14 Lecture 15: Spanning, Independence and Basis

14.1 Vector Space

A vector space is a nonempty set of vectors that is closed under the formation of linear combinations. That is, $V \subset \mathbb{R}^n$ is a real vector space if and only if

$$\{\mathbf{u}_1, \ldots, \mathbf{u}_p\} \subset V \text{ then } \alpha_1\mathbf{u}_1 + \ldots \alpha_p\mathbf{u}_p \in \mathbf{V}, \quad \text{for all } \alpha_1, \ldots, \alpha_p \in \mathbb{R}. \tag{14.1}$$

A vector space that is a subset of another vector space is called a **subspace.**

Example 14.1.
- *$\mathbb{R}^n$ is a vector space.*
- *The singleton set $\{\mathbf{0}\} \subset \mathbb{R}^n$ is a vector space and is a subspace of $\mathbb{R}^n$.*
- *The solution set of a homogeneous linear system of n unknown is a vector space, and is a subspace of $\mathbb{R}^n$.*

14.2 Spanning set

Let $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\} \subset \mathbb{R}^n$, the set of p vectors in $\mathbb{R}^n$. The set of all linear combinations of S:

$$V := \{\mathbf{v} = \alpha_1\mathbf{u}_1 + \ldots \alpha_p\mathbf{u}_p | \, \alpha_1, \ldots, \alpha_p \in \mathbb{R}\}$$

is called the span of S, denoted by $\mathrm{Span}(S)$.

Remark:

- $\mathrm{Span}(S)$ is a vector space for any set $S \subset \mathbb{R}^n$, including the empty set.
- $\mathrm{Span}(\emptyset) = \{\mathbf{0}\}$.

For each vector space V, the set S such that $V = \mathrm{Span}(S)$ is called the **spanning set** of V.

14.3 Dependence/Independence

The set of p vectors $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\} \subset \mathbb{R}^n$ is called **linearly independent** of the vector equation

$$\alpha_1\mathbf{u}_1 + \ldots \alpha_p\mathbf{u}_p = \mathbf{0} \tag{14.2}$$

has only trivial solution.

If a set of vectors is **not linearly independent**, it is said to be **linearly dependent** (or simply **dependent**). Equivalently, if the equation (14.2) has non-trivial solution, the set $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is **linear dependent**.

Equivalent conditions for the linearly dependence/independence

Denote $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\} \subset \mathbb{R}^n$ and $\mathbf{A} = \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_p\end{bmatrix}$ an $n \times p$ matrix.

- If $\mathbf{A}$ has p pivot columns, i.e., $rank(\mathbf{A}) = p$, then S is linearly independent.
- $\mathbf{A}$ has p pivot columns, then S is linearly dependent.

Some special case for the linearly dependence/independence

Denote $S = \{\mathbf{u}_1, \dots, \mathbf{u}_p\} \subset \mathbb{R}^n$.

- If S contains the zero vector, then S is linearly dependent.
- If S contains more than n vectors, i.e., $p > n$, then S is linearly dependent.
- If there is a vector in S is a linear combination of the others, then S is linearly dependent.
- If S is the set of one nonzero vector, then S is linearly independent.

Example 14.2.

- *The set* $S = \left\{ \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ *is linearly dependent due to S contains the zero vector, so* $0 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + \alpha \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ *for any $\alpha \in \mathbb{R}$.*
- *The set* $S = \left\{ \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$ *is linearly dependent due to matrix* $\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 3 & 3 & 1 & 0 \\ 4 & 2 & 1 & 2 \end{bmatrix}$ *has at least one column which is not pivot.*
- *The set* $S = \left\{ \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 6 \end{bmatrix} \right\}$ *is linearly dependent due to* $\begin{bmatrix} 3 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$, *which implies* $\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.
- *The set* $S = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ *is linearly independent since the equation* $x \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ *has only trivial solution $x = 0$.*

Example 14.3. *Check whether the set* $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$ *is dependent or independent.*

Solution: We find the rank of matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{bmatrix}.$$

Row reduce matrix $\mathbf{A}$,

$$\mathbf{A} \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & -3 \\ 0 & -4 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & -17 \end{bmatrix}. \tag{14.3}$$

$\mathbf{A}$ has three pivot columns, so $rank(\mathbf{A}) = 3$, which is the number of column of $\mathbf{A}$. Thus, the set S is linearly independent.

14.4 Basis

If S is a spanning set of a vector space V and S is linearly independent then S is a basis of V. In another word, a basis of a vector space V is a linearly independent set that spans V.

Note: The basis of a vector space is not unique, that is, a vector space $V \neq \{\mathbf{0}\}$ has more than one basis. However, the number of vectors in all bases of a vector space is the same.

Example 14.4.

- $S_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ *is a basis of* $\mathbb{R}^2$. *Indeed,* S_1 *is a linearly independent set, and* $\text{Span}(S_1) \equiv \mathbb{R}^2$.

- $S_2 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$ *is also a basis of* $\mathbb{R}^2$. *Indeed,* S_2 *is a linearly independent set (due to the matrix* $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ *is invertible, so all columns of* $\mathbf{A}$ *are pivots), and* $\text{Span}(S_2) \equiv \mathbb{R}^2$. *To verify*

$$\text{Span}(S_2) \equiv \mathbb{R}^2,$$

we see that $S_2 \subset \mathbb{R}^2$, *so* $\text{Span}(S_2) \subset \mathbb{R}^2$. *On the other hand, for each* $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$, *the equation* $\mathbf{Ax} = b$ *always has a unique solution, which implies,* $\mathbf{b}$ *is a linear combination of vectors in* S_2. *Thus,* $\mathbb{R}^2 \subset \text{Span}(S_2)$. *Combining* $\text{Span}(S_2) \subset \mathbb{R}^2$ *and* $\mathbb{R}^2 \subset \text{Span}(S_2)$ *we have* $\text{Span}(S_2) \equiv \mathbb{R}^2$.

- *More general, a basis of* $\mathbb{R}^2$ *has 2 vectors. In addition, a set* $A = \{\mathbf{u}_1, \mathbf{u}_2\}$ *where* $\mathbf{u}_1$ *is not a scalar multiplication of* $\mathbf{u}_2$ *is a basis of* $\mathbb{R}^2$.