# 1 Lecture 1: Preview

## 1.1 Vectors

A **column vector** or simply a **vector** with $n-$entries (or $n-$components) is written of the form $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$. If all components $v_1, v_2, \ldots, v_n)$ of the vector $\mathbf{v}$ are real numbers, we say $\mathbf{v} \in \mathbb{R}^n$. If at least one component of $\mathbf{v}$ is a complex number, we say $\mathbf{v} \in \mathbb{C}^n$.

**Note:**

- Notation $\mathbb{R}$ is to denote the set of all real numbers; $b \in \mathbb{R}$ means $b$ is a real number.
- A complex number, $z$, is an expression of the form

$$z = a + ib,$$

where $a, b \in \mathbb{R}$ and $i$ is a symbol satisfying $i^2 = -1$.

- $C$ denotes the set of all complex numbers.

**Example 1.1.** *(Vector in $\mathbb{R}^2$)*

- *General form of a vector* $\mathbf{v} \in \mathbb{R}^2$ *is* $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$*; for example,* $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

**Example 1.2.** *(Vector in $\mathbb{R}^3$)*

- *General form of a vector* $\mathbf{v} \in \mathbb{R}^3$ $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$*; for example,* $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

**Example 1.3.**

*The zero vector in $\mathbb{R}^n$ (or $\mathbb{C}^n$) is the one whose components are 0:* $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

*The $\ell^{th}$ standard basis vector in $\mathbb{R}^n$ (or $\mathbb{C}^n$) is the column vector whose the $\ell^{th}$ entry is 1 and other entries are 0. For example,* $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, $\ldots$, $\mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$.

## 1.2 Vector operations

### 1.2.1 Sum of two vectors

Given two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ (or $\mathbb{C}^n$), their sum is the vector $\mathbf{u}+\mathbf{v}$, which is a vector in $\mathbb{R}^n$ (or $\mathbb{C}^n$) whose entries equal the sum of the corresponding entries of $\mathbf{u}$ and $\mathbf{v}$.

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \dots \\ u_n+v_n \end{bmatrix},$$

**Example 1.4.** *(Sum of two vector in $\mathbb{R}^2$)*

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1+2 \\ -2+5 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

### 1.2.2 Scalar multiple of a vector

Given a vector $\mathbf{u}$ in $\mathbb{R}^n$ (or $\mathbb{C}^n$), the scalar multiple of a vector $\mathbf{u}$ by $c$ is the vector $c\mathbf{u}$ in $\mathbb{R}^n$ (or $\mathbb{C}^n$) whose entries equal the corresponding entries of $\mathbf{u}$ multiply by $c$.

$$c\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}$$

**Example 1.5.** *If* $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ *and* $c=5$, *then*

$$c\mathbf{u} = 5\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 15 \\ -5 \end{bmatrix}$$

**Example 1.6.** *Let denote* $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, *then*

- $3\mathbf{a}+2\mathbf{b} = 3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} + \begin{bmatrix} 8 \\ 10 \\ 12 \end{bmatrix} = \begin{bmatrix} 3+8 \\ 6+10 \\ 9+12 \end{bmatrix} = \begin{bmatrix} 11 \\ 16 \\ 21 \end{bmatrix}$
- $\mathbf{a}+\mathbf{w}$: *Not compatible.*

### 1.2.3 Algebraic Properties of $\mathbb{R}^n$

For all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$ and all scalars $c$ and $d$:

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(ii) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

(iii) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

(iv) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$

(v) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$

(vi) $c(d\mathbf{u})=(cd)\mathbf{u}$

(vii) $\mathbf{u}+(-\mathbf{u})=-\mathbf{u}+\mathbf{u}=\mathbf{0}$, where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$

(viii) $1\mathbf{u}=\mathbf{u}, \quad 0\mathbf{u}=\mathbf{0}$.

## 1.3 Linear combinations

A *linear combination* constructed from a set of two or more **parameters** is an expression formed by multiplying each parameter by a constant and adding them together. These parameters can be scalars or vectors;... For instance,

If $x_1$ and $x_2$ are two parameters, a linear combination of $x_1$ and $x_2$ is an expression of the form

$$\alpha_1 x_1 + \alpha_2 x_2,$$

where $\alpha_1$ and $\alpha_2$, called the weights, are two numbers not depending on $x_1$ and $x_2$.

**Example 1.7.** *(i)* $2x_1+3x_2$ *is a linear combination of* $x_1$ *and* $x_2$*.*

*(ii)* $2x_1^2+3x_2$ *is* **not** *a linear combination of* $x_1$ *and* $x_2$*.*

*(iii)* $\sin(2)x_1+\cos(3)x_2$ *is a linear combination of* $x_1$ *and* $x_2$*.*

*(iv)* $\sin(2x_1)+\cos(3x_2)$ *is* **not** *a linear combination of* $x_1$ *and* $x_2$*.*

A linear combination of $n$ parameters $x_1, x_2, \cdots, x_n$ is an expression of the form

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots \alpha_n x_n$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are $n$ numbers not depending on $x_1, x_2, \ldots, x_n$.

**Linear combination of two vectors**

Given two vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$ and two scalars $\alpha$ and $\beta$. The vector $\mathbf{w}$ defined by

$$\mathbf{w} = \alpha\mathbf{x} + \beta\mathbf{y} = \alpha \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} + \beta \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \\ \vdots \\ \alpha x_m + \beta y_m \end{bmatrix}$$

is a linear combination of two vectors $\mathbf{x}$ and $\mathbf{y}$.

**Linear combination of $n$ vectors**

Given $n$ vectors $\mathbf{x}_1 = \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{m1} \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{m2} \end{bmatrix}$, $\ldots$, $\mathbf{x}_n = \begin{bmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{mn} \end{bmatrix}$ and $n$ scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$. The vector $\mathbf{w}$ defined by

$$\mathbf{w} = \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \ldots \alpha_n\mathbf{x}_n = \begin{bmatrix} \alpha_1 x_{11} + \alpha_2 x_{12} + \ldots \alpha_n x_{1n} \\ \alpha_1 x_{21} + \alpha_2 x_{22} + \ldots \alpha_n x_{2n} \\ \vdots \\ \alpha_1 x_{m1} + \alpha_2 x_{m2} + \ldots \alpha_n x_{mn} \end{bmatrix}, \tag{1.1}$$

is a linear combination of the $n$ vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$.

If the sum $\alpha_1 + \alpha_2 + \ldots + \alpha_n = 1$, then the linear combination defined in (1.1) is called an **affine combination**.

**Remark 1.1.** *As a result, a vector* $\mathbf{b}$ *is a linear combination of* $n$ *vectors* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ *if and only if there exists* $n$ *constants* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *such that*

$$\mathbf{b} = \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \ldots \alpha_n\mathbf{x}_n.$$

**Example 1.8.** *(i) Vector* $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ *is a linear combination of* $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ *and* $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, *since*

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

*(ii) Vector* $\mathbf{b} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ *is NOT a linear combination of* $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ *and* $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$*, since there does not exist* $\alpha_1$ *and* $\alpha_2$ *such that*

$$\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{1.2}$$

*Indeed, if* $\alpha_1$ *and* $\alpha_2$ *such that the identity (1.3) holds, then*

$$4 = 2\alpha_1 \quad \text{and} \quad 5 = 3\alpha_1,$$

*which does not exist.*

*(iii) Vector* $\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ *is NOT a linear combination of* $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ *and* $\mathbf{x}_2 = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$*, since there does not exist* $\alpha_1$ *and* $\alpha_2$ *such that*

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}. \tag{1.3}$$

*Indeed, if* $\alpha_1$ *and* $\alpha_2$ *such that the identity (1.3) holds, then*

$$1 = \alpha_1 + 2\alpha_2 \quad \text{and} \quad -1 = 3\alpha_1 + 4\alpha_2,$$

*which implies* $\alpha_1 = -3$ *and* $\alpha_2 = 2$*. However,* $1 \neq (-3)1 + 2(1)$*. Thus, the identity (1.3) is not true for any* $\alpha_1$ *and* $\alpha_2$*.*

### 1.4 Span

Given the set $S$ of $n$ vector $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, i.e., $S = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$. The collection of all linear combination of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is called the span of $S$, and denoted by "Span($S$)". Specifically, the span of $S = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is formulated as

$$\mathrm{Span}(\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}) = \{\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \ldots \alpha_n\mathbf{x}_n | \, \alpha_1, \ldots, \alpha_n \text{ are scalars}\}$$

**Remark 1.2.** *The span of the empty set contains only the zero vector, i.e.,* $\mathrm{Span}(\emptyset) = \{\mathbf{0}\}$*.*

**Remark 1.3.**

- *The span of a set of vectors* $S$ *always contains the zeros vector. As a result,* $\mathrm{Span}(S)$ *is a non-empty set of vectors.*
- $\mathrm{Span}(S)$ *is closed under the formation of linear combinations (i.e., forming a linear combination of vectors from* $\mathrm{Span}(S)$ *always results in a vector in* $\mathrm{Span}(S)$*).*

**Example 1.9.** *(i) If* $\mathbf{u}$ *is a non-zero vector in* $\mathbb{R}^n$*,* $\mathrm{Span}(\{\mathbf{u}\})$ *is a line that passes through the origin and of the direction* $\mathbf{u}$*.*

*(ii)* $\mathbb{R}^2 = \text{Span}\left(\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}\right)$

*(iii)* $\mathbb{R}^2 = \text{Span}\left(\left\{\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\}\right)$

*(iv)* $\mathbb{R}^2 = \text{Span}(\{\mathbf{u}, \mathbf{v}\})$, *where* $\mathbf{0} \neq \mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ *and* $\mathbf{u} \notin \text{Span}\{\mathbf{v}\}$

## 1.5 Length and dot product

1. **The length** of vector $\mathbf{u}$, denoted by $\|\mathbf{u}\|$ is a non-negative constant and is defined by

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2} \geq 0.$$

The length of a vector is zero if and only if this is a zero vector, i.e., $\|\mathbf{u}\| = 0$ is and only if $\mathbf{u} = \mathbf{0}$.

2. The **dot product** of two vector $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, denoted by $\mathbf{u} \cdot \mathbf{v}$ is a real constant and defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \ldots + u_nv_n.$$

3. (Angle in $\mathbb{R}^2$ and $\mathbb{R}^3$) Denote by $\theta$ the angle between two vector $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$ (or $\mathbb{R}^3$) then

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta. \tag{1.4}$$

The relation (1.4) is extended to $\mathbb{R}^n$ and used to define the angle of two vector in $\mathbb{R}^n$ for $n > 3$.

**The Cauchy-Schwarz inequality:** For $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$:

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|. \tag{1.5}$$

## 1.6 Matrix

A $m \times n$ matrix $\mathbf{A}$ is an array of $mn-$entries of the form

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \tag{1.6}$$

Note that, if we denote $\mathbf{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}$, $\ldots$, $\mathbf{a}_n = \begin{bmatrix} v_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$, then the matrix $\mathbf{A}$ is written as a row vector whose each entry is a column vector, i.e.,

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{bmatrix}.$$

The vector $\mathbf{a}_j := \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$ is called the $j-$th column of the matrix $\mathbf{A}$.

**Remark 1.4.** *An $m \times n$ matrix $\mathbf{A}$, defined in (1.6), is sometimes written as $\mathbf{A} = [a_{ij}]_{m\times n}$. The number of rows and columns gives the size of the matrix, i.e., the size of the matrix $\mathbf{A}$ is $m \times n$.*

**Remark 1.5.**

- *If all $n$ column of $\mathbf{A}$ (i.e., $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$) are in $\mathbb{R}^m$, we say $\mathbf{A} \in \mathbb{R}^{m\times n}$.*
- *If at least one column of $\mathbf{A}$ is in $\mathbb{C}^m$, we say $\mathbf{A} \in \mathbb{C}^{m\times n}$.*

**Example 1.10.**

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \in \mathbb{R}^{2\times 3}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 2-i & 3 \\ 4 & 5 & 6 \end{bmatrix} \in \mathbb{C}^{2\times 3}.$$

The **zero matrix** in $\mathbb{R}^{m\times n}$ is the $m \times n$ matrix whose components are 0.

## Sum of two matrices

Two compatible matrices for sum are two matrices of the same size. The sum of two matrices is a matrix of the same size whose entries equal the sum of the corresponding entries.

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ b_{m1} & b_{m2} & \ldots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \ldots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \ldots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \ldots & a_{mn}+b_{mn} \end{bmatrix}$$

**Example 1.11.** *Let*

$$A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$$

*Then*

$$A + B = \begin{bmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{bmatrix},$$

*but $A + C$ is not defined because $A$ and $C$ have different sizes.*

## Multiplying a matrix with a constant:

A multiplication matrix with a constant is a matrix of the same size whose entries equal the corresponding entries multiplied by the constant.

$$\alpha \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \dots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \dots & \alpha a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ \alpha a_{m1} & \alpha a_{m2} & \dots & \alpha a_{mn} \end{bmatrix}$$

**Example 1.12.** *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are the matrices in Example 1.11, then*

$$2\mathbf{B} = 2\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{bmatrix}$$

$$3\mathbf{A} = 3\begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 0 & 15 \\ -3 & 9 & 6 \end{bmatrix}$$

$$3\mathbf{A} - 2\mathbf{B} = \begin{bmatrix} 12 & 0 & 15 \\ -3 & 9 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{bmatrix} = \begin{bmatrix} 10 & -2 & 31 \\ -9 & -1 & -8 \end{bmatrix}$$

**Example 1.13.**

$$3\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 1 \end{bmatrix} + 2\begin{bmatrix} 4 & -5 & 2 \\ 5 & 2 & 1 \\ 6 & 4 & -9 \end{bmatrix} = \begin{bmatrix} 3(1)+2(4) & 3(2)+2(-5) & 3(3)+2(2) \\ 3(2)+2(5) & 3(3)+2(2) & 3(4)+2(1) \\ 3(3)+2(6) & 3(4)+2(4) & 3(1)+2(-9) \end{bmatrix} = \begin{bmatrix} 11 & -4 & 13 \\ 16 & 13 & 14 \\ 21 & 20 & -15 \end{bmatrix}$$

**Example 1.14.**

- $3\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} + 2\begin{bmatrix} 4 & -5 & 2 \\ 5 & 2 & 1 \\ 6 & 4 & -9 \end{bmatrix}$ *: Not compatible*

- $3\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} + 2\begin{bmatrix} 4 & 2 \\ 5 & 2 \\ 6 & -9 \end{bmatrix}$ *: Not compatible.*

**Properties 1.1.** *Let* $\mathbf{A}, \mathbf{B}$, *and* $\mathbf{C}$ *be matrices of the same size, and let* $r$ *and* $s$ *be scalars.*

*(a)* $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

*(b)* $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$

*(c)* $\mathbf{A} + \mathbf{0} = \mathbf{A}$

*(d)* $r(\mathbf{A} + \mathbf{B}) = r\mathbf{A} + r\mathbf{B}$

*(e)* $(r + s)\mathbf{A} = r\mathbf{A} + s\mathbf{A}$

*(f)* $r(s\mathbf{A}) = (rs)\mathbf{A}$

# 2 Lecture 2: Small Linear Systems

## 2.1 Some examples of simple linear systems

Finding the solution set of a system of two linear equations in two variables is easy because it amounts to finding the intersection of two lines. We here consider some examples

**Example 2.1.** *Consider the linear system*

$$\begin{cases} x + y = 3 \\ -x + y = 5 \end{cases} \tag{2.1}$$

To solve the system (2.1), we can substitute $y = 3 - x$ from the first equation and then plug it into the second equation to have

$$-x + 3 - x = 5,$$

which implies $x = -1$, $y = 3 - (-1) = 4$. Thus, the system has the unique solution $(-1, 4)$. The solution set is $\{(-1, 4)\}$

The graphs of these equations are lines, which we denote by $\ell_1$ and $\ell_2$. A pair of numbers $(x_1, x_2)$ satisfies both equations in the system if and only if the point $(x_1, x_2)$ lies on both $\ell_1$ and $\ell_2$. In the system (2.1), the solution is the single point $(-1, 4)$ (see Figure 1), as you can easily verify.

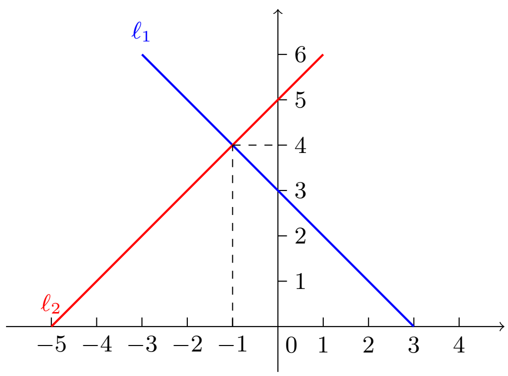


Figure 1: Unique solution

**Example 2.2.** *Consider the linear system*

$$\begin{cases} x + y = 3 \\ 2x + 2y = 5 \end{cases} \tag{2.2}$$

The graph that corresponds to the system (2.2) is shown in Figure 2, which indicates that there is no intersection between two lines $\ell_1$ and $\ell_2$. Thus, system (2.2) has no solution. It is easy to double this by solving the system. Indeed, substituting $y = 3 - x$ into the second equation of system (2.2), we have $6 = 5$, which is not valid for any $x$. The solution set is $\emptyset$.

**Example 2.3.** *Consider the linear system*

$$\begin{cases} x + y = 3 \\ 2x + 2y = 6 \end{cases} \tag{2.3}$$

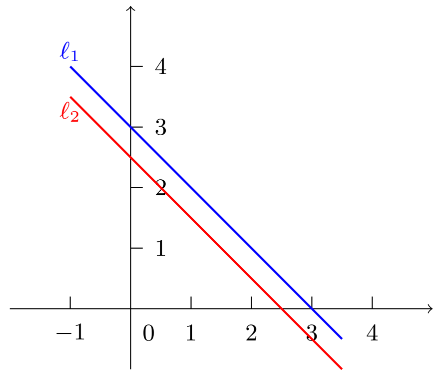


Figure 2: No solution

The graph that corresponds to the system (2.3) is shown in Figure 3, which indicates that two lines $\ell_1$ and $\ell_2$ coincide. Thus, system (2.2) has infinitely many solutions. We can check it by solving the

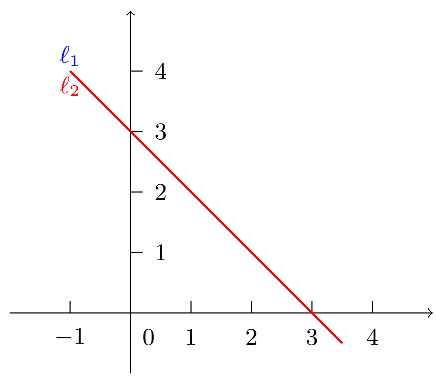


Figure 3: Infinitely many solutions

system (2.3). Substituting $y = 3 - x$ into the second equation of system (2.2), we have $5 = 5$, which is true for all values $x$. Thus, the solution set of system (2.3) is $\{(x, 3 - x) | x \in \mathbb{R}\}$.

**Example 2.4.**

$$\left\{ \; x + y = 3 \right. \tag{2.4}$$

*is a linear system. This system is equivalent to system (2.3) since they have the same solution set, which is* $\{(x, 3 - x) | x \in \mathbb{R}\}$.

**Remark 2.1.** *A system of linear equations has*

1. *no solution, or*
2. *exactly one solution, or*
3. *infinitely many solutions.*

**Remark 2.2.** *If a linear system has a solution (at least one solution), this system is said to be* **consistent**. *The linear system which has no solution is said to be* **inconsistent**.

**Example 2.5.** *The systems (2.1), (2.3), (2.4) are consistent. The system (2.2) is inconsistent.*

## 2.2 Linear equations and linear systems

**A linear equation in the variables** $x_1, x_2, \dots, x_n$ is an equation of the form

$$\alpha_1 x_1 + \alpha_2 x_2 + \dots \alpha_n x_n = \beta, \tag{2.5}$$

where $\beta$ and the **coefficients** $\alpha_1, \alpha_2, \dots, \alpha_n$ are real or complex numbers and do not depend on the variables $x_1, x_2, \dots, x_n$. The coefficients are usually given.

**A system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same variable - say, $x_1, x_2, \dots, x_n$. The general form of a linear system is given by

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n = b_2, \\ \qquad\qquad\qquad\qquad\qquad \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots a_{mn}x_n = b_m, \end{cases} \tag{2.6}$$

**Note:** $m$ can be 1, smaller than $n$, equals to $n$, or greater than $n$.

**Example 2.6.**

$$\begin{cases} 3x_1 + 9x_2 + 7x_3 &= 6 \\ \qquad\quad x_1 + 3x_2 &= -5 \end{cases} \tag{2.7}$$

*is a linear system of three variables* $x_1, x_2, x_3$.

A **solution** of the linear equation (2.5) is a list $(s_1, s_2, \dots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \dots, s_n$ are substituted for $x_1, \dots, x_n$, respectively. For instance, $(1, -2, 3)$ is a solution of system (2.7) since when substituting these values for $(x_1, x_2, x_3)$, respectively, the system becomes $6 = 6$ and $-5 = -5$.

**Note:** Solutions of a linear system can be described in the form of a column vector. That will be detailed in Lecture 5 and used from there on.

The set of all possible solutions is called the **solution set** of the linear system. Two linear systems are called **equivalent** if they have the same solution set. Solving a linear system is to find the solution set of this linear system.

**Definition 2.1. (Affine set):** *A set of vectors is called* **affine** *when it has the property that whenever two distinct points, say* $\mathbf{x}$ *and* $\mathbf{y}$*, are in the set then every point on the line through those two points is also in the set.*

In addition, the linear combination given by

$$\alpha\mathbf{x} + \beta\mathbf{y}, \quad \text{with} \;\; \alpha + \beta = 1$$

is called an **affine combination** of $\mathbf{x}$ and $y$.

If $\mathbf{x}$ and $\mathbf{y}$ denote two endpoints of vectors $\mathbf{x}$ and $\mathbf{y}$, then the line passing through two points $\mathbf{x}$ and $\mathbf{y}$ is a set of all affine combinations.

## 2.3 Matrix notations

The matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

(formed by coefficients of the systems) is called the **coefficient matrix** (or **matrix of coefficients**) of the linear system (2.6). The column vector

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

(formed by the constants from the respective right sides of the equations) is called the **right hand side (RHS)**. The matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

which consists of the coefficient matrix with adding the RHS as the last column, is called the **augmented matrix** of the linear system (2.6). The solution of system (2.6) is now written as a vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$ (instead of writing as the form $(x_1, x_2, \dots, x_m)$). A linear system is represented by its augmented matrix.