**Problem 1.1.** (30 points total)

(a) (20 points) Clearly documenting your extremely careful work, compute the reduced row-echelon form of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 0 \\ 2 & 4 & 4 & 10 & -2 \\ 1 & 2 & 5 & 11 & -4 \end{bmatrix}$$

(b) (1 points) Which columns of $A$ have pivots? $\boxed{1,\ 3}$ (list column numbers)

$$\begin{array}{c} \bullet \\ 2 \\ 1 \end{array} \begin{bmatrix} 1 & 2 & 1 & 3 & 0 \\ 2 & 4 & 4 & 10 & -2 \\ 1 & 2 & 5 & 11 & -4 \end{bmatrix} \sim \begin{array}{c} \\ \bullet \\ 2 \end{array} \begin{bmatrix} 1 & 2 & 1 & 3 & 0 \\ 0 & 0 & 2 & 4 & -2 \\ 0 & 0 & 4 & 8 & -4 \end{bmatrix} \sim \begin{array}{c} \\ \times\ 1/2 \\ \end{array} \begin{bmatrix} 1 & 2 & 1 & 3 & 0 \\ 0 & 0 & 2 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{c} 1 \\ \bullet \\ \\ \end{array} \begin{bmatrix} 1 & 2 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \boxed{\begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

(c) (9 points) If possible, find the solution of the linear system below in parametric vector form.

$$\begin{cases} x_1 + 2x_2 + x_3 + 3x_4 = 0 \\ 2x_1 + 4x_2 + 4x_3 + 10x_4 = -2 \\ x_1 + 2x_2 + 5x_3 + 11x_4 = -4 \end{cases}$$

Since the augmented matrix for this system is the matrix given in part (a), the rref computed in (a) gives the information needed for the parametric vector form:

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}}$$

**Problem 1.2.** (6 points) Carefully simplify each valid expression. For any invalid expression, explain why it is invalid.

(*a*) $$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \boxed{\text{is undefined}}$$

Here there is a $5 \times 4$ matrix times a $5 \times 1$ matrix.

(*b*) $$\begin{bmatrix} 1 & 3 & 1 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 5 \\ 9 & 8 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}}$$

Alternatively, one may recognize the given expression as a product minus the first two terms of its outer product expansion. Hence, the expression is equal to the third term in the outer product expansion $\begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix}$.

(*c*) $$\begin{bmatrix} 1 & 3 & 1 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 4 & 2 & 3 \end{bmatrix}^T - \begin{bmatrix} 1 & 3 & 1 \\ 4 & 2 & 3 \end{bmatrix}^T \begin{bmatrix} 1 & 3 & 1 \\ 4 & 2 & 3 \end{bmatrix} \quad \text{is } \boxed{\text{undefined}}.$$

The first product in the expression is a $2 \times 3$ matrix times a $3 \times 2$ matrix giving a $2 \times 2$ matrix.

The second product in the expresion is a $3 \times 2$ matrix times a $2 \times 3$ matrix giving a $3 \times 3$ matrix.

Since these two products have different dimensions their difference is undefined.

$$(d) \qquad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \boxed{\text{is undefined}}$$

The product given is a $1 \times 5$ matrix times a $5 \times 4$ matrix by a $5 \times 1$ matrix.

$$\text{(e)} \qquad \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}^T = \boxed{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

The given product of four matrices is a $1 \times 2$ matrix times a $2 \times 2$ matrix times a $2 \times 1$ matrix times a $1 \times 5$ matrix; hence, the product is defined and is a $1 \times 5$ matrix. Since one of the factors is a zero matrix, the product is also a zero matrix.

$$(f) \qquad \begin{bmatrix} \sqrt{3} & -1 \\ 3 & -\sqrt{3} \end{bmatrix}^2 = \begin{bmatrix} \sqrt{3} & -1 \\ 3 & -\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{3} & -1 \\ 3 & -\sqrt{3} \end{bmatrix} = \begin{bmatrix} 3-3 & -\sqrt{3}+\sqrt{3} \\ 3\sqrt{3}-3\sqrt{3} & -3+3 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}$$

Hence, we find an unexpected square root of the $2 \times 2$ zero matrix.

**Problem 1.3.** (6 points) For each matrix below, compute the matrix inverse, if it exists. If it fails to exist explain why.

$$A_1 = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$$

$$\begin{array}{c}1\\ \bullet\end{array}\begin{bmatrix}1 & 1 & 1 & 0\\ 0 & 1 & 0 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 1 & -1\\ 0 & 1 & 0 & 1\end{bmatrix} \quad \Rightarrow \quad A_1^{-1} = \boxed{\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}}$$

$$A_2 = \begin{bmatrix}1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\begin{array}{c}\\ \\ 1\\ \bullet\end{array}\begin{bmatrix}1 & 1 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\end{bmatrix} \sim \begin{array}{c}\\ 1\\ \bullet\\ \\ \end{array}\begin{bmatrix}1 & 1 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\end{bmatrix} \sim$$

$$\begin{array}{c}1\\ \bullet\\ \\ \\ \end{array}\begin{bmatrix}1 & 1 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1\\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0 & 0 & 1 & -1 & 1 & -1\\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1\\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow \quad A_2^{-1} = \boxed{\begin{bmatrix}1 & -1 & 1 & -1\\ 0 & 1 & -1 & 1\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1\end{bmatrix}}$$

$$A_3 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{array}{c} \bullet \\ {\scriptstyle 1} \\ \phantom{0} \\ \phantom{0} \end{array} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence the second column has no pivot and the inverse $\boxed{\text{does not exist.}}$

$$A_4 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\begin{array}{c} \bullet \\ {\scriptstyle 4} \\ {\scriptstyle 7} \end{array} \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{bmatrix} \sim \begin{array}{c} \phantom{0} \\ \bullet \\ {\scriptstyle 2} \end{array} \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -12 & -7 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

Hence the third column has no pivot and the inverse $\boxed{\text{does not exist.}}$

$$A_5 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$$

$$\begin{array}{c} \bullet \\ 4 \\ 7 \end{array} \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 10 & 0 & 0 & 1 \end{bmatrix} \sim \begin{array}{c} \\ \bullet \\ 2 \end{array} \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -11 & -7 & 0 & 1 \end{bmatrix} \sim \begin{array}{c} \\ \times\ -1/3 \\ \\ \end{array} \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{bmatrix} \sim$$

$$\begin{array}{c} 3 \\ 2 \\ \bullet \end{array} \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 4/3 & -1/3 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{bmatrix} \sim \begin{array}{c} 2 \\ \bullet \\ \\ \end{array} \begin{bmatrix} 1 & 2 & 0 & -2 & 6 & -3 \\ 0 & 1 & 0 & -2/3 & 11/3 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2/3 & -4/3 & 1 \\ 0 & 1 & 0 & -2/3 & 11/3 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{bmatrix}$$

$$A_5^{-1} = \boxed{\begin{bmatrix} -2/3 & -4/3 & 1 \\ -2/3 & 11/3 & -2 \\ 1 & -2 & 1 \end{bmatrix}}$$

$$A_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{c} \bullet \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -4 & 0 & 0 & 1 \end{bmatrix}$$

$$A_6^{-1} = \boxed{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \end{bmatrix}}$$

**Problem 1.4.** (4 points) If possible, find the LU factorization of each matrix below:

$$A_1 = \begin{bmatrix} 0 & 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 4 & 0 \end{bmatrix}$$

$\boxed{\text{not possible}}$ as row exchange is necessary in the forward phase.

$$A_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix}$$

$$\begin{array}{c} \bullet \\ 2 \\ 3 \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \text{pref}(A_2)$$

$$A_2 = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{\text{not possible}}$ as row exchange is necessary in the forward phase.

$$A_4 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 \end{bmatrix}$$

$$\begin{array}{c} \bullet \\ 2 \end{array} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \text{pref}(A_4)$$

$$A_4 = \boxed{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

**Problem 1.5.** (4 points) If possible, find the CR factorization of each matrix below:

$$A_1 = \begin{bmatrix} 0 & 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 4 & 0 \end{bmatrix}$$

$$\begin{array}{c} \downarrow \\ \uparrow \end{array}\begin{bmatrix} 0 & 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 4 & 0 \end{bmatrix} \sim \begin{array}{c} {\scriptstyle 2} \\ \bullet \end{array}\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{bmatrix} = \mathrm{rref}(A_1)$$

$$A_1 = \boxed{\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{bmatrix}}$$

$$A_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix}$$

$$\begin{array}{c} \bullet \\ {\scriptstyle 2} \\ {\scriptstyle 3} \end{array}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathrm{rref}(A_2)$$

$$A_2 = \boxed{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}}$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{c} \downarrow \\ \\ \\ \uparrow \end{array} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{array}{c} \\ \downarrow \\ \uparrow \\ \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \text{rref}(A_3)$$

$$A_3 = \boxed{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

$$A_4 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 \end{bmatrix}$$

$$\begin{array}{c} \bullet \\ {\scriptstyle 2} \end{array} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \text{rref}(A_4)$$

$$A_4 = \boxed{\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{bmatrix}}$$

**Problem 2.** (10 points) Evaluate the truth of each statement below. If the statement is true write $T$ in the box preceeding the statment. Otherwise, write $F$.

(a) $\boxed{\text{F}}$ A homogeneous linear system must have infinitely many solutions.

For example, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as its unique solution.

(b) $\boxed{\text{F}}$ If $A$ is $3 \times 3$ and $B$ is $3 \times 3$ then it must be that $AB \neq BA$.

For example, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(c) $\boxed{\text{T}}$ A sum of two symmetric $3 \times 3$ matrices is symmetric.

(d) $\boxed{\text{T}}$ The product of two invertible $3 \times 3$ matrices is invertible.

$$(AB)^{-1} = B^{-1}A^{-1}$$

(e) $\boxed{\text{F}}$ Every matrix may be factored as $LU$ where $L$ is lower unitriangular and $U$ is proto-row-echelon.

Matrices requiring row exchange have no $LU$ factorization.

(a) $\boxed{\text{F}}$ A linear system with one or more free variables has infinitely many solutions.

The linear system with augmented matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ has a free variable but no solutions.

(b) $\boxed{\text{F}}$ Two different matrices cannot have the same reduced row-echelon form.

$$\text{rref}\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) = \text{rref}\left(\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(c) $\boxed{\text{F}}$ An inconsistent system must have more equations than unknowns.

The linear system with the augmented matrix $[0 \quad 0 \quad 1]$ has one equation with two unknowns but is inconsistent.

(d) $\boxed{\text{T}}$ The linear combination of two solutions of a homogeneous linear system is also a solution.

If $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{y} = \mathbf{0}$, then $A(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha A\mathbf{x} + \beta A\mathbf{y} = \alpha\mathbf{0} + \beta\mathbf{0} = \mathbf{0}$.

(e) $\boxed{\text{F}}$ If $A$ and $B$ are $2 \times 2$ matrices and $AB = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, then either $A$ or $B$ must be equal to $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

**Problem 3.** (10 points) [Sample Problem A] What is the result of the MATLAB commands below.

```
>> A = [1 2 3; 2 3 4];
>> rref(A)

ans =


     1     0    -1
     0     1     2

>> rref([2 2; 0 2]*A)
```

```
ans =
         1     0    -1
         0     1     2
```

**The long way.**

$$\begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} = \begin{array}{r} \bullet \\ {\scriptstyle 2/3} \end{array}\begin{bmatrix} 6 & 10 & 14 \\ 4 & 6 & 8 \end{bmatrix} \sim \begin{array}{r} {\scriptstyle \times\ 1/6} \\ {\scriptstyle \times\ -3/2} \end{array}\begin{bmatrix} 6 & 10 & 14 \\ 0 & -2/3 & -4/3 \end{bmatrix} \sim \begin{array}{r} {\scriptstyle 5/3} \\ \bullet \end{array}\begin{bmatrix} 1 & 5/3 & 7/3 \\ 0 & 1 & 2 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

**The short way.** Premultiplying by an invertible matrix does not change the rref.

**Problem 3.** (10 points) [Sample Problem B] Suppose that the MATLAB function `initialzeros` is defined by

```
function zerocount=initialzeros(A)
   [m n]=size(A);
   zerocount=zeros(m,1);   % column vector of m zeros
   for r=1:m
      for c=1:n
         if A(r,c) ~= 0
            break;
         end
         zerocount(r)=zerocount(r)+1;
      end
   end
end
```

Assuming `initialzeros` is in the current MATLAB search path, what is the output obtained by entering the following MATLAB commands:

```
>> A = [ 0 0 2 3 4; 0 0 0 0 0; 1 2 3 4 5; 0 0 2 0 0; 0 1 2 3 2; 0 0 0 0 0];
>> initialzeros(A)
```

`ans =`

```
2
5
0
2
1
5
```

Zerocount counts the number of zeros at the beginning of each row.

**Problem 4.** (15 points) [Sample Problem A] The matrices listed below constitute all possible $3\times 3$ reduced row-echelon matrices. Note that some of these matrices contain parameters which are aribitrary scalar values.

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 1 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix} \quad A_6 = \begin{bmatrix} 1 & a & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad A_7 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad A_8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) Determine the rank of each matrix above.

$$\operatorname{rank}(A_1) = 0, \qquad \operatorname{rank}(A_2) = \operatorname{rank}(A_3) = \operatorname{rank}(A_4) = 1,$$

$$\operatorname{rank}(A_5) = \operatorname{rank}(A_6) = \operatorname{rank}(A_7) = 2, \qquad \operatorname{rank}(A_8) = 3.$$

(b) For each matrix above, give the inverse, if possible.

Every matrix, except $A_8$, has a column without a pivot and hence has no inverse.

$$A_8^{-1} = A_8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

.

(c) Which of the matrices above, considered as augmented matrices, represent an inconsistent system of linear equations?

$\boxed{A_4, A_6, A_7, A_8}$ represent inconsistent systems because each has a pivot in the last column.

(d) For each matrix above that represents a linear system having multiple solutions, provide the solution set in parametric vector form.

$$A_1: \quad \mathbf{x} = \boxed{\alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}} \qquad \text{the whole plane}$$

$$A_2: \quad \mathbf{x} = \boxed{\begin{bmatrix} b \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -a \\ 1 \end{bmatrix}} \qquad \text{nonhorizontal lines}$$

$$A_3: \quad \mathbf{x} = \boxed{\begin{bmatrix} 0 \\ a \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix}} \qquad \text{horizontal lines}$$

**Problem 4.** (15 points) [Sample Problem B] Suppose

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 3 & 2 & 1 & 1 \\ 1 & 3 & 3 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} 1 & 1 & 0 & -3 \\ -2 & -1 & 0 & 5 \\ 3 & 2 & 1 & -10 \\ -2 & -2 & -1 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

(a) Solve the linear system

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 3 & 2 & 1 & 1 \\ 1 & 3 & 3 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$\mathbf{x} = A^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & -3 \\ -2 & -1 & 0 & 5 \\ 3 & 2 & 1 & -10 \\ -2 & -2 & -1 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ 2 \\ -4 \\ 4 \end{bmatrix}}$$

(b) Solve the linear system

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 3 & 2 & 1 & 1 \\ 1 & 3 & 3 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$\mathbf{x} = B^{-1}A^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & -3 \\ -2 & -1 & 0 & 5 \\ 3 & 2 & 1 & -10 \\ -2 & -2 & -1 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} =$$

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -4 \\ 4 \end{bmatrix} = \begin{bmatrix} -2-2 \\ 1+4+4 \\ -2-8-4 \\ 4+4 \end{bmatrix} = \boxed{\begin{bmatrix} -4 \\ 9 \\ -14 \\ 8 \end{bmatrix}}$$

(c) Simplify

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}^{-1} \left( \begin{bmatrix} 1 & 1 & 2 & 2 \\ 3 & 2 & 1 & 1 \\ 1 & 3 & 3 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}^{-1} \right)^{-1} \begin{bmatrix} 1 & 1 & 2 & 2 \\ 3 & 2 & 1 & 1 \\ 1 & 3 & 3 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1}$$

$$= B^{-1}\left(A^{-1}B^{-1}\right)^{-1}A^{-1} = B^{-1}(BA)A^{-1} = (B^{-1}B)(AA^{-1}) = \boxed{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

**Problem 4.** (15 points) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}$. Fully explain your responses.

(a) Determine the rank (number of pivots) of the matrix $A$.

$$\begin{array}{c} \bullet \\ 3 \\ 5 \\ 7 \end{array}\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix} \sim \begin{array}{c} \\ \bullet \\ 2 \\ 3 \end{array}\begin{bmatrix} 1 & 2 \\ 0 & -2 \\ 0 & -4 \\ 0 & -6 \end{bmatrix} \sim \begin{array}{c} \\ \times -1/2 \\ \\ \\ \end{array}\begin{bmatrix} 1 & 2 \\ 0 & -2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \sim \begin{array}{c} 2 \\ \bullet \\ \\ \\ \end{array}\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

There are two pivots; hence, $\boxed{\operatorname{rank}(A) = 2}$.

(b) Determine the rank of the matrix $A^T A$.

$$A^T A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1+9+25+49 & 2+12+30+56 \\ 2+12+30+56 & 4+16+36+64 \end{bmatrix} = \begin{bmatrix} 84 & 100 \\ 100 & 120 \end{bmatrix}$$

$$84 \cdot 120 - 100 \cdot 100 \neq 0$$

So $A^T A$ is invertible and must therefore have two pivots. $\boxed{\operatorname{rank}(A^T A) = 2}$.

(c) Determine the rank of the matrix $AA^T$.

**The long way.**

$$AA^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix} = \begin{array}{c} \bullet \\ 11/5 \\ 17/5 \\ 23/5 \end{array}\begin{bmatrix} 5 & 11 & 17 & 23 \\ 11 & 25 & 39 & 53 \\ 17 & 39 & 61 & 83 \\ 23 & 53 & 83 & 113 \end{bmatrix} \sim \begin{array}{c} \\ \bullet \\ 2 \\ 3 \end{array}\begin{bmatrix} 5 & 11 & 17 & 23 \\ 0 & 4/5 & 8/5 & 12/5 \\ 0 & 8/5 & 16/5 & 24/5 \\ 0 & 12/5 & 24/5 & 36/5 \end{bmatrix} \sim$$

$$\begin{bmatrix} 5 & 11 & 17 & 23 \\ 0 & 4/5 & 8/5 & 12/5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \boxed{\operatorname{rank}(AA^T) = 2}$$

**The short way.** Premultiplying by the same elementary matrices that would bring $A$ to $\operatorname{rref}(A)$ shows

$$AA^T \sim \operatorname{rref}(A)A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix} = \begin{array}{c} \bullet \\ 2 \\ \\ \\ \end{array}\begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -2 & -4 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $A^T A$ also has two pivots; hence, $\boxed{\operatorname{rank}(A^T A) = 2}$.

Some morals to the story:

(1) A row operation on a product may be performed on the first factor before multiplying. This follows from the fact that a row operation may be represented by premultiplication by an elementary matrix $E$. $E(AB) = (EA)B$

(2) $A$, $A^T$, $A^T A$, and $AA^T$ all share the same rank.

**Problem 5.** (15 points) [Sample Problem A] An $n \times n$ matrix $A$ is congruent to an $n \times n$ matrix $B$ if and only if there is a nonsingular $n \times n$ matrix $S$ such that

$$SAS^T \; = \; B.$$

(a) Explain why every $n \times n$ matrix is congruent to itself.

Taking $S = I$ shows $SAS^T = IAI = A$.

(b) Explain why if $A$ is congruent to $B$ then $B$ is congruent to $A$.

Since $A$ is congruent to $B$ there is a nonsignular matrix $S$ such that $SAS^T = B$. To show that $B$ is congruent to $A$ we must find a nonsignular matrix $\tilde{S}$ such that $\tilde{S}B\tilde{S}^T = A$. We mulitiply the known equality $SAS^T = B$ on the left by $S^{-1}$ and on the right by $S^{-T}$:

$$\begin{aligned} & S^{-1}SAS^TS^{-T} &&= S^{-1}BS^{-T} \\ \Rightarrow \quad & IAI &&= S^{-1}BS^{-T} \\ \Rightarrow \quad & \tilde{S}B\tilde{S}^T &&= A \qquad \text{where} \quad \tilde{S} \; = \; S^{-1} \end{aligned}$$

(c) Explain why if $A$ is congruent to $B$ and $B$ is congruent to $C$ then $A$ is congruent to $C$.

Since $A$ is congruent to $B$ there is $S \in \mathrm{GL}(n)$ such that $SAS^T = B$ and since $B$ is congruent to $C$ there is $\tilde{S} \in \mathrm{GL}(n)$ such that $\tilde{S}B\tilde{S}^T = C$. To show that $A$ is congruent to $C$ we must find $\check{S} \in \mathrm{GL}(n)$ such that $\check{S}A\check{S}^T = C$. Multiplying the equality $SAS^T = B$ on the left by $\tilde{S}$ and on the right by $\tilde{S}^T$ gives

$$\tilde{S}SAS^T\tilde{S}^T \; = \; \tilde{S}B\tilde{S}^T \; = \; C.$$

Hence, $\check{S}A\check{S}^T = C$ where $\check{S} = \tilde{S}S$ (since $\check{S}^T = (\tilde{S}S)^T = S^T\tilde{S}^T$).

**Problem 5.** (15 points) [Sample Problem B]

(a) Suppose that $\mathbf{a} \in \mathbb{R}^m$ and $\mathbf{b} \in \mathbb{R}^n$ what is the rank of $\mathbf{a}\mathbf{b}^T$? Explain why.

The rank is either $\boxed{0 \text{ or } 1}$. If $\mathbf{a}$ or $\mathbf{b}$ is a zero vector than $\mathbf{a}\mathbf{b}^T$ is a zero matrix and has a rank of zero. If neither $\mathbf{a} = \mathbf{0}$ nor $\mathbf{b} = \mathbf{0}$ then every row of $\mathbf{a}\mathbf{b}^T$ is a multiple of $\mathbf{b}^T$. Hence, $\mathbf{a}\mathbf{b}^T$ has only one pivot.

(b) Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 8 & 12 \\ -3 & -6 & -9 \end{bmatrix}$. Find $\mathbf{a} \in \mathbb{R}^3$ and $\mathbf{b} \in \mathbb{R}^3$ such that $A = \mathbf{a}\mathbf{b}^T$.

$$\boxed{\mathbf{a} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} \qquad \text{alternatively} \qquad \mathbf{a} = \begin{bmatrix} 10 \\ 40 \\ -30 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1/10 \\ 2/10 \\ 3/10 \end{bmatrix}$$

(c) Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Find vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_1, \mathbf{b}_2$ such that

$$A = \mathbf{a}_1\mathbf{b}_1^T + \mathbf{a}_2\mathbf{b}_2^T.$$

An idea: Select $\mathbf{a}_1$ and $\mathbf{b}_1$ to make $A - \mathbf{a}_1\mathbf{b}_1^T$ have rank 1. Eliminating the first row and column with $\mathbf{a}_1 = \begin{bmatrix} 1 & 4 & 7 \end{bmatrix}^T$ and $\mathbf{b}_1 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$ would seem to have some promise. Indeed,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 4 & 8 & 12 \\ 7 & 14 & 21 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 0 & -3 & -6 \end{bmatrix}$$

Hence,

$$\boxed{\mathbf{a}_1 = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ -3 \\ -6 \end{bmatrix}}$$

Alternatively, because the matrix has rank 2, the outer product expansion of the CR factorization contains two terms:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}$$

**Problem 5.** (15 points) [Sample Problem C] Carefully explain your answers to each part of this question.

(a) How many $2 \times 2$ real symmetric matrices are square roots of $I = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$?

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}^2 = \begin{bmatrix} a^2+b^2 & b(a+c) \\ b(a+c) & b^2+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Rightarrow \quad a^2+b^2 = 1,\ b(a+c) = 0, \text{ and } b^2+c^2 = 1.$$

Since $b(a+c) = 0$ either $b = 0$ or $a + c = 0$. If $b = 0$ then $a^2 = 1$ and $c^2 = 1$. Hence, there are four square roots when $b = 0$:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

If $a + c = 0$ then $a = \pm\sqrt{1-b^2}$ so the square roots are

$$\begin{bmatrix} \pm\sqrt{1-b^2} & b \\ b & \mp\sqrt{1-b^2} \end{bmatrix} \qquad -1 \le b \le 1.$$

There are infinitely many symmetric square roots of the $2 \times 2$ identity matrix.

(b) How many $2 \times 2$ real antisymmetric matrices are square roots of $I = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$?

$$\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}^2 = \begin{bmatrix} -a^2 & 0 \\ 0 & -a^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence, there are no real antisymmetric square roots of $I$.

(c) How many $2 \times 2$ real antisymmetric matrices are fourth roots of $I = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$?

$$\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}^4 = \begin{bmatrix} a^4 & 0 \\ 0 & a^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence, $a = \pm 1$ (assuming $a \in \mathbb{R}$). The two real, antisymmetric fourth roots of I are

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

**Problem 5.** (15 points) [Sample Problem D] Consider Gauss-Jordan applied to $A = \begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$:

$A \qquad\qquad\qquad\qquad \mathrm{pref}(A) \qquad\qquad \mathrm{nref}(A) \qquad\qquad \mathrm{rref}(A)$

$$\begin{array}{c} \bullet \\ 4 \\ 7 \end{array}\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix} \sim \begin{array}{c} \\ \bullet \\ 2 \end{array}\begin{bmatrix}1&2&3\\0&-3&-6\\0&-6&-12\end{bmatrix} \sim \begin{array}{c} \\ \times -1/3 \\ \\ \end{array}\begin{bmatrix}1&2&3\\0&-3&-6\\0&0&0\end{bmatrix} \sim \begin{array}{c} 2 \\ \bullet \\ \\ \end{array}\begin{bmatrix}1&2&3\\0&1&2\\0&0&0\end{bmatrix} \sim \begin{bmatrix}1&0&-1\\0&1&2\\0&0&0\end{bmatrix}$$

(a) Find a lower unitriangular matrix $L$, a diagonal matrix $D$, and an upper unitriangular matrix $U$ such that

$$A = L\ \mathrm{pref}(A), \qquad \mathrm{pref}(A) = D\ \mathrm{nref}(A), \qquad \text{and} \qquad \mathrm{nref}(A) = U\ \mathrm{rref}(A).$$

$$L = \left(\begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-7&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-4&1&0\\0&0&1\end{bmatrix}\right)^{-1} = \begin{bmatrix}1&0&0\\4&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\7&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&2&1\end{bmatrix}$$

$$D = \begin{bmatrix}1&0&0\\0&-1/3&0\\0&0&1\end{bmatrix}^{-1}$$

$$U = \begin{bmatrix}1&-2&0\\0&1&0\\0&0&1\end{bmatrix}^{-1}$$

$$L = \boxed{\begin{bmatrix}1&0&0\\4&1&0\\7&2&1\end{bmatrix}} \qquad D = \boxed{\begin{bmatrix}1&0&0\\0&-3&0\\0&0&1\end{bmatrix}} \qquad U = \boxed{\begin{bmatrix}1&2&0\\0&1&0\\0&0&1\end{bmatrix}}$$

Because that matrix is not of full rank (there is a row of zeros in the row-echelon forms), there are multiple correct answers for $D$ and $U$:

$$D = \begin{bmatrix}1&0&0\\0&-3&0\\0&0&a\end{bmatrix} \qquad \text{and} \qquad U = \begin{bmatrix}1&2&b\\0&1&c\\0&0&1\end{bmatrix}$$

Note that $D$ is required to be diagonal and $U$, upper unitriangular.

(b) Find a matrix $C$ such that $A = C\begin{bmatrix}1&0&-1\\0&1&2\end{bmatrix}$.

From the CR factorization, we may select $C$ having the pivot columns of $A$ as its columns (this solution is unique).

$$C = \boxed{\begin{bmatrix}1&2\\4&5\\7&8\end{bmatrix}}$$

(c) Find all matrices $S$ such that $A = S\ \mathrm{rref}(A)$.

We seek $S = \begin{bmatrix} a & x & u \\ b & y & v \\ c & z & w \end{bmatrix}$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} a & x & u \\ b & y & v \\ c & z & w \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a & x & 2x-a \\ b & y & 2y-b \\ c & z & 2z-c \end{bmatrix}$$

Hence, we obtain a system of 9 linear equations for the components of $S$: $a = 1$, $b = 4$, $c = 7$, $x = 2$, $y = 5$, $z = 8$, $2x - a = 3$, $2y - b = 6$ $2z - c = 9$. The first 6 equations determine the values of $a$, $b$, $c$, $x$, $y$, $z$. The last 3 equations are consistent with the determinations. The last 3 variables $u$, $v$, $w$ are unconstrained (i.e. free variables). Thus,

$$S = \boxed{\begin{bmatrix} 1 & 2 & u \\ 4 & 5 & v \\ 7 & 8 & w \end{bmatrix} \qquad \text{where} \quad u,\ v,\ w \quad \text{are free parameters.}}$$

Check:

$$\begin{bmatrix} 1 & 2 & u \\ 4 & 5 & v \\ 7 & 8 & w \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

**Extra Credit.** (5 points) [Sample Problem B] Showing your work, find the smallest positive integers $x_1$, $x_2$, $x_3$ and $x_4$ that balances the chemical equation for the combustion of ethanol:

$$(x_1)C_2H_6O \ + \ (x_2)O_2 \ \rightarrow \ (x_3)CO_2 \ + \ (x_4)H_2O.$$

$$\boxed{\quad 1 \quad} C_2H_6O \ + \boxed{\quad 3 \quad} O_2 \ \rightarrow \boxed{\quad 2 \quad} CO_2 \ + \boxed{\quad 3 \quad} H_2O.$$

$$\begin{array}{c} \mathrm{C} \\ \mathrm{H} \\ \mathrm{O} \end{array} \qquad x_1 \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix} \quad + \quad x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \quad = \quad x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad + \quad x_4 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{array}{c} \bullet \\ 3 \\ \frac{1}{2} \end{array} \begin{bmatrix} 2 & 0 & -1 & 0 & 0 \\ 6 & 0 & 0 & -2 & 0 \\ 1 & 2 & -2 & -1 & 0 \end{bmatrix} \sim \begin{array}{c} \\ \downarrow \\ \uparrow \end{array} \begin{bmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & -2 & 0 \\ 0 & 2 & -\frac{3}{2} & -1 & 0 \end{bmatrix} \sim \begin{array}{c} \times \frac{1}{2} \\ \times \frac{1}{2} \\ \times \frac{1}{3} \end{array} \begin{bmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & -\frac{3}{2} & -1 & 0 \\ 0 & 0 & 3 & -2 & 0 \end{bmatrix} \sim$$

$$\begin{array}{c} -\frac{1}{2} \\ -\frac{3}{4} \\ \bullet \end{array} \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{3}{4} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} \frac{1}{3} \\ 1 \\ \frac{2}{3} \\ 1 \end{bmatrix} \qquad\qquad x_4 = 3 \qquad \text{efficiently clears denominators}$$

$$\Rightarrow \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 3 \\ 2 \\ 3 \end{bmatrix}}$$