The $f(x)$ notation may be used to indicate which element in the codomain is associated with an element x in the domain or to define a function. For example, for the function $f = \{(a, 1), (b, 3), (c, 2), (d, 1)\}$, we could write $f(a) = 1, f(b) = 3$, and so on. Assuming that the domain of definition is the positive integers, the equation $g(n) = n+2$ defines the function $\{(n, n+2) \mid n \text{ is a positive integer}\}$ from the set of positive integers to the set of positive integers.

To prove that a function f from X to Y is one-to-one, show that for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

To prove that a function f from X to Y is *not* one-to-one, find $x_1, x_2 \in X$, $x_1 \neq x_2$, such that $f(x_1) = f(x_2)$.

To prove that a function f from X to Y is onto, show that for all $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

To prove that a function f from X to Y is *not* onto, find $y \in Y$ such that $f(x) \neq y$ for all $x \in X$.

3.1 Review Exercises

1. What is a function from X to Y?
2. Explain how to use an arrow diagram to depict a function.
3. What is the graph of a function?
4. Given a set of points in the plane, how can we tell whether it is a function?
5. What is the value of $x \bmod y$?
6. What is a hash function?
7. What is a collision for a hash function?
8. What is a collision resolution policy?
9. What are pseudorandom numbers?
10. Explain how a linear congruential random number generator works, and give an example of a linear congruential random number generator.
11. What is the floor of x? How is the floor denoted?
12. What is the ceiling of x? How is the ceiling denoted?
13. Define *one-to-one function*. Give an example of a one-to-one function. Explain how to use an arrow diagram to determine whether a function is one-to-one.
14. Define *onto function*. Give an example of an onto function. Explain how to use an arrow diagram to determine whether a function is onto.
15. What is a bijection? Give an example of a bijection. Explain how to use an arrow diagram to determine whether a function is a bijection.
16. Define *inverse function*. Give an example of a function and its inverse. Given the arrow diagram of a function, how can we find the arrow diagram of the inverse function?
17. Define *composition of functions*. How is the composition of f and g denoted? Give an example of functions f and g and their composition. Given the arrow diagrams of two functions, how can we find the arrow diagram of the composition of the functions?
18. What is a binary operator? Give an example of a binary operator.
19. What is a unary operator? Give an example of a unary operator.

3.1 Exercises

In Exercises 1–6, determine which credit card numbers have correct check digits.

1. 5366-2806-9965-4138
2. 5194-1132-8860-3905
3. 4004-6067-3429-0019
4. 3419-6888-7169-444
5. 3016-4773-7532-21
6. 4629-9521-3698-0203
7. Show that when 82 in the valid credit card number 4690-3582-1375-4657 is transposed to 28, the check digit changes.

Determine whether each set in Exercises 8–12 is a function from $X = \{1, 2, 3, 4\}$ *to* $Y = \{a, b, c, d\}$. *If it is a function, find its do-*

main and range, draw its arrow diagram, and determine if it is one-to-one, onto, or both. If it is both one-to-one and onto, give the description of the inverse function as a set of ordered pairs, draw its arrow diagram, and give the domain and range of the inverse function.

8. $\{(1, a), (2, a), (3, c), (4, b)\}$

9. $\{(1, c), (2, a), (3, b), (4, c), (2, d)\}$

10. $\{(1, c), (2, d), (3, a), (4, b)\}$

11. $\{(1, d), (2, d), (4, a)\}$

12. $\{(1, b), (2, b), (3, b), (4, b)\}$

Draw the graphs of the functions in Exercises 13–16. The domain of each function is the set of real numbers. The codomain of each function is also the set of real numbers.

13. $f(x) = \lceil x \rceil - \lfloor x \rfloor$

14. $f(x) = x - \lfloor x \rfloor$

15. $f(x) = \lceil x^2 \rceil$

16. $f(x) = \lfloor x^2 - x \rfloor$

Determine whether each function in Exercises 17–22 is one-to-one, onto, or both. Prove your answers. The domain of each function is the set of all integers. The codomain of each function is also the set of all integers.

17. $f(n) = n + 1$

18. $f(n) = n^2 - 1$

19. $f(n) = \lceil n/2 \rceil$

20. $f(n) = |n|$

21. $f(n) = 2n$

22. $f(n) = n^3$

Determine whether each function in Exercises 23–28 is one-to-one, onto, or both. Prove your answers. The domain of each function is $\mathbf{Z} \times \mathbf{Z}$*. The codomain of each function is* $\mathbf{Z}$*.*

23. $f(m, n) = m - n$

24. $f(m, n) = m$

25. $f(m, n) = mn$

26. $f(m, n) = m^2 + n^2$

27. $f(m, n) = n^2 + 1$

28. $f(m, n) = m + n + 2$

29. Prove that the function f from $\mathbf{Z}^+ \times \mathbf{Z}^+$ to $\mathbf{Z}^+$ defined by $f(m, n) = 2^m 3^n$ is one-to-one but not onto.

Determine whether each function in Exercises 30–35 is one-to-one, onto, or both. Prove your answers. The domain of each function is the set of all real numbers. The codomain of each function is also the set of all real numbers.

30. $f(x) = 6x - 9$

31. $f(x) = 3x^2 - 3x + 1$

32. $f(x) = \sin x$

33. $f(x) = 2x^3 - 4$

34. $f(x) = 3^x - 2$

35. $f(x) = \dfrac{x}{1 + x^2}$

36. Give an example of a function different from those presented in the text that is one-to-one but not onto, and prove that your function has the required properties.

37. Give an example of a function different from those presented in the text that is onto but not one-to-one, and prove that your function has the required properties.

38. Give an example of a function different from those presented in the text that is neither one-to-one nor onto, and prove that your function has the required properties.

39. Write the definition of "one-to-one" using logical notation (i.e., use $\forall$, $\exists$, etc.).

40. Use De Morgan's laws of logic to negate the definition of "one-to-one."

41. Write the definition of "onto" using logical notation (i.e., use $\forall$, $\exists$, etc.).

42. Use De Morgan's laws of logic to negate the definition of "onto."

Each function in Exercises 43–48 is one-to-one on the specified domain X. By letting Y = *range of* f*, we obtain a bijection from X to Y. Find each inverse function.*

43. $f(x) = 4x + 2$, X = set of real numbers

44. $f(x) = 3^x$, X = set of real numbers

45. $f(x) = 3 \log_2 x$, X = set of positive real numbers

46. $f(x) = 3 + \dfrac{1}{x}$, X = set of nonzero real numbers

47. $f(x) = 4x^3 - 5$, X = set of real numbers

48. $f(x) = 6 + 2^{7x-1}$, X = set of real numbers

49. Given

$$g = \{(1, b), (2, c), (3, a)\},$$

a function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c, d\}$, and

$$f = \{(a, x), (b, x), (c, z), (d, w)\},$$

a function from Y to $Z = \{w, x, y, z\}$, write $f \circ g$ as a set of ordered pairs and draw the arrow diagram of $f \circ g$.

50. Let f and g be functions from the positive integers to the positive integers defined by the equations

$$f(n) = 2n + 1, \qquad g(n) = 3n - 1.$$

Find the compositions $f \circ f$, $g \circ g$, $f \circ g$, and $g \circ f$.

51. Let f and g be functions from the positive integers to the positive integers defined by the equations

$$f(n) = n^2, \qquad g(n) = 2^n.$$

Find the compositions $f \circ f$, $g \circ g$, $f \circ g$, and $g \circ f$.

52. Let f and g be functions from the nonnegative real numbers to the nonnegative real numbers defined by the equations

$$f(x) = \lfloor 2x \rfloor, \qquad g(x) = x^2.$$

Find the compositions $f \circ f$, $g \circ g$, $f \circ g$, and $g \circ f$.

53. A store offers a (fixed, nonzero) percentage off the price of certain items. A coupon is also available that offers a (fixed, nonzero) amount off the price of the same items. The store will honor both discounts. Show that regardless of the price of an item, the percentage off the price, and amount off the price, it is always cheapest to use the coupon first.

In Exercises 54–59, decompose the function into simpler functions as in Example 3.1.46.

54. $f(x) = \log_2(x^2 + 2)$

55. $f(x) = \dfrac{1}{2x^2}$

56. $f(x) = \sin 2x$

57. $f(x) = 2\sin x$

58. $f(x) = (3 + \sin x)^4$

59. $f(x) = \dfrac{1}{(\cos 6x)^3}$

60. Given

$$f = \{(x, x^2) \mid x \in X\},$$

a function from $X = \{-5, -4, \ldots, 4, 5\}$ to the set of integers, write f as a set of ordered pairs and draw the arrow diagram of f. Is f one-to-one or onto?

61. How many functions are there from $\{1, 2\}$ to $\{a, b\}$? Which are one-to-one? Which are onto?

62. Given

$$f = \{(a, b), (b, a), (c, b)\},$$

a function from $X = \{a, b, c\}$ to X:

(a) Write $f \circ f$ and $f \circ f \circ f$ as sets of ordered pairs.

(b) Define

$$f^n = f \circ f \circ \cdots \circ f$$

to be the n-fold composition of f with itself. Write f^9 and f^{623} as sets of ordered pairs.

63. Let f be the function from $X = \{0, 1, 2, 3, 4\}$ to X defined by

$$f(x) = 4x \bmod 5.$$

Write f as a set of ordered pairs and draw the arrow diagram of f. Is f one-to-one? Is f onto?

64. Let f be the function from $X = \{0, 1, 2, 3, 4, 5\}$ to X defined by

$$f(x) = 4x \bmod 6.$$

Write f as a set of ordered pairs and draw the arrow diagram of f. Is f one-to-one? Is f onto?

65. An International Standard Book Number (ISBN) is a code of 13 characters separated by dashes, such as 978-1-59448-950-1. An ISBN consists of five parts: a product code, a group code, a publisher code, a code that uniquely identifies the book among those published by the particular publisher, and a check digit. For 978-1-59448-950-1, the product code 978 identifies the product as a book (the same scheme is used for other products). The group code is 1, which identifies the book as one from an English-speaking country. The publisher code 59448 identifies the book as one published by Riverhead Books, Penguin Group. The code 950 uniquely identifies the book among those published by Riverhead Books, Penguin Group (Hosseini: *A Thousand Splendid Suns*, in this case). The check digit is 1.

Let S equal the sum of the first digit, plus three times the second digit, plus the third digit, plus three times the fourth digit, . . . , plus three times the twelfth digit. The check digit is equal to

$$[10 - (S \bmod 10)] \bmod 10.$$

Universal Product Codes (UPC), the bar codes that are scanned at the grocery store for example, use a similar method to compute the check digit.

Verify the check digit for this book.

In Exercises 66–71, determine which ISBNs (see Exercise 65) have correct check digits.

66. 978-1-61374-376-9

67. 978-0-8108-8139-2

68. 978-0-939460-91-5

69. 978-0-8174-3593-6

70. 978-1-4354-6028-7

71. 978-0-684-87018-0

72. Show that if a single digit of an ISBN is changed, the check digit will change. Thus, any single-digit error will be detected.

For each hash function in Exercises 73–76, show how the data would be inserted in the order given in initially empty cells. Use the collision resolution policy of Example 3.1.15.

73. $h(x) = x \bmod 11$; cells indexed 0 to 10; data: 53, 13, 281, 743, 377, 20, 10, 796

74. $h(x) = x \bmod 17$; cells indexed 0 to 16; data: 714, 631, 26, 373, 775, 906, 509, 2032, 42, 4, 136, 1028

75. $h(x) = x^2 \bmod 11$; cells and data as in Exercise 73

76. $h(x) = (x^2 + x) \bmod 17$; cells and data as in Exercise 74

77. Suppose that we store and retrieve data as described in Example 3.1.15. Will any problem arise if we delete data? Explain.

78. Suppose that we store data as described in Example 3.1.15 and that we never store more than 10 items. Will any problem arise when retrieving data if we stop searching when we encounter an empty cell? Explain.

79. Suppose that we store data as described in Example 3.1.15 and retrieve data as described in Exercise 78. Will any problem arise if we delete data? Explain.

Let g be a function from X to Y and let f be a function from Y to Z. For each statement in Exercises 80–87, if the statement is true, prove it; otherwise, give a counterexample.

80. If g is one-to-one, then $f \circ g$ is one-to-one.

81. If f is onto, then $f \circ g$ is onto.

82. If g is onto, then $f \circ g$ is onto.

83. If f and g are onto, then $f \circ g$ is onto.

84. If f and g are one-to-one and onto, then $f \circ g$ is one-to-one and onto.

85. If $f \circ g$ is one-to-one, then f is one-to-one.

86. If $f \circ g$ is one-to-one, then g is one-to-one.

87. If $f \circ g$ is onto, then f is onto.

If f is a function from X to Y and $A \subseteq X$ and $B \subseteq Y$, we define

$$f(A) = \{f(x) \mid x \in A\}, \qquad f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

We call $f^{-1}(B)$ the inverse image of B under f.

88. Let

$$g = \{(1, a), (2, c), (3, c)\}$$

be a function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c, d\}$. Let $S = \{1\}$, $T = \{1, 3\}$, $U = \{a\}$, and $V = \{a, c\}$. Find $g(S)$, $g(T)$, $g^{-1}(U)$, and $g^{-1}(V)$.

★89. Let f be a function from X to Y. Prove that f is one-to-one if and only if

$$f(A \cap B) = f(A) \cap f(B)$$

for all subsets A and B of X. [When S is a set, we define $f(S) = \{f(x) \mid x \in S\}$.]

★90. Let f be a function from X to Y. Prove that f is one-to-one if and only if whenever g is a one-to-one function from any set A to X, $f \circ g$ is one-to-one.

★91. Let f be a function from X to Y. Prove that f is onto Y if and only if whenever g is a function from Y onto any set Z, $g \circ f$ is onto Z.

92. Let f be a function from X onto Y. Let

$$\mathcal{S} = \{f^{-1}(\{y\}) \mid y \in Y\}.$$

Show that $\mathcal{S}$ is a partition of X.

Let $\mathbf{R}^{\mathbf{R}}$ denote the set of functions from $\mathbf{R}$ to $\mathbf{R}$. We define the evaluation function E_a, *where $a \in \mathbf{R}$, from $\mathbf{R}^{\mathbf{R}}$ to $\mathbf{R}$ as*

$$E_a(f) = f(a).$$

93. Is E_1 one-to-one? Prove your answer.

94. Is E_1 onto? Prove your answer.

95. Let f be a function from $\mathbf{R}$ to $\mathbf{R}$ such that for some $r \in \mathbf{R}$, $f(rx) = rf(x)$ for all $x \in \mathbf{R}$. Prove that $f(r^n x) = r^n f(x)$ for all $x \in \mathbf{R}$, $n \in \mathbf{Z}^+$.

Exercises 96–100 use the following definitions. Let $X = \{a, b, c\}$. Define a function S from $\mathcal{P}(X)$ to the set of bit strings of length 3 as follows. Let $Y \subseteq X$. If $a \in Y$, set $s_1 = 1$; if $a \notin Y$, set $s_1 = 0$. If $b \in Y$, set $s_2 = 1$; if $b \notin Y$, set $s_2 = 0$. If $c \in Y$, set $s_3 = 1$; if $c \notin Y$, set $s_3 = 0$. Define $S(Y) = s_1 s_2 s_3$.

96. What is the value of $S(\{a, c\})$?

97. What is the value of $S(\varnothing)$?

98. What is the value of $S(X)$?

99. Prove that S is one-to-one.

100. Prove that S is onto.

Exercises 101–107 use the following definitions. Let U be a universal set and let $X \subseteq U$. Define

$$C_X(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{if } x \notin X. \end{cases}$$

We call C_X the characteristic function of X (in U). *(A look ahead at the next Problem-Solving Corner may help in understanding the following exercises.)*

101. Prove that $C_{X \cap Y}(x) = C_X(x) C_Y(x)$ for all $x \in U$.

102. Prove that $C_{X \cup Y}(x) = C_X(x) + C_Y(x) - C_X(x) C_Y(x)$ for all $x \in U$.

103. Prove that $C_{\overline{X}}(x) = 1 - C_X(x)$ for all $x \in U$.

104. Prove that $C_{X-Y}(x) = C_X(x)[1 - C_Y(x)]$ for all $x \in U$.

105. Prove that if $X \subseteq Y$, then $C_X(x) \le C_Y(x)$ for all $x \in U$.

106. Find a formula for $C_{X \triangle Y}$. ($X \triangle Y$ is the symmetric difference of X and Y. The definition is given before Exercise 101, Section 1.1.)

107. Prove that the function f from $\mathcal{P}(U)$ to the set of characteristic functions in U defined by

$$f(X) = C_X$$

is one-to-one and onto.

108. Let X and Y be sets. Prove that there is a one-to-one function from X to Y if and only if there is a function from Y onto X.

A binary operator f on a set X is commutative *if $f(x, y) = f(y, x)$ for all $x, y \in X$. In Exercises 109–113, state whether the given function f is a binary operator on the set X. If f is not a binary operator, state why. State whether or not each binary operator is commutative.*

109. $f(x, y) = x + y, \quad X = \{1, 2, \ldots\}$

110. $f(x, y) = x - y, \quad X = \{1, 2, \ldots\}$

111. $f(x, y) = x \cup y, \quad X = \mathcal{P}(\{1, 2, 3, 4\})$

112. $f(x, y) = x/y, \quad X = \{0, 1, 2, \ldots\}$

113. $f(x, y) = x^2 + y^2 - xy, \quad X = \{1, 2, \ldots\}$

In Exercises 114 and 115, give an example of a unary operator [different from $f(x) = x$, for all x] on the given set.

114. $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$

115. The set of all finite subsets of $\{1, 2, 3, \ldots\}$

116. Prove that if f is a one-to-one, onto function from X to Y, then

$$\{(y, x) \mid (x, y) \in f\}$$

is a one-to-one, onto function from Y to X.

In Exercises 117–119, if the statement is true for all real numbers, prove it; otherwise, give a counterexample.

117. $\lceil x + 3 \rceil = \lceil x \rceil + 3$

118. $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$

119. $\lfloor x + y \rfloor = \lfloor x \rfloor + \lceil y \rceil$

120. Prove that if n is an odd integer,

$$\left\lfloor \frac{n^2}{4} \right\rfloor = \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right).$$

3.2 Review Exercises

1. Define *sequence*.
2. What is an index in a sequence?
3. Define *increasing sequence*.
4. Define *decreasing sequence*.
5. Define *nonincreasing sequence*.
6. Define *nondecreasing sequence*.
7. Define *subsequence*.
8. What is $\sum_{i=m}^{n} a_i$?
9. What is $\prod_{i=m}^{n} a_i$?
10. Define *string*.
11. Define *null string*.
12. If X is a finite set, what is X^*?
13. If X is a finite set, what is X^+?
14. Define *length of a string*. How is the length of the string α denoted?
15. Define *concatenation of strings*. How is the concatenation of strings α and β denoted?
16. Define *substring*.

3.2 Exercises

Answer 1–3 for the sequence $\{s_n\}_{n=1}^{6}$ *defined by*

$$c, d, d, c, d, c.$$

1. Find s_1.
2. Find s_4.
3. Write s as a string.
4. Let s be a sequence whose domain D is a set of consecutive integers. Prove that if $s_i < s_{i+1}$ for all i for which i and $i+1$ are in D, then s is increasing. *Hint:* Let $i \in D$. Use induction on j to show that $s_i < s_j$ for all j, $i < j$ and $j \in D$.

In Exercises 5–9, tell whether the sequence s defined by $s_n = 2^n - n^2$ *is*

(a) *increasing*
(b) *decreasing*
(c) *nonincreasing*
(d) *nondecreasing*

for the given domain D.

5. $D = \{0, 1\}$
6. $D = \{0, 1, 2, 3\}$
7. $D = \{1, 2, 3\}$
8. $D = \{1, 2, 3, 4\}$
9. $D = \{n \mid n \in \mathbf{Z},\ n \geq 3\}$

Answer 10–22 for the sequence t defined by

$$t_n = 2n - 1, \qquad n \geq 1.$$

10. Find t_3.
11. Find t_7.
12. Find t_{100}.
13. Find t_{2077}.
14. Find $\displaystyle\sum_{i=1}^{3} t_i$.
15. Find $\displaystyle\sum_{i=3}^{7} t_i$.
16. Find $\displaystyle\prod_{i=1}^{3} t_i$.
17. Find $\displaystyle\prod_{i=3}^{6} t_i$.
18. Find a formula that represents this sequence as a sequence whose lower index is 0.
19. Is t increasing?
20. Is t decreasing?
21. Is t nonincreasing?
22. Is t nondecreasing?

Answer 23–30 for the sequence v defined by

$$v_n = n! + 2, \qquad n \geq 1.$$

23. Find v_3.
24. Find v_4.
25. Find $\displaystyle\sum_{i=1}^{4} v_i$.
26. Find $\displaystyle\sum_{i=3}^{3} v_i$.
27. Is v increasing?
28. Is v decreasing?
29. Is v nonincreasing?
30. Is v nondecreasing?

Answer 31–36 for the sequence

$$q_1 = 8, \quad q_2 = 12, \quad q_3 = 12, \quad q_4 = 28, \quad q_5 = 33.$$

31. Find $\displaystyle\sum_{i=2}^{4} q_i$.
32. Find $\displaystyle\sum_{k=2}^{4} q_k$.
33. Is q increasing?
34. Is q decreasing?
35. Is q nonincreasing?
36. Is q nondecreasing?

Answer 37–40 for the sequence

$$\tau_0 = 5, \quad \tau_2 = 5.$$

37. Is τ increasing?
38. Is τ decreasing?
39. Is τ nonincreasing?
40. Is τ nondecreasing?

Answer 41–44 for the sequence

$$\Upsilon_2 = 5.$$

41. Is Υ increasing?
42. Is Υ decreasing?

43. Is Υ nonincreasing?

44. Is Υ nondecreasing?

Answer 45–56 for the sequence a defined by

$$a_n = n^2 - 3n + 3, \qquad n \geq 1.$$

45. Find $\sum_{i=1}^{4} a_i$.

46. Find $\sum_{j=3}^{5} a_j$.

47. Find $\sum_{i=4}^{4} a_i$.

48. Find $\sum_{k=1}^{6} a_k$.

49. Find $\prod_{i=1}^{2} a_i$.

50. Find $\prod_{i=1}^{3} a_i$.

51. Find $\prod_{n=2}^{3} a_n$.

52. Find $\prod_{x=3}^{4} a_x$.

53. Is a increasing?

54. Is a decreasing?

55. Is a nonincreasing?

56. Is a nondecreasing?

Answer 57–64 for the sequence b defined by $b_n = n(-1)^n$, $n \geq 1$.

57. Find $\sum_{i=1}^{4} b_i$.

58. Find $\sum_{i=1}^{10} b_i$.

59. Find a formula for the sequence c defined by

$$c_n = \sum_{i=1}^{n} b_i.$$

60. Find a formula for the sequence d defined by

$$d_n = \prod_{i=1}^{n} b_i.$$

61. Is b increasing?

62. Is b decreasing?

63. Is b nonincreasing?

64. Is b nondecreasing?

Answer 65–72 for the sequence Ω *defined by* $\Omega_n = 3$ *for all* n.

65. Find $\sum_{i=1}^{3} \Omega_i$.

66. Find $\sum_{i=1}^{10} \Omega_i$.

67. Find a formula for the sequence c defined by

$$c_n = \sum_{i=1}^{n} \Omega_i.$$

68. Find a formula for the sequence d defined by

$$d_n = \prod_{i=1}^{n} \Omega_i.$$

69. Is Ω increasing?

70. Is Ω decreasing?

71. Is Ω nonincreasing?

72. Is Ω nondecreasing?

Answer 73–79 for the sequence x defined by

$$x_1 = 2, \quad x_n = 3 + x_{n-1}, \qquad n \geq 2.$$

73. Find $\sum_{i=1}^{3} x_i$.

74. Find $\sum_{i=1}^{10} x_i$.

75. Find a formula for the sequence c defined by

$$c_n = \sum_{i=1}^{n} x_i.$$

76. Is x increasing?

77. Is x decreasing?

78. Is x nonincreasing?

79. Is x nondecreasing?

Answer 80–87 for the sequence w defined by

$$w_n = \frac{1}{n} - \frac{1}{n+1}, \qquad n \geq 1.$$

80. Find $\sum_{i=1}^{3} w_i$.

81. Find $\sum_{i=1}^{10} w_i$.

82. Find a formula for the sequence c defined by

$$c_n = \sum_{i=1}^{n} w_i.$$

83. Find a formula for the sequence d defined by

$$d_n = \prod_{i=1}^{n} w_i.$$

84. Is w increasing?

85. Is w decreasing?

86. Is w nonincreasing?

87. Is w nondecreasing?

Answer 88–100 for the sequence a defined by

$$a_n = \frac{n-1}{n^2(n-2)^2}, \qquad n \geq 3$$

and the sequence z defined by $z_n = \sum_{i=3}^{n} a_i$.

88. Find a_3.

89. Find a_4.

90. Find z_3.

91. Find z_4.

★**92.** Find z_{100}. *Hint:* Show that

$$a_n = \frac{1}{4}\left[\frac{1}{(n-2)^2} - \frac{1}{n^2}\right]$$

and use this form in the sum. Write out $a_3 + a_4 + a_5 + a_6$ to see what is going on.

93. Is a increasing?

★**94.** Is a decreasing?

★**95.** Is a nonincreasing?

96. Is a nondecreasing?

97. Is z increasing?

98. Is z decreasing?

99. Is z nonincreasing?

100. Is z nondecreasing?

Let X be the set of positive integers that are not perfect squares. (A perfect square *m is an integer of the form* $m = i^2$ *where i is an integer.) Exercises 101–107 concern the sequence s from X to* **Z** *defined as follows. If* $n \in X$, *let* s_n *be the least integer* a_k *for* *which there exist integers $a_1, \ldots, a_k$ with $n < a_1 < a_2 < \cdots < a_k$ such that $n \cdot a_1 \cdots a_k$ is a perfect square. As an example, consider s_2. None of $2 \cdot 3, 2 \cdot 4, 2 \cdot 3 \cdot 4$ is a perfect square, so $s_2 \neq 3$ and $s_2 \neq 4$. If $s_2 = 5$, $2 \cdot a_1 \cdots a_k \cdot 5$ would be a perfect square for some $a_1, \ldots, a_k$, $2 < a_1 < a_2 < \cdots < a_k < 5$. But then one of the a_i would be a multiple of 5, which is impossible. Therefore, $s_2 \neq 5$. However, $2 \cdot 3 \cdot 6$ is a perfect square, so $s_2 = 6$.*

101. Show that s_n is defined for every $n \in X$, that is, that for any $n \in X$, there exist integers $a_1, \ldots, a_k$ with $n < a_1 < a_2 < \cdots < a_k$ such that $n \cdot a_1 \cdots a_k$ is a perfect square.

102. Show that $s_n \leq 4n$ for all $n \in X$.

103. Find s_3.

104. Find s_5.

105. Find s_6.

★**106.** Prove that if p is a prime, $s_p = 2p$ for all $p \geq 5$.

107. Prove that s is not increasing.

108. Let u be the sequence defined by

$$u_1 = 3, \quad u_n = 3 + u_{n-1}, \qquad n \geq 2.$$

Find a formula for the sequence d defined by

$$d_n = \prod_{i=1}^{n} u_i.$$

Exercises 109–112 refer to the sequence $\{s_n\}$ defined by the rule

$$s_n = 2n - 1, \qquad n \geq 1.$$

109. List the first seven terms of s.

Answer 110–112 for the subsequence of s obtained by taking the first, third, fifth, ... terms.

110. List the first seven terms of the subsequence.

111. Find a formula for the expression n_k as described before Example 3.2.15.

112. Find a formula for the kth term of the subsequence.

Exercises 113–116 refer to the sequence $\{t_n\}$ defined by the rule

$$t_n = 2^n, \qquad n \geq 1.$$

113. List the first seven terms of t.

Answer 114–116 for the subsequence of t obtained by taking the first, second, fourth, seventh, eleventh, ... terms.

114. List the first seven terms of the subsequence.

115. Find a formula for the expression n_k as described before Example 3.2.15.

116. Find a formula for the kth term of the subsequence.

Answer 117–120 using the sequences y and z defined by

$$y_n = 2^n - 1, \qquad z_n = n(n-1).$$

117. Find $\left(\sum_{i=1}^{3} y_i\right)\left(\sum_{i=1}^{3} z_i\right)$.

118. Find $\left(\sum_{i=1}^{5} y_i\right)\left(\sum_{i=1}^{4} z_i\right)$.

119. Find $\sum_{i=1}^{3} y_i z_i$.

120. Find $\left(\sum_{i=3}^{4} y_i\right)\left(\prod_{i=2}^{4} z_i\right)$.

Answer 121–128 for the sequence r defined by

$$r_n = 3 \cdot 2^n - 4 \cdot 5^n, \qquad n \geq 0.$$

121. Find r_0.

122. Find r_1.

123. Find r_2.

124. Find r_3.

125. Find a formula for r_p.

126. Find a formula for r_{n-1}.

127. Find a formula for r_{n-2}.

128. Prove that $\{r_n\}$ satisfies

$$r_n = 7r_{n-1} - 10r_{n-2}, \qquad n \geq 2.$$

Answer 129–136 for the sequence z defined by

$$z_n = (2+n)3^n, \qquad n \geq 0.$$

129. Find z_0.

130. Find z_1.

131. Find z_2.

132. Find z_3.

133. Find a formula for z_i.

134. Find a formula for z_{n-1}.

135. Find a formula for z_{n-2}.

136. Prove that $\{z_n\}$ satisfies

$$z_n = 6z_{n-1} - 9z_{n-2}, \qquad n \geq 2.$$

137. Find b_n, $n = 1, \ldots, 6$, where

$$b_n = n + (n-1)(n-2)(n-3)(n-4)(n-5).$$

138. Rewrite the sum

$$\sum_{i=1}^{n} i^2 r^{n-i},$$

replacing the index i by k, where $i = k + 1$.

139. Rewrite the sum

$$\sum_{k=1}^{n} C_{k-1} C_{n-k},$$

replacing the index k by i, where $k = i + 1$.

140. Let a and b be sequences, and let

$$s_k = \sum_{i=1}^{k} a_i.$$

Prove that

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} s_k(b_k - b_{k+1}) + s_n b_{n+1}.$$

This equation, known as the *summation-by-parts formula*, is the discrete analog of the integration-by-parts formula in calculus.

141. Sometimes we generalize the notion of sequence as defined in this section by allowing more general indexing. Suppose that $\{a_{ij}\}$ is a sequence indexed over pairs of positive integers. Prove that

$$\sum_{i=1}^{n}\left(\sum_{j=i}^{n} a_{ij}\right) = \sum_{j=1}^{n}\left(\sum_{i=1}^{j} a_{ij}\right).$$

142. Compute the given quantity using the strings

$$\alpha = baab, \qquad \beta = caaba, \qquad \gamma = bbab.$$

(a) $\alpha\beta$	(b) $\beta\alpha$	(c) $\alpha\alpha$
(d) $\beta\beta$	(e) $\lvert\alpha\beta\rvert$	(f) $\lvert\beta\alpha\rvert$
(g) $\lvert\alpha\alpha\rvert$	(h) $\lvert\beta\beta\rvert$	(i) $\alpha\lambda$
(j) $\lambda\beta$	(k) $\alpha\beta\gamma$	(l) $\beta\beta\gamma\alpha$

143. List all strings over $X = \{0, 1\}$ of length 2.

144. List all strings over $X = \{0, 1\}$ of length 2 or less.

145. List all strings over $X = \{0, 1\}$ of length 3.

146. List all strings over $X = \{0, 1\}$ of length 3 or less.

147. Find all substrings of the string *babc*.

148. Find all substrings of the string *aabaabb*.

149. Use induction to prove that

$$\sum \frac{1}{n_1 \cdot n_2 \cdots n_k} = n,$$

for all $n \geq 1$, where the sum is taken over all nonempty subsets $\{n_1, n_2, \ldots, n_k\}$ of $\{1, 2, \ldots, n\}$.

150. Suppose that the sequence $\{a_n\}$ satisfies $a_1 = 0$, $a_2 = 1$, and

$$a_n = (n-1)(a_{n-1} + a_{n-2}) \quad \text{for all } n \geq 3.$$

Use induction to prove that

$$\frac{a_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \quad \text{for all } n \geq 1.$$

In Exercises 151–153, $x_1, x_2, \ldots, x_n$, $n \geq 2$, are real numbers satisfying $x_1 < x_2 < \cdots < x_n$, and x is an arbitrary real number.

151. Prove that if $x_1 \leq x \leq x_n$, then

$$\sum_{i=1}^{n} |x - x_i| = \sum_{i=2}^{n-1} |x - x_i| + (x_n - x_1),$$

for all $n \geq 3$.

152. Prove that if $x < x_1$ or $x > x_n$, then

$$\sum_{i=1}^{n} |x - x_i| > \sum_{i=2}^{n-1} |x - x_i| + (x_n - x_1),$$

for all $n \geq 3$.

153. A *median* of $x_1, \ldots, x_n$ is the middle value of $x_1, \ldots, x_n$ when n is odd, and any value between the two middle values of $x_1, \ldots, x_n$ when n is even. For example, if $x_1 < x_2 < \cdots < x_5$, the median is x_3. If $x_1 < x_2 < x_3 < x_4$, a median is any value between x_2 and x_3, including x_2 and x_3.

Use Exercises 151 and 152 and mathematical induction to prove that the sum

$$\sum_{i=1}^{n} |x - x_i|, \tag{3.2.9}$$

$n \geq 1$, is minimized when x is equal to a median of $x_1, \ldots, x_n$.

If we repeat an experiment n times and observe the values $x_1, \ldots, x_n$, the sum (3.2.9) can be interpreted as a measure of the error in assuming that the correct value is x. This exercise shows that this error is minimized by choosing x to be a median of the values $x_1, \ldots, x_n$. The requested inductive argument is attributed to J. Lancaster.

154. Prove that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(i-j)^2 = \frac{n^2(n^2-1)}{6}.$$

155. Let $X = \{a, b\}$. Define a function from X^* to X^* as $f(\alpha) = \alpha ab$. Is f one-to-one? Is f onto X^*? Prove your answers.

156. Let $X = \{a, b\}$. Define a function from X^* to X^* as $f(\alpha) = \alpha\alpha$. Is f one-to-one? Is f onto X^*? Prove your answers.

157. Let $X = \{a, b\}$. A *palindrome over X* is a string α for which $\alpha = \alpha^R$ (i.e., a string that reads the same forward and backward). An example of a palindrome over X is *bbaabb*. Define a function from X^* to the set of palindromes over X as $f(\alpha) = \alpha\alpha^R$. Is f one-to-one? Is f onto? Prove your answers.

Let L be the set of all strings, including the null string, that can be constructed by repeated application of the following rules:

- *If $\alpha \in L$, then $a\alpha b \in L$ and $b\alpha a \in L$.*
- *If $\alpha \in L$ and $\beta \in L$, then $\alpha\beta \in L$.*

For example, ab is in L, for if we take $\alpha = \lambda$, then $\alpha \in L$ and the first rule states that $ab = a\alpha b \in L$. Similarly, $ba \in L$. As another example, aabb is in L, for if we take $\alpha = ab$, then $\alpha \in L$; by the first rule, $aabb = a\alpha b \in L$. As a final example, aabbba is in L, for if we take $\alpha = aabb$ and $\beta = ba$, then $\alpha \in L$ and $\beta \in L$; by the second rule, $aabbba = \alpha\beta \in L$.

158. Show that *aaabbb* is in L.

159. Show that *baabab* is in L.

160. Show that *aab* is *not* in L.

161. Prove that if $\alpha \in L$, α has equal numbers of a's and b's.

★162. Prove that if α has equal numbers of a's and b's, then $\alpha \in L$.

163. Let $\{a_n\}_{n=1}^{\infty}$ be a nondecreasing sequence, which is bounded above, and let L be the least upper bound of the set $\{a_n \mid n = 1, 2, \ldots\}$. Prove that for every real number $\varepsilon > 0$, there exists a positive integer N such that $L - \varepsilon < a_n \leq L$ for every $n \geq N$. In calculus terminology, a nondecreasing sequence, which is bounded above, converges to the limit L, where L is the least upper bound of the set of elements of the sequence.