Inter-IIT Tech Meet Prep

Week 2: Graph Theory

MathSoc IIT Delhi

Contents

1 Graphs and Terminology

Definition. A (simple) graph $G = (V, E)$ consists of a set of vertices V and a set of edges E, where each edge is an unordered pair $\{u, v\}$ with $u, v \in V$ and $u \neq v$.

Graphs can be:

- **Simple:** No loops (edges from a vertex to itself) and no multiple edges between the same pair of vertices.
- **Multigraphs:** Graphs that may include multiple edges between the same pair of vertices.
- **Loops:** Edges that connect a vertex to itself.
- **Directed Graphs (Digraphs):** Edges are ordered pairs (u, v), indicating direction.
- **Weighted Graphs:** Edges carry a numerical weight or cost.
- **Degree:** The degree of a vertex is the number of edges incident to it.
- **Subgraph:** A graph formed from a subset of the vertices and edges of another graph.

Some common types of graphs:

- **Complete Graph** K_n: Every pair of n vertices is connected by an edge.
- **Cycle Graph** C_n: Vertices form a single cycle with n edges.

2 Graph Representations

Graphs can be represented as:

- **Adjacency Matrix:** An $n \times n$ matrix where entry a_{ij} is 1 if there is an edge between i and j.
- **Adjacency List:** A list where each vertex stores a list of its adjacent vertices.

Example. For a graph with vertices $\{1, 2, 3\}$ and edges $\{(1, 2), (2, 1), (2, 3), (3, 2)\}$, the adjacency list is:

$$1 \to [2], \quad 2 \to [1, 3], \quad 3 \to [2]$$

3 Trees and Forests

Definition. A **tree** is a connected graph with no cycles. A **forest** is a graph with no cycles; it may be disconnected and is simply a collection of trees.

Theorem. A graph is a tree $\iff$ it is connected and has $n - 1$ edges (where n is the number of vertices).

Trees appear frequently in algorithms (e.g., DFS, BFS, MSTs).

4 Paths and Walks

Walk: A sequence of vertices such that each pair of consecutive vertices is connected by an edge.

Path: A walk with no repeated vertices.

Cycle: A path that starts and ends at the same vertex.

Theorem (Handshake Lemma). The sum of degrees of all vertices in a graph is twice the number of edges:

$$\sum_{v \in V} \deg(v) = 2|E|$$

5 Connectedness and Components

Definition. A graph is connected if there exists a path between every pair of vertices.

A **connected component** is a maximal set of connected vertices.

6 Bipartite Graphs

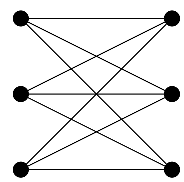
Definition. A graph is **bipartite** if its vertex set V can be divided into two disjoint sets A and B such that every edge connects a vertex from A to a vertex from B, and there are no edges between vertices within the same set.

An easy way to check if a graph is bipartite is to color it using 2 colors, such that no two nodes of the same color are adjacent (share a common edge).

Example: A tree is always bipartite.

Theorem. A graph is bipartite $\iff$ it contains no odd-length cycles.

Complete Bipartite Graph. A complete bipartite graph $K_{m,n}$ has m vertices in set A and n in set B, with every vertex in A connected to every vertex in B.



$K_{3,3}$

7 Eulerian Paths and Circuits

Definition. An *Eulerian Path* is a path that visits every edge exactly once.

An *Eulerian Circuit* is an Eulerian path that starts and ends at the same vertex.

Theorem. A connected graph has:

- An Eulerian circuit $\iff$ all vertices have even degree.
- An Eulerian path but not a circuit $\iff$ exactly two vertices have odd degree.

8 Hamiltonian Paths and Cycles

Definition. A *Hamiltonian Path* visits every vertex exactly once.

A *Hamiltonian Cycle* is a Hamiltonian path that starts and ends at the same vertex.

Note. Unlike Eulerian cycles, there is no simple necessary and sufficient condition for Hamiltonian cycles.

Theorem (Dirac's Theorem). If G is a simple graph with $n \geq 3$ vertices and every vertex has degree $\geq n/2$, then G has a Hamiltonian cycle.

9 Planarity and Kuratowski's Theorem

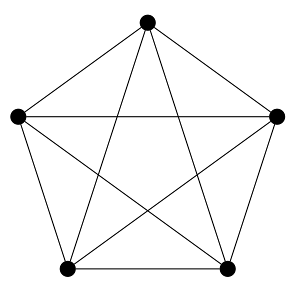
Some graphs can be drawn on paper such that no edges cross, these are called **planar graphs**.

Definition. A graph is **planar** if it can be drawn in the plane without any edges crossing.

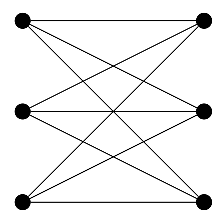
The complete graph on 5 vertices K_5, and the complete bipartite graph $K_{3,3}$, are **not** planar.

Theorem (Kuratowski's Theorem). A graph is non-planar if and only if it contains a subgraph that can be transformed into K_5 or $K_{3,3}$ by replacing edges with paths (called a subdivision).

These two graphs are the "forbidden minors" of planar graphs — spotting them inside a larger graph is a powerful way to show that planarity is impossible.



K_5



$K_{3,3}$

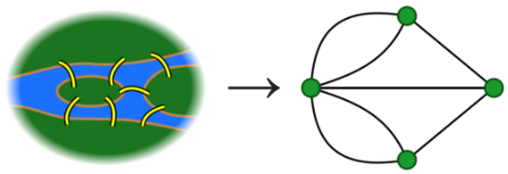
Fun Fact: The famous "three houses and three utilities" puzzle — where each house must connect to each utility without crossing lines — is actually just an attempt to draw $K_{3,3}$ on the plane!

10 Königsberg Bridges

In the city of Königsberg, there were seven bridges connecting four landmasses. The challenge: can one walk through the city crossing each bridge exactly once?

Hint: Think of Eulerian Path

Euler showed the answer is **no**, and in doing so, laid the foundation of graph theory. Can you prove it?



11 This Week's Problem

Problem 1

Prove that in any graph, the number of vertices with odd degree is even.

Problem 2

Show that at any party, there are always at least two people with exactly the same number of friends at the party.

Problem 3

There are $2n$ people in a room where each person is enemies with at most $n-1$ people in the room. Prove that the $2n$ people can sit at a circular table so that no two enemies are sitting next to each other.

Problem 4

Prove or disprove: every connected graph with an Eulerian path has at most two vertices of odd degree.

Problem 5

Let G be a graph with n vertices and more than $\frac{(n-1)(n-2)}{2}$ edges. Prove that G is connected.

Problem 6

Let G be a connected graph with all vertices of even degree. Prove that for any edge $e \in G$, there exists an Eulerian circuit containing e.

Problem 7 (Construction)

Construct a simple graph with $2k$ vertices where every vertex has degree k, but the graph contains no Hamiltonian cycle.

Puzzle

Given n vertices, how many edges must a graph have to guarantee it contains a triangle? (Hint: Explore Turán's theorem.)