23 Lecture 23: Discrete Linear Dynamical Systems and Markov Chains

23.1 Discrete Linear Dynamical Systems

Definition 23.1. *A (first-order homogeneous) matrix* **difference equation** *is an equation of the form*

$$\mathbf{x}_{\ell+1} = \mathbf{A}\mathbf{x}_\ell,$$

where $\mathbf{A}$ *is an* $n \times n$ *matrix. An* **initial condition** $\mathbf{x}_0$ *is a vector* $\mathbf{x}_0 \in \mathbf{R}^n$*. Taken together, the difference equation and the initial condition determine a sequence of vectors* $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ *such that* $\mathbf{x}_{\ell+1} = \mathbf{A}\mathbf{x}_\ell$*, for all* $\ell = 0, 1, 2, 3, \ldots$*. This is called a (linear)* **discrete dynamical system.**

Solving a discrete linear dynamical system is to find the sequence of vectors $\mathbf{x}_\ell$, for all $\ell = 1, 2, 3, \ldots$ given the difference equation $\mathbf{x}_{\ell+1} = \mathbf{A}\mathbf{x}_\ell$ and the initial condition $\mathbf{x}_\ell$.

From the difference equation, and the initial condition, we have

$$\begin{aligned}
\mathbf{x}_1 &= \mathbf{A}\mathbf{x}_0 \\
\mathbf{x}_2 &= \mathbf{A}\mathbf{x}_1 = \mathbf{A}\mathbf{A}\mathbf{x}_0 = \mathbf{A}^2\mathbf{x}_0 \\
\mathbf{x}_3 &= \mathbf{A}\mathbf{x}_2 = \mathbf{A}\mathbf{A}^2\mathbf{x}_0 = \mathbf{A}^3\mathbf{x}_0 \\
&\ldots \\
\mathbf{x}_\ell &= \mathbf{A}\mathbf{x}_{\ell-1} = \mathbf{A}\mathbf{A}^{\ell-1}\mathbf{x}_0 = \mathbf{A}^\ell\mathbf{x}_0
\end{aligned}$$

This shows $\mathbf{x}_\ell = \mathbf{A}^\ell\mathbf{x}_0$, for all $\ell = 0, 1, 2, \ldots$. The ℓ^{th} vector of the sequence is calculated from matrix $\mathbf{A}$ and the initial condition $\mathbf{x}_0$.

Assume that matrix $\mathbf{A}$ is diogonalizable and the diagonalization

$$\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1},$$

where $\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$, $\mathbf{S} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix}$, and $(\lambda_j, \mathbf{v}_j)$ are eigenpairs for all $j = 1, 2, \ldots, n$. Then, for all $\ell = 1, 2, \ldots$

$$\mathbf{A}^\ell = (\mathbf{S}\mathbf{D}\mathbf{S}^{-1})(\mathbf{S}\mathbf{D}\mathbf{S}^{-1})\ldots(\mathbf{S}\mathbf{D}\mathbf{S}^{-1}) = \mathbf{S}\mathbf{D}(\mathbf{S}^{-1}\mathbf{S})\mathbf{D}(\mathbf{S}^{-1}\ldots\mathbf{S})\mathbf{D}\mathbf{S}^{-1} = \mathbf{S}\mathbf{D}^\ell\mathbf{S}^{-1},$$

where D^ℓ is a diagonal matrix given by

$$\mathbf{D}^\ell = \begin{bmatrix} \lambda_1^\ell & 0 & \ldots & 0 \\ 0 & \lambda_2^\ell & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \lambda_n^\ell \end{bmatrix}.$$

From this diagonalization of $\mathbf{A}^\ell$, we can calculate $\mathbf{x}_\ell$, which is

$$\mathbf{x}_\ell = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1^\ell & 0 & \dots & 0 \\ 0 & \lambda_2^\ell & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_n^\ell \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix}^{-1} \mathbf{x}_0 \tag{23.1}$$

Remark 23.1. *If we denote* $\mathbf{s} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix}^{-1} \mathbf{x}_0$, *then* $\mathbf{s}$ *is a vector in* $\mathbf{R}^n$, *which is formed by coefficient of* $\mathbf{x}_0$ *as a linear combination of eigenvectors. That is,*

$$\mathbf{x}_0 = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix} \mathbf{s} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \dots + s_n\mathbf{v}_n.$$

As a result, $\mathbf{x}_\ell$ *can be written as*

$$\mathbf{x}_\ell = s_1\lambda_1^\ell\mathbf{v}_1 + s_2\lambda_2^\ell\mathbf{v}_2 + \dots + s_n\lambda_n^\ell\mathbf{v}_n.$$

Limits as infinity

- If $0 < \lambda < 1$, $\lambda^\ell \to 0$ as $\ell \to \infty$
- If $\lambda > 1$, $\lambda^\ell \to +\infty$ as $\ell \to \infty$.

Example 23.1. *The population of frogs and midges in a pond satisfy the difference equation*

$$\mathbf{x}_{\ell+1} = \begin{bmatrix} 0.7 & 0.1 \\ -0.5 & 1.3 \end{bmatrix} \mathbf{x}_\ell, \quad \textit{where } \mathbf{x}_\ell := \begin{bmatrix} f_\ell \\ m_\ell \end{bmatrix};$$

f_ℓ *and* m_ℓ *are populations of frogs and midges at the year* ℓ, *respectively. Assume that at the beginning* $\ell = 0$ *the population of frogs and midges are* $f_0 = 7$ *and* $m_0 = 9$.

- *Find the population of frogs and midges at the year* $\ell = 100$*?*
- *Find the ratio between the population of frogs and midges when the time tends to infinity?*

Solution: From the problem statement, we have that

$$\mathbf{x}_{\ell+1} = \mathbf{A}\mathbf{x}_\ell, \quad \mathbf{A} = \begin{bmatrix} 0.7 & 0.1 \\ -0.5 & 1.3 \end{bmatrix} \mathbf{x}_\ell$$

and $\mathbf{x}_0 = \begin{bmatrix} f_0 \\ m_0 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$. From the difference equation, we obtain

$$\mathbf{x}_\ell = \mathbf{A}^\ell\mathbf{x}_0 = \mathbf{S}\mathbf{D}^\ell\mathbf{S}^{-1}\mathbf{x}_0,$$

where $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$ is the diagonalization of $\mathbf{A}$.

To find the diagonalization of $\mathbf{A}$, we find the eigenvalues and the associated eigenvector, which are

$$\lambda_1 = 1.2,\ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \quad \text{and} \quad \lambda_2 = 0.8,\ \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$\mathbf{A}$ has a diagonalization $\mathbf{A} = \mathbf{SDS}^{-1}$, where

$$\mathbf{S} = \begin{bmatrix} 1 & 1 \\ 5 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1.2 & 0 \\ 0 & 0.8 \end{bmatrix}, \quad \text{and} \quad \mathbf{S}^{-1} = \frac{-1}{4}\begin{bmatrix} 1 & -1 \\ -5 & 1 \end{bmatrix}.$$

Thus,

$$\begin{aligned}
\mathbf{x}_\ell = \mathbf{SD}^\ell\mathbf{S}^{-1}\mathbf{x}_0 &= -\frac{1}{4}\begin{bmatrix} 1 & 1 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 1.2^\ell & 0 \\ 0 & 0.8^\ell \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 7 \\ 9 \end{bmatrix} \\
&= \frac{-1}{4}\begin{bmatrix} 1.2^\ell & 0.8^\ell \\ 5(1.2)^\ell & 0.8^\ell \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 7 \\ 9 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} 1.2^\ell - 5(0.8)^\ell & -(1.2)^\ell + 0.8^\ell \\ 5(1.2^\ell) - 5(0.8)^\ell & -5(1.2)^\ell + 0.8^\ell \end{bmatrix}\begin{bmatrix} 7 \\ 9 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 1.2^\ell + 13(0.8)^\ell \\ 5(1.2)^\ell + 13(0.8)^\ell \end{bmatrix},
\end{aligned}$$

which implies, the population of frogs and midges at the time ℓ are

$$f_\ell = \frac{1.2^\ell + 13(0.8)^\ell}{2} \quad \text{and} \quad m_\ell = \frac{5(1.2^\ell) + 13(0.8)^\ell}{2}.$$

At $\ell = 100$, $\quad f_{100} = \frac{1.2^{100}+13(0.8)^{100}}{2}$ and $m_{100} = \frac{5(1.2^{100})+13(0.8)^{100}}{2}$.

At $\ell \to \infty$,

$$\lim_{\ell\to\infty} \frac{f_\ell}{m_\ell} = \lim_{\ell\to\infty} \frac{1.2^\ell + 13(0.8)^\ell}{5(1.2^\ell) + 13(0.8)^\ell} = \lim_{\ell\to\infty} \frac{1 + 13(0.8/1.2)^\ell}{5 + 13(0.8/1.2)^\ell} = \frac{1}{5}$$

23.1.1 Application to Fibonacci sequence

The Fibonacci sequence is the series of numbers defined by: $F_0 = 0$, $F_1 = 1$, and the recurrence relation

$$F_{n+2} = F_{n+1} + F_n, \quad \text{for all } n = 0, 1, 2, \ldots$$

A method to find the general formula for F_n is to describe the problem under the form of a discrete dynamical system mentioned below. To do so, we introduce the system

$$\begin{cases} F_{n+2} &= F_{n+1} + F_n \\ F_{n+1} &= \qquad F_{n+1} \end{cases}$$

and define $\mathbf{x} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$. Then, the system is written as the difference equation $\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The general formulation for $\mathbf{x}_n$ is

$$\mathbf{x}_n = \mathbf{A}^n\mathbf{x}_0 = \mathbf{S}\mathbf{D}^n\mathbf{S}^{-1}\mathbf{x}_0.$$

Matrix $\mathbf{A}$ has eigenvalues $\lambda_1 = \frac{1-\sqrt{5}}{2}$ and $\lambda_2 = \frac{1+\sqrt{5}}{2}$, with eigenvectors $\mathbf{v}_1 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$ (respectively). We define $\mathbf{S} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} \frac{1-\sqrt{5}}{2} & 0 \\ 0 & \frac{1+\sqrt{5}}{2} \end{bmatrix}$, then $\mathbf{S}^{-1} = \begin{bmatrix} -\frac{\sqrt{5}}{5} & \frac{5+\sqrt{5}}{10} \\ \frac{\sqrt{5}}{5} & \frac{5-\sqrt{5}}{10} \end{bmatrix}$ and

$$\begin{aligned}
\mathbf{x}_n = \mathbf{S}\mathbf{D}^n\mathbf{S}^{-1}\mathbf{x}_0 &= \begin{bmatrix} \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \left(\frac{1-\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1+\sqrt{5}}{2}\right)^n \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{5}}{5} & \frac{5+\sqrt{5}}{10} \\ \frac{\sqrt{5}}{5} & \frac{5-\sqrt{5}}{10} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} + \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} \\ -\frac{\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^{n} + \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^{n} \end{bmatrix}
\end{aligned}$$

Thus,

$$F_n = \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

General second-order linear difference equation

The problem of finding y_n from the given of y_0, y_1, and the recurrence relation

$$y_{n+2} = ay_{n+1} + by_n$$

is rewritten in the matrix form

$$\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n,$$

where $\mathbf{A} = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}$ and $\mathbf{x}_n = \begin{bmatrix} y_{n+1} \\ y_n \end{bmatrix}$. Obviously, $\mathbf{x}_0 = \begin{bmatrix} y_1 \\ y_0 \end{bmatrix}$.

General third-order linear difference equation

The problem of finding y_n from the given of y_0, y_1, y_2 and the recurrence relation

$$y_{n+3} = ay_{n+2} + by_{n+1} + cy_n$$

is rewritten in the matrix form

$$\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n,$$

where $\mathbf{A} = \begin{bmatrix} a & b & c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $\mathbf{x}_n = \begin{bmatrix} y_{n+2} \\ y_{n+1} \\ y_n \end{bmatrix}$. Obviously, $\mathbf{x}_0 = \begin{bmatrix} y_2 \\ y_1 \\ y_0 \end{bmatrix}$.

23.1.2 Graphical Description of Solutions

See the .mlx file on Canvas - common course

23.2 Markov Chains

Definitions 23.1. *A vector with nonnegative entries summing to 1 is called a* **probability vector**. *A* **stochastic matrix** *is a square matrix whose columns are probability vectors.*

A Markov chain is a sequence of probability vectors $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2 \ldots$ *together with a stochastic matrix* $\mathbf{P}$ *such that*

$$\mathbf{x}_{\ell+1} = \mathbf{P}\mathbf{x}_\ell, \quad \text{for all} \quad \ell = 0, 1, 2, \ldots$$

Matrix $\mathbf{P} = [p_{ij}]$ *is called the* **transition matrix** *of the system, where* p_{ij} *indicates the probability of state* j^{th} *moves to state* i^{th}. *Vector* $\mathbf{x}_\ell$ *is called a* **state vector**.

Example 23.2. *The sequence* $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ *such that*

$$\mathbf{x}_\ell = \begin{bmatrix} 1-p & 1-q & 1-r & 0 \\ p & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & r & 1 \end{bmatrix} \mathbf{x}_{\ell-1}, \quad 0 \le p, q, r \le 1$$

is a Markov chain and $\mathbf{P} = \begin{bmatrix} 1-p & 1-q & 1-r & 0 \\ p & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & r & 1 \end{bmatrix}$ *is the transition matrix.*

23.2.1 Absorbing states

An **absorbing state** is the state from which it is impossible to leave. Mathematically, the i^{th} state is an absorbing state if and only if $p_{ii} = 1$.

Example 23.3. *Consider a transition matrix*

$$\mathbf{P} = \begin{bmatrix} 0 & 1-p & p & 0 & 0 \\ p & 0 & 0 & 0 & 0 \\ 1-p & 0 & 0 & 0 & 0 \\ 0 & p & 0 & 1 & 0 \\ 0 & 0 & 1-p & 0 & 1 \end{bmatrix}$$

Then, states 4 and 5 are absorbing states due to $p_{44} = p_{55} = 1$.

Example 23.4. *Consider the transition matrix*

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, state 3 is an absorbing matrix due to $p_{33} = 1$*, but states 1 and 2 are not absorbing states since* $p_{11} = 0 \neq 1$ *and* $p_{22} = 0 \neq 1$.

23.2.2 Steady distributions and Predicting the Distant Future

The most interesting aspect of Markov chains is the study of a chain's long-term behavior. We consider example below.

Example 23.5. *Let* $\mathbf{P} = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix}$ *and* $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. *Consider a system whose state is described by the Markov chain* $\mathbf{x}_{k+1} = \mathbf{P}\mathbf{x}_k$*, for* $k = 0, 1, \ldots$ *What happens to the system as time passes? Compute the state vectors* $\mathbf{x}_1, \ldots, \mathbf{x}_{15}$ *to find out.*

Solution:

$$\mathbf{x}_1 = \mathbf{P}\mathbf{x}_0 = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} .5 \\ .3 \\ .2 \end{bmatrix}$$

$$\mathbf{x}_2 = \mathbf{P}\mathbf{x}_1 = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix} \begin{bmatrix} .5 \\ .3 \\ .2 \end{bmatrix} = \begin{bmatrix} .37 \\ .45 \\ .18 \end{bmatrix}$$

$$\mathbf{x}_3 = \mathbf{P}\mathbf{x}_2 = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix} \begin{bmatrix} .37 \\ .45 \\ .18 \end{bmatrix} = \begin{bmatrix} .329 \\ .525 \\ .146 \end{bmatrix}$$

The results of further calculations are shown below, with entries rounded to four or five significant figures.

$$\mathbf{x}_4 = \begin{bmatrix} .3133 \\ .5625 \\ .1242 \end{bmatrix}, \quad \mathbf{x}_5 = \begin{bmatrix} .3064 \\ .5813 \\ .1123 \end{bmatrix}, \quad \mathbf{x}_6 = \begin{bmatrix} .3032 \\ .5906 \\ .1062 \end{bmatrix}, \quad \mathbf{x}_7 = \begin{bmatrix} .3016 \\ .5953 \\ .1031 \end{bmatrix}$$

$$\mathbf{x}_8 = \begin{bmatrix} .3008 \\ .5977 \\ .1016 \end{bmatrix}, \quad \mathbf{x}_9 = \begin{bmatrix} .3004 \\ .5988 \\ .1008 \end{bmatrix}, \quad \mathbf{x}_{10} = \begin{bmatrix} .3002 \\ .5994 \\ .1004 \end{bmatrix}, \quad \mathbf{x}_{11} = \begin{bmatrix} .3001 \\ .5997 \\ .1002 \end{bmatrix}$$

$$\mathbf{x}_{12} = \begin{bmatrix} .30005 \\ .59985 \\ .10010 \end{bmatrix}, \quad \mathbf{x}_{13} = \begin{bmatrix} .30002 \\ .59993 \\ .10005 \end{bmatrix}, \quad \mathbf{x}_{14} = \begin{bmatrix} .30001 \\ .59996 \\ .10002 \end{bmatrix}, \quad \mathbf{x}_{15} = \begin{bmatrix} .30001 \\ .59998 \\ .10001 \end{bmatrix}$$

These vectors seem to be approaching $\mathbf{q} = \begin{bmatrix} .3 \\ .6 \\ .1 \end{bmatrix}$. The probabilities are hardly changing from one value of k to the next. Observe that the following calculation is exact (with no rounding error):

$$P\mathbf{q} = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix} \begin{bmatrix} .3 \\ .6 \\ .1 \end{bmatrix} = \begin{bmatrix} .15 + .12 + .03 \\ .09 + .48 + .03 \\ .06 + 0 + .04 \end{bmatrix} = \begin{bmatrix} .30 \\ .60 \\ .10 \end{bmatrix} = \mathbf{q}$$

When the probability vector is $\mathbf{q}$, it is preserved through the system. This vector is called a **stationary distribution** (or a "**steady-state vector**").

A **stationary distribution** (also known as a "**steady-state vector**") is a probability vector which is an eigenvector of a transition matrix $\mathbf{P}$ with eigenvalue 1. That is, a probability vector $\mathbf{v}$ such that

$$\mathbf{Pv} = \mathbf{v}.$$

Stationary distributions of an stochastic matrix always exist due to the following theorem.

Theorem 12. *If $\mathbf{P}$ is a stochastic matrix, then 1 is an eigenvalue of $\mathbf{P}$.*

We say that a stochastic matrix $\mathbf{P}$ is **regular** if some matrix power $\mathbf{P}^k$ contains only strictly positive entries. For example, for matrix $\mathbf{P}$ in Example 23.5,

$$P^2 = \begin{bmatrix} .37 & .26 & .33 \\ .45 & .70 & .45 \\ .18 & .04 & .22 \end{bmatrix}$$

Since every entry in $\mathbf{P}^2$ is strictly positive, $\mathbf{P}$ is a regular stochastic matrix.

Also, we say that a sequence of vectors, $\mathbf{x}_k$ for $k = 1, 2, \ldots$, **converges** to a vector $\mathbf{q}$ as $k \to \infty$, if the entries in $\mathbf{x}_k$ can be made as close as desired to the corresponding entries in $\mathbf{q}$ by taking k sufficiently large.

Theorem 13. *If $\mathbf{P}$ is an $n \times n$ regular stochastic matrix, then $\mathbf{P}$ has a unique steady-state vector $\mathbf{v}$. Further, if $\mathbf{x}_0$ is any initial state vector and $\mathbf{x}_{k+1} = \mathbf{P}\mathbf{x}_k$ for $k = 0, 1, 2, \ldots$, then the Markov chain $\{\mathbf{x}_k\}$ converges to $\mathbf{v}$ as $k \to \infty$.*

23.3 Suggested problems

Problem 23.1. *Suppose*

$$x_{n+2} = \frac{1}{3}x_{n+1} + \frac{2}{3}x_n, \;\; x_1 = 1, \;\; x_0 = 0$$

(a) Evaluate x_n, for all $n = 2, 3, \ldots$

(b) Calculate

$$\lim_{n\to\infty} x_n$$

Problem 23.2. *Let*

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

(a) Diagonalize $\mathbf{A}^n$*, for all* $n = 2, 3, \ldots$

(b) Calculate

$$\lim_{n\to\infty} \mathbf{A}^n$$

Problem 23.3. *Consider the Markov chain having four states and the transition matrix*

$$\mathbf{P} = \begin{bmatrix} 0 & 1-a & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & a & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \textit{where } 0 \le \ell \le 1.$$

Assuming that $\mathbf{p}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$*, find*

$$\lim_{\ell\to\infty} \mathbf{p}_\ell.$$

Problem 23.4. *Let* $\mathbf{A} = \begin{bmatrix} \frac{7}{8} & \frac{1}{8} \\ \frac{1}{9} & \frac{8}{9} \end{bmatrix}$.

a) Find $\mathbf{P} = \lim_{\ell\to\infty} \mathbf{A}^\ell$

b) Show that $\mathbf{P}$ *is a projection matrix.*