Quick Overview of Stage 3 for Math 337 in F24

I. Gram-Schmidt/QR. Converts a basis for $\mathbb{R}^n$ to an orthonormal basis for $\mathbb{R}^n$ and factors the matrix as orthogonal times upper triangular.

$$\begin{aligned}
A &= [\,\mathbf{u}_1 \quad \cdots \quad \mathbf{u}_n\,] \\
\mathbf{v}_\ell &= \mathbf{u}_\ell - \sum_{j=1}^{\ell-1} \frac{\mathbf{v}_j \cdot \mathbf{u}_\ell}{\mathbf{v}_j \cdot \mathbf{v}_j}\mathbf{v}_j \qquad \ell = 1, \ldots, n \\
\mathbf{w}_\ell &= \frac{1}{||\mathbf{v}_\ell||}\mathbf{v}_\ell \qquad \ell = 1, \ldots, n \\
\\
Q &= [\,\mathbf{w}_1 \quad \cdots \quad \mathbf{w}_n\,] \\
R &= Q^T A
\end{aligned}$$

II. Linear Dynamics. (application of diagonalization)

- discrete: $\mathbf{x}_{\ell+1} = A\mathbf{x}_\ell \quad \Rightarrow \quad \mathbf{x}_\ell = A^\ell \mathbf{x}_0$
- continuous: $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) \quad \Rightarrow \quad \mathbf{x}(t) = e^{At}\mathbf{x}(0)$

III. Orthogonal diagonalization (for symmetric matrices $A^T = A$).

- $A = SDS^{-1}$ ordinary diagonalization
- $S = QR$ orthonormalize the eigenvectors
- $A = QDQ^T$ orthogonal diagonalization

IV. Singular Value Decomposition (SVD).

$$\underset{m \times n}{A} = \underset{m \times m}{U} \quad \underset{m \times n}{\Sigma} \quad \underset{n \times n}{V^T}$$

- U is $m \times m$ orthogonal.
- Σ is $m \times n$ "diagonal."
- V is $n \times n$ orthogonal.