**Problem 1.1.** (8 points) Showing your work, compute the determinants below:

(a) $\begin{vmatrix} 0 & 0 & 1 \\ 1 & 4 & 5 \\ 1 & 3 & 6 \end{vmatrix} = +\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = 3 - 4 = \boxed{-1}$

After Laplace expansion by the first row, the determinant is easily computed as the determinante of a $2 \times 2$ matrix.

(b) $\begin{vmatrix} 3 & 8 & -1 & -2 & -5 \\ 0 & 0 & 1 & -7 & -8 \\ 0 & 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 2 & -3 & -4 & -6 \end{vmatrix} =$

$$3\begin{vmatrix} 0 & 1 & -7 & -8 \\ 0 & 0 & 7 & 1 \\ 0 & 0 & 4 & 0 \\ 2 & -3 & -4 & -6 \end{vmatrix} = -2(3)\begin{vmatrix} 1 & -7 & -8 \\ 0 & 7 & 1 \\ 0 & 4 & 0 \end{vmatrix} = -6\begin{vmatrix} 7 & 1 \\ 4 & 0 \end{vmatrix} = (-6)(-4) = \boxed{24}$$

Laplace expansion by the first column, followed by Laplace exapansion by the (new) first column, followed by expansion by the (next new) first column, followed by computation of the determinant of the remaining $2 \times 2$ matrix.

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For a broader range of practice problems with determinants consult the guide for Common Quiz 7.

**Problem 1.2.** (10 points) Showing your work, diagonalize the matrix $A$ (if possible):

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 2 \end{bmatrix}$$

$$p_A(\lambda) = \lambda^2 - 3\lambda + 6 \qquad \lambda_\pm = \frac{3 \pm \sqrt{9 - 4(6)}}{2} = \frac{3 \pm i\sqrt{15}}{2}$$

$$\lambda_\pm = \frac{3 \pm i\sqrt{15}}{2}: \qquad A - \lambda_\pm I = \begin{bmatrix} 1 - \lambda_\pm & -2 \\ 2 & 2 - \lambda_\pm \end{bmatrix} \sim \begin{bmatrix} 1 - \lambda_\pm & -2 \\ 0 & 0 \end{bmatrix} \qquad \Rightarrow \qquad \mathbf{v}_\pm = \begin{bmatrix} 2 \\ 1 - \lambda_\pm \end{bmatrix}$$

$$S = [\,\mathbf{v}_+ \quad \mathbf{v}_-\,] = \begin{bmatrix} 2 & 2 \\ 1 - \lambda_+ & 1 - \lambda_- \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 2 \\ \frac{-1-i\sqrt{15}}{2} & \frac{-1+i\sqrt{15}}{2} \end{bmatrix}} \quad \text{or} \quad S = \boxed{\begin{bmatrix} 4 & 4 \\ -1 - i\sqrt{15} & -1 + i\sqrt{15} \end{bmatrix}}$$

$$D = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix} = \boxed{\begin{bmatrix} \frac{3+i\sqrt{15}}{2} & 0 \\ 0 & \frac{3-i\sqrt{15}}{2} \end{bmatrix}}$$

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For a broader range of practice problems with diagonalization of $2 \times 2$ matrices consult the guide for Common Quiz 8 as well as Daily Diag problems.

**Problem 1.3.** (9 points) Showing your work, determine if each of the the sets of vectors below is linearly dependent or linearly independent:

(a) $\left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix} \right\}$

$$\begin{array}{c} \bullet \\ 2 \\ 3 \end{array}\begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix} \sim \begin{array}{c} \\ \bullet \\ 2 \end{array}\begin{bmatrix}1&4&7\\0&-3&-6\\0&-6&-12\end{bmatrix} \sim \begin{bmatrix}1&4&7\\0&-3&-6\\0&0&0\end{bmatrix} \qquad \boxed{\text{linearly dependent}}$$

(b) $\left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix} \right\}$

Set contains the zero vector; hence, it is $\boxed{\text{linearly dependent}}$.

(c) $\left\{ \begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\\1\end{bmatrix} \right\}$

$$\begin{vmatrix}1&0&0&0&1\\1&1&0&0&0\\0&1&1&0&0\\0&0&1&1&0\\0&0&0&1&1\end{vmatrix} = \begin{vmatrix}1&0&0&0\\1&1&0&0\\0&1&1&0\\0&0&1&1\end{vmatrix} + \begin{vmatrix}1&1&0&0\\0&1&1&0\\0&0&1&1\\0&0&0&1\end{vmatrix} = 1+1=2\neq 0$$

Since the determinant of the matrix in nonzero, the columns must be $\boxed{\text{linearly independent}}$.

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For a broader range of practice problems for determining the independents of a set of vectors consult the guide for Common Quiz 9.

**Problem 1.4.** (8 points) Give a basis for each of the four fundamental subspaces of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}.$$

$$[\,A \quad | \quad I\,] = \begin{array}{c} \bullet \\ {\scriptstyle 1} \\ {\scriptstyle 1} \end{array} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 1 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 0 & 1 & 0 \\ 1 & 2 & 3 & 4 & 5 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} = [\,\mathrm{rref}(A) \quad | \quad W\,]$$

$$\text{basis for row}(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \right\} \qquad \text{basis for col}(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\text{basis for nul}(A) = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad \text{basis for nul}(A^T) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Some easy checks:

(1) $\mathrm{rank}(A) = 1$ vectors in bases for $\mathrm{row}(A)$ and $\mathrm{col}(A)$.
(2) row space orthogonal to null space

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0, \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0, \qquad \ldots$$

(3) column space orthogonal to left null space

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = 0, \qquad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = 0$$

---

For a broader range of practice problems for finding bases for the four fundamental subspaces of a matrix consult the guide for Common Quiz 10.

**Problem 1.5.** (11 points) If possible, diagaonalize the matrix $\begin{bmatrix} -1 & 2 & 2 \\ -2 & 3 & 2 \\ -1 & 0 & 4 \end{bmatrix}$.

$$\begin{vmatrix} -1 & 2 & 2 \\ -2 & 3 & 2 \\ -1 & 0 & 4 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 2 \\ 0 & -1 & -2 \\ 0 & -2 & 2 \end{vmatrix} = -\begin{vmatrix} -1 & -2 \\ -2 & 2 \end{vmatrix} = -(-2-4) = 6$$

$$\begin{aligned} p_A(\lambda) &= -\lambda^3 + (-1+3+4)\lambda^2 - \left( \begin{vmatrix} 3 & 2 \\ 0 & 4 \end{vmatrix} + \begin{vmatrix} -1 & 2 \\ -1 & 4 \end{vmatrix} + \begin{vmatrix} -1 & 2 \\ -2 & 3 \end{vmatrix} \right) \lambda + 6 \\ &= -\lambda^3 + 6\lambda^2 - (12-2+1)\lambda + 6 \\ &= -\lambda^3 + 6\lambda^2 - 11\lambda + 6 \end{aligned}$$

$$\begin{array}{l|rrrr} \underline{1} & -1 & 6 & -11 & 6 \\ & & -1 & 5 & -6 \\ \hline & -1 & 5 & -6 & 0 \end{array}$$

$$p_A(\lambda) = (\lambda-1)(-\lambda^2+5\lambda-6) = -(\lambda-1)(\lambda^2-5\lambda+6) = -(\lambda-1)(\lambda-2)(\lambda-3)$$

$\underline{\lambda_1 = 1}$ : $$A - \lambda_1 I = \begin{bmatrix} -2 & 2 & 2 \\ -2 & 2 & 2 \\ -1 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$\underline{\lambda_2 = 2}$ : $$A - \lambda_2 I = \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$\underline{\lambda_3 = 3}$ : $$A - \lambda_3 = \begin{bmatrix} -4 & 2 & 2 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ -4 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$S = \boxed{\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}} \quad \text{and} \quad D = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}}$$

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For a broader range of practice problems with diagonalization of $3 \times 3$ matrices consult the guide for Common Quiz 11 as well as Daily Diag problems.

**Problem 1.6.** (9 points) Find the least squares solution(s) of each of the following linear systems:

(a) $$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Premultiply by the coefficient matrix to obtain the normal equations:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 & 0 & 5 \\ 0 & 4 & 0 \\ 5 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 & 5 & 2 \\ 0 & 4 & 0 & 0 \\ 5 & 0 & 5 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2/5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boxed{\begin{bmatrix} 2/5 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$$

---

(b) $$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \end{bmatrix}$$ Augmented matrix: $\begin{bmatrix} 1 & 1 & 1 & 1 & 4 \end{bmatrix}$.

Since solutions exist, least squares solutions equal ordinary solutions.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \boxed{\begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}$$

---

(c) $$\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Premultiplying the augmented matrix by the transpose of the coefficient matrix gives the augmented matrix of the normal equations:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 5 & 8 \\ 5 & 6 & 7 \end{bmatrix} \quad \Rightarrow$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 7 \end{bmatrix} = \frac{1}{36-25} \begin{bmatrix} 6 & -5 \\ -5 & 6 \end{bmatrix} \begin{bmatrix} 8 \\ 7 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 48-35 \\ -40+42 \end{bmatrix} = \boxed{\begin{bmatrix} 13/11 \\ 2/11 \end{bmatrix}}$$

**Problem 2.** (10 points) Evaluate the truth of each statement below. If the statement is true write $T$ in the box preceding the statement. Otherwise, write $F$.

(a) $\boxed{F}$ The determinant of a projection matrix must be zero.

The identity matrix is a projection matrix ($I^2 = I$) having a determinant of 1.

(b) $\boxed{T}$ For a $5 \times 5$ matrix, the row space is unequal to the null space.

The dimension of the row space plus the dimension of the null space must be 5, but no two equal integers have a sum of 5. Thus, the row space and the null space have different dimensions and are thus clearly unequal.

(c) $\boxed{T}$ If $\mathbf{x}$ and $\mathbf{y}$ are orthogonal vectors in $\mathbb{R}^n$, then

$$||\mathbf{x} + \mathbf{y}||^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2.$$

$$||\mathbf{x} + \mathbf{y}||^2 = (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) = \mathbf{x} \cdot \mathbf{x} + \mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} = ||\mathbf{x}||^2 + ||\mathbf{y}||^2$$

(d) $\boxed{F}$ A real $n \times n$ matrix must have $n$ eigenvalues each with a different value.

All the eigenvalues of the $n \times n$ identity matrix have the same value, which is 1.

(e) $\boxed{T}$ The Gram matrix of a real matrix must be symmetric.

$$(A^T A)^T = A^T \left(A^T\right)^T = A^T A$$

**Supplemental Practice for Problem 2.**

(a) $\boxed{F}$ Every projection matrix is invertible.

For example, a square zero matrix is a projection matrix but hardly invertible.

(b) $\boxed{F}$ Every matrix with orthogonal columns is an orthogonal matrix.

Orthogonal columns do not necessarily make a matrix square which an othogonal matrix must be.

(c) $\boxed{F}$ If $A^TA = I$ then $A$ is orthogonal.

Notice $\begin{bmatrix} 1&0&0\\0&1&0 \end{bmatrix}\begin{bmatrix} 1&0\\0&1\\0&0 \end{bmatrix} = \begin{bmatrix} 1&0\\0&1 \end{bmatrix}$. But alas, $\begin{bmatrix} 1&0\\0&1\\0&0 \end{bmatrix}$ is not an orthogonal matrix.

(d) $\boxed{T}$ If $A$ is square and $A^TA = I$ then $A$ is orthogonal.

(e) $\boxed{F}$ Every invertible matrix is orthogonal.

(f) $\boxed{T}$ If $U$ is an $n \times n$ orthogonal matrix then, for $\mathbf{x} \in \mathbb{R}^n$, $|\mathbf{x} \cdot U\mathbf{x}| \leq ||\mathbf{x}||^2$.

$|\mathbf{x} \cdot U\mathbf{x}| \leq ||\mathbf{x}||\,||U\mathbf{x}|| = ||\mathbf{x}||\,||\mathbf{x}|| = ||\mathbf{x}||^2$ as $||U\mathbf{x}|| = ||\mathbf{x}||$ since $U$ is orthogonal.

(g) $\boxed{T}$ If the sets of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ are linearly independent and

$$\mathrm{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_n\} = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$$

then $m = n$.

All linearly independent spanning sets (bases) of a subspace must have the same number of elements (the dimension of the subspace).

(h) $\boxed{T}$ If $A$ and $B$ have the same characteristic polynomials then $\sigma(A) = \sigma(B)$.

(i) $\boxed{F}$ If $\sigma(A) = \sigma(B)$ then $p_A(\lambda) = p_B(\lambda)$.

Conside $A = \begin{bmatrix} 1&0&0\\0&2&0\\0&0&2 \end{bmatrix}$ and $B = \begin{bmatrix} 1&0&0\\0&1&0\\0&0&2 \end{bmatrix}$ – an issue of multiplicity.

(j) $\boxed{T}$ A real $3 \times 3$ matrix must have a real eigenvalue.

A real polynomial of odd degree must have a real root.

(k) $\boxed{F}$ A real $2 \times 2$ matrix must have a real eigenvalue.

A real polynomial of even degree may not have any real roots, e.g., $p(\lambda) = \lambda^2 + 1$.

(l) $\boxed{F}$ The set of $n \times n$ elementary matrices is closed under multiplication.

For example, $\begin{bmatrix} 1&0\\1&1 \end{bmatrix}\begin{bmatrix} 1&1\\0&1 \end{bmatrix} = \begin{bmatrix} 1&1\\1&2 \end{bmatrix}$

(m) $\boxed{T}$ The set of $n \times n$ elementary matrices is closed under matrix inversion.

The inverse of an elementary matrix is indeed an elementary matrix.

**Problem 3.** (10 points) Show the output of the last MATLAB command below.

```
>>  A = [1 2 3; 4 5 6; 7 8 9];
>>  [F, pivots] = rref(A)       % rref and pivot list for A
F =
       1   0   -1
       0   1    2
       0   0    0
pivots =
       1   2
>>  r = rank(A);                % rank of A
>>  SS = A(:,pivots)            % pivot columns of A
SS =
       1   2
       4   5
       7   8
>>  FF=F(1:r,:)                 % nonzero rows of rref(A)
FF =
       1   0   -1
       0   1    2
>>  SS*FF
ans =
```

```
     1     2     3
     4     5     6
     7     8     9
```

The matrix `SS` is the pivot column matrix of `A` (that is, $C$ of the CR factorization) and the matrix `FF` consists of the nonzero rows of rref($A$) (that is, $R$ of the CR factorizaton). Hence `SS*FF` is $CR = A$.

**Supplemental Practice for Problem 3.** What is the result of the MATLAB commands below. Explain the basis of your solution.

```
>> A = [1 2 3 4; 5 6 7 8; 9 10 11 12; 13 14 15 16];
>> rref(A)

ans =


     1     0    -1    -2
     0     1     2     3
     0     0     0     0
     0     0     0     0


>> S=[1 1 0 0; 0 1 1 0; 0 0 1 1; 0 0 0 1];
>> det(S)

ans =

   1

>> rref(S*A)


ans =


     1     0    -1    -2
     0     1     2     3
     0     0     0     0
     0     0     0     0
```

Because $S$ is nonsingular, $A$ and $SA$ share the same reduced row-echelon form.

**Problem 4.** (15 points) The vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

are eigenvectors of the matrix $A = \begin{bmatrix} -1 & -4 & -2 & 6 \\ 1 & 0 & -2 & 0 \\ -2 & -4 & -1 & 6 \\ 1 & -1 & -2 & 1 \end{bmatrix}$.

(a) Find the eigenvalues for each of the eigenvectors above.

$$A \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2+0-2+6 \\ * \\ * \\ * \end{bmatrix} = \begin{bmatrix} 2 \\ * \\ * \\ * \end{bmatrix} \Rightarrow \lambda_1 = \boxed{1} \qquad A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1+4-2+0 \\ * \\ * \\ * \end{bmatrix} = \begin{bmatrix} 1 \\ * \\ * \\ * \end{bmatrix} \Rightarrow \lambda_2 = \boxed{1}$$

$$A \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2-4-4+6 \\ * \\ * \\ * \end{bmatrix} = \begin{bmatrix} -4 \\ * \\ * \\ * \end{bmatrix} \Rightarrow \lambda_3 = \boxed{-2} \qquad A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1-4-2+6 \\ * \\ * \\ * \end{bmatrix} = \begin{bmatrix} -1 \\ * \\ * \\ * \end{bmatrix} \Rightarrow \lambda_4 = \boxed{-1}$$

(b) Diagonalize $A$. That is, find an invertible matrix $S$ and a diagonal matrix $D$ such that $A = SDS^{-1}$; evaluating $S^{-1}$ is not required.

$$S = \boxed{\begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & -1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}} \quad \text{and} \quad D = \boxed{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}$$

(c) Diagonalize $A^3$. That is, find an invertible matrix $\tilde{S}$ and a diagonal matrix $\tilde{D}$ such that $A^3 = \tilde{S}\tilde{D}\tilde{S}^{-1}$; evaluating $\tilde{S}^{-1}$ is not required.

$$\tilde{S} = \boxed{\begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & -1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}} \quad \text{and} \quad \tilde{D} = \boxed{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}$$

**Supplemental Practice for Problem 4.**

(a) Find the $3 \times 3$ matrix $P$ such that $P\mathbf{x}$ is the orthogonal projection of $\mathbf{x}$ into

$$V = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\}.$$

$$P = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix} \left( \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 5 & 13 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 13 & -5 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & -2 & 0 \\ -1 & 1 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

(b) Explain why the vectors above whose span defines $V$ are linearly independent eigenvectors of $P$. What are the eigenvalues?

The two vectors are independent as neither is a multiple of the other. Since $P$ projects into $V$ if $\mathbf{x} \in V$ then $P\mathbf{x} = \mathbf{x}$, i.e., the nonzero vectors in $V$ are eigenvectors of $P$ with an eigenvalue of 1.

(c) Find an additional linearly independent eigenvector of $P$. What is the eigenvalue?

The orthogonal complement of the column space of $P$, the null space, is the $z$-axis. Hence $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is a third independent eigenvector. The eigenvalue is 0 since $\mathbf{v}_3$ is in the null space of $P$.

**More Supplemental Practice for Problem 4.** (15 points) Diagonalize each matrix, if possible.

(a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ $\boxed{\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3}$

since a triangular matrix displays its eigenvalues on its diagonal.

$\underline{\lambda_1 = 1}:$ $A - \lambda_1 I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ $\mathbf{v}_1 = \boxed{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$ (A column of zeros is a give away.)

$\underline{\lambda_2 = 2}:$ $A - \lambda_2 I = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\mathbf{v}_2 = \boxed{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}$

$\underline{\lambda_3 = 3}:$ $A - \lambda_3 I = \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -2 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $\mathbf{v}_3 = \boxed{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$

$A = SDS^{-1}$ where $S = \boxed{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}$ and $D = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}}$

---

(b) $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ $\boxed{\lambda_1 = \lambda_2 = \lambda_3 = 1}$

since a triangular matrix displays its eigenvalues on its diagonal.

$\underline{\lambda_1 = \lambda_2 = \lambda_3 = 1}:$ $A - \lambda_1 I = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\mathbf{v}_1 = \boxed{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$

No additional independent eigenvectors can be found as the nullity of $A - \lambda_1 I$ is only 1.

$\boxed{\text{There is no diagonalization.}}$

(c) $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix}$$

Using the structure of this matrix $I - 2\mathbf{x}\mathbf{x}^T$ is very helpful. Note that adding the identity to any matrix leaves the eigenvectors unchanged but increases the eigenvalues by 1. Hence, we first seek the eigenvalues and eigenvectors of the rank 1 matrix $-2\mathbf{x}\mathbf{x}^T$. We observe that multiplying this matrix by any vector $\mathbf{v}$ gives $-2\mathbf{x}\mathbf{x}^T\mathbf{v} = -2(\mathbf{v}\cdot\mathbf{x})\mathbf{x}$. Hence, we see that $\mathbf{x}$ is an eigenvector with eigenvalues -2. And any vector orthogonal to $\mathbf{x}$ is an eigenvector with eigenvalue 0. The basis for the space of vectors orthogonal to $\mathbf{x}$ is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Adding the identity, we have

$\underline{\lambda_1 = -2 + 1 = -1}$ : $$\boxed{\mathbf{v}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}}$$

$\underline{\lambda_2 = \lambda_3 = \lambda_4 = 0 + 1 = 1}$ : $$\boxed{\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}$$

Hence,

$$S = \boxed{\begin{bmatrix} 1/2 & -1 & -1 & -1 \\ 1/2 & 1 & 0 & 0 \\ 1/2 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 1 \end{bmatrix}} \quad \text{and} \quad D = \boxed{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

**Problem 5.** (10 points) Let $\mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ with $\theta \in \mathbb{R}$ and define

$$A = \mathbf{a}\mathbf{a}^T, \qquad B = \mathbf{b}\mathbf{b}^T, \qquad U = I - 2A, \quad \text{and} \quad V = I - 2B.$$

(a) Show that $A$ and $B$ are orthogonal projection matrices.

$$A^2 = (\mathbf{a}\mathbf{a}^T)(\mathbf{a}\mathbf{a}^T) = \mathbf{a}(\mathbf{a}^T\mathbf{a})\mathbf{a}^T = \mathbf{a}\mathbf{a}^T = A \qquad \text{and} \qquad A^T = \left(\mathbf{a}\mathbf{a}^T\right)^T = \mathbf{a}\mathbf{a}^T = A$$

$$B^2 = (\mathbf{b}\mathbf{b}^T)(\mathbf{b}\mathbf{b}^T) = \mathbf{b}(\mathbf{b}^T\mathbf{b})\mathbf{b}^T = \mathbf{b}\mathbf{b}^T = B \qquad \text{and} \qquad B^T = \left(\mathbf{b}\mathbf{b}^T\right)^T = \mathbf{b}\mathbf{b}^T = B$$

since $\mathbf{a}^T\mathbf{a} = \mathbf{b}^T\mathbf{b} = 1$.

(b) Show that $U$ and $V$ are orthogonal matrices.

$U$ and $V$ are $2 \times 2$ matrices,

$$U^TU = (I-2A)^T(I-2A) = (I-2A)^2 = I - 2A - 2A + 4A^2 = I - 4A + 4A = I,$$

and

$$V^TV = (I-2B)^T(I-2B) = (I-2B)^2 = I - 2B - 2B + 4B^2 = I - 4B + 4B = I.$$

**Problem 5 continues on the next page.**

(c) Find $\phi$ such that $UV = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$ or explain why this is impossible.

$$U = I - 2A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$V = I - 2B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$
$$= \begin{bmatrix} 1 - 2\cos^2\theta & -2\cos\theta\sin\theta \\ -2\cos\theta\sin\theta & 1 - 2\sin^2\theta \end{bmatrix} = \begin{bmatrix} \sin^2\theta - \cos^2\theta & -2\cos\theta\sin\theta \\ -2\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$$
$$= \begin{bmatrix} -\cos(2\theta) & -\sin(2\theta) \\ -\sin(2\theta) & \cos(2\theta) \end{bmatrix}$$

Hence,

$$UV = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -\cos(2\theta) & -\sin(2\theta) \\ -\sin(2\theta) & \cos(2\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ -\sin(2\theta) & \cos(2\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \quad \text{where} \quad \phi = \boxed{-2\theta}$$

**Supplemental Practice for Problem 5.** Suppose $P$ is an $n \times n$ orthogonal projection matrix with $\text{rank}(P) = r$ with $r < n$.

(a) Show that $U = I - 2P$ is a orthogonal matrix.

$$U^T U = (I - 2P)(I - 2P) = I - 2P - 2P + 4P^2 = I - 4P + 4P = I$$

(b) What are the eigenvalues of $U$?

For a vector $\mathbf{x}$ in the $r$-dimensional column space of $P$, $U\mathbf{x} = (I - 2P)\mathbf{x} = \mathbf{x} - 2\mathbf{x} = -\mathbf{x}$. That is, a basis for $\text{col}(P)$ is a linearly independent collection of $r$ eigenvectors of $U$ with eigenvalue of $-1$. A basis for the nontrivail null space of $P$, the orthogonal complement of $\text{col}(P)$ since $P$ is symmetric, gives a collectionn of eigenvectors of $U$ having eigenvalues 1 ($U\mathbf{v} = (I - P)\mathbf{v} = \mathbf{v} - \mathbf{0} = \mathbf{v}$). Hence, $\boxed{\sigma(U) = \{-1, 1\}}$.

(c) Explaining your reasoning, find the determininant of $U$.

The determinant is the product of the eigenvalues (counted according to their multiplicities): $1^{n-r}(-1)^r = \boxed{(-1)^r}$

**More Supplemental Practice for Problem 5.** Consider the one-dimensional subspace $V \subset \mathbb{R}^2$ with the orthonormal basis $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ ($V$ is also known as the $x$-axis).

(a) Find the matrix $P_a$ such that for $\mathbf{x} \in \mathbb{R}^2$, $\mathbf{p} = P_a\mathbf{x}$ is the point in $V$ that minimizes the function

$$D_a(\mathbf{u}) = (\mathbf{x} - \mathbf{u})^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (\mathbf{x} - \mathbf{u})$$

where $\mathbf{u} \in V$.

$D_a(\mathbf{u})$ is $||\mathbf{x} - \mathbf{u}||^2$. Hence, $P_a$ implements orthogonal projection into the $x$-axis. That is,

$$P_a = \boxed{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}$$

(b) Find the matrix $P_b$ such that for $\mathbf{x} \in \mathbb{R}^2$, $\mathbf{p} = P_b\mathbf{x}$ is the point in $V$ that minimizes the function

$$D_b(\mathbf{u}) = (\mathbf{x} - \mathbf{u})^T \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} (\mathbf{x} - \mathbf{u})$$

where $\mathbf{u} \in V$.

For convenience let $\mathbf{e} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{a} = A\mathbf{e} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Note that $\mathbf{x}^T A\mathbf{e} = \mathbf{e}^T A\mathbf{x}$ since $A$ is symmetric. The point $\mathbf{p} = s\mathbf{e}$ where $s$ minimizes

$$f(s) = D_b(s\mathbf{e}) = (\mathbf{x} - s\mathbf{e})^T A(\mathbf{x} - s\mathbf{e}) = \mathbf{x}^T A\mathbf{x} - 2s\mathbf{e}^T A\mathbf{x} + s^2\mathbf{e}^T A\mathbf{e}.$$

Setting the derivative of $f(s)$ equal to zero (to find the minimum)

$$0 = -2\mathbf{e}^T A\mathbf{x} + 2s\mathbf{e}^T A\mathbf{e} \quad \Rightarrow \quad s = \frac{\mathbf{e}^T A\mathbf{x}}{\mathbf{e} \cdot A\mathbf{e}} \quad \Rightarrow \quad \mathbf{p} = \frac{1}{2}\mathbf{e}\mathbf{a}^T\mathbf{x} \Rightarrow P_b = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} [2 \quad 1] = \boxed{\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}}$$

(c) Is $P_b$ a projection matrix? Is $P_b$ a symmetric matrix?

$P_b^2 = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} = P_b$; hence, $\boxed{P_b \text{ is a projection}}$.
However, $\boxed{P_b \text{ is not symmetric}}$; hence, it is not an orthogonal projection.

**Even More Supplemental Practice for Problem 5.** Suppose $P_a$ and $P_b$ are $2 \times 2$ orthogonal projection matrices projection onto the subspaces $V_a = \text{Span}\{\mathbf{a}\}$ and $V_b = \text{Span}\{\mathbf{b}\}$ respectively. Assume $||\mathbf{a}|| = ||\mathbf{b}|| = 1$.

(a) Find formulas for $P_a$ and $P_b$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

$$P_a = \mathbf{a}\mathbf{a}^T \qquad \text{and} \qquad P_b = \mathbf{b}\mathbf{b}^T$$

(b) Find and simplify formulas for $P_aP_b$ and $P_bP_a$.

$$P_aP_b = \mathbf{a}\mathbf{a}^T\mathbf{b}\mathbf{b}^T = (\mathbf{a}\cdot\mathbf{b})\mathbf{a}\mathbf{b}^T \qquad \text{and} \qquad P_bP_a = \mathbf{b}\mathbf{b}^T\mathbf{a}\mathbf{a}^T = (\mathbf{a}\cdot\mathbf{b})\mathbf{b}\mathbf{a}^T$$

(c) Find the eigenvalues for $P_aP_b$.

Vectors in the column space of $P_aP_b$ are multiples of $\mathbf{a}$. Hence, $\mathbf{a}$ is necessarily an eigenvector:

$$P_aP_b\mathbf{a} = (\mathbf{a}\cdot\mathbf{b})\mathbf{a}\mathbf{b}^T\mathbf{a} = (\mathbf{a}\cdot\mathbf{b})^2\mathbf{a}$$

and the eigenvalue is $\lambda = (\mathbf{a}\cdot\mathbf{b})^2$.

The null space of $P_aP_b$ includes the vectors perpendicular to $\mathbf{b}$. Suppose that $\mathbf{v} \neq \mathbf{0}$ is such a vector. Then

$$P_aP_b\mathbf{v} = (\mathbf{a}\cdot\mathbf{b})\mathbf{a}\mathbf{b}^T\mathbf{v} = \mathbf{0} \qquad (\text{since} \quad \mathbf{b}^T\mathbf{v} = \mathbf{b}\cdot\mathbf{v} = 0),$$

that is, $\mathbf{v}$ is an eigenvector of $P_aP_b$ with eigenvalue zero.

**Extra Credit.** (5 points) Find the point on the plane $z = 2x + 3y$ closest to the point (4,5,6).

The plane is the null space of the matrix $\begin{bmatrix} 2 & 3 & -1 \end{bmatrix}$ which has $\left\{ \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$ as a basis.

Orthogonal projection into the plane is given by

$$P = A(A^TA)^{-1}A^T \qquad \text{where} \qquad A = \begin{bmatrix} 3 & 1 \\ -2 & 0 \\ 0 & 2 \end{bmatrix}.$$

$$\begin{aligned}
A^TA &= \begin{bmatrix} 3 & -2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 3 \\ 3 & 5 \end{bmatrix} \\
(A^TA)^{-1} &= \frac{1}{65-9} \begin{bmatrix} 5 & -3 \\ -3 & 13 \end{bmatrix} = \frac{1}{56} \begin{bmatrix} 5 & -3 \\ -3 & 13 \end{bmatrix} \\
(A^TA)^{-1}A^T &= \frac{1}{56} \begin{bmatrix} 5 & -3 \\ -3 & 13 \end{bmatrix} \begin{bmatrix} 3 & -2 & 0 \\ 1 & 0 & 2 \end{bmatrix} = \frac{1}{56} \begin{bmatrix} 12 & -10 & -6 \\ 4 & 6 & 26 \end{bmatrix} \\
A(A^TA)^{-1}A^T &= \frac{1}{56} \begin{bmatrix} 3 & 1 \\ -2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 12 & -10 & -6 \\ 4 & 6 & 26 \end{bmatrix} = \frac{1}{56} \begin{bmatrix} 40 & -24 & 8 \\ -24 & 20 & 12 \\ 8 & 12 & 52 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 10 & -6 & 2 \\ -6 & 5 & 3 \\ 2 & 3 & 13 \end{bmatrix}
\end{aligned}$$

$$P \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 10 & -6 & 2 \\ -6 & 5 & 3 \\ 2 & 3 & 13 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 40 - 30 + 12 \\ -24 + 25 + 18 \\ 8 + 15 + 78 \end{bmatrix} = \boxed{\frac{1}{14} \begin{bmatrix} 22 \\ 19 \\ 101 \end{bmatrix}}$$

**Extra Extra Credit.** (5 points) Find the orthogonal projection matrix $P$ that projects onto $\operatorname{col}(A)$ where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \end{bmatrix}.$$

From the given CR factorization of $A$, a basis for $\operatorname{col}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. Hence,

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \left( \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$

$$= \boxed{\frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}}$$