Homework 10 - Cardinalities

CS241

Finite sets

1. Let X, Y be finite sets. $|x| = m$, $|Y| = n$. $f : X \to Y$.
   What can you conclude about the m, n in the following cases? Match one of the statements $(a) - (e)$. Explain you answers.
   - f is surjective
   - f is injective
   - f is bijective
   - f is surjective, not bijective
   - f is injective, not surjective

   $(a)\, m < n$, $(b)\, m \leq n$, $(c)\, m = n$, $(d)\, m \geq n$, $(e)\, m > n$

Enumerable sets

2. We saw a bijection $B : \mathbb{N} \cup \{0\} \to \mathbb{Z}$

$$B : \mathbb{N} \cup \{0\} \to \mathbb{Z}$$

$$B(n) = \begin{cases} \frac{n+1}{2} & \text{if n odd} \\ -\frac{n}{2} & \text{if n even} \end{cases}$$

   This is the bijection that maps the odd numbers to the positive integers, and the even numbers to the negative integers (and 0 to 0)

   (a) Find $B(0)$, $B(1)$, $B(2)$, $B(3)$, $B(4)$,$B(5)$, $B(6)$,$B(7)$

   (b) Complete the following function definition, such that B^{-1} is the inverse bijection $\mathbb{Z} \to \mathbb{N} \cup \{0\}$

$$B^{-1} : \mathbb{Z} \to \mathbb{N} \cup \{0\}$$

$$B^{-1}(y) = \begin{cases} \underline{\qquad\quad} & \text{if } y > 0 \\ \underline{\qquad\quad} & \text{if } y \leq 0 \end{cases}$$

(c) $B^{-1}(-3)$, $B^{-1}(3)$, $B^{-1}(-10)$, $B^{-1}(10)$

3. Prove that the set $\mathbb{N} \setminus \{3, 10, 50\}$ is enumerable (Hint: use a split notation like in the bijections above)

4. Prove that the set $A = \{5\} \times \mathbb{N}$ is enumerable.

5. The set T is the set of all non-negative integers that have reminder of 3 when divided by 4.

   (a) What are the 5 smallest elements of the set T?

   (b) Find a bijection between $\mathbb{N} \cup \{0\}$ and the set T

   (c) Find a bijection between $\mathbb{N}$ and the set T above.

6. Consider the enumerable set $\mathbb{N}$

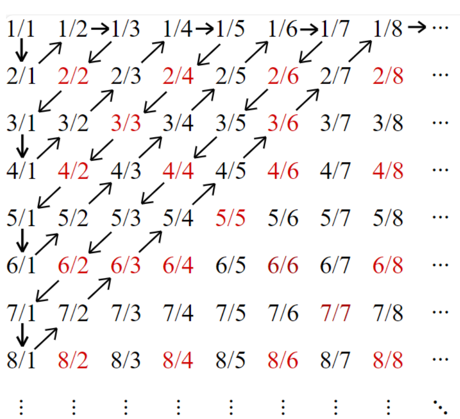
   (a) Find a bijection $\mathbb{N} \to \mathbb{N}$ that is NOT the identity function on $\mathbb{N}$ (so not the function $f(x) = x$)

   (b) Find a function $\mathbb{N} \to \mathbb{N}$ that is injection but NOT surjective

   (c) Find a function $\mathbb{N} \to \mathbb{N}$ that is surjective but NOT injective.

   (d) Answer the above questions for the finite set $\{1, 2, 3, 4\}$. If no such function possible, explain why.

7. Consider the function $b_2 : \mathbb{N} \to \mathbb{Q}$ that we saw, illustrated by:



   (a) Find $b_2(4)$, $b_2(10)$

   (b) Is this function injective? surjective? bijective?

   (c) Let $a, b, m, n \in \mathbb{N}$. Find a condition to tell if $\dfrac{a}{b} = \dfrac{m}{n}$

Continuity of $\mathbb{R}$

8. Prove: the composition of 2 bijections is also a bijection:
   Let $f : A \to B$,
   $g : B \to C$ be bijections.
   Then $g \circ f : A \to C$ is a bijection.

9. Find bijections as described, graph these functions in their domain and co-domain:

   (a) Find a bijection between the open interval $(0, 1)$ and the open interval $(0, 3)$ (Hint; $f(x) = 2x$ is a bijection $(0, 1) \to (0, 2)$)

   (b) Find a bijection $(0, 1) \to (2, 3)$.

   (c) Find a bijection $(0, 1) \to (5, 8)$

   (d) Find a bijection $(-10, 10) \to (-0.1, 0.1)$

   (e) Find a bijection between the closed interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and the closed interval: $[-1, 1]$

   *Hint: Consider linear and trigonometric functions. Use a graphing calculator for visualization: https://www.desmos.com/calculator.

10. Find bijections as described:

    (a) Find a bijection $(0, 1) \to (-1, 1)$

    (b) Find a bijection $(-1, 1) \to (-\frac{\pi}{2}, \frac{\pi}{2})$

    (c) Find a bijection $(-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$

    (d) Using your previous answers, compose a bijection $(-1, 1) \to \mathbb{R}$

    (e) Using your previous answers, compose a bijection $(0, 1) \to \mathbb{R}$

    (f) Use a graphing calculator to verify you answers!

11. Watch the following video: `https://www.youtube.com/watch?v=OxGsU8oIWjY`
    Which bijections that we saw in class appear in this video? How is the last part of the video different then presented in lecture?