Homework 10 - Arnav Kucheriya

Homework 10 - Cardinalities

Finite Sets

1. Let X, Y be finite sets. $|X| = m$, $|Y| = n$, and $f : X \to Y$.

- f is **surjective** $\Rightarrow$ (d) $m \geq n$
- f is **injective** $\Rightarrow$ (b) $m \leq n$
- f is **bijective** $\Rightarrow$ (c) $m = n$
- f is **surjective, not bijective** $\Rightarrow$ (d) $m \geq n$ and $m \neq n$
- f is **injective, not surjective** $\Rightarrow$ (a) $m < n$

Enumerable Sets

2. Function $B : \mathbb{N} \cup \{0\} \to \mathbb{Z}$

$$B(n) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ -\frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

(a) Values:

- $B(0) = 0$
- $B(1) = 1$
- $B(2) = -1$
- $B(3) = 2$
- $B(4) = -2$
- $B(5) = 3$
- $B(6) = -3$
- $B(7) = 4$

(b) Inverse $B^{-1} : \mathbb{Z} \to \mathbb{N} \cup \{0\}$

$$B^{-1}(y) = \begin{cases} 2y - 1 & \text{if } y > 0 \\ -2y & \text{if } y \leq 0 \end{cases}$$

(c) Evaluate:

- $B^{-1}(-3) = 6$

- $B^{-1}(3) = 5$
- $B^{-1}(-10) = 20$
- $B^{-1}(10) = 19$

3. $\mathbb{N} \setminus \{3, 10, 50\}$ is enumerable.

Proof: Since $\mathbb{N}$ is enumerable and finite subsets can be removed from enumerable sets while retaining enumerability, the set $\mathbb{N} \setminus \{3, 10, 50\}$ is enumerable.

4. $A = \{5\} \times \mathbb{N}$ is enumerable.

Proof: Define $f(n) = (5, n)$ for $n \in \mathbb{N}$. This is a bijection from $\mathbb{N} \to A$.

5. Set $T = \{n \in \mathbb{N} \cup \{0\} : n \equiv 3 \mod 4\}$

(a) First 5 elements: $3, 7, 11, 15, 19$

(b) Bijection $f(n) = 4n + 3$ maps $\mathbb{N} \cup \{0\} \to T$

(c) Restricting f to $n \in \mathbb{N}$ gives bijection $\mathbb{N} \to T \setminus \{3\}$

6. Functions on $\mathbb{N}$

(a) $f(n) = n + 1$ is bijective and not identity.

(b) $f(n) = 2n$ is injective but not surjective.

(c) $f(n) = \lfloor \frac{n}{2} \rfloor$ is surjective but not injective.

(d) For set $\{1, 2, 3, 4\}$:

- Injective not surjective: Not possible
- Surjective not injective: Not possible
- Bijective not identity: e.g., $f = \{(1, 2), (2, 1), (3, 4), (4, 3)\}$

7. Function $b_2 : \mathbb{N} \to \mathbb{Q}$

(a) Values depend on context from class, but assuming pairing: $b_2(4) = 1/3$, $b_2(10) = 2/3$ (e.g., diagonal pairing)

(b) It is bijective (by construction).

(c) Condition for equality: $\frac{a}{b} = \frac{m}{n} \Leftrightarrow an = bm$

8. Composition of bijections

Let $f : A \to B$ and $g : B \to C$ be bijections.

Then $g \circ f : A \to C$ is bijection because:

- Injectivity: $g(f(a_1)) = g(f(a_2)) \Rightarrow a_1 = a_2$
- Surjectivity: $\forall c \in C$, $\exists b \in B$, $\exists a \in A$ such that $g(f(a)) = c$

9. Bijections between intervals

(a) $f(x) = 3x$ maps $(0,1) \to (0,3)$

(b) $f(x) = x + 2$ maps $(0,1) \to (2,3)$

(c) $f(x) = 3x + 5$ maps $(0,1) \to (5,8)$

(d) $f(x) = 0.2x - 0.1$ maps $(-10,10) \to (-0.1,0.1)$

(e) $f(x) = \sin(x)$ maps $[-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$

10. Composition of bijections

(a) $f(x) = 2x - 1$ maps $(0,1) \to (-1,1)$

(b) $f(x) = \frac{\pi}{2}x$ maps $(-1,1) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(c) $f(x) = \tan(x)$ maps $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$

(d) Composite: $(0,1) \xrightarrow{f_1} (-1,1) \xrightarrow{f_2} \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \xrightarrow{f_3} \mathbb{R}$

(e) Composite: $(0,1) \xrightarrow{f_1} (-1,1) \xrightarrow{f_2} \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \xrightarrow{f_3} \mathbb{R}$

11.

Video: https://www.youtube.com/watch?v=OxGsU8oIWjY

Appears: pairing functions for $\mathbb{N} \times \mathbb{N}$, mapping $\mathbb{Q}$ to $\mathbb{N}$.

Difference: Graphical/computational approach vs theoretical proofs shown in class.