15 Lecture 16: Four Fundamental Subspaces

15.1 Dimension of a subspace in $\mathbb{R}^n$:

The dimension of a subspace$V$ in $\mathbb{R}^n$, denoted by $\dim(V)$, is the number of vectors in its basis.

Example 15.1.

- $\dim(\mathbb{R}^2) = 2$, *due to* $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ *is a basis of* $\mathbb{R}^2$.
- $\dim(\mathbb{R}^n) = n$, *due to* $\{\mathbf{e}_1, \mathbf{e}_2, \dots \mathbf{e}_n\}$ *is a basis of* $\mathbb{R}^n$.
- $\dim(\{0\}) = 0$, *because there is no vector in its basis.*

15.2 Column space

Let $\mathbf{A}$ be an $m \times n$ matrix. The **column space** of $\mathbf{A}$, denoted by $\mathrm{Col}(\mathbf{A})$, is a set of all linear combination of column of $\mathbf{A}$.

Theorem 5. *The column space of an $m \times n$ matrix $\mathbf{A}$ is a subspace of $\mathbb{R}^m$. The set of pivot columns of $\mathbf{A}$ form a basis of $\mathrm{Col}(\mathbf{A})$ and $\dim(\mathrm{Col}(\mathbf{A})) = rank(\mathbf{A})$.*

Example 15.2. *Find a basis of* $\mathrm{Col}(\mathbf{A})$ *with* $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Solution: The basis of $\mathbf{A}$ is the set of pivot columns of $\mathbf{A}$. We first row reduce matrix $\mathbf{A}$:

$$\mathbf{A} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

From the $nref(\mathbf{A})$, we see that the first and the second column of $\mathbf{A}$ are pivot. Thus, the basis of $\mathrm{Col}(\mathbf{A})$ is $\left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \right\}$.

15.3 Null space:

The null space of an $m \times n$ matrix $\mathbf{A}$,denoted by $\mathrm{Nul}(\mathbf{A})$, is the solution set of the homogeneous system $\mathbf{Ax} = \mathbf{0}$. The dimension of the null space of $\mathbf{A}$ is called the **nullity** of $\mathbf{A}$.

Theorem 6. *The column space of an $m \times n$ matrix $\mathbf{A}$ is a subspace of $\mathbb{R}^m$.*

Theorem 7. *If $\mathbf{A}$ is an $m \times n$ matrix, then*

$$\dim(\mathrm{Col}(\mathbf{A})) + \dim(\mathrm{Nul}(\mathbf{A})) = n.$$

Note that, the nullity of $\mathbf{A}$ equals the number of free variable of the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$. Thus, the nullity of $\mathbf{A}$ can be determine using the proto-row-echelon form of $\mathbf{A}$. However, to find the basis of $\text{Nul}(\mathbf{A})$, we must solve the linear system $\mathbf{Ax} = \mathbf{0}$, and then describe the solution in the parametric vector form.

Example 15.3. *Find a basis of* $\text{Nul}(\mathbf{A})$*, with* $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

Solution: We first solve the homogeneous system $\mathbf{Ax} = \mathbf{0}$. Row reducing matrix $\mathbf{A}$ gives

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, x_3 is the free variable. The solution is

$$x_1 = x_3, \quad x_2 = -2x_3, \quad x_3 \text{ is free.}$$

Under the parametric vector form, the solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

The basis of $\text{Nul}(\mathbf{A})$ is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$ and the nullity is 1.

Example 15.4. *Find a basis of* $\text{Nul}(\mathbf{A})$*, with* $\mathbf{A} = \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \\ -3 & 6 & 1 & 1 & -7 \end{bmatrix}$.

Solution: We first solve the linear system $\mathbf{Ax} = \mathbf{0}$. Row reducing the matrix $\begin{bmatrix} \mathbf{A} & \mathbf{0} \end{bmatrix}$ we found

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}, \quad \begin{cases} x_1 \quad -2x_2 \qquad\qquad +2x_5 = 0 \\ \qquad\qquad x_3 \qquad\qquad\quad = 0 \\ \qquad\qquad\qquad +x_4 \quad -x_5 = 0 \end{cases}.$$

The solution is $x_1 = 2x_2 - 2x_5, \quad x_3 = 0, \quad x_4 = x_5$, x_2 and x_5 are free. The parametric vector form of the solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 - 2x_5 \\ x_2 \\ 0 \\ x_5 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Thus, as basis of $\text{Nul}(\mathbf{A})$ is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

15.4 Row space

The row space of a real $m \times n$ matrix $\mathbf{A}$, denoted by $\text{Row}(\mathbf{A})$ is the column space of $\mathbf{A}^T$. That is, the row space is the set of all linear combinations of the rows of $\mathbf{A}$ (transposed to become columns of $\mathbf{A}^T$).

Note: $rank(\mathbf{A}^T) = rank(\mathbf{A})$. Equivalently, $\dim(\text{Col}(\mathbf{A})) = \dim(\text{Row}(\mathbf{A}))$.

Theorem 8. *If two matrices $\mathbf{A}$ and $\mathbf{B}$ are row equivalent, then their row spaces are the same. If $\mathbf{B}$ is in echelon form, the nonzero rows of $\mathbf{B}$ form a basis for the row space of $\mathbf{A}$ as well as for that of $\mathbf{B}$.*

Remark 15.1. *The row space of an $m \times n$ matrix $\mathbf{A}$ is a subspace of $\mathbb{R}^n$. The set of nonzero row of the $pref(\mathbf{A})$ forms a basis of $\text{Row}(\mathbf{A})$. Note that, the set of nonzero row of the $nref$ or $rref(\mathbf{A})$ also forms a basis of $\text{Row}(\mathbf{A})$*

Example 15.5. *Find a basis of $\text{Row}(\mathbf{A})$ with* $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Solution: We have that

$$\mathbf{A} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

So, a basis of $\text{Row}(\mathbf{A})$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$.

15.5 Left null space

The left null space of the real $m \times n$ matrix $\mathbf{A}$ written is the null space of $\mathbf{A}^t$, written $\mathrm{Nul}(\mathbf{A}^T)$. This is a subspace of $\mathbb{R}^m$.

Method to find the basis of $\mathrm{Nul}(\mathbf{A}^T)$**:**

- **Standard method (follow the definition)**: Solve the homogeneous system $\mathbf{A}^T\mathbf{x} = \mathbf{0}$ and to similarly as finding basis of $\mathrm{Nul}(\mathbf{A})$.
- **An alternate method:** Row reduce the matrix $\begin{bmatrix}\mathbf{A} & \mathbf{I}\end{bmatrix}$ to the matrix of the form $\begin{bmatrix}\mathrm{rref}(\mathbf{A}) & \mathbf{B}\end{bmatrix}$. A basis of $\mathrm{Nul}(\mathbf{A}^T)$ consists of $m - rank(\mathbf{A})$ last columns of $\mathbf{B}^T$ (i.e., $m - rank(\mathbf{A})$ last row of $\mathbf{B}$)

Example 15.6. *Find a basis of* $\mathrm{Nul}(\mathbf{A}^T)$, *with* $\mathbf{A} = \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}$

Solution: We row reduce the matrix $\begin{bmatrix}\mathbf{A} & \mathbf{I}\end{bmatrix}$ and obtain

$$\begin{bmatrix}\mathbf{A} & \mathbf{I}\end{bmatrix} = \begin{bmatrix}1 & 2 & 3 & 1 & 0 & 0\\4 & 5 & 6 & 0 & 1 & 0\\7 & 8 & 9 & 0 & 0 & 1\end{bmatrix} \sim \dots \sim \begin{bmatrix}1 & 0 & -1 & 0 & -8/3 & 5/3\\0 & 1 & 2 & 0 & 7/3 & -4/3\\0 & 0 & 0 & 1 & -2 & 1\end{bmatrix}$$

$rank(\mathbf{A}) = 2$, so the last row of the matrix $\mathbf{B} := \begin{bmatrix}0 & -8/3 & 5/3\\0 & 7/3 & -4/3\\1 & -2 & 1\end{bmatrix}$ forms the basis of $\mathrm{Nul}(\mathbf{A}^T)$.

Thus the basis is $\left\{\begin{bmatrix}1\\-2\\1\end{bmatrix}\right\}$

Example 15.7. *Find the basis of four subspace* $\mathrm{Col}(\mathbf{A})$, $\mathrm{Nul}(\mathbf{A})$, $\mathrm{Row}(\mathbf{A})$, $\mathrm{Nul}(\mathbf{A}^T)$ *for the matrix* $\mathbf{A} = \begin{bmatrix}1 & 2 & 3 & 4\\5 & 6 & 7 & 8\\9 & 10 & 11 & 12\end{bmatrix}$

Solution:
We start by row reducing matrix $\begin{bmatrix}\mathbf{A} & \mathbf{I}_3\end{bmatrix}$

$$\begin{bmatrix}\mathbf{A} & \mathbf{I}_3\end{bmatrix} = \begin{bmatrix}1 & 2 & 3 & 4 & 1 & 0 & 0\\5 & 6 & 7 & 8 & 0 & 1 & 0\\9 & 10 & 11 & 12 & 0 & 0 & 1\end{bmatrix} \sim \dots \sim \begin{bmatrix}1 & 0 & -1 & -2 & 0 & -5/2 & 3/2\\0 & 1 & 2 & 3 & 0 & 9/4 & -5/4\\0 & 0 & 0 & 0 & 1 & -2 & 1\end{bmatrix}.$$

We see that $\text{rref}(\mathbf{A}) = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, which gives:

- Columns 1 and 2 of $\mathbf{A}$ are pivots, so, basis for $\text{Col}(\mathbf{A})$ is $\left\{ \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 10 \end{bmatrix} \right\}$ and $rank(\mathbf{A}) = 2$.

- The solution to the system $\mathbf{Ax} = \mathbf{0}$ is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 + 2x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$. Thus the basis of $\text{Nul}(\mathbf{A})$ is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

- Basis of $\text{Row}(\mathbf{A})$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ (nonzero rows of $\text{rref}(\mathbf{A})$).

- Basis for $\text{Nul}(\mathbf{A}^T)$ is the last $m - rank(\mathbf{A}) = 3 - 2 = 1$ last row of the matrix $\mathbf{B} = \begin{bmatrix} 0 & -5/2 & 3/2 \\ 0 & 9/4 & -5/4 \\ 1 & -2 & 1 \end{bmatrix}$. Thus, the basis of $\text{Nul}(\mathbf{A}^T)$ is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Example 15.8. *Find the basis of four subspace* $\text{Col}(\mathbf{A})$, $\text{Nul}(\mathbf{A})$, $\text{Row}(\mathbf{A})$, $\text{Nul}(\mathbf{A}^T)$ *for the matrix* $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 5 & 6 \\ 9 & 10 \end{bmatrix}$

Solution: We row reduce matrix $\begin{bmatrix} \mathbf{A} & \mathbf{I}_3 \end{bmatrix}$

$$\begin{bmatrix} \mathbf{A} & \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 5 & 6 & 0 & 1 & 0 \\ 9 & 10 & 0 & 0 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & -5/2 & 3/2 \\ 0 & 1 & 0 & 9/4 & -5/4 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}.$$

We see that $\text{rref}(\mathbf{A}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, which gives:

- Both columns of $\mathbf{A}$ are pivots, so, basis for $\text{Col}(\mathbf{A})$ is $\left\{ \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 10 \end{bmatrix} \right\}$ and $rank(\mathbf{A}) = 2$.
- The nullity of $\mathbf{A}$ is zeros, so $\text{Nul}(\mathbf{A}) = \{\mathbf{0}\}$. The basis of $\mathbf{A}$ is an empty set.
- Basis of $\text{Row}(\mathbf{A})$ is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ (nonzero rows of $\text{rref}(\mathbf{A})$).
- Basis for $\text{Nul}(\mathbf{A}^T)$ is the last $m - rank(\mathbf{A}) = 3 - 2 = 1$ last row of the matrix $\mathbf{B} = \begin{bmatrix} 0 & -5/2 & 3/2 \\ 0 & 9/4 & -5/4 \\ 1 & -2 & 1 \end{bmatrix}$. Thus, the basis of $\text{Nul}(\mathbf{A}^T)$ is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

The relation between these four subspace

Table of Fundamental Subspaces for an $m \times n$ Matrix A with $r = \text{rank}(A)$

Name	Notation	Standard Basis	Dimension
column space	$\text{col}(A)$	pivot columns of A	r
null space	$\text{nul}(A)$	basic solutions of $A\mathbf{x} = \mathbf{0}$	$n - r$
row space	$\text{row}(A)$	nonzero rows of $\text{rref}(A)$	r
left null space	$\text{nul}(A^T)$	last $m - r$ rows of W $\text{rref}\big([A\ I]\big) = [\text{rref}(A)\ W]$	$m - r$

Please see the lecture note on Canvas for more references