This is known as the flux of fluid, and is defined as the mass of fluid crossing a given surface per unit time. For the fluid having a density ρ and the radius of the inclined surface being R, can you obtain the expression for the mass-flux through the right-hand side exit? Assume that the direction of the fluid velocity $\boldsymbol{v}$ remains unchanged as it exits the pipe. (Hint: rely on the "dot" product. First evaluate the volumetric flow rate.)

6. Prove the following relations:
   - $\boldsymbol{a} \wedge (\boldsymbol{b} \wedge \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$ for three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$.
   - $\boldsymbol{a} = (\boldsymbol{a} \cdot \boldsymbol{e})\boldsymbol{e} + \boldsymbol{e} \wedge (\boldsymbol{a} \wedge \boldsymbol{e})$ for a unit vector $\boldsymbol{e}$ and an arbitrary vector $\boldsymbol{a}$.

Tutorial 1

1) $\underset{\sim}{\delta_x}, \underset{\sim}{\delta_y}, \underset{\sim}{\delta_z}$

$$\underset{\sim}{\delta_x} = 1\,\underset{\sim}{\delta_1} + 0\,\underset{\sim}{\delta_2} + 0\,\underset{\sim}{\delta_3} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\underset{\sim}{\delta_y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} ; \quad \underset{\sim}{\delta_z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

c) $$\underset{\approx}{\delta} = \delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \underset{\sim}{\delta_1} \cdot \underset{\sim}{\delta_1} & \underset{\sim}{\delta_1} \cdot \underset{\sim}{\delta_2} & \underset{\sim}{\delta_1} \cdot \underset{\sim}{\delta_3} \\ \underset{\sim}{\delta_1} \cdot \underset{\sim}{\delta_2} & \underset{\sim}{\delta_2} \cdot \underset{\sim}{\delta_2} & \underset{\sim}{\delta_2} \cdot \underset{\sim}{\delta_3} \\ \underset{\sim}{\delta_1} \cdot \underset{\sim}{\delta_3} & \underset{\sim}{\delta_3} \cdot \underset{\sim}{\delta_3} & \underset{\sim}{\delta_3} \cdot \underset{\sim}{\delta_3} \end{bmatrix}$$

d) $$\underset{\approx}{\delta} \cdot \underset{\sim}{a} = \delta_{ij}\, \underset{\sim}{\delta_i}\, \underset{\sim}{\delta_j} \cdot a_k \underset{\sim}{\delta_k}$$

$$= \delta_{ij}\, \underset{\sim}{\delta_i}\, a_k \delta_{jk} = \delta_{ij}\, a_j\, \underset{\sim}{\delta_i}$$

e) $\underset{\sim}{v}$ $\quad |\underset{\sim}{v}| = \sqrt{v_i^2} = \sqrt{\underset{\sim}{v} \cdot \underset{\sim}{v}}$

$$\underset{\sim}{v} = v_1 \underset{\sim}{\delta_1} + v_2 \underset{\sim}{\delta_2} + v_3 \underset{\sim}{\delta_3}$$

$$\underset{\sim}{v} = |v| \, \hat{v}$$

speed

$$\hat{v} = \underset{\sim}{\nabla} v$$

$$= \underset{\sim}{\delta_i} \frac{\partial}{\partial x_i}\left(\sqrt{v_1^2 + v_2^2 + v_3^2}\right)$$

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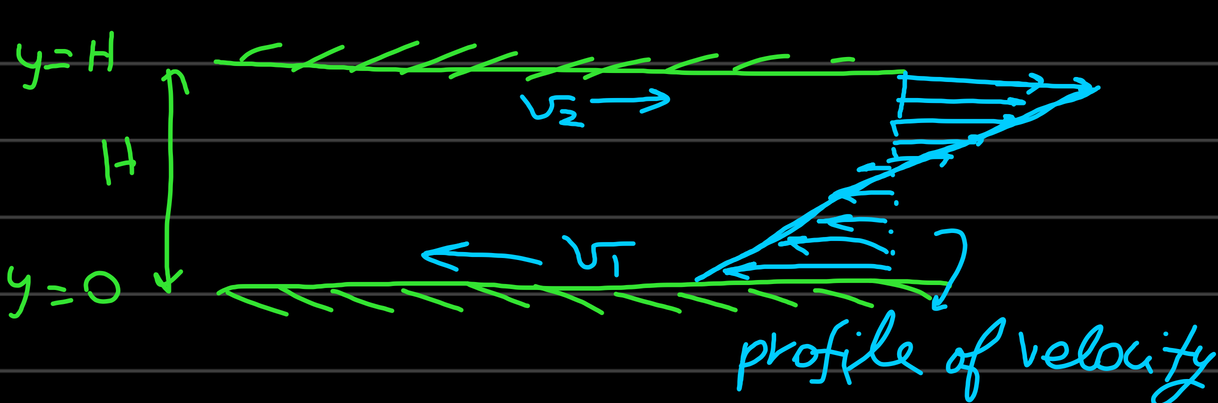
4)



$$v_x(y) = ay + b$$

$$-v_1 = b \quad \{ y = 0$$

$$v_2 = aH + b \qquad a = \frac{v_2 + v_1}{H}$$

$$v_x(y) = \left(\frac{v_2 + v_1}{H}\right) y - v_1$$

Newtonian fluid in Laminar flow.

$$\underset{\approx}{\tau} = \mu \underset{\sim}{\nabla} v$$