Problem 1.1. (30 points total)

(a) (20 points) Clearly documenting your extremely careful work, compute the reduced row-echelon form of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 0 \\ 2 & 4 & 4 & 10 & -2 \\ 1 & 2 & 5 & 11 & -4 \end{bmatrix}$$

(b) (1 points) Which columns of A have pivots? $\boxed{}$ (list column numbers)

(c) (9 points) If possible, find the solution of the linear system below in parametric vector form.

$$\begin{cases} x_1 + 2x_2 + x_3 + 3x_4 = 0 \\ 2x_1 + 4x_2 + 4x_3 + 10x_4 = -2 \\ x_1 + 2x_2 + 5x_3 + 11x_4 = -4 \end{cases}$$

Problem 1.2. (6 points) Carefully simplify each valid expression. For any invalid expression, explain why it is invalid.

$$(a)\qquad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$(b)\qquad \begin{bmatrix} 1 & 3 & 1 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$(c)\qquad \begin{bmatrix} 1 & 3 & 1 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 4 & 2 & 3 \end{bmatrix}^T - \begin{bmatrix} 1 & 3 & 1 \\ 4 & 2 & 3 \end{bmatrix}^T \begin{bmatrix} 1 & 3 & 1 \\ 4 & 2 & 3 \end{bmatrix}$$

$$(d) \qquad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$(e) \qquad \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}^T$$

$$(f) \qquad \begin{bmatrix} \sqrt{3} & -1 \\ 3 & -\sqrt{3} \end{bmatrix}^2$$

Problem 1.3. (6 points) For each matrix below, compute the matrix inverse, if it exists. If it fails to exist explain why.

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{bmatrix}$$

Problem 1.4. (4 points) If possible, find the LU factorization of each matrix below:

$$A_1 = \begin{bmatrix} 0 & 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 4 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 \end{bmatrix}$$

Problem 1.5. (4 points) If possible, find the CR factorization of each matrix below:

$$A_1 = \begin{bmatrix} 0 & 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 4 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 \end{bmatrix}$$

Problem 2. (10 points) Evaluate the truth of each statement below. If the statement is true write T in the box preceeding the statment. Otherwise, write F.

(a) $\boxed{}$ A homogeneous linear system must have infinitely many solutions.

(b) $\boxed{}$ If A is 3×3 and B is 3×3 then it must be that $AB \neq BA$.

(c) $\boxed{}$ A sum of two symmetric 3×3 matrices is symmetric.

(d) $\boxed{}$ The product of two invertible 3×3 matrices is invertible.

(e) $\boxed{}$ Every matrix may be factored as LU where L is lower unitriangular and U is proto-row-echelon.

(a) $\square$ A linear system with one or more free variables has infinitely many solutions.

(b) $\square$ Two different matrices cannot have the same reduced row-echelon form.

(c) $\square$ An inconsistent system must have more equations than unknowns.

(d) $\square$ The linear combination of two solutions of a homogeneous linear system is also a solution.

(e) $\square$ If A and B are 2×2 matrices and $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. then either A or B must be equal to $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Problem 3. (10 points) [Sample Problem A] What is the result of the MATLAB commands below.

```
>> A = [1 2 3; 2 3 4];
>> rref(A)

ans =


     1     0    -1
     0     1     2

>> rref([2 2; 0 2]*A)


ans =
```

Problem 3. (10 points) [Sample Problem B] Suppose that the MATLAB function `initialzeros` is defined by

```
function zerocount=initialzeros(A)
    [m n]=size(A);
    zerocount=zeros(m,1);   % column vector of m zeros
    for r=1:m
        for c=1:n
            if A(r,c) ~= 0
                break;
            end
            zerocount(r)=zerocount(r)+1;
        end
    end
end
```

Assuming `initialzeros` is in the current MATLAB search path, what is the output obtained by entering the following MATLAB commands:

```
>> A = [ 0 0 2 3 4; 0 0 0 0 0; 1 2 3 4 5; 0 0 2 0 0; 0 1 2 3 2; 0 0 0 0 0];
>> initialzeros(A)
```

ans =

Problem 4. (15 points) [Sample Problem A] The matrices listed below constitute all possible 3×3 reduced row-echelon matrices. Note that some of these matrices contain parameters which are aribitrary scalar values.

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 0 & 1 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix} \qquad A_6 = \begin{bmatrix} 1 & a & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad A_7 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad A_8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) Determine the rank of each matrix above.

(b) For each matrix above, give the inverse, if possible.

(c) Which of the matrices above, considered as augmented matrices, represent an inconsistent system of linear equations?

(d) For each matrix above that represents a linear system having multiple solutions, provide the solution set in parametric vector form.

Problem 4. (15 points) [Sample Problem B] Suppose

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 3 & 2 & 1 & 1 \\ 1 & 3 & 3 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} 1 & 1 & 0 & -3 \\ -2 & -1 & 0 & 5 \\ 3 & 2 & 1 & -10 \\ -2 & -2 & -1 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

(a) Solve the linear system

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 3 & 2 & 1 & 1 \\ 1 & 3 & 3 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(b) Solve the linear system

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 3 & 2 & 1 & 1 \\ 1 & 3 & 3 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(c) Simplify

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}^{-1} \left(\begin{bmatrix} 1 & 1 & 2 & 2 \\ 3 & 2 & 1 & 1 \\ 1 & 3 & 3 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}^{-1} \right)^{-1} \begin{bmatrix} 1 & 1 & 2 & 2 \\ 3 & 2 & 1 & 1 \\ 1 & 3 & 3 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1}$$

Problem 4. (15 points) [Sample Problem C]

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}$. Fully explain your responses.

(a) Determine the rank (number of pivots) of the matrix A.

(b) Determine the rank of the matrix $A^T A$.

(c) Determine the rank of the matrix AA^T.

Problem 5. (15 points) [Sample Problem A] An $n \times n$ matrix A is congruent to an $n \times n$ matrix B if and only if there is a nonsingular $n \times n$ matrix S such that

$$SAS^T = B.$$

(a) Explain why every $n \times n$ matrix is congruent to itself.

(b) Explain why if A is congruent to B then B is congruent to A.

(c) Explain why if A is congruent to B and B is congruent to C then A is congruent to C.

Problem 5. (15 points) [Sample Problem B]

(a) Suppose that $\mathbf{a} \in \mathbb{R}^m$ and $\mathbf{b} \in \mathbb{R}^n$ what is the rank of $\mathbf{a}^T\mathbf{b}$? Explain why.

(b) Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 8 & 12 \\ -3 & -6 & -9 \end{bmatrix}$. Find $\mathbf{a} \in \mathbb{R}^3$ and $\mathbf{b} \in \mathbb{R}^3$ such that $A = \mathbf{a}^T\mathbf{b}$.

(c) Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Find vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_1, \mathbf{b}_2$ such that

$$A = \mathbf{a}_1^T\mathbf{b}_1 + \mathbf{a}_2^T\mathbf{b}_2.$$

Problem 5. (15 points) [Sample Problem C] Carefully explain your answers to each part of this question.

(a) How many 2×2 real symmetric matrices are square roots of $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$?

(b) How many 2×2 real antisymmetric matrices are square roots of $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$?

(c) How many 2×2 real antisymmetric matrices are fourth roots of $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$?

Problem 5. (15 points) [Sample Problem D] Consider Gauss-Jordan applied to $A = \begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$:

$$\begin{array}{ccccccccc} A & & & & \text{pref}(A) & & \text{nref}(A) & & \text{rref}(A) \\ \begin{array}{c} \bullet \\ 4 \\ 7 \end{array}\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix} & \sim & \begin{array}{c} \\ \bullet \\ 2 \end{array}\begin{bmatrix}1&2&3\\0&-3&-6\\0&-6&-12\end{bmatrix} & \sim & \begin{array}{c} \\ \times -1/3 \\ \\ \end{array}\begin{bmatrix}1&2&3\\0&-3&-6\\0&0&0\end{bmatrix} & \sim & \begin{array}{c} 2 \\ \bullet \\ \\ \end{array}\begin{bmatrix}1&2&3\\0&1&2\\0&0&0\end{bmatrix} & \sim & \begin{bmatrix}1&0&-1\\0&1&2\\0&0&0\end{bmatrix} \end{array}$$

(a) Find a lower unitriangular matrix L, a diagonal matrix D and an upper unitriangular matrix U such that

$$A = L\ \text{pref}(A), \qquad \text{pref}(A) = D\ \text{nref}(A), \qquad \text{and} \qquad \text{nref}(A) = U\ \text{rref}(A).$$

$L =$ $\boxed{}$ $D =$ $\boxed{}$ $U =$ $\boxed{}$

(b) Find a matrix C such that $A = C \begin{bmatrix}1&0&-1\\0&1&2\end{bmatrix}$.

$C =$ $\boxed{}$

(c) Find all matrices S such that $A = S\ \mathrm{rref}(A)$.

Extra Credit. (5 points) *Showing your work*, find the smallest positive integers x_1, x_2, x_3 and x_4 that balances the chemical equation for the combustion of ethanol:

$$(x_1)\ C_2H_6O\ +\ (x_2)\ O_2\ \rightarrow\ (x_3)\ CO_2\ +\ (x_4)\ H_2O.$$

$\boxed{}\ C_2H_6O + \boxed{}\ O_2 \rightarrow \boxed{}\ CO_2 + \boxed{}\ H_2O.$