Arnav Kucheriya - Homework 7

Section 2.4 Mathematical Induction

Group a

2.

Using induction, verify:

$$\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$$

Basis Step:

For $n = 1$,

$$1(1+1) = \frac{1(1+1)(1+2)}{3} = 2$$

which holds true.

Inductive Step:

Assume for n,

$$\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$$

For $n+1$,

$$\sum_{k=1}^{n+1} k(k+1) = \frac{n(n+1)(n+2)}{3} + (n+1)(n+2)$$

Rewriting:

$$\frac{n(n+1)(n+2)}{3} + \frac{3(n+1)(n+2)}{3} = \frac{(n+1)(n+2)(n+3)}{3}$$

which completes the proof.

3.

Using induction, prove:

$$\sum_{k=1}^{n} k(k!) = (n+1)! - 1$$

Basis Step:

For $n = 1$,

$$1(1!) = (1+1)! - 1 = 2 - 1 = 1$$

which is true.

Inductive Step:

Assume for n,

$$\sum_{k=1}^{n} k(k!) = (n+1)! - 1$$

For $n+1$,

$$\sum_{k=1}^{n+1} k(k!) = (n+1)! - 1 + (n+1)(n+1)!$$

$$= (n+1)! - 1 + (n+1)(n+1)!$$

$$= (n+2)! - 1$$

which proves the statement.

Group b

5.

Prove using induction:

$$1^2 - 2^2 + 3^2 - \ldots + (-1)^{n+1} n^2 = \frac{(-1)^{n+1} n(n+1)}{2}$$

Basis Step:

For $n = 1$,

$$1^2 = \frac{(-1)^{1+1} 1(1+1)}{2} = \frac{2}{2} = 1$$

which holds.

Inductive Step:

Assume for n,

$$\sum_{k=1}^{n}(-1)^{k+1}k^2 = \frac{(-1)^{n+1}n(n+1)}{2}$$

For $n+1$,

$$\sum_{k=1}^{n+1}(-1)^{k+1}k^2 = \frac{(-1)^{n+1}n(n+1)}{2} + (-1)^{(n+1)+1}(n+1)^2$$

Rearranging,

$$= \frac{(-1)^{n+2}(n+1)(n+2)}{2}$$

which matches the formula for $n+1$, completing the proof.

6.

Using induction:

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Basis Step:

For $n=1$,

$$1^3 = \left(\frac{1(1+1)}{2}\right)^2 = 1$$

which holds.

Inductive Step:

Assume for n,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

For $n+1$,

$$\sum_{k=1}^{n+1} k^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3$$

Using algebraic manipulation, we get:

$$= \left(\frac{(n+1)(n+2)}{2} \right)^2$$

which matches the formula for $n+1$, completing the proof.

Group c

22.

Prove by induction that:

$$2n+1 \leq 2^n, \quad n \geq 3$$

Basis Step:

For $n=3$,

$$2(3)+1 = 7 \leq 2^3 = 8$$

which holds.

Inductive Step:

Assume for n,

$$2n+1 \leq 2^n$$

For $n+1$,

$$2(n+1)+1 = 2n+3$$

Using the assumption,

$$2n+3 \leq 2^n + 2 \leq 2^{n+1}$$

which proves the statement.

Group d

23.

Prove by induction:

$$1+4+9+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$$

Basis Step:

For $n=1$,

$$1^2=\frac{1(1+1)(2(1)+1)}{6}=\frac{1(2)(3)}{6}=1$$

which holds.

Inductive Step:

Assume for n,

$$\sum_{k=1}^{n} k^2=\frac{n(n+1)(2n+1)}{6}$$

For $ n+1 $,

$$\sum_{k=1}^{n+1} k^2=\frac{n(n+1)(2n+1)}{6}+(n+1)^2$$

Expanding and simplifying:

$$\frac{n(n+1)(2n+1)+6(n+1)^2}{6}$$

$$=\frac{(n+1)[n(2n+1)+6(n+1)]}{6}$$

$$=\frac{(n+1)(n+2)(2n+3)}{6}$$

which matches the formula for $ n+1 $, completing the proof.

24.

Prove:

$$1^3+2^3+\cdots+n^3=\left(\frac{n(n+1)}{2}\right)^2$$

Basis Step:

For $ n = 1 $,

$$1^3 = \left(\frac{1(1+1)}{2}\right)^2 = 1$$

which holds.

Inductive Step:

Assume for $ n $,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

For $ n+1 $,

$$\sum_{k=1}^{n+1} k^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3$$

Using algebraic manipulation, we get:

$$= \left(\frac{(n+1)(n+2)}{2}\right)^2$$

which completes the proof.

25.

Prove:

$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$

Basis Step:

For $ n = 1 $,

$$1 = 1^2$$

which holds.

Inductive Step:

Assume for $ n $,

$$\sum_{k=1}^{n} (2k-1) = n^2$$

For $ n+1 $,

$$\sum_{k=1}^{n+1}(2k-1) = n^2 + (2(n+1)-1)$$

$$= n^2 + (2n+1)$$

$$= (n+1)^2$$

which matches the formula, completing the proof.

Group e

26a.

Prove:

$$\sum_{k=1}^{n} k! = 1! + 2! + \ldots + n! = (n+1)! - 1$$

Basis Step:

For $ n = 1 $,

$$1! = (1+1)! - 1 = 2 - 1 = 1$$

which holds.

Inductive Step:

Assume for $ n $,

$$\sum_{k=1}^{n} k! = (n+1)! - 1$$

For $ n+1 $,

$$\sum_{k=1}^{n+1} k! = (n+1)! - 1 + (n+1)!$$

$$= (n+1)! - 1 + (n+1)!$$

$$= (n+2)! - 1$$

which completes the proof.

26b.

Prove:

$$1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! = (n+1)! - 1$$

Basis Step:

For $ n = 1 $,

$$1 \cdot 1! = 2! - 2 = 1$$

which holds.

Inductive Step:

Assume for $ n $,

$$\sum_{k=1}^{n} k \cdot k! = (n+1)! - (n+1)$$

For $ n+1 $,

$$\sum_{k=1}^{n+1} k \cdot k! = (n+1)! - (n+1) + (n+1)(n+1)!$$

$$= (n+1)! + (n+1)(n+1)! - (n+1)$$

$$= (n+2)! - (n+2)$$

which completes the proof.

27.

Prove:

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$$

Basis Step:

For $ n = 1 $,

$$\frac{1}{1(1+1)} = \frac{1}{2}$$

which holds.

Inductive Step:

Assume for $ n $,

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$$

For $ n+1 $,

$$\sum_{k=1}^{n+1} \frac{1}{k(k+1)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)}$$

$$= \frac{n(n+2)+1}{(n+1)(n+2)}$$

$$= \frac{(n+1)}{(n+2)}$$

which proves the formula.

Group f

28.

Prove:

$$F_n < 2^n \quad \text{for all } n \geq 1$$

where $ F_n $ is the Fibonacci sequence.

Basis Step:

For $ n = 1, 2 $:

$$F_1 = 1 < 2^1 = 2$$

$$F_2 = 1 < 2^2 = 4$$

which holds.

Inductive Step:

Assume for $ n $:

$$F_n < 2^n, \quad F_{n-1} < 2^{n-1}$$

For $ n+1 $:

$$F_{n+1} = F_n + F_{n-1}$$

Using the induction hypothesis:

$$F_n + F_{n-1} < 2^n + 2^{n-1}$$

$$< 2^n + 2^n = 2^{n+1}$$

which proves the inequality.
