9 Lecture 9: Linear Transformations

9.1 Definitions and examples

Let $\mathbf{U} \subset \mathbb{R}^n$ and $\mathbf{V} \subset \mathbb{R}^m$ be two subset in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. A **transformation** (or **function** or **mapping**) f from $\mathbf{U}$ to $\mathbf{V}$, denoted $f : \mathbf{U} \to \mathbf{V}$, is a rule that assigns to each vector $\mathbf{x} \in \mathbf{U}$ a vector $f(\mathbf{x})$ in $\mathbf{V}$. The set $\mathbf{U}$ is called the **domain**, and $\mathbf{V}$ is called the **codomain**. For each $\mathbf{x} \in \mathbf{U}$, the vector $f(\mathbf{x})$ is called the **image** of $\mathbf{x}$ (under the action of f). The set of all images $f(\mathbf{x})$ is called the **range** of f.

Definition 9.1. *A transformation f is* **linear** *if*

(i) $f(\mathbf{u}+\mathbf{v}) = f(u) + f(\mathbf{v})$, *for all* $\mathbf{u}, \mathbf{v}$ *in the domain of* f

(ii) $f(c\mathbf{u}) = f(\mathbf{u})$, *for all constant* c *and for all* $\mathbf{u}$ *in the domain of* f.

Note, two condition (i) and (ii) in the definition of linear transformations can be written equivalently, as

$$f(c_1\mathbf{x}_1 + c_2\mathbf{x}_2) = c_1 f(\mathbf{x}_1) + c_2 f(\mathbf{x}_2), \quad \text{and} \quad f(\mathbf{0}) = \mathbf{0}.$$

Example 9.1. *For each matrix* $\mathbf{A} \in \mathbb{R}^{m\times n}$, *the transformation* $f : \mathbb{R}^n \to \mathbb{R}^m$ *given by*

$$f(\mathbf{x}) = \mathbf{A}\mathbf{x}$$

is a linear transformation.

Indeed, by the linearity of the matrix vector multiplications, for any $c_1, c_2 \in \mathbb{R}$ and $\mathbf{x}_1, x_2 \in \mathbb{R}^n$,

$$\begin{aligned} f(c_1\mathbf{x}_1 + c_2\mathbf{x}_2) &= \mathbf{A}(c_1\mathbf{x}_1 + c_2\mathbf{x}_2) = c_1\mathbf{A}\mathbf{x}_1 + c_2\mathbf{A}\mathbf{x}_2 = c_1 f(\mathbf{x}_1) + c_2 f(\mathbf{x}_2) \\ f(\mathbf{0}) &= \mathbf{A}\mathbf{0} = \mathbf{0} \end{aligned}$$

Example 9.2. *Given the linear transformation* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *such that* $f\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\3\end{bmatrix}$ *and* $f\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix}$. *Calculate* $f\left(\begin{bmatrix}3\\3\end{bmatrix}\right)$

Solution: Since

$$\begin{bmatrix}3\\3\end{bmatrix} = 3\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix},$$

by the linearity of the transformation f:

$$\begin{aligned} f\left(\begin{bmatrix}3\\3\end{bmatrix}\right) = f\left(3\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix}\right) &= 3f\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + 3f\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \\ &= 3\begin{bmatrix}1\\3\end{bmatrix} + 3\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}3\\9\end{bmatrix} + \begin{bmatrix}6\\9\end{bmatrix} = \begin{bmatrix}9\\18\end{bmatrix}. \end{aligned}$$

9.2 The matrix of a linear transformation

For each linear transformation $f : \mathbb{R}^n \to \mathbb{R}^m$, there exists a unique matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ such that

$$f(\mathbf{x}) = \mathbf{A}\mathbf{x}, \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^n.$$

Moreover, the matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} f(\mathbf{e}_1) & f(\mathbf{e}_2) & \dots & f(\mathbf{e}_n) \end{bmatrix}, \tag{9.1}$$

where $\mathbf{e}_j$ is the j^{th} column of the identity matrix in $\mathbb{R}^n$. The matrix $\mathbf{A}$ defined in (9.1) is called the **standard matrix for the linear transformation** f.

Example 9.3. *Find the standard matrix for the linear transformation $f(\mathbf{x}) = 2\mathbf{x}$ for $x \in \mathbb{R}^2$.*

Solution: We write

$$f(\mathbf{e}_1) = 2\mathbf{e}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad f(\mathbf{e}_2) = 2\mathbf{e}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Thus

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Matrix of composite transformation

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation given by $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$ with $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $g : \mathbb{R}^p \to \mathbb{R}^n$ be a linear transformation given by $g(\mathbf{x}) = \mathbf{B}\mathbf{x}$ with $\mathbf{B} \in \mathbb{R}^{n\times p}$. Define the composite transformation (or composition of transformation)

$$h(\mathbf{x}) = f \circ g(\mathbf{x}) = f(g(\mathbf{x})).$$

Then $h : \mathbb{R}^p \to \mathbb{R}^m$ and h is given by $h(\mathbf{x}) = \mathbf{A}\mathbf{B}\mathbf{x}.$

Thus, a matrix-matrix multiplication is the standard matrix of a composite transformation.

9.3 Geometric Linear Transformations of $\mathbb{R}^2$

We here consider the geometry of some special kind of linear transformations.

9.3.1 Horizonal dilation (or expansion) and constriction

The linear transformation if of the form

$$f(\mathbf{x}) = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x}, \tag{9.2}$$

where,

- $k > 1$ for dilation
- $0 < k < 1$ for constriction.

That is, the geometric of an object in $\mathbb{R}^2$ is scaled by k along x_1- direction under the transformation f given by (9.2).

9.3.2 Vertical dilation (or expansion) and constriction

The linear transformation if of the form

$$f(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \mathbf{x}, \tag{9.3}$$

where,

- $k > 1$ for dilation
- $0 < k < 1$ for constriction.

That is, the geometric of an object in $\mathbb{R}^2$ is scaled by k along x_2- direction under the transformation f given by (9.3).

9.3.3 Rotation

The linear transformation producing **counterclockwise rotation** through an angle θ is of the form

$$f(\mathbf{x}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \mathbf{x}, \tag{9.4}$$

9.3.4 Reflection

The linear transformation producing the reflection through a line, which passes through the origin and makes an angle θ with the x_1- axis is of the form

$$f(\mathbf{x}) = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \mathbf{x}, \tag{9.5}$$

See more examples and graphics in the Matlab script file "L_I9_F24.mlx" on Canvas (the common course).