3 Lecture 3: The Gauss-Jordan Elimination Algorithm: Preliminaries

3.1 Matrix Operations Method to Solve a Linear System

Substituting one variable from one equation and plugging it into other equations for large systems with more than two variables (or unknowns) is inefficient. To solve such large systems, we use a systematic procedure. The basic strategy is to replace one system with an equivalent system (i.e., one with the same solution set) that is easier to solve. Three basic operations are used to simplify a linear system:

1. Replace one equation by the sum of itself and a multiple of another equation,
2. Interchange two equations,
3. Multiply all the terms in an equation by a nonzero constant.

Example 3.1. *Solve the linear system*

$$\begin{cases} x_1 & -2x_2 & +x_3 = & 0 \\ x_1 & & -x_3 = & 2 \\ & x_2 & -4x_3 = & 4 \end{cases} \tag{3.1}$$

We first see that the augmented matrix of the linear system (3.1) is

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 1 & 0 & -1 & 2 \\ 0 & 1 & -4 & 4 \end{bmatrix}$$

We now apply the three basic operations mentioned above to solve the linear system (3.1) and write the system and the associated augmented matrix side-by-side to easily compare.

Keep the first equation, and eliminate x_1 from the second equation by adding (-1) multiplies by the first equation to the second equation, we obtain an equivalent system. This system and its augmented matrix is given by

$$\begin{cases} x_1 & -2x_2 & +x_3 = & 0 \\ & 2x_2 & -2x_3 = & 2 \\ & x_2 & -4x_3 = & 4 \end{cases} \quad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -2 & 2 \\ 0 & 1 & -4 & 4 \end{bmatrix} \tag{3.2}$$

Multiply the second equation of the system (3.2) by 1/2 we obtain and equivalent system

$$\begin{cases} x_1 & -2x_2 & +x_3 = & 0 \\ & 1x_2 & -1x_3 = & 1 \\ & x_2 & -4x_3 = & 4 \end{cases} \quad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -4 & 4 \end{bmatrix} \tag{3.3}$$

Keep the first and the second equations of the system (3.3), and eliminate x_2 from the last equation by (-1) multiplies by the second equation to the last equation, we obtain an equivalent system

$$\begin{cases} x_1 & -2x_2 & +x_3 = & 0 \\ & 1x_2 & -1x_3 = & 1 \\ & & -3x_3 = & 3 \end{cases} \quad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -3 & 3 \end{bmatrix} \tag{3.4}$$

Now, from the last equation of the system (3.4), we can find $x_3 = -1$. Plugging $x_3 = -1$ into the second equation of the system (3.4), we can find $x_2 = 0$. Finally, plugging $x_3 = -1$ and $x_2 = 0$ into the first equation of the system (3.4), we have $x_1 = 1$. So, the solution of the linear system is

$$\begin{cases} x_1 = 1 \\ x_2 = 0 \\ x_3 = -1 \end{cases} . \tag{3.5}$$

On the other hand, (3.5) can be seen as a linear system and it augmented matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \tag{3.6}$$

The augmented matrix of such equivalent systems can also be obtained by applying some of **elementary row operations** *(see Definition 3.1) to the augmented matrix of the original system. That leads to the* **Gauss-Jordan Elimination Algorithm** *(or* **Row Reduction Algorithm** *) that is detailed in Lecture 4). Moreover, the matrix in (3.4) is called a proto-row-echelon form of* $\mathbf{A}$ *and the matrix in (3.6) is called the reduced row echelon form of* $\mathbf{A}$*. The section below will discuss in more details.*

Definition 3.1. (Elementary row operations).

The elementary row operations (also known as **row operations***) consist of the following*

1. *(Interchange) Interchange two rows.*
2. *(Replacement) Replace one row by the sum of itself and a multiple of another row.*
3. *(Scaling) Multiply all entries in a row by a nonzero constant.*

3.2 Row Reduction and Echelon Forms

Definition 3.2. (Leading entries)

The leading entry of a nonzero row is the leftmost nonzero entry.

Example 3.2. *Consider the matrix*

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 0 & 4 & 16 \\ 0 & 3 & 1 & 3 \end{bmatrix}$$

The leading entry in the first, the second, and the third rows of matrix $\mathbf{A}$ are 1, 4, and 3, respectively.

Definition 3.3. *A matrix is in* **proto-row-echelon form (pref)** *if it has three of the three following properties*

1. *All nonzero rows are above any rows of all zeros.*
2. *Each leading entry of a row is in a column to the right of the leading entry of the row above it.*
3. *All entries in a column below a leading entry are zeros.*

Example 3.3. *The following matrices are in proto-row-echelon form. The leading entries (■) may have any nonzero value; the starred entries ($*$) may have any value (including zero).*

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}$$

Definition 3.4. *A matrix is in* **normalized-row-echelon form (nref)** *if it is in proto-row-echelon form and*

4. *The leading entry in each nonzero row is 1.*

Example 3.4. *The following matrices are in normalized-row-echelon form. The starred entries ($*$) may have any value (including zero).*

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Definition 3.5. *A matrix is in* **reduced-row-echelon form (rref)** *if it is in normalized-row-echelon form and*

5. *Each leading entry 1 is the only non-zero entry in its column (i.e., all entries in a column above and below the leading entry are 0).*

Example 3.5. *The following matrices are in reduced-row-echelon form. The starred entries ($*$) may have any value (including zero).*

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \textit{and} \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & * & 0 & 0 & * & 0 & * & * & 0 \\ 0 & 0 & 0 & 1 & 0 & * & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 & * & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Definition 3.6. *Two matrices are called* **row equivalent** *if a sequence of elementary row operations transforms one matrix into the other. A reduced row echelon form matrix equivalent to a matrix* $\mathbf{A}$ *is called the reduced echelon form of* $\mathbf{A}$.

Remark 3.1.

- *The augmented matrix of two equivalent linear systems are row equivalent.*
- *If the augmented of two linear systems are row equivalent, the two systems have the same solution set (i.e., equivalent).*

3.3 Pivot positions, pivot columns

Definition 3.7. *A location in a matrix* $\mathbf{A}$ *corresponding to a leading 1 in the reduced echelon form of* A *is called a* **pivot position** *of* $\mathbf{A}$. *A pivot column of* A *that contains a pivot position is called a* **pivot column**.

Example 3.6. *Suppose that the reduced echelon form of a matrix* $\mathbf{A}$ *is of the form*

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & 0 & * & * & 0 \\ 0 & 0 & 0 & 1 & 0 & * & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 & * & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

then pivot position of $\mathbf{A}$ *is* $(1,2)$ *(which is the position in the first row and the second column),* $(2,4)$, $(3,5)$, $(4,7)$, $(5,10)$. *Pivot columns are:* $2,4,5,7,10$.

Remark 3.2. *If a matrix is in row echelon form, row operations to obtain the reduced echelon form do not change the positions of the leading entries. Thus, pivot positions of a matrix are also positions of a row echelon form of* $\mathbf{A}$ *(i.e., a matrix in the row echelon form that is row equivalent to* $\mathbf{A}$*). To find the pivot positions of a matrix, we just need to find a row echelon form of* $\mathbf{A}$. *The procedure to find a row echelon form of* $\mathbf{A}$ *is called* **row reduce** *matrix* $\mathbf{A}$.

4 Lecture 4: The Gauss-Jordan Elimination Algorithm: Implementation

4.1 Row Reduction Algorithm

This algorithm consists of five steps to row reduce a matrix into its reduced row echelon form, where, the **forward phase** consists of the first four steps to row reduce a matrix into its proto-row-echelon form and the **backward phase** consists of the last step is to row reduce the proto-row-echelon form into the reduced row echelon form.

The forward phase

Step 1 Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

Step 2 Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

(**Optional:** *Try to make the pivot equal to 1 if possible. If necessary, use row replacement operations or scaling*).

Step 3 Use row replacement operations to make zeros in all positions below the pivot.

Step 4 Ignore the row containing the pivot position and all rows above it, if any. Apply Steps 1 – 3 to the sub-matrix formed by the remainder rows. Repeat the process until there are no more nonzero rows to modify.

The backward phase

Step 5 Beginning with the rightmost pivot column and working upward and to the left. Use row replacement operations to make zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

Example 4.1. *Row reduce the following matrix first into echelon form and then into reduced echelon form:*

$$\mathbf{A} = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Instructions

Step 1: The first pivot column is column 1. The pivot position of this column is at the first row.
Step 2: The entry in the first row and the first column of matrix $\mathbf{A}$ is 0, which can not be a pivot, but entries in rows 2 and 3 are non-zero. Interchange rows 1 and 3 to obtain matrix $\mathbf{A}_1$. (We could have interchanged rows 1 and 2 instead.) The symbol $\sim$ before a matrix indicates that the matrix is a row equivalent to the preceding matrix.

$$\mathbf{A} \sim \mathbf{A}_1 = \begin{array}{c} \bullet \\ 1 \\ \end{array} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Step 3: Keep the first row of matrix $\mathbf{A}_1$, add -1 times row 1 to row to obtain matrix $\mathbf{A}_2$

$$\sim \mathbf{A}_2 = \begin{array}{c} \\ \bullet \\ 3/2 \end{array} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

(All entries below the pivot position of the first column are 0).

Step 4: (Ignore row 1 of matrix $\mathbf{A}_2$ and apply Steps 1 – 3 to the sub-matrix formed by rows 2 and 3). The next pivot column is column 2. The top entry, which is 2, is non-zero, so no interchange is needed. Keep the first row of the sub-matrix, then add $-3/2$ times the first row of the sub-matrix to the row below

$$\sim \mathbf{A}_3 = \begin{array}{c} 6 \\ 2 \\ \bullet \end{array} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Ignore Row 2 of $\mathbf{A}_3$ and all rows about it (row 1); the remainder is a sub-matrix formed by only row 3. No work in Steps 1-3 is required for this matrix. Matrix $\mathbf{A}_3$ is in echelon form.

(**Note:** We can normalize rows 1 and 2 of matrix $\mathbf{A}_3$, but it is not mandatory when the row is reducing into an echelon form matrix.)

Step 5: To row reduce the echelon form matrix $\mathbf{A}_3$ into the reduced row echelon form, we begin with the rightmost pivot column, the fifth column. The rightmost pivot is in row 3. We keep this row and make zeros above the pivot by adding -2 times row 3 to row 2 and -6 times row 3 to row 1.

$$\sim \mathbf{A}_4 = \begin{array}{c} \\ 1/2 \\ \\ \end{array} \begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

The next pivot column is column 2, and the pivot is in row 2. Scale this row, dividing by the pivot.

$$\sim \mathbf{A}_5 = \begin{array}{c} -9 \\ \bullet \\ \\ \end{array} \begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

We keep row 2 and make zeros above the pivot by adding 9 times row 2 to row 1.

$$\sim \mathbf{A}_6 = \begin{array}{c} 1/3 \\ \\ \\ \end{array} \begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Finally, scale row 1, dividing by the pivot, 3.

$$\sim \mathbf{A}_7 = \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Matrix $\mathbf{A}_7$ is the reduced echelon form of the original matrix, $\mathbf{A}$.

4.2 Some Examples

Example 4.2. *Row reduce the following matrix first into echelon form and then into reduced echelon form:*

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

Solution: The forward phase:

$$\mathbf{A} = \begin{array}{c} \bullet \\ 1 \\ \end{array} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix} \sim \begin{array}{c} \\ \bullet \\ -1 \end{array} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix} \sim \begin{array}{c} \\ -1 \\ \end{array} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix $\mathbf{A}_4 := \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is in row echelon form thus it is a row echelon form of $\mathbf{A}$.

Backward phase:

$$\mathbf{A}_4 = \begin{array}{c} 2 \\ \bullet \\ \end{array} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix $\mathbf{A}_5 := \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is in reduced row echelon form thus it is a reduced echelon form of $\mathbf{A}$.

Theorem 1. (Uniqueness of the Reduced Row Echelon Form)

Each matrix is a row equivalent to one and only one reduced echelon from the matrix. That is, the reduced echelon form of a matrix exists and is unique.

4.3 Solution to Linear Systems

We first consider the example:

Example 4.3. *Find the solution to the linear system whose augmented matrix is*

$$\begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The augmented matrix has four columns, so the linear system has three variables: x_1, x_2, x_3. The augmented matrix is in reduced row echelon form, so we now write back the linear system, which is given by

$$\begin{cases} x_1 \quad\quad -3x_3 &= 1 \\ \quad\quad x_2 \; +x_3 &= 2 \\ \quad\quad\quad 0 &= 0 \end{cases}$$

The variables corresponding to pivot columns in the matrix, x_1 and x_2, are called **basic variables**. The remainder variable, x_3, is called a **free variable**. Writing x_1 and x_2 in terms of x_3 we obtain

$$\begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free} \end{cases} \tag{4.1}$$

Note: "x_3 is free" means x_3 can take any value. For each specific value of x_3, we have a specific value of x_1, x_2, and then one specific solution. There are infinitely many possible values of x_3 (such as all real numbers). Therefore, the system is consistent, and it has infinitely many solutions.

Example 4.4. *Find the general solution of the linear system whose augmented matrix has been reduced to*

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \tag{4.2}$$

Solution: The matrix is in echelon form, but we need the reduced echelon form before solving for the basic variables. The row reduction is completed next.

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

There are five variables, say x_1, x_2, x_3, x_4, x_5, because the augmented matrix has six columns. The associated system is

$$\begin{cases} x_1 + 6x_2 + 3x_4 = 0 \\ x_3 - 4x_4 = 5 \\ x_5 = 7 \end{cases} \tag{4.3}$$

The pivot columns of the matrix are 1,3, and 5, so the basic variables are x_1, x_3, and x_5. The remaining variables, x_2 and x_4, must be free. Solve for the basic variables to obtain the general solution:

$$\begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 4x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{cases}$$

This system has infinitely many solutions.

4.4 Existence and Uniqueness questions

Example 4.5. *Find the general solution of the linear system whose augmented matrix has been reduced to*

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -4 \\ 0 & 0 & 1 & -4 & 2 \\ 0 & 0 & 2 & -8 & 5 \end{bmatrix} \tag{4.4}$$

Solution: We first row reduce the augmented matrix.

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -4 \\ 0 & 0 & 1 & -4 & 2 \\ 0 & 0 & 2 & -8 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 2 & -5 & -4 \\ 0 & 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

There are four variables, say x_1, x_2, x_3, x_4, because the augmented matrix has five columns. Usually, we need the reduced echelon form before solving for the basic variables. However, in this case, we quickly observe that the last row of the matrix produce the equation $0 = 1$, which never happens for any x_1, x_2, x_3, x_4. Thus, the system has no solution, or the system is **inconsistent**.

Example 4.6. *Find the solution to the linear system whose augmented matrix is*

$$\begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 3 & 0 \end{bmatrix}$$

Solution: We first row reduce the augmented matrix of the linear system

$$\begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

The system has the variable since the augmented matrix has four columns. All three variables are basic variables because three columns are pivot columns. Thus, there is no free variable. In this case, the solution is obtained directly from the reduced echelon form matrix, which is

$$\begin{cases} x_1 = -11 \\ x_2 = 6 \\ x_3 = -4. \end{cases}$$

The solution is unique.

Theorem 2. *A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column-that is, if and only if an echelon form of the augmented matrix has no row of the form.*

$$\begin{bmatrix} 0 & \cdots & 0 & b \end{bmatrix} \quad \textit{with } b \textit{ nonzero}$$

If a linear system is **consistent***, then the solution set contains either*

(i) a unique solution when there are no free variables, or

(ii) infinitely many solutions when there is at least one free variable.

Algorithm: Row Reduction to Solve a Linear Systems

The row reduction method to solve a linear system generally consists of five steps, summarized below:

1. Write the augmented matrix of the system. (*If the system is given by its augmented matrix, this step is skipped.*)
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

4.5 Conceptual questions

1. (T/F) In some cases, a matrix may be row reduced to more than one matrix in reduced echelon form, using different sequences of row operations.
2. (T/F) The proto-row-echelon form of a matrix is unique.
3. (T/F) The normalized-row-echelon form of a matrix is unique.
4. (T/F) The row reduction algorithm applies only to augmented matrices for a linear system.
5. (T/F) The pivot positions in a matrix depend on whether row interchanges are used in the row reduction process.