Problem 4. (10 points) [Sample Problem A] Suppose that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal bases for $\mathbb{R}^2$. Define the 2×2 matrix

$$A \;=\; 2\mathbf{u}_1\mathbf{u}_1^T \;-\; \mathbf{u}_2\mathbf{u}_2^T.$$

(a) Find an orthogonal diagonalization of A.

(b) Find a formula for A^{-1} (if it exists) in terms of the given information.

(c) Find a singular value decomposition (SVD) for A.

(a)

$$\begin{aligned} A \;&=\; 2\mathbf{u}_1\mathbf{u}_1^T - \mathbf{u}_2\mathbf{u}_2^T \;=\; [\,\mathbf{u}_1 \quad \mathbf{u}_2\,]\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}[\,\mathbf{u}_1 \quad \mathbf{u}_2\,]^T \\ &= UDU^T \qquad \text{where} \qquad U = [\,\mathbf{u}_1 \quad \mathbf{u}_2\,] \;\; \text{and} \;\; D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

(b)

$$\begin{aligned} A^{-1} \;&=\; [\,\mathbf{u}_1 \quad \mathbf{u}_2\,]\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -1 \end{bmatrix}[\,\mathbf{u}_1 \quad \mathbf{u}_2\,]^T \\ &= \tilde{U}\tilde{D}\tilde{U}^T \qquad \text{where} \qquad \tilde{U} = [\,\mathbf{u}_1 \quad \mathbf{u}_2\,] \;\; \text{and} \;\; \tilde{D} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

(c)

$$\begin{aligned} A \;&=\; 2\mathbf{u}_1\mathbf{u}_1^T - \mathbf{u}_2\mathbf{u}_2^T \;=\; [\,\mathbf{u}_1 \quad \mathbf{u}_2\,]\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}[\,\mathbf{u}_1 \quad \mathbf{u}_2\,]^T \\ &= [\,\mathbf{u}_1 \quad \mathbf{u}_2\,]\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}[\,\mathbf{u}_1 \quad \mathbf{u}_2\,]^T \\ &= [\,\mathbf{u}_1 \quad -\mathbf{u}_2\,]\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}[\,\mathbf{u}_1 \quad \mathbf{u}_2\,]^T \\ &= U\Sigma V^T \qquad \text{where} \;\; U = [\,\mathbf{u}_1 \quad -\mathbf{u}_2\,], \quad \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad V = [\,\mathbf{u}_1 \quad \mathbf{u}_2\,] \end{aligned}$$

Problem 4. (10 points) [Sample B] Consider the 3×3 matrix $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$.

(a) Diagonalize the matrix A.

$$p_A(\lambda) = \begin{vmatrix} -1-\lambda & 0 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = -(1+\lambda)\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = -(1+\lambda)\big[(1-\lambda)^2+1\big] = -(1+\lambda)(\lambda^2-2\lambda+2)$$

$$\lambda_1 = -1, \qquad \lambda_{2,3} = \frac{2 \pm \sqrt{4-4\cdot 2}}{2} = 1 \pm i$$

$$\underline{\lambda_1 = -1:} \qquad A - \lambda_1 I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{\lambda_2 = 1+i:} \qquad A - \lambda_2 I = \begin{bmatrix} -2-i & 0 & 0 \\ 0 & -i & -1 \\ 0 & 1 & -i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -i \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

$$\underline{\lambda_3 = 1-i:} \qquad A - \lambda_2 I = \begin{bmatrix} -2+i & 0 & 0 \\ 0 & i & -1 \\ 0 & 1 & i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & i \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$$

$A = SDS^{-1}$

where

$$S = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & i & -i \\ 0 & 1 & 1 \end{bmatrix}} \quad \text{and} \quad D = \boxed{\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1+i & 0 \\ 0 & 0 & 1-i \end{bmatrix}}.$$

(b) Compute and simplify e^{At} where t is a real number. Simplification should eliminate any explicit appearance of the complex number $i = \sqrt{-1}$. Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ may prove helpful.

$$e^{At} = Se^{Dt}S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & -i \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{(1+i)t} & 0 \\ 0 & 0 & e^{(1-i)t} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & -i \\ 0 & 1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & ie^{(1+i)t} & -ie^{(1-i)t} \\ 0 & e^{(1+i)t} & e^{(1-i)t} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2i} & \frac{i}{2i} \\ 0 & -\frac{1}{2i} & \frac{i}{2i} \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & \frac{1}{2}\left(e^{(1+i)t} + e^{(1-i)t}\right) & \frac{i}{2}\left(e^{(1+i)t} - e^{(1-i)t}\right) \\ 0 & \frac{1}{2i}\left(e^{(1+i)t} - e^{(1-i)t}\right) & \frac{1}{2}\left(e^{(1+i)t} + e^{1-i)t}\right) \end{bmatrix}$$

$$\begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & \frac{1}{2}e^t\left(e^{it} + e^{-it}\right) & \frac{i}{2}e^t\left(e^{it} - e^{-it}\right) \\ 0 & \frac{1}{2i}e^t\left(e^{it} - e^{-it}\right) & \frac{1}{2}e^t\left(e^{it} + e^{it}\right) \end{bmatrix} \quad \text{and since} \quad \cos t = \frac{e^{it} + e^{-it}}{2} \quad \text{and} \quad \sin t = \frac{e^{it} - e^{-it}}{2i}$$

$$e^{At} = \boxed{\begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^t\cos t & -e^t\sin t \\ 0 & e^t\sin t & e^t\cos t \end{bmatrix}}$$

(c) Find the singular value decomposition (SVD) of A.

$$AA^T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^T = VDV^T$$

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\mathbf{u}_3 = \frac{1}{\sigma_3} A\mathbf{v}_3 = \frac{1}{1} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$A = U\Sigma V^T$
where

$$U = \boxed{\begin{bmatrix} 0 & 0 & -1 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}} \qquad \Sigma = \boxed{\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}} \qquad V = \boxed{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}$$

Problem 4. (10 points) [Sample Problem C] Suppose $A = QR$ where the orthogonal matrix Q and the upper trapezoidal matrix R are given by

$$Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \qquad \text{and} \qquad R = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) Explaining your solution process, find rref(A).

(b) Find an orthonormal basis for the left null space of A.

(c) Find the SVD of A.

(a) Since Q is invertible, A and R share the same reduced row-echelon form. Hence,

$$R = \begin{matrix} \\ \times\ 1/2 \\ \\ \\ \end{matrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{rref}(A)$$

(b) If $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$\begin{aligned} [\,A \mid I\,] &= [\,QR \mid I\,] \\ &\sim Q^T[\,QR \mid I\,] = [\,R \mid Q^T\,] \\ &\sim E[\,R \mid Q^T\,] = [\,ER \mid EQ^T\,] \\ &= [\,\text{rref}(A) \mid EQ^T\,] \end{aligned}$$

and the last two rows of rref(A) are all zeros

$$EQ^T = \frac{1}{2}\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

An orthonormal basis for nul(A^T) is $\boxed{\left\{ \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}}$.

These vectors are the last two columns of the orthonormal matrix Q; hence, they are orthonormal.

(c) $A = QR$

$$A^TA = R^TR = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$p_{A^TA}(\lambda) = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 4-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = (4-\lambda)(\lambda^2-5\lambda) = \lambda(4-\lambda)(\lambda-5)$$

$$\underline{\lambda_1 = 5}: \quad A^TA - \lambda_3 I = \begin{bmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}_1 = \frac{2}{\sqrt{5}} \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\underline{\lambda_2 = 4}: \quad A^TA - \lambda_2 I = \begin{bmatrix} -3 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\underline{\lambda_3 = 0}: \quad A^TA - \lambda_1 I = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}_3 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{u}_1 = \frac{1}{\sigma_1} QR\mathbf{v}_1 = Q\frac{1}{\sqrt{5}} \begin{bmatrix} \frac{5}{\sqrt{5}} \\ 0 \\ 0 \\ 0 \end{bmatrix} = Q \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{u}_2 = \frac{1}{\sigma_2} QR\mathbf{v}_2 = Q\frac{1}{2} \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} = Q \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

The last two columns of U are an orthonormal basis for nul(A^T) which was found in part (b).

$$V = \boxed{\begin{bmatrix} \frac{\sqrt{5}}{5} & 0 & -\frac{2\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & 0 & \frac{\sqrt{5}}{5} \\ 0 & 1 & 0 \end{bmatrix}} \qquad \Sigma = \boxed{\begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}} \qquad U = \boxed{\frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}}$$

Note: $U = Q$.

Problem 4. (10 points) [Sample Problem D] (a) Suppose that A is a square matrix with characteristic polynomial

$$p_A(\lambda) = (1-\lambda)^n.$$

Furthermore, suppose that $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a set of linearly indepoendent eigenvectors of A. Compute and simplify A.

$$\begin{aligned} A &= SIS^{-1} \qquad \text{where} \qquad S = [\,\mathbf{v}_1 \quad \cdots \quad \mathbf{v}_n\,] \\ &= SS^{-1} \\ &= \boxed{I} \qquad \text{(the } n \times n \text{ identity matrix)} \end{aligned}$$

(b) Suppose that R is not a diagonal matrix but is a diagonalizable matrix which is a square root of I ($R^2 = I$). What is the spectrum (set of eigenvalues) of R?

Since R is diagonalizable, $R = SDS^{-1}$. Hence, $R^2 = I \quad \Rightarrow \quad I = SD^2S^{-1}$. Multiplying this equation on the left by S^{-1} and on the right by S gives $S^{-1}IS = S^{-1}SD^2S^{-1}S$ and therefore $D^2 = I$. Thus, each diagonal element of D is either -1 or 1. Both values must be present as otherwise, R would be diagonal. Thus,

$$\boxed{\sigma(R) = \{1, -1\}}$$

Problem 5. (10 points) [Sample Problem A] Suppose that $\{\mathbf{u}_1, \mathbf{u}_2\}$ and $\{\mathbf{v}_1, \mathbf{v}_2\}$ are two orthonormal bases for $\mathbb{R}^2$. Define the 2×2 matrix

$$A \; = \; \mathbf{u}_1\mathbf{u}_1^T \; - \; 2\mathbf{u}_2\mathbf{u}_2^T.$$

(a) Find the singular value decomposition (SVD) for A.

(b) Find a formula for A^{-1} (if it exists) in terms of the given information.

(c) Find the singular value decomposition (SVD) of A^{-1}.

(a)

$$\begin{aligned}
A \; &= \; \mathbf{u}_1\mathbf{u}_1^T - 2\mathbf{u}_2\mathbf{u}_2^T \; = \; [\,\mathbf{u}_1 \quad \mathbf{u}_2\,]\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}[\,\mathbf{u}_1 \quad \mathbf{u}_2\,]^T \\
&= [\,\mathbf{u}_2 \quad \mathbf{u}_1\,]\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}[\,\mathbf{u}_2 \quad \mathbf{u}_1\,]^T \\
&= [\,\mathbf{u}_2 \quad \mathbf{u}_1\,]\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}[\,\mathbf{u}_2 \quad \mathbf{u}_1\,]^T \\
&= [\,-\mathbf{u}_2 \quad \mathbf{u}_1\,]\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}[\,\mathbf{u}_2 \quad \mathbf{u}_1\,]^T \\
&= U\Sigma V^T \qquad \text{where} \quad U = \boxed{[\,-\mathbf{u}_2 \quad \mathbf{u}_1\,]}, \quad \Sigma = \boxed{\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}}, \quad V = \boxed{[\,\mathbf{u}_2 \quad \mathbf{u}_1\,]}
\end{aligned}$$

(b)

$$\begin{aligned}
A^{-1} \; &= \; \left([\,\mathbf{u}_1 \quad \mathbf{u}_2\,]\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}[\,\mathbf{u}_1 \quad \mathbf{u}_2\,]^T \right)^{-1} \\
&= [\,\mathbf{u}_1 \quad \mathbf{u}_2\,]\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}[\,\mathbf{u}_1 \quad \mathbf{u}_2\,]^T \\
&= \boxed{\mathbf{u}_1\mathbf{u}_1^T - \frac{1}{2}\mathbf{u}_2\mathbf{u}_2^T}
\end{aligned}$$

(c)

$$\begin{aligned}
A^{-1} &= [\,\mathbf{u}_1 \quad \mathbf{u}_2\,]\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}[\,\mathbf{u}_1 \quad \mathbf{u}_2\,]^T \\
&= \boxed{[\,\mathbf{u}_1 \quad -\mathbf{u}_2\,]\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}[\,\mathbf{u}_1 \quad \mathbf{u}_2\,]^T}
\end{aligned}$$

Problem 5. (10 points) [Sample Problem B] Suppose θ is real and $s \geq t > 0$. Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

(a) Find a full singular value decomposition (SVD) for A.

(b) Find an orthogonal diagonalization of $A^T A$.

(c) Find an orthogonal diagonalization of AA^T.

(a) Following standard operating procedure:

$$\begin{aligned} A^T A &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 2s^2 & 0 \\ 0 & 2t^2 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\ &= VDV^T \quad \text{were} \quad V = \boxed{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}} \quad \text{and} \quad D = \boxed{\begin{bmatrix} 2s^2 & 0 \\ 0 & 2t^2 \end{bmatrix}} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \end{aligned}$$

$$[\,\sigma_1 \mathbf{u}_1 \quad \sigma_2 \mathbf{u}_2\,] = [\,A\mathbf{v}_1 \quad A\mathbf{v}_2\,] = AV = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix} V^T V = \left[\begin{array}{cc|} s & t \\ s & -t \\ 0 & 0 \end{array}\right] \quad \Rightarrow$$

$$[\,\mathbf{u}_1 \quad \mathbf{u}_2\,] = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$U = \boxed{\begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \quad \text{and} \quad \Sigma = \boxed{\begin{bmatrix} s\sqrt{2} & 0 \\ 0 & t\sqrt{2} \\ 0 & 0 \end{bmatrix}}$$

Hence, the SVD for A is $\boxed{U\Sigma V^T}$ where U, Σ and V are given above.

(b) $A^T A = VDV^T$ where V and D are given in part (a).

(c) Also from part (a),

$$AA^T = U\Sigma V^T (U\Sigma V^T)^T = U\Sigma V^T V \Sigma^T U = U^T(\Sigma\Sigma^T)U^T = \boxed{U\tilde{D}U^T}$$

where

$$\tilde{D} = \Sigma\Sigma^T = \begin{bmatrix} s\sqrt{2} & 0 \\ 0 & t\sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s\sqrt{2} & 0 & 0 \\ 0 & t\sqrt{2} & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 2s^2 & 0 & 0 \\ 0 & 2t^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

Problem 5. (10 points) [Sample Problem C]

(a) Find the orthogonal diagonalization of $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$.

(b) Find the orthogonal diagonalization of $B = \begin{bmatrix} 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & -3 \\ 3 & 4 & 0 & 0 \\ 4 & -3 & 0 & 0 \end{bmatrix}$.

Hint: If $\mathbf{v}$ is an eigenvector of A, then $\begin{bmatrix} \mathbf{v} \\ \mathbf{v} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{v} \\ -\mathbf{v} \end{bmatrix}$ are eigenvectors of B.

(c) Find the SVD of $B = \begin{bmatrix} 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & -3 \\ 3 & 4 & 0 & 0 \\ 4 & -3 & 0 & 0 \end{bmatrix}$.

(a) $p_A(\lambda) = \lambda^2 - (3-3)\lambda + (-9-16) = \lambda^2 - 25 = (\lambda-5)(\lambda+5)$

$$\underline{\lambda_1 = 5:} \qquad A - \lambda_1 I = \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \quad \Rightarrow \quad \mathbf{w}_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad \Rightarrow \quad \mathbf{u}_1 = \frac{1}{||\mathbf{w}_1||}\mathbf{w}_1 = \frac{\sqrt{5}}{5}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\underline{\lambda_2 = -5:} \qquad A - \lambda_2 I = \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \quad \Rightarrow \quad \mathbf{w}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \quad \Rightarrow \quad \mathbf{u}_2 = \frac{1}{||\mathbf{w}_2||}\mathbf{w}_2 = \frac{\sqrt{5}}{5}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$A = UDU^T \qquad \text{where} \qquad U = \boxed{\begin{bmatrix} \frac{2\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \end{bmatrix}} \qquad \text{and} \qquad D = \boxed{\begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}}$$

(b) From each eigenvector of A we obtain two mutually orthogonal eigenvectors of B:

$$B\begin{bmatrix} \mathbf{v} \\ \pm\mathbf{v} \end{bmatrix} = \begin{bmatrix} O & A \\ A & O \end{bmatrix}\begin{bmatrix} \mathbf{v} \\ \pm\mathbf{v} \end{bmatrix} = \begin{bmatrix} \pm A\mathbf{v} \\ A\mathbf{v} \end{bmatrix} = \begin{bmatrix} \pm\lambda\mathbf{v} \\ \lambda\mathbf{v} \end{bmatrix} = \pm\lambda\begin{bmatrix} \mathbf{v} \\ \pm\mathbf{v} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{v} \\ \mathbf{v} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{v} \\ -\mathbf{v} \end{bmatrix} = \mathbf{v}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{v} = 0$$

Normalizing the constructed eigenvectors provides 4 orthonormal eigenvetors of B:

$$\boxed{\tilde{\mathbf{u}}_1 = \frac{\sqrt{2}}{2}\begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_1 \end{bmatrix}, \quad \tilde{\mathbf{u}}_2 = \frac{\sqrt{2}}{2}\begin{bmatrix} \mathbf{u}_1 \\ -\mathbf{u}_1 \end{bmatrix}, \quad \tilde{\mathbf{u}}_3 = \frac{\sqrt{2}}{2}\begin{bmatrix} \mathbf{u}_2 \\ \mathbf{u}_2 \end{bmatrix}, \quad \tilde{\mathbf{u}}_1 = \frac{\sqrt{2}}{2}\begin{bmatrix} \mathbf{u}_2 \\ -\mathbf{u}_2 \end{bmatrix}}.$$

Hence,

$$B = \tilde{U}\tilde{D}\tilde{U}^T \quad \text{where} \quad \tilde{U} = \boxed{\begin{bmatrix} \tilde{\mathbf{u}}_1 & \tilde{\mathbf{u}}_2 & \tilde{\mathbf{u}}_3 & \tilde{\mathbf{u}}_4 \end{bmatrix}} \quad \text{and} \quad \tilde{D} = \boxed{\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}}$$

(c) $B = \breve{U}\breve{\Sigma}\breve{V}^T$ where

$$\breve{U} = \boxed{\begin{bmatrix} \tilde{\mathbf{u}}_1 & -\tilde{\mathbf{u}}_2 & -\tilde{\mathbf{u}}_3 & \tilde{\mathbf{u}}_4 \end{bmatrix}}, \quad \breve{\Sigma} = \boxed{\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}}, \quad \text{and} \quad \breve{V} = \boxed{\begin{bmatrix} \tilde{\mathbf{u}}_1 & \tilde{\mathbf{u}}_2 & \tilde{\mathbf{u}}_3 & \tilde{\mathbf{u}}_4 \end{bmatrix}}$$

Problem 5. (10 points) [Sample Problem D]

Suppose $A = \mathbf{u}_1\mathbf{v}_1^T$ where $\mathbf{u}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$ with $a^2+b^2 = 1$ and $\mathbf{v}_1 = \begin{bmatrix} c \\ d \end{bmatrix}$ with $c^2+d^2 = 1$.

(a) Find vectors $\mathbf{u}_2$ and $\mathbf{v}_2$ such that $\{\mathbf{u}_1, \mathbf{u}_2\}$ and $\{\mathbf{v}_1, \mathbf{v}_2\}$ are orthonormal bases for $\mathbb{R}^2$.

$$\mathbf{u}_2 = \begin{bmatrix} -b \\ a \end{bmatrix} \quad \text{since} \quad \mathbf{u}_1 \cdot \mathbf{u}_2 = a(-b) + b(a) = 0 \quad \text{and} \quad ||\mathbf{u}_2||^2 = (-b)^2 + a^2 = 1$$

$$\mathbf{v}_2 = \begin{bmatrix} -d \\ c \end{bmatrix} \quad \text{since} \quad \mathbf{v}_1 \cdot \mathbf{v}_2 = c(-d) + d(c) = 0 \quad \text{and} \quad ||\mathbf{v}_2||^2 = (-d)^2 + c^2 = 1$$

(b) Find the SVD of A.

The Gram matrix of A is $A^TA = (\mathbf{u}_1\mathbf{v}_1^T)^T(\mathbf{u}_1\mathbf{v}_1^T) = \mathbf{v}_1\mathbf{u}_1^T\mathbf{u}_1\mathbf{v}_1^T = (\mathbf{u}_1 \cdot \mathbf{u}_1)\mathbf{v}_1\mathbf{v}_1^T = \mathbf{v}_1\mathbf{v}_1^T$. The eigenvectors of A^TA are $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\begin{aligned} A^TA\mathbf{v}_1 &= (\mathbf{v}_1\mathbf{v}_1^T)\mathbf{v}_1 = (\mathbf{v}_1 \cdot \mathbf{v}_1)\mathbf{v}_1 = 1\mathbf{v}_1 \\ A^TA\mathbf{v}_2 &= (\mathbf{v}_1\mathbf{v}_1^T)\mathbf{v}_2 = (\mathbf{v}_1 \cdot \mathbf{v}_2)\mathbf{v}_1 = 0\mathbf{v}_1 = 0\mathbf{v}_2 \end{aligned}$$

Hence, the orthogonal diagonalization of A is $A = VDV^T$ where

$$V = \boxed{[\,\mathbf{v}_1 \quad \mathbf{v}_2\,]} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Moreover, the singular value $\sigma_1 = +\sqrt{1} = 1$. Hence,

$$\Sigma = \boxed{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}.$$

The left singular vector associated paired with $\mathbf{v}_1$ is

$$\tilde{\mathbf{u}}_1 = \frac{1}{\sigma_1}A\mathbf{v}_1 = \frac{1}{1}\mathbf{u}_1\mathbf{v}_1^T\mathbf{v}_1 = \mathbf{u}_1$$

The second singular vector is a basis for the left null space of A, but most easily identified as being orthogonal to $\mathbf{u}_1$ (as $\{\mathbf{u}_1\}$ is a basis for the column space of A). Hence, $\tilde{\mathbf{u}}_2 = \mathbf{u}_2$, and $U = \boxed{[\,\mathbf{u}_1 \quad \mathbf{u}_2\,]}$.

$$A = [\,\mathbf{u}_1 \quad \mathbf{u}_2\,]\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}[\,\mathbf{v}_1 \quad \mathbf{v}_2\,]^T$$

Extra Credit. (8 points) [Sample Problem A] Suppose

$$\begin{aligned} \ddot{y} + \dot{y} &= 0 \\ y(0) &= 0 \\ \dot{y}(0) &= 1 \end{aligned}$$

Find and simplify

$$\lim_{t \to \infty} y(t).$$

$$\begin{bmatrix} \ddot{y} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{y} \\ y \end{bmatrix} \qquad \text{with} \qquad \begin{bmatrix} \dot{y}(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Hence,

$$\begin{bmatrix} \dot{y}(t) \\ y(t) \end{bmatrix} = e^{At} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{where} \qquad A = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$p_A(\lambda) = \lambda^2 + \lambda = \lambda(\lambda + 1)$$

$$\begin{aligned} \underline{\lambda_1 = 0}: &\qquad A - \lambda_1 I = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \qquad \Rightarrow \qquad \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \underline{\lambda_2 = -1}: &\qquad A - \lambda_2 I = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \qquad \Rightarrow \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} \dot{y}(t) \\ y(t) \end{bmatrix} &= e^{At} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \exp\left(t \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \right) \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -e^{-t} \end{bmatrix} \\ &= \begin{bmatrix} e^{-t} \\ 1 - e^{-t} \end{bmatrix} \end{aligned}$$

Thus,

$$\lim_{t \to \infty} y(t) = \lim_{t \to \infty} \; 1 - e^{-t} \; = \boxed{1}.$$

Extra Credit. (8 points) [Sample Problem B] Showing your work, derive an explicit formula (the Binet formula) for the n^{th} Fibonacci number F_n where

$$F_{\ell+2} = F_{\ell+1} + F_\ell \qquad \text{for} \quad \ell = 0, 1, \ldots$$

and $F_0 = 0$ and $F_1 = 1$.

$$\begin{bmatrix} F_{\ell+2} \\ F_{\ell+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{\ell+1} \\ F_\ell \end{bmatrix} \qquad \text{with} \qquad \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Hence,

$$\begin{bmatrix} F_{\ell+1} \\ F_\ell \end{bmatrix} = A^\ell \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{where} \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

$$p_A(\lambda) = \lambda^2 - \lambda - 1 \quad \Rightarrow \quad \lambda_\pm = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$\underline{\lambda_\pm = \frac{1 \pm \sqrt{5}}{2}}: \qquad A - \lambda_\pm = \begin{bmatrix} \frac{1 \mp \sqrt{5}}{2} & 1 \\ 1 & \frac{-1 \mp \sqrt{5}}{2} \end{bmatrix} \qquad \Rightarrow \qquad \mathbf{v}_\pm = \begin{bmatrix} 1 \pm \sqrt{5} \\ 2 \end{bmatrix}$$

$$\begin{aligned}
\begin{bmatrix} F_{\ell+1} \\ F_\ell \end{bmatrix} &= A^\ell \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \lambda_+^\ell & 0 \\ 0 & \lambda_-^\ell \end{bmatrix} \begin{bmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \lambda_+^\ell & 0 \\ 0 & \lambda_-^\ell \end{bmatrix} \frac{1}{4\sqrt{5}} \begin{bmatrix} 2 & -1+\sqrt{5} \\ -2 & 1+\sqrt{5} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{4\sqrt{5}} \begin{bmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \lambda_+^\ell & 0 \\ 0 & \lambda_-^\ell \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} \\
&= \frac{1}{4\sqrt{5}} \begin{bmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2\lambda_+^\ell \\ -2\lambda_-^\ell \end{bmatrix} \\
&= \frac{1}{4\sqrt{5}} \begin{bmatrix} 2(1+\sqrt{5})\lambda_+^\ell + 2(1-\sqrt{5})\lambda_-^\ell \\ 4\lambda_+^\ell - 4\lambda_-^\ell \end{bmatrix}
\end{aligned}$$

Thus,

$$F_\ell = \frac{1}{4\sqrt{5}}(4\lambda_+^\ell - 4\lambda_-^\ell) \ = \ \boxed{\frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^\ell - \left(\frac{1-\sqrt{5}}{2} \right)^\ell \right]}$$

Extra Extra Credit. (8 points) Suppose $0 \le a \le 1$. Consider the stochastic matrix

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\ 0 & \frac{2}{3} & 0 & 0 & 0 & 1 \end{bmatrix}.$$

And define the iteration $\mathbf{p}_{\ell+1} = A\mathbf{p}_\ell$ where $\mathbf{p}_0 = \begin{bmatrix} a \\ 1-a \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Find $\lim_{\ell\to\infty} \mathbf{p}_\ell$.

For the stochastic matrix M states 3, 4, 5 and 6 are absorbing states. Moreover, from any initial state, an absorbing state is reached in one step. Hence,

$$\lim_{\ell\to\infty} \mathbf{p}_\ell = \mathbf{p}_1 = M\mathbf{p}_0 = \boxed{\begin{bmatrix} 0 \\ 0 \\ \frac{1}{2}a \\ \frac{1}{2}a \\ \frac{1}{3}(1-a) \\ \frac{2}{3}(1-a) \end{bmatrix}}$$

Checking $\mathbf{p}_1 = M\mathbf{p}_1$:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\ 0 & \frac{2}{3} & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2}a \\ \frac{1}{2}a \\ \frac{1}{3}(1-a) \\ \frac{2}{3}(1-a) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2}a \\ \frac{1}{2}a \\ \frac{1}{3}(1-a) \\ \frac{2}{3}(1-a) \end{bmatrix}$$