Problem 1.1. (8 points) Showing your work, compute the determinants below:

(a) $\begin{vmatrix} 0 & 0 & 1 \\ 1 & 4 & 5 \\ 1 & 3 & 6 \end{vmatrix}$

(b) $\begin{vmatrix} 0 & 0 & 1 & -7 & -8 \\ 0 & 0 & 0 & 7 & 1 \\ 3 & 8 & -1 & -2 & -5 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 2 & -3 & -4 & -6 \end{vmatrix}$

Problem 1.2. (10 points) Showing your work, diagonalize the matrix A (if possible):

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 2 \end{bmatrix}$$

Problem 1.3. (9 points) Showing your work, determine if each of the the sets of vectors below is linearly dependent or linearly independent:

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Problem 1.4. (8 points) Give a basis for each of the four fundamental subspaces of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}.$$

Problem 1.5. (11 points) If possible, diagaonalize the matrix $\begin{bmatrix} -1 & 2 & 2 \\ -2 & 3 & 2 \\ -1 & 0 & 4 \end{bmatrix}$.

Problem 1.6. (9 points) Find the least squares solution(s) of each of the following linear systems:

(a) $$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(b) $$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \end{bmatrix}$$

(c) $$\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Problem 2. (10 points) Evaluate the truth of each statement below. If the statement is true write T in the box preceding the statement. Otherwise, write F.

(a) $\boxed{}$ The determinant of a projection matrix must be zero.

(b) $\boxed{}$ For a 5×5 matrix, the row space is unequal to the null space.

(c) $\boxed{}$ If $\mathbf{x}$ and $\mathbf{y}$ are orthogonal vectors in $\mathbb{R}^n$, then

$$||\mathbf{x} + \mathbf{y}||^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2.$$

(d) $\boxed{}$ A real $n \times n$ matrix must have n eigenvalues each with a different value.

(e) $\boxed{}$ The Gram matrix of a real matrix must be symmetric.

Problem 3. (10 points) Show the output of the last MATLAB command below.

```
>> A = [1 2 3; 4 5 6; 7 8 9];
>> [F, pivots] = rref(A)    % rref and pivot list for A
F =
       1   0   -1
       0   1    2
       0   0    0
pivots =
       1   2
>> r = rank(A);             % rank of A
>> SS = A(:,pivots)         % pivot columns of A
SS =
       1   2
       4   5
       7   8
>> FF=F(1:r,:)              % nonzero rows of rref(A)
FF =
       1   0   -1
       0   1    2
>> SS*FF
ans =
```

Problem 4. (15 points) The vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

are eigenvectors of the matrix $A = \begin{bmatrix} -1 & -4 & -2 & 6 \\ 1 & 0 & -2 & 0 \\ -2 & -4 & -1 & 6 \\ 1 & -1 & -2 & 1 \end{bmatrix}$.

(a) Find the eigenvalues for each of the eigenvectors above.

(b) Diagonalize A. That is, find an invertible matrix S and a diagonal matrix D such that $A = SDS^{-1}$; evaluating S^{-1} is not required.

(c) Diagonalize A^3. That is, find an invertible matrix $\tilde{S}$ and a diagonal matrix $\tilde{D}$ such that $A^3 = \tilde{S}\tilde{D}\tilde{S}^{-1}$; evaluating $\tilde{S}^{-1}$ is not required.

Problem 5. (10 points) Let $\mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ with $\theta \in \mathbb{R}$ and define

$$A = \mathbf{a}\mathbf{a}^T, \qquad B = \mathbf{b}\mathbf{b}^T, \qquad U = I - 2A, \quad \text{and} \quad V = I - 2B.$$

(a) Show that A and B are orthogonal projection matrices.

(b) Show that U and V are orthogonal matrices.

Problem 5 continues on the next page.

(c) Find ϕ such that $UV = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$ or explain why this is impossible.

Extra Credit. (5 points) Find the point on the plane $z = 2x + 3y$ closest to the point (4,5,6).

Extra Extra Credit. (5 points) Find the orthogonal projection matrix P that projects onto col(A) where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \end{bmatrix}.$$