10 Lecture 11: Families of Matrices

Families of Matrices Already Seen

- The $m \times n$ matrices of real entries, $\mathbb{R}^{m\times n}$,
- The $m \times n$ matrices of complex entries, $\mathbb{C}^{m\times n}$,
- The $n \times n$ real square matrices, $\mathbb{R}^{n\times n}$,
- The $n \times n$ invertible matrices, $\mathbf{GL}_n$,
- The $n \times n$ upper (or lower) triangular matrices,
- The $n \times n$ upper (or lower) unitriangular matrices,
- The $n \times n$ diagonal matrices,
- The $n \times n$ symmetric (or Hermitian symmetric) matrices,
- The $n \times n$ antisymmetric matrices,
- The $m \times n$ proto-row-echelon matrices,
- The $m \times n$ normalized row-echelon matrices,
- The $m \times n$ reduced row-echelon matrices,
- The elementary matrices (replacement, scaling and exchange).

10.1 Gram matrix

Definition 10.1. *The Gram matrix of a matrix* $\mathbf{A} \in \mathbb{C}^{m\times n}$ *is*

$$\mathbf{A}^H\mathbf{A},$$

where $\mathbf{A}^H$ *is the Hermitian transpose of* $\mathbf{A}$.

Note that the Gram matrix of $\mathbf{A} \in \mathbb{C}^{m\times n}$ is a matrix in $\in \mathbb{C}^{n\times n}$. In particular, if $\mathbf{A} \in \mathbb{R}^{m\times n}$ then the Gram matrix of $\mathbf{A}$ is in $\mathbb{R}^{n\times n}$ and given by

$$\mathbf{A}^T\mathbf{A}.$$

Properties:

- The Gram matrix is Hermitian symmetric, i.e., $(\mathbf{A}^H\mathbf{A})^H = \mathbf{A}^H\mathbf{A}$.
- The entries of the Gram matrix of $\mathbf{A} = \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n\end{bmatrix}$ are the dot products of each column of $\mathbf{A}$ with every other column of $\mathbf{A}$ including itself, i.e.,

$$\mathbf{A}^H\mathbf{A} = \begin{bmatrix} \mathbf{a}_1^H\mathbf{a}_1 & \mathbf{a}_1^H\mathbf{a}_2 & \dots & \mathbf{a}_1^H\mathbf{a}_n \\ \mathbf{a}_2^H\mathbf{a}_1 & \mathbf{a}_2^H\mathbf{a}_2 & \dots & \mathbf{a}_2^H\mathbf{a}_n \\ \vdots & \vdots & \dots & \vdots \\ \mathbf{a}_n^H\mathbf{a}_1 & \mathbf{a}_n^H\mathbf{a}_2 & \dots & \mathbf{a}_n^H\mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1\cdot\mathbf{a}_1 & \mathbf{a}_1\cdot\mathbf{a}_2 & \dots & \mathbf{a}_1\cdot\mathbf{a}_n \\ \mathbf{a}_2\cdot\mathbf{a}_1 & \mathbf{a}_2\cdot\mathbf{a}_2 & \dots & \mathbf{a}_2\cdot\mathbf{a}_n \\ \vdots & \vdots & \dots & \vdots \\ \mathbf{a}_n\cdot\mathbf{a}_1 & \mathbf{a}_n\cdot\mathbf{a}_2 & \dots & \mathbf{a}_n\cdot\mathbf{a}_n \end{bmatrix}$$

10.2 Matrix with orthornormal columns

A set of n vector $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ is called **orthornormal** if and only if

$$\begin{aligned} \mathbf{a}_i \cdot \mathbf{a}_i &= 1, \quad \text{for } \ i = 1, 2, \ldots, n \\ \mathbf{a}_i \cdot \mathbf{a}_j &= 0, \quad \text{for } \ i \neq j \end{aligned} \tag{10.1}$$

If $\mathbf{A}$ is an $m \times n$ matrix and the Gram matrix of $\mathbf{A}$ is the $n \times n$ identity matrix, i.e., $\mathbf{A}^T\mathbf{A} = \mathbf{I}_n$, then the columns of $\mathbf{A}$ form an orthonormal set. In this case, we say, $\mathbf{A}$ is a **matrix with orthornormal columns**.

Example 10.1. $\mathbf{A} = \begin{bmatrix} 1/2 & 1/2 & 1/2; 1/2 & -1/2 & 1/2; 1/2 & 1/2 & -1/2; 1/2 & -1/2 & -1/2 \end{bmatrix}$

is a matrix with orthornormal columns.

Remark 10.1. *If an $m \times n$ matrix $\mathbf{A}$ is a a matrix with orthornormal columns, then $n \leq m$.*

10.2.1 Orthogonal matrices

A *real square matrix* that has the identity as its Gram matrix is known as an orthogonal matrix. A matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is orthogonal if and only if

$$\mathbf{A}^T\mathbf{A} = \mathbf{I}_n$$

Example 10.2. *The standard matrix of rotation and reflection in $\mathbb{R}^2$ are orthogonal matrix. That is, $\mathbf{A} = \begin{bmatrix} \cos\theta & -sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $\begin{bmatrix} \cos 2\theta & sin2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$ are orthogonal matrices.*

Inverse of an orthogonal matrix: If $\mathbf{A}$ is orthogonal matrix then $\mathbf{A}^{-1} = \mathbf{A}^T$.

10.3 Permutation matrices

An $n \times n$ matrix formed by reordering the rows (or columns) of the *identity* matrix is known as a permutation matrix. We denote the set of all $n \times n$ permutation matrices S_n. Every permutation matrix is an orthogonal matrix. The set S_n contains $n!$ elements.

10.4 Unitary matrices

An $n \times n$ matrix (which may have complex entries) is **unitary** if and only if $\mathbf{A}^H\mathbf{A} = \mathbf{I}_n$. The set of all $n \times n$ unitary matrices is denoted $U(n)$.

10.5 Projection matrices.

A square matrix $\mathbf{P}$ is a **projection matrix** if and only if

$$\mathbf{P}^2 = \mathbf{P}$$

If $\mathbf{u}$ and $\mathbf{v}$ are two vectors in $\mathbb{C}^n$ such that $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^H\mathbf{v} = 1$, then the matrix

$$\mathbf{P} = \mathbf{u}\mathbf{v}^H$$

is a projection matrix of rank 1.

Indeed, $\mathbf{P}^2 = \mathbf{P}\mathbf{P} = (\mathbf{u}\mathbf{v}^H)(\mathbf{u}\mathbf{v}^H) = \mathbf{u}(\mathbf{v}^H\mathbf{u})\mathbf{v}^H = \mathbf{u}(\mathbf{v}^H\mathbf{u})^T\mathbf{v}^H\mathbf{u}(\mathbf{u}^H\mathbf{v})\mathbf{v}^H = \mathbf{u}\mathbf{v}^H = \mathbf{P}$.

10.6 Orthogonal projection matrices

Orthogonal projection matrices are projection matrices that are real symmetric ($\mathbf{P}^T = \mathbf{P}$).

The rank 1 projection matrices are of the form $\mathbf{u}\mathbf{u}^T$, where $\mathbf{u}$ is a real column vector. In this case,$\mathbf{u}$ is a unit vector since $\mathbf{u} \cdot \mathbf{u} = 1$. A particular property of an orthogonal projection matrix $\mathbf{P}$ is that it decomposes any compatible vector $\mathbf{x}$ into two orthogonal components: $\mathbf{x} = \mathbf{x}_{\parallel} + \mathbf{x}_{\perp}$, where $\mathbf{x}_{\parallel} = \mathbf{P}\mathbf{x}$ and $\mathbf{x}_{\perp} \cdot \mathbf{x}_{\parallel} = 0$.

10.7 Stochastic matrices

A matrix in $\mathbb{R}^{n \times n}$ having nonnegative entries and having sum of the entries in each column equal to 1 is known as a stochastic or Markov matrix.

11 Lecture 12: Determinants

The determinant is a scalar-valued function that can be computed from the elements of a square matrix. The determinant of a matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ is denoted by $\det(\mathbf{A})$, or $\det \mathbf{A}$, or $|\mathbf{A}|$.

A particularly significant property of the determinant is that the matrix is nonsingular (i.e., invertible) if and only if its determinant is nonzero.

11.1 Calculation of the determinant

11.1.1 For a 2×2 matrix:

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$\det(A) = ad - bc.$$

Example 11.1.

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1(4) - 2(3) = -2.$$

For $n \times n$ matrix with $n \geq 3$: Using Laplace expansion. Denote $\mathbf{A} - [a_{ij}]$ an $n \times n$ matrix. To each entries a_{ij} in the $i^{th}-$row and the $j^{th}-$column, we denote by $\mathbf{A}_{ij}$, the $(n-1)\times(n-1)$ matrix obtained by removing the i^{th}row and the j^{th} column of $\mathbf{A}$, and define

$$\mathrm{cof}(\mathbf{A})_{ij} = (-1)^{i+j}\det(\mathbf{A}_{ij})$$

The determinant of matrix $\mathbf{A}$ is calculated using the Laplace expansion by the i^{th}row or the j^{th} column. Particularly,

- The Laplace expansion by the i^{th}row:

  $$\det(\mathbf{A}) = \sum_{\ell=1}^{n} a_{i\ell}\mathrm{cof}(\mathbf{A})_{i\ell},$$

  where i is any value from 1 to n, but is fixed for all terms in the sum.

- The Laplace expansion by the j^{th}column:

  $$\det(\mathbf{A}) = \sum_{\ell=1}^{n} a_{\ell j}\mathrm{cof}(\mathbf{A})_{\ell j},$$

  where j is any value from 1 to n, but is fixed for all terms in the sum.

Example 11.2. *Calculate the determinant of matrix* $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$

Solution: We use the Laplace expansion by the first column:

$$|A| = a_{11}\mathrm{cof}(\mathbf{A})_{11} + a_{21}\mathrm{cof}(\mathbf{A})_{21} + a_{31}\mathrm{cof}(\mathbf{A})_{31},$$

here

$$\mathrm{cof}(\mathbf{A})_{11} = (-1)^{1+1}\begin{vmatrix} 5 & 6 \\ 8 & 10 \end{vmatrix} = 2$$
$$\mathrm{cof}(\mathbf{A})_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 3 \\ 8 & 10 \end{vmatrix} = (-1)(-4) = 4$$
$$\mathrm{cof}(\mathbf{A})_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = -3$$

Thus,

$$|\mathbf{A}| = 1(2) + 4(4) + 7(-3) = -3$$

Determinant of triangular matrices: If $\mathbf{A} = [a_{ij}]$ is a triangular matrix (either upper triangular or lower triangular), then the determinant of $\mathbf{A}$ equals the product of all entries in the main diagonal, e.i.,

$$\det(\mathbf{A}) = \prod_{\ell=1}^{n} a_{\ell\ell}, \quad \text{if } \mathbf{A} \text{ is triangular.}$$

Example 11.3.

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 10 \end{vmatrix} = (1)(5)(10) = 50 \quad \textit{and} \quad \begin{vmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 1 & 0 & 10 \end{vmatrix} = 10(1)(5)(10) = 50$$

11.2 Some Properties

11.2.1 Effect of elementary row operations on the determinant.

- Exchanging two rows of a matrix reverses the sign of the determinant.
- Scaling a matrix by multiplication of one row by a factor α multiplies the determinant by α
- Subtracting a multiple of one row from another row leaves the determinant unchanged.

11.2.2 Determinant of a product:

Determinant of a product equals the product of determinants, i.e., if $\mathbf{A}$ and $\mathbf{B}$ are two square matrix of the same size, then

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$$

11.2.3 Determinant of a transpose:

Transposing a matrix does not change its determinant, i.e.,

$$\det(A^T) = \det(\mathbf{A})$$

11.3 Alternate method to calculate the determinant:

Using the row reduction method and be aware of the affection of elementary row operations on the determinant mentioned in subsection 11.2.1.

Example 11.4.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -11 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{vmatrix} = (1)(-3)(1) = -3.$$

11.4 Application of the determinant:

11.4.1 Area of the parallelogram

If $\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$. The parallelogram S generated by the columns of $\mathbf{A}$, i.e., the set of all linear combinations $\alpha\mathbf{a}_1 + \beta\mathbf{a}_2$, where $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$ is $|\det(A)|$

11.4.2 Volume of the parallelepiped

If $\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$. The parallelepiped V generated by the columns of $\mathbf{A}$, i.e., the set of all linear combinations $\alpha\mathbf{a}_1 + \beta\mathbf{a}_2 + \gamma\mathbf{a}_3$, where $0 \leq \alpha, \beta, \gamma \leq 1$ is $|\det(A)|$