19 Review for exam 2

Topics to review

1. Calculate determinants

- Using Laplace expansion
- Using row reduction: Be aware of the change of the determinant when applying elementary row operations.

2. Find eigenvalues and eigenvectors of an $n \times n$ matrix $\mathbf{A}$:

- In general, to find eigenvalue, we find the roots of the characteristic polynomial $p_{\mathbf{A}}(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I})$, that is, solve equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$. To find eigenvector associated with an eigenvalue λ (λ is given), we solve the linear system $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$.
- **If matrix A and an eigenvector v is given**, then to find the eigenvalue associated with $\mathbf{v}$, we calculate $\mathbf{Av}$. The eigenvalue λ is such that $\mathbf{Av} = \lambda\mathbf{v}$.

3. Diagonalize an $n \times n$ matrix A, $(n = 2, 3, \ldots)$

- We first first all eigenvalues and the associated eigenvectors of $\mathbf{A}$.
- Matrix $\mathbf{A}$ is diagonalizable (i.e., the diagonalization of $\mathbf{A}$ exists) if and only if, $\mathbf{A}$ has n linearly independent eigenvectors.
- If $\mathbf{A}$ is is diagonalizable, the diagonalization of $\mathbf{A}$ is

$$\mathbf{A} = \mathbf{SDS}^T,$$

where,

$$\mathbf{S} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}, \quad \text{with } \mathbf{Av}_\ell = \lambda_\ell \mathbf{v}_\ell, \;\; \ell = 1, 2, \ldots, n.$$

4. Determine if a set of vectors is dependent or independent: To check if the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\} \subset \mathbb{R}^n$ are linearly dependent or independent, we consider the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + \mathbf{x}_p\mathbf{v}_p = \mathbf{0} \tag{19.1}$$

- If equation (19.1) has only trivial solution, i.e.., $x_1 = x_2 = \ldots = x_0 = 0$, the set S is linearly independent.
- If equation (19.1) has nontrivial solution, i.e., exists p constants $\alpha_1, \alpha_2, \ldots, \alpha_p$ not all zeros such that $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_p\mathbf{v}_p = \mathbf{0}$, then the set S is linearly dependent.

In practice, equation (19.1) is equivalent to

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_p \end{bmatrix}\mathbf{x} = \mathbf{0} \tag{19.2}$$

- If the linear system (19.2) has only trivial solution (i.e., all columns of the matrix $\mathbf{A} = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_p\end{bmatrix}$ are pivots), then the set S is linearly independent.
- If the linear system (19.2) has only nontrivial solution (i.e., equation (19.2) has at least one free variable, or at least one columns of the matrix $\mathbf{A} = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_p\end{bmatrix}$ is not pivot), then the set S is linearly dependent.

5. **Find basis of four fundamental spaces:** To find a basis for four fundamental subspace $\text{Col}(\mathbf{A})$, $\text{Nul}(\mathbf{A})$, $\text{Row}(\mathbf{A})$, $\text{Nul}(\mathbf{A}^T)$ of an $m \times n$ matrix $\mathbf{A}$, we row reduce the matrix $\begin{bmatrix}\mathbf{A} & \mathbf{I}_m\end{bmatrix}$ to the matrix of the form $\begin{bmatrix}\text{rref}(\mathbf{A}) & \mathbf{B}\end{bmatrix}$, where $\mathbf{I}_m$ is the $m \times m$ identity matrix.
   - The set of all pivot columns of $\mathbf{A}$ is a basis for $\text{Col}(\mathbf{A})$.
   - Solve equation $\mathbf{Ax} = \mathbf{0}$ (equivalent to $\text{rref}(\mathbf{A})\mathbf{x} = \mathbf{0}$), then write the solution under the parametric vector form to find a basis for $\text{Nul}(\mathbf{A})$.
   - The set of transpose of all nonzero rows of $\text{rref}(\mathbf{A})$ is a basis of $\text{Row}(\mathbf{A})$.
   - The set of transpose of $(m - rank(\mathbf{A}))$ last rows of the matrix $\mathbf{B}$ is a basis for $\text{Nul}(\mathbf{A}^T)$.

6. **Find least-square solution of a linear system:** The least square solution of the linear system $\mathbf{Ax} = \mathbf{b}$ is the solution to the new linear system

$$\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}.$$

   Calculate $\widetilde{\mathbf{A}} = \mathbf{A}^T\mathbf{A}$, $\widetilde{\mathbf{b}} = \mathbf{A}^T\mathbf{b}$ then solve the system $\widetilde{\mathbf{A}}\mathbf{x} = \widetilde{\mathbf{b}}$ by row reducing the augmented matrix $\begin{bmatrix}\widetilde{\mathbf{A}} & \widetilde{\mathbf{b}}\end{bmatrix}$.

7. **Gram matrix and properties:** Let $\mathbf{A} = \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n\end{bmatrix}$ an $m \times n$ matrix. The Gram matrix of $\mathbf{A}$ is $\mathbf{A}^T\mathbf{A}$. Then,

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix}\mathbf{a}_1 \cdot \mathbf{a}_1 & \mathbf{a}_1 \cdot \mathbf{a}_2 & \dots & \mathbf{a}_1 \cdot \mathbf{a}_n \\ \mathbf{a}_2 \cdot \mathbf{a}_1 & \mathbf{a}_2 \cdot \mathbf{a}_2 & \dots & \mathbf{a}_2 \cdot \mathbf{a}_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_n \cdot \mathbf{a}_1 & \mathbf{a}_n \cdot \mathbf{a}_2 & \dots & \mathbf{a}_n \cdot \mathbf{a}_n\end{bmatrix}$$

   If we write the Gram matrix of $\mathbf{A}$ as the form $\mathbf{A}^T\mathbf{A} = [g_{ij}]$, then $g_{ij} = \mathbf{a}_i \cdot \mathbf{a}_j$. Obviously, the Gram matrix is symmetric.
   - An $n \times n$ matrix $\mathbf{A}$ is an **orthogonal matrix** if and only if $\mathbf{A}^T\mathbf{A} = \mathbf{I}$.
   - A square matrix $\mathbf{A}$ is a projection matrix if and only if $\mathbf{A}^2 = \mathbf{A}$.
   - A symmetric, projection matrix is called an **orthogonal projection matrix**.

8. **Some statements:**
   - An $n \times n$ symmetric real matrix has n real eigenvalues. As a consequence, an $n \times n$ orthogonal projection matrix has n real eigenvalues.
   - The row space is the orthogonal complement of the null space.

- The column space is the orthogonal complement of the left null space.
- A real matrix may have complex eigenvalues. Furthermore, if λ is a complex eigenvalue then its complex conjugate also an eigenvalue.
- An $n \times n$ real matrix with n is an odd number, always has at least one real eigenvalue.
- If $\mathbf{A}$ is a real matrix then the rank of $\mathbf{A}$ and the rank of its corresponding Gram matrix, $\mathbf{A}^T\mathbf{A}$ are equal. Thus, for real matrices

$$rank(\mathbf{A}^T\mathbf{A}) = rank(\mathbf{A}\mathbf{A}^T) = rank(\mathbf{A}^T) = rank(\mathbf{A})$$

- $\mathbf{A}^T\mathbf{A}$ is invertible if and only if columns of $\mathbf{A}$ are linearly independent.

Please note: If $\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix}$, then $\mathbf{A}^T = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix}$, and

$$\mathbf{A}\mathbf{A}^T = \mathbf{a}_1\mathbf{a}_1^T + \mathbf{a}_2\mathbf{a}_2^T + \ldots + \mathbf{a}_n\mathbf{a}_n^T.$$

In addition, please review your quizzes, lecture notes and examples in lecture notes.