Chapter 15 (Lecture 11)

Shilpa Gondhali

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BITS Pilani
K K Birla Goa Campus

Extra lecture on 05/11/24 (Tuesday) from 06.15 PM to 07.15 PM in DLT8
by Shilpa Gondhali - Sunday, 3 November 2024, 9:56 AM

Dear All,

I hope this email finds you well. We will be discussing double and triple integration this week and along with regular lectures, we will meet on **05/11/24 (Tuesday) from 06.15 PM to 07.15 PM in DLT8.**

Please note that it will not be possible for me to take lectures in last week of November hence need for extra lecture now.

Hoping for your understanding and cooperation.

Sincerely,

Shilpa Gondhali.

$$S_K = \lim_{n\to\infty} \sum_{k=1}^{n} f(x,y)\,\Delta A_k$$

Integration using Polar Coordinates

A version of Fubini's Theorem says that the limit approached by these sums can be evaluated by repeated single integrations with respect to r and θ as

$$\iint_R f(r,\theta)\, r\, dr\, d\theta = \int_{\theta=\alpha}^{\beta} \int_{r=g_1(\theta)}^{g_2(\theta)} f(r,\theta)\, r\, dr\, d\theta.$$

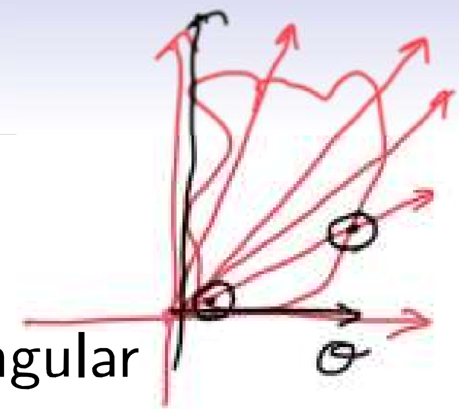
polar.

in r,θ coordinates/plane.

θ = outer limit

r = inner limit

Finding Limits of Integration



The procedure for finding limits of integration in rectangular coordinates also works for polar coordinates.

1. **Sketch.** Sketch the region and label the bounding curves.
2. **Find the r-limits of integration.** Imagine a ray L from the origin cutting through R in the direction of increasing r. Mark the r-values where L enters and leaves R. These are the r-limits of integration. They usually depend on the angle θ that L makes with the positive x-axis.
3. **Find the θ-limits of integration.** Find the smallest and largest θ-values that bound R. These are the θ-limits of integration.

innovate achieve lead

Area in Polar Coordinates

The area of a closed and bounded region R in the polar coordinate plane is $A = \iint\limits_R r\, dr\, d\theta$.

Changing Cartesian Integrals $\iint\limits_R f(x, y)dxdy$ into Polar Integrals

1. Substitute $x = r\cos\theta$ and $y = r\sin\theta$, and replace $dx\,dy$ by $r\,dr\,d\theta$ in the Cartesian integral.
2. Supply polar limits of integration for the boundary of R.

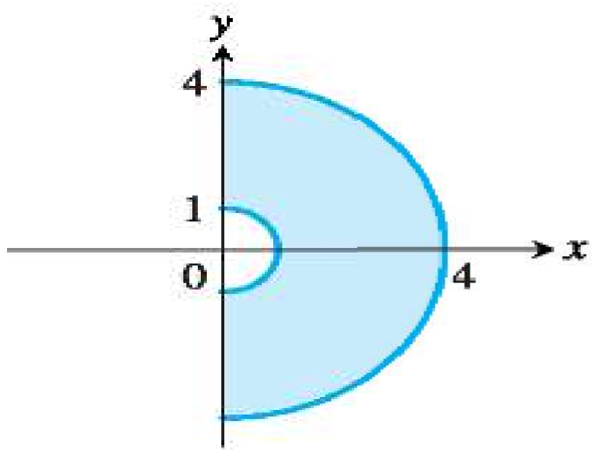
Then, we have

$$\iint\limits_{\mathbf{R}} \mathbf{f}(\mathbf{x}, \mathbf{y})\,\mathbf{dx}\,\mathbf{dy} = \iint\limits_{\mathbf{G}} \mathbf{f}(\mathbf{r}\cos\theta, \mathbf{r}\sin\theta)\,\mathbf{r}\,\mathbf{dr}\,\mathbf{d}\theta,$$

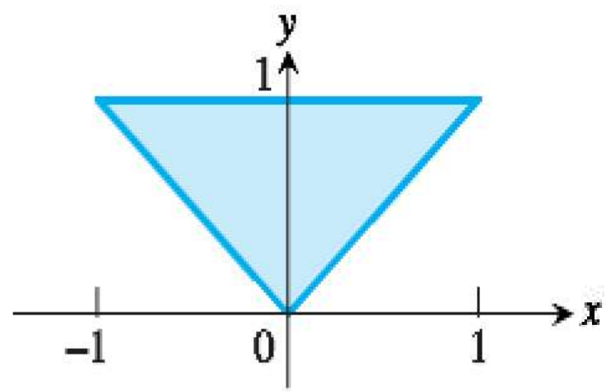
where G denotes the same region of integration now described in polar coordinates.

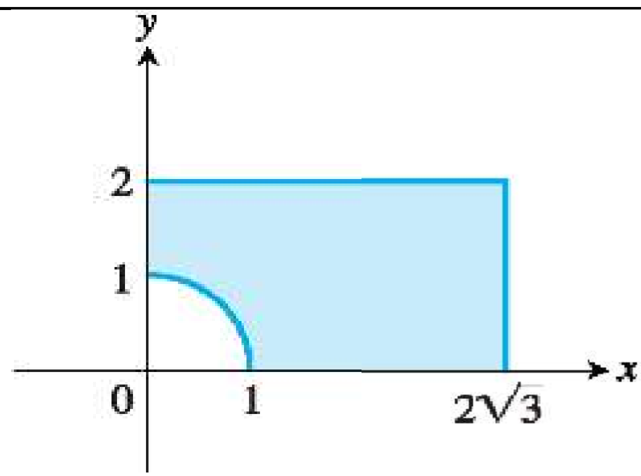
Describe region in polar coordinates

R1



R2



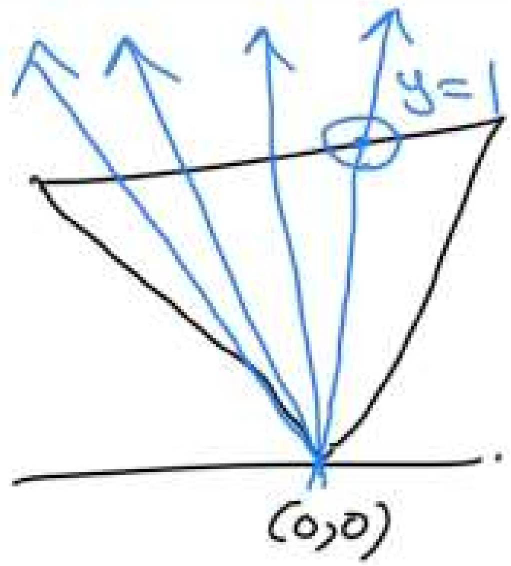


R3

R4

Region enclosed by $x^2 + y^2 = 2y, \ y \geq 0$

$R1 : -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \quad 1 \leq r \leq 4$



$R2 : \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$

Lower limit for r is 0

& upper limit is $y=1$

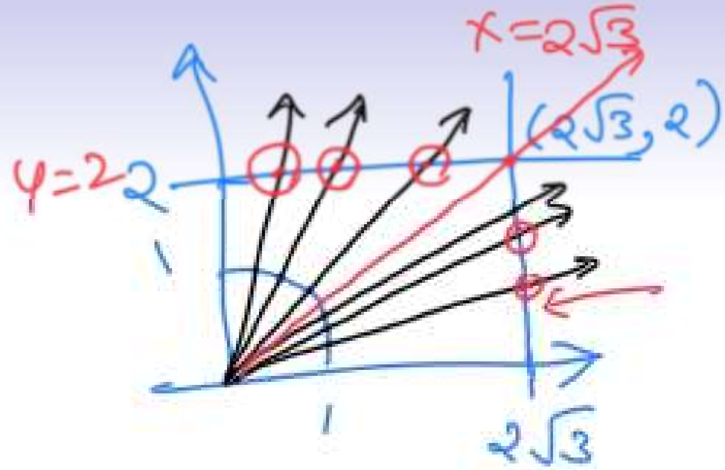
$r \sin\theta = 1$

$\therefore r = \operatorname{cosec}\theta$

$0 \leq r \leq \operatorname{cosec}\theta$

$\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$

R3 :



$$0 \le \theta \le \frac{\pi}{2}$$

$(2\sqrt{3}, 2)$

$$\tan\theta = \frac{y}{x}$$

$$\theta = \pi/6$$

$x = 2\sqrt{3}$

$r\cos\theta = 2\sqrt{3}$

$r = 2\sqrt{3}\,\sec\theta$

$1 \le r \le 2\sqrt{3}\sec\theta$

$0 \le \theta \le \pi/6$

$y = 2$

$r\sin\theta = 2$

$r = 2\,\operatorname{cosec}\theta$

$1 \le r \le 2\operatorname{cosec}\theta$

$\pi/6 \le \theta \le \pi/2$

4. $y \ge 0$

$x^2 + y^2 = 2y$

$r^2\sin^2\theta + r^2\cos^2\theta = 2r\sin\theta$

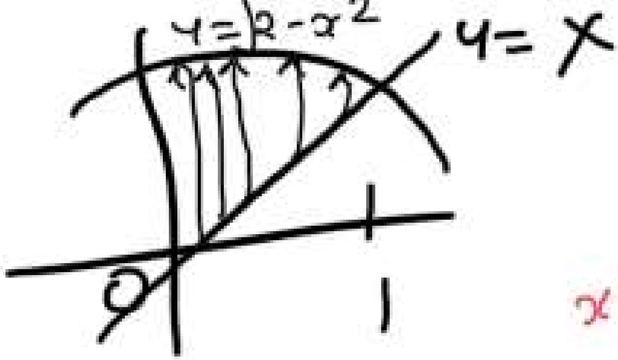
$\therefore\ r^2 = 2r\sin\theta$

$\therefore\ r = 2\sin\theta$

$$0 \leq r \leq 2\sin\theta$$
$$0 \leq \theta \leq \pi$$

Examples

1. Convert into equivalent polar integral and solve:



$$\int_{x=0}^{1} \int_{y=x}^{\sqrt{2-x^2}} (x + 2y)\, dy\, dx$$

$0 \leq r \leq \sqrt{2}$

$\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$

2. Convert to Cartesian integral. Do not evaluate the integral.

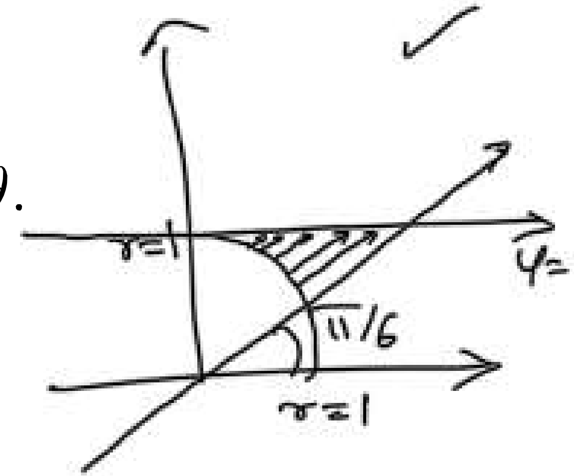
$r = \operatorname{cosec} \theta$

$r = \frac{1}{\sin \theta}$

$r \sin \theta = 1$

$y = 1$

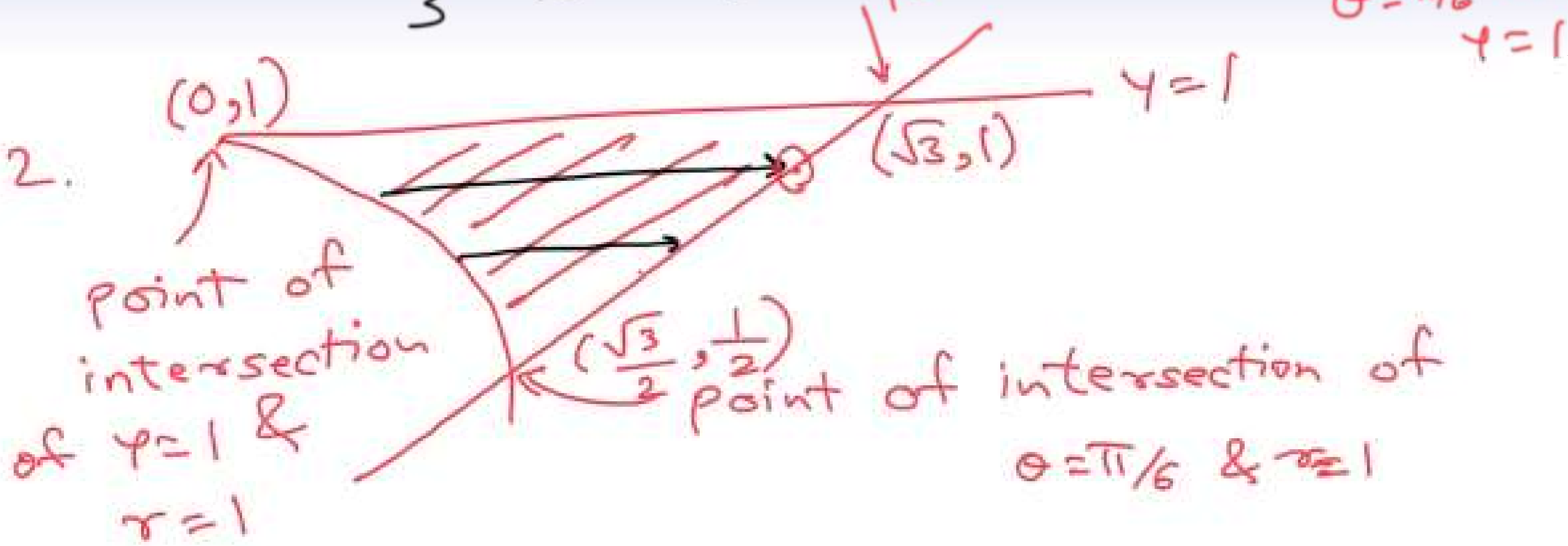
$$\int_{\pi/6}^{\pi/2} \int_{1}^{\csc \theta} r^2 \cos \theta \, dr\, d\theta.$$



3. Calculate $I = \int_0^{\infty} e^{-x^2}\, dx.$

1. Ans $\frac{2}{3}(\sqrt{2}+1)$

2.



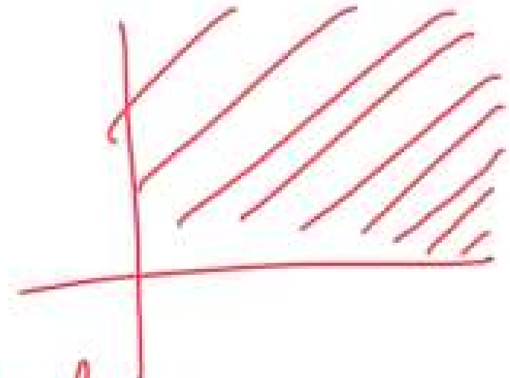
$$\sqrt{1-y^2} \le x \le \sqrt{3}\,y$$
$$\frac{1}{2} \le y \le 1$$

Order y

3. $$I = \int_0^\infty e^{-x^2}\,dx$$

Consider $$I^2 = \int_0^\infty e^{-x^2}\,dx \int_0^\infty e^{-y^2}\,dy$$

$$= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$$



Convert to polar

$$I^2 = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2}\, r\, dr\, d\theta$$

Substitute $r^2 = u$ & proceed.