# 5 Lecture 5: Parametric Vector Form

## 5.1 Solution to Linear Systems

We first consider the example:

**Example 5.1.** *Find the solution to the linear system whose augmented matrix is*

$$\begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The augmented matrix has four columns, so the linear system has three variables: $x_1$, $x_2$, $x_3$. The augmented matrix is in reduced row echelon form, so we now write back the linear system, which is given by

$$\begin{cases} x_1 \qquad -3x_3 &= 1 \\ \qquad x_2 + x_3 &= 2 \\ \qquad\quad 0 &= 0 \end{cases}$$

The variables corresponding to pivot columns in the matrix, $x_1$ and $x_2$, are called **basic variables**. The remainder variable, $x_3$, is called a **free variable**. Writing $x_1$ and $x_2$ in terms of $x_3$ we obtain

$$\begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free} \end{cases} \tag{5.1}$$

**Note:** "$x_3$ is free" means $x_3$ can take any value. For each specific value of $x_3$, we have a specific value of $x_1$, $x_2$, and then one specific solution. There are infinitely many possible values of $x_3$ (such as all real numbers). Therefore, the system is consistent, and it has infinitely many solutions.

**Example 5.2.** *Find the general solution of the linear system whose augmented matrix has been reduced to*

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \tag{5.2}$$

**Solution:** The matrix is in echelon form, but we need the reduced echelon form before solving for the basic variables. The row reduction is completed next.

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

There are five variables, say $x_1, x_2, x_3, x_4, x_5$, because the augmented matrix has six columns. The associated system is

$$\begin{cases} x_1 + 6x_2 + 3x_4 = 0 \\ x_3 - 4x_4 = 5 \\ x_5 = 7 \end{cases} \tag{5.3}$$

The pivot columns of the matrix are 1,3, and 5, so the basic variables are $x_1, x_3$, and $x_5$. The remaining variables, $x_2$ and $x_4$, must be free. Solve for the basic variables to obtain the general solution:

$$\begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 4x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{cases}$$

This system has infinitely many solutions.

## 5.2 Existence and Uniqueness questions

**Example 5.3.** *Find the general solution of the linear system whose augmented matrix has been reduced to*

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -4 \\ 0 & 0 & 1 & -4 & 2 \\ 0 & 0 & 2 & -8 & 5 \end{bmatrix} \tag{5.4}$$

**Solution:** We first row reduce the augmented matrix.

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -4 \\ 0 & 0 & 1 & -4 & 2 \\ 0 & 0 & 2 & -8 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 2 & -5 & -4 \\ 0 & 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

There are four variables, say $x_1, x_2, x_3, x_4$, because the augmented matrix has five columns. Usually, we need the reduced echelon form before solving for the basic variables. However, in this case, we quickly observe that the last row of the matrix produce the equation $0 = 1$, which never happens for any $x_1, x_2, x_3, x_4$. Thus, the system has no solution, or the system is **inconsistent**.

**Example 5.4.** *Find the solution to the linear system whose augmented matrix is*

$$\begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 3 & 0 \end{bmatrix}$$

**Solution:** We first row reduce the augmented matrix of the linear system

$$\begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

The system has the variable since the augmented matrix has four columns. All three variables are basic variables because three columns are pivot columns. Thus, there is no free variable. In this case, the solution is obtained directly from the reduced echelon form matrix, which is

$$\begin{cases} x_1 = -11 \\ x_2 = 6 \\ x_3 = -4. \end{cases}$$

The solution is unique.

**Theorem 2.** *A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column-that is, if and only if an echelon form of the augmented matrix has no row of the form.*

$$\begin{bmatrix} 0 & \cdots & 0 & b \end{bmatrix} \quad \textit{with } b \textit{ nonzero}$$

*If a linear system is* **consistent***, then the solution set contains either*

*(i) a unique solution when there are no free variables, or*

*(ii) infinitely many solutions when there is at least one free variable.*

**Algorithm: Row Reduction to Solve a Linear Systems**

The row reduction method to solve a linear system generally consists of five steps, summarized below:

1. Write the augmented matrix of the system. (*If the system is given by its augmented matrix, this step is skipped.*)
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

### 5.3 Parametric description of Solution Sets

This subsection discusses how to write the solution set of a linear system in a vector form.

**Example 5.5.** *We consider the solution set of the linear system whose augmented matrix is*

$$\begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From Example 5.1, we found that the solution is

$$\begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free} \end{cases} \tag{5.5}$$

The solution is written in the vector form of $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Replacing $x_1$ and $x_2$ in the vector form by $1+5x_3$ and $4-x_3$ (respectively) and then splitting the vector as a sum of two vectors that multiplying and not multiplying with $x_3$, we reformulate the vector $\mathbf{x}$ as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1+5x_3 \\ 4-x_3 \\ 0+x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 5x_3 \\ -x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix}$$

The final representation $\mathbf{x} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix}$ is called the parametric vector form of the solution set of the system considered in Example 5.1.

**Example 5.6.** *Write the solution to the system in Example 5.2 in the parametric vector form.*

**Solution:** It has been found in Example 5.2 that the solution is

$$\begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 4x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{cases} \tag{5.6}$$

To find the parametric vector form, we replace $x_1$, $x_3$, $x_5$ (basic variables) by the expression on the right-hand side, then decompose the vector solution as a sum of vectors that multiply with free variables and a free vector, i.e.,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -6x_2 - 3x_4 \\ x_2 \\ 5 + 4x_4 \\ x_4 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 0 \\ 7 \end{bmatrix} + \begin{bmatrix} -6x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_4 \\ 0 \\ 4x_4 \\ x_4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 0 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}$$

The parametric vector form is

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 0 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}.$$

### Homogeneous and non-homogeneous system

**Definition 5.1.** *A linear system is said to be* **homogeneous** *if the RHS is a zero-column vector. Conversely, if at least one entry of the RHS is non-zero, the system is said to be non-homogeneous.*

Given a non-homogeneous system, the new system obtained by replacing its RHS with a zero vector of the same size is called the **associated homogeneous system**

**Example 5.7.** *The associated homogeneous system of the non-homogeneous system*

$$\begin{cases} x_1 + 4x_2 + x_3 &= 5 \\ \qquad\; x_2 - x_3 &= -4 \end{cases}$$

*is*

$$\begin{cases} x_1 + 4x_2 + x_3 = 0 \\ \qquad\; x_2 - x_3 = 0 \end{cases}$$

**Remark 5.1.**

*The augmented matrix of a homogeneous linear system is* $\begin{bmatrix} \mathbf{A} & \mathbf{0} \end{bmatrix}$, *where* **A** *is the coefficient matrix.*

*Homogeneous linear systems are always consistent since the zero vector (of the appropriate size) is a solution of the homogeneous linear system. This solution is called the* **trivial solution**. *A solution other than the zero vector is* **nontrivial solution**.

### 5.4 Application to Balancing Chemical Equations

**Example 5.8.** *Balancing the chemical equation for the combustion of ethanol below*

$$C_2H_6O + O_2 \to CO_2 + H_2O \tag{5.7}$$

Balancing equation (5.7) aims to find 4 values $x_1, x_2, x_3, x_4 > 0$ such that

$$x_1C_2H_6O + x_2O_2 \to x_3CO_2 + x_4H_2O$$

where the amount of $C, H, O$ on the left and the right sides are the same. These relations produce a linear system

$$\begin{cases} 2x_1 & = x_3 \text{ (balancing C)} \\ 6x_1 & = 2x_4 \text{ (balancing H)} \\ x_1 + 2x_2 & = 2x_3 + x_4 \text{ (balancing O)} \end{cases},$$

which is equivalent to

$$\begin{cases} 2x_1 & & -x_3 & & = 0 \\ 6x_1 & & & -2x_4 & = 0 \\ x_1 & +2x_2 & -2x_3 & -x_4 & = 0 \end{cases}$$

Solving this homogeneous linear system (row reducing the augmented matrix):

$$\begin{bmatrix} 2 & 0 & -1 & 0 & 0 \\ 6 & 0 & 0 & -2 & 0 \\ 1 & 2 & -2 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & -2 & 0 \\ 0 & 2 & -3/2 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & -3/2 & -1 & 0 \\ 0 & 0 & 3 & -2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1/2 & 0 & 0 \\ 0 & 1 & -3/4 & -1/2 & 0 \\ 0 & 0 & 1 & -2/3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -2/3 & 0 \end{bmatrix},$$

which, given the solution

$$\begin{cases} x_1 & = 1/3x_4 \\ x_2 & = x_4 \\ x_3 & = 2/3x_4 \\ x_4 & \text{is free} \end{cases}, \quad \text{or} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \frac{x_4}{3}\begin{bmatrix} 1 \\ 3 \\ 2 \\ 3 \end{bmatrix}.$$

Choosing $x_4 = 3$, we have $x_1 = 1, x_2 = 3, x_3 = 2, x_4 = 3$. Thus, the balanced equation is

$$C_2H_6O + 3O_2 \to 2CO_2 + 3H_2O$$

### 5.5 Conceptual questions

1. (T/F) A basic variable in a linear system is a variable that corresponds to a pivot column in the coefficient matrix.

2. (T/F) Reducing a matrix to the proto row echelon form is called the forward phase of the row reduction process.

3. (T/F) Finding a parametric description of the solution set of a linear system is the same as solving the system.

4. (T/F) Whenever a system has free variables, the solution set contains a unique solution.

5. (T/F) If one row in an echelon form of an augmented matrix is $\begin{bmatrix} 0 & 0 & 0 & 0 & 5 \end{bmatrix}$, then the associated linear system is inconsistent.

6. (T/F) A general solution of a system is an explicit description of all solutions of the system.

7. Suppose a $3\times 5$ coefficient matrix for a system has three pivot columns. Is the system consistent? Why or why not?

8. Suppose a system of linear equations has a $3 \times 5$ augmented matrix whose fifth column is a pivot column. Is the system consistent? Why (or why not)?

# 6 Lecture 6: Vectors and Matrices

## 6.1 Rank of a matrix

There are many way to define the rank of a matrix. Based on material we have learned so far, the **rank of a matrix A** is the number of pivot column in the matrix $\mathbf{A}$, and denoted by $rank(\mathbf{A})$. For example,

- Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{bmatrix}$, then $rank(\mathbf{A}) = 2$.
- Let $\mathbf{A} = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 \end{bmatrix}$, then $rank(\mathbf{A}) = 2$.
- Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix}$, then $rank(\mathbf{A}) = 1$.
- Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, then $rank(\mathbf{A}) = 1$.

## 6.2 Matrix-vector multiplication

### 6.2.1 Introduction and examples

If $\mathbf{A} \in \mathbb{R}^{m\times n}$ is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the product of $\mathbf{A}$ and $\mathbf{x}$, denoted by $\mathbf{Ax}$, is the linear combination of the columns of $\mathbf{A}$ using the corresponding entries in $\mathbf{x}$ as weights; that is,

$$\mathbf{Ax} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

**Example 6.1.** *1. Let* $\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix}$ *and* $\mathbf{x} = \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$, *then*

$$\mathbf{Ax} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -15 \end{bmatrix} + \begin{bmatrix} -7 \\ 21 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

*2. Let* $\mathbf{A} = \begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix}$ *and* $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 3 \\ -2 \end{bmatrix}$*, then*

$$\mathbf{A}\mathbf{x} = 1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (-1)\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + 3\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} + (-2)\begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -4 \\ -5 \\ -6 \end{bmatrix} + \begin{bmatrix} 21 \\ 24 \\ 27 \end{bmatrix} + \begin{bmatrix} -20 \\ -22 \\ -24 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}.$$

*3. The multiplication* $\begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ *is undefined.*

#### 6.2.2 Linearity properties

- $\mathbf{A}(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha\mathbf{A}\mathbf{x} + \beta\mathbf{A}\mathbf{y}$, for any $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, and $\alpha, \beta \in \mathbb{R}$.
- $(\alpha\mathbf{A} + \beta\mathbf{B})\mathbf{x} = \alpha\mathbf{A}\mathbf{x} + \beta\mathbf{B}\mathbf{x}$, for any $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times n}$, $x \in \mathbb{R}^n$, and $\alpha, \beta \in \mathbb{R}$.

#### 6.2.3 Application to reformulate linear systems

If $\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix} \in \mathbb{R}^{m\times n}$ is the coefficient matrix and $\mathbf{b} \in \mathbb{R}^m$ is the RHS of a linear system, then the linear system can be written equivalently in terms of matrix-vector multiplication as

$$\mathbf{A}\mathbf{x} = \mathbf{b},$$

here the unknown vector $\mathbf{x}$ is in $\mathbb{R}^n$.
Note that the augmented matrix of this linear system is $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$, or $\begin{bmatrix} \mathbf{A} & \mathbf{b} \end{bmatrix}$.

**Theorem 3.** *Let* $\mathbf{A}$ *be an* $m \times n$ *matrix. Then, the following statements are logically equivalent. That is, for a particular* $\mathbf{A}$*, they are all true statements or all false.*

*(a) For each* $\mathbf{b}$ *in* $\mathbb{R}^m$*, the equation* $A\mathbf{x} = \mathbf{b}$ *has a solution.*

*(b) Each* $\mathbf{b}$ *in* $\mathbb{R}^m$ *is a linear combination of the columns of* $\mathbf{A}$*.*

*(c) The columns of* $\mathbf{A}$ *span* $\mathbb{R}^m$*, i.e.,* $\mathbb{R}^m = \mathrm{Span}(\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\})$

*(d)* $\mathbf{A}$ *has a pivot position in every row.*

### 6.3 Matrix-matrix multiplication

Given two matrices $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times p}$. The multiplication $\mathbf{A}\mathbf{B}$ is a matrix in $\mathbb{R}^{m\times p}$ and is defined by

$$\mathbf{A}\mathbf{B} = \begin{bmatrix} \mathbf{A}\mathbf{b}_1 & \mathbf{A}\mathbf{b}_2 & \dots & \mathbf{A}\mathbf{b}_p \end{bmatrix},$$

where $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_p$ are $p$ column of $\mathbf{B}$, i.e., $\mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_p \end{bmatrix}$.

**Example 6.2.** *Let* $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ *and* $\mathbf{B} = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$, *compute* $\mathbf{AB}$.

**Solution:** *Write* $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}$, *and compute:*

$$\mathbf{Ab}_1 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \end{bmatrix}$$

$$\mathbf{Ab}_2 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 13 \end{bmatrix}$$

$$\mathbf{Ab}_3 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 21 \\ -9 \end{bmatrix}.$$

$$\mathbf{AB} = \begin{bmatrix} \mathbf{Ab}_1 & \mathbf{Ab}_2 & \mathbf{Ab}_3 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$$

**Row-column rule for computing matrix-matrix multiplication**

Given two matrices $\mathbf{A} = [a_{ij}]_{m\times n}$ and $\mathbf{B} = [b_{ij}]_{n\times p}$, the product $\mathbf{AB}$ is defined, then the entry in row $i$ and column $j$ of $\mathbf{AB}$, denoted by $(AB)_{ij}$, is the sum of the products of corresponding entries from row $i$ of $\mathbf{A}$ and column $j$ of $\mathbf{B}$, i.e.,

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{\ell=1}^{n} a_{j\ell}b_{\ell j}$$

**Example 6.3.** *Use the row-column rule to compute the entry in the second row and the second column of the matrix* $\mathbf{AB}$ *for the matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *in Example 6.2*

**Solution:** To calculate the entry in the second row and the second column of the matrix $\mathbf{AB}$, we use row 2 of $\mathbf{A}$ and column 2 of $\mathbf{B}$:

$$(AB)_{22} = 1(3) + (-5)(-2) = 13.$$

## Warnings:

1. In general, $\mathbf{AB} \neq \mathbf{BA}$ (non-commutativity of matrix multiplication). Furthermore, $\mathbf{BA}$ may not be defined, although $\mathbf{AB}$ is defined.

2. The cancellation laws do not hold for matrix multiplication. That is, if $\mathbf{AB} = \mathbf{AC}$, then it is not generally true that $\mathbf{B} = \mathbf{C}$.

   Example: Let $\mathbf{A} = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix}$, and $\mathbf{C} = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$, then $\mathbf{AB} = \mathbf{AC}$ and but $\mathbf{B} \neq \mathbf{C}$.

3. If a product $\mathbf{AB}$ is the zero matrix, it cannot generally conclude that either $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$.

   Example: Let $\mathbf{A} = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$, then $\mathbf{AB}$ is the zero matrix but both matrix $\mathbf{A}$ and $\mathbf{B}$ are non-zeros.

4. **Powers of a matrix**

$$\mathbf{A}^n = \underbrace{\mathbf{A}\mathbf{A}\ldots\mathbf{A}}_{\text{n times}}$$

## Properties

Let $\mathbf{A}$ be an $m \times n$ matrix, and let $\mathbf{B}$ and $\mathbf{C}$ have sizes for which the indicated sums and products are defined.

1. $\mathbf{A}(\mathbf{B}\mathbf{C}) = (\mathbf{A}\mathbf{B})\mathbf{C}$ (associative law of multiplication)
2. $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{A}\mathbf{B}+\mathbf{A}\mathbf{C}$ (left distributive law)
3. $(\mathbf{B}+\mathbf{C})\mathbf{A} = \mathbf{B}\mathbf{A}+\mathbf{C}\mathbf{A}$ (right distributive law)
4. $r(\mathbf{A}\mathbf{B}) = (r\mathbf{A})\mathbf{B} = \mathbf{A}(r\mathbf{B})$, for any scalar $r$ (identity for matrix multiplication)

## 6.4 Matrix transpose

### 6.4.1 Definition and examples

**Definition 6.1.** *The transpose of an matrix $\mathbf{A}$ (denoted by $\mathbf{A}^T$) is matrix that simple converts each row to a column.*

If matrix $\mathbf{A}$ is written as $\mathbf{A} = [a_{ij}]$, and $\mathbf{B} = [b_{ij}]$ is the transpose of $\mathbf{A}$, i.e., $\mathbf{B} = \mathbf{A}^T$, then $b_{ij}a_{ji}$.

**Example 6.4.**

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \textit{ (the transpose of a column vector is called a row vector)}$$

In particular, $(\mathbf{A}^T)^T = \mathbf{A}$.

### 6.4.2 The transpose of a product

1. $(\mathbf{A}\mathbf{B})^T = \mathbf{B}^T\mathbf{A}^T$.
2. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ (two column vectors) then
   - $\mathbf{x}^T\mathbf{y} = \mathbf{x} \cdot \mathbf{y} = \sum_{j=1}^n x_j y_j$, is a scalar (known as an inner product).
   - $xy^T$ is an outer product, denote by $\mathbf{x} \otimes \mathbf{y}$, is a $n \times n$ matrix. Furthermore,

$$xy^T = [x_i y_j] = \begin{bmatrix} x_1y_1 & x_1y_2 & \ldots & x_1y_n \\ x_2y_1 & x_2y_2 & \ldots & x_2y_n \\ \vdots & \vdots & \ldots & \vdots \\ x_ny_1 & x_ny_2 & \ldots & x_ny_n \end{bmatrix}$$

### 6.4.3 Symmetric and antisymmetric matrices

A real matrix $\mathbf{A}$ is said to be **symmetric** if $\mathbf{A}^T = \mathbf{A}$. For example

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 9 \\ 3 & 9 & 16 \end{bmatrix} \text{ is symmetric due to } \mathbf{A}^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 9 \\ 3 & 9 & 16 \end{bmatrix} = \mathbf{A}.$$

Note, a $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is symmetric if and only if $a_{ij} = a_{ji}$ for $1 \leq i, j \leq n$.

A real matrix $\mathbf{A}$ is said to be **antisymmetric** if $\mathbf{A}^T = -\mathbf{A}$. For example

$$\mathbf{A} = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 9 \\ -3 & -9 & 0 \end{bmatrix} \text{ is symmetric due to } \mathbf{A}^T = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & -9 \\ 3 & 9 & 0 \end{bmatrix} = -\mathbf{A}.$$

Note, a $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is antisymmetric if and only if $a_{ij} = -a_{ji}$ for $1 \leq i, j \leq n$. Entries in the main diagonal of an antisymmetric matrix must be 0.

### 6.4.4 Gram matrix and Hermitian transpose

- For any real matrix $\mathbf{A}$, the matrix $\mathbf{A}^T\mathbf{A}$ is called the **Gram matrix** of $\mathbf{A}$. In particular, if $\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & a_n \end{bmatrix}$ then

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{a}_1 & \mathbf{a}_1 \cdot \mathbf{a}_2 & \dots & \mathbf{a}_1 \cdot \mathbf{a}_n \\ \mathbf{a}_2 \cdot \mathbf{a}_1 & \mathbf{a}_2 \cdot \mathbf{a}_2 & \dots & \mathbf{a}_2 \cdot \mathbf{a}_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_n \cdot \mathbf{a}_1 & \mathbf{a}_n \cdot \mathbf{a}_2 & \dots & \mathbf{a}_n \cdot \mathbf{a}_n \end{bmatrix}$$

- If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a vector in $\mathbb{C}^n$, the **complex conjugate** of $\mathbf{x}$, denoted by $\mathbf{x}^*$, is the vector of the same size whose entries equal the complex conjugate of $\mathbf{x}$. That is, $\mathbf{x}^* = \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \vdots \\ \overline{x}_n \end{bmatrix}$

- If $x \in \mathbb{C}^n$ is a complex-valued vector, then the **Hermitian transpose** of $\mathbf{x}$, denoted by $\mathbf{x}^H$, is the vector given by the complex conjugate of the transpose of vector $\mathbf{x}$.