13 Lecture 14: Euclidean Vector Spaces

13.1 Affine sets in $\mathbb{R}^3$

A set $V \in \mathbb{R}^n$ is affine if and only if for each $\mathbf{x} \in \mathbf{V}$ and $\mathbf{y} \in \mathbf{V}$,

$$\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in \mathbf{V}, \quad \text{for all } \alpha \in \mathbb{R}.$$

The linear combination "$\alpha\mathbf{x} + (1-\alpha)\mathbf{y}$ for $\alpha \in \mathbb{R}$ " is called the affine combination.

Remark 13.1.
- *The solution of a linear system is an affine set.*
- *An affine set is the solution set of a linear system. The linear system represents the affine set* **implicitly**. *If the linear system is solved, then the parametric vector form of the solution is the* **explicit** *representation of the affine set.*

13.2 Parametric equation of a line

Let ℓ be a line in $\mathbb{R}^3$, which passes through a point $\mathbf{x}_0 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and has the direction vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$,

then any point $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in the line ℓ is written as

$$\begin{cases} x_1 = a + tv_1 \\ x_2 = b + tv_2 \\ x_3 = c + tv_3 \end{cases}, \quad t \in \mathbb{R}.$$

In another word, the line ℓ is described as $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}, \quad t \in \mathbb{R}$.

Recall

1. Inner product of two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ is

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T\mathbf{y} = x_1y_1 + x_2y_2 + \ldots + x_ny_n$$

2. Length of a vector $\mathbf{x} \in \mathbb{R}^n$ is

$$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{x_1^2 + x_2 2^2 + \ldots + x_n^2}.$$

Cauchy-Schwarz inequality $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Angle of two vectors:

Let $\mathbf{u}$ and $\mathbf{v}$ be two vector in $\mathbb{R}^n$ and θ be the included angle, then

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta \tag{13.1}$$

Cauchy-Schwarz inequality can be deduced directly from (13.1) due to $|\cos\theta| \leq 1$.

Distance between two vectors:

The distance between two vectors**u** and **v**, denoted by $dist(\mathbf{x}, \mathbf{y})$, is the length of the vector $\|\mathbf{u} - \mathbf{v}\|$, i.e.,

$$dist(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2}$$

- **Triangle inequality:**

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$$

Orthogonal vectors:

Two vectors **u** and **v** in $\mathbb{R}^n$ are orthogonal (to each others) if $\mathbf{u} \cdot \mathbf{v} = 0$.

- **The Pythagorean Theorem**

  Theorem 4. *Two vectors* **u** *and* **v** *are orthogonal if and only if* $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$

- **Orthogonal projection:** Let **u** and **v** be two vector in $\mathbb{R}^n$. The (orthogonal) projection of **v** onto **u** is given by

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u}$$

13.3 Plane in $\mathbb{R}^3$

- **Normal vector:** A nonzero vector **n** in $\mathbb{R}^n$ is called a **normal vector** of a plane $P \subset \mathbb{R}^n$ is $\mathbf{n} \cdot \mathbf{v} = 0$ for all vector $\mathbf{v} \in P$.

- **Description of a plane:** A plane is unique determined knowing its normal vector and a point belongs to it. Let $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be a normal vector of a plane which contains a point $\mathbf{x}_0 = \begin{bmatrix} x_{10} \\ x_{20} \\ x_{30} \end{bmatrix}$, then the plane is given implicitly a linear system

$$ax_1 + bx_2 + cx_3 = d,$$

  where $d = ax_{10} + bx_{20} + cx_{30}$. The latter equation is equivalent to

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = 0. \tag{13.2}$$

  Note that, the normal vector **n** is nonzero, thus the linear system (13.2) has two free variable. If explicit description of the plane contains two parameters.

Please read the lecture note II.9 on Canvas for more references