Final Exam Review Exercise I

This exercise reviews the core problems applying Gauss-Jordan to general matrices matrices (no determinants, no inverse matrices, no diagonalization) and the interrelationships among those problems.

Instructions. For each matrix at the bottom of this page perform the following list of operations.

(1) Denote the matrix A. (This is just giving the matrix a name.)

(2) Find the pref(A). (This is the proto-row-echelon form resulting from the forward phase of Gauss-Jordan algorithm as taught in this course.)

(3) Find the LU factorization of A, if it exists.

(4) Find (a) rank(A), (b) the list of pivot columns, (c) nullity(A), (d) the number of rows of zero in rref(A), and (e) the dimensions of each of the four fundamental subspaces of A.

(5) Find nref(A). (This is the normalized row-echelon form obtained by dividing each nonzero row of pref(A) by its pivot value.)

(6) Find rref(A).

(7) In rref(A), (a) circle each of the pivots and (b) box the values from the nonzero rows of each column not having a pivot.

(8) Find the CR factorization of A.

(9) Find a basis for each of the four fundamental subspaces of A. Determine the dimension of each.

(10) (a) Calculate the angles between the basis elements of row(A) and the basis elements of nul(A). (b) Calculate the angles between the basis elements of col(A) and the basis elements of nul(A^T).

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 5 & 17 \\ -2 & -3 & -8 & -29 \\ 4 & 5 & 14 & 53 \end{bmatrix}, \quad \begin{bmatrix} 0 & 4 & 4 & 4 & 0 \\ 0 & 4 & 5 & 4 & 0 \\ 0 & 4 & 5 & 4 & 0 \\ 0 & 4 & 5 & 4 & 0 \end{bmatrix}$$

(1&2) $A = \begin{array}{c} \bullet \\ 3 \\ 5 \end{array}\begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 4 & 6 & 8 \\ 5 & 6 & 10 & 12 \end{bmatrix} \sim \begin{array}{c} \\ \bullet \\ 2 \end{array}\begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & -2 & 0 & -4 \\ 0 & -4 & 0 & -8 \end{bmatrix} \sim \boxed{\begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & -2 & 0 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}} = \text{pref}(A)$

(3) $A = LU$ where $L = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 2 & 1 \end{bmatrix}}$ and $U = \text{pref}(A)$

(4a) $\text{rank}(A) = 2$, (4b)= pivot cols $= \{1, 2\}$, (4c) $\text{nullity}(A) = 2$, (4d) #zero rows $= 1$

(4e) $\dim(\text{row}(A)) = \dim(\text{col}(A)) = 2$, $\dim(\text{nul}(A)) = 2$, $\dim(\text{nul}(A^T)) = 1$

(5) $\text{pref}(A) = \begin{array}{c} \\ \times\ -1/2 \\ \\ \end{array}\begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & -2 & 0 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \boxed{\begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}} = \text{nref}(A)$

(6&7) $\text{nref}(A) = \begin{array}{c} 2 \\ \bullet \\ \\ \end{array}\begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \boxed{\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}} = \text{rref}(A)$

(8) Standard procedure requires W where $[\,A \quad | \quad I\,] \sim [\text{rref})A \mid W\,]$. To obtain W we need only apply to I the row operations used above to obtain $\text{rref}(A)$:

$$\begin{array}{c} \bullet \\ 3 \\ 5 \end{array}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{array}{c} \\ \bullet \\ 2 \end{array}\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} \sim \begin{array}{c} \\ \times\ -1/2 \\ \\ \end{array}\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \sim \begin{array}{c} 2 \\ \bullet \\ \\ \end{array}\begin{bmatrix} 1 & 0 & 0 \\ 3/2 & -1/2 & 0 \\ 1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} -3 & 1 & 0 \\ 3/2 & -1/2 & 0 \\ 1 & -2 & 1 \end{bmatrix} = W$$

It is noteworthy that only the operations for the forward phase of Gauss-Jordan are actually necessary to obtain the needed rows of W.

$\text{row}(A)$: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} \right\}$ $\text{col}(A)$: $\left\{ \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right\}$

$\text{nul}(A)$: $\left\{ \begin{bmatrix} -2 \\ -0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ $\text{nul}(A^T)$: $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$

(9) There are six angels under consideration. All are right angles, i.e., $\theta = \frac{\pi}{2}$. This result follows as the dot products of all indicated pairs of vectors are 0:

$$\begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -0 \\ 1 \\ 0 \end{bmatrix} = 0, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -0 \\ -2 \\ 0 \\ 1 \end{bmatrix} = 0, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -0 \\ 1 \\ 0 \end{bmatrix} = 0, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -0 \\ -2 \\ 0 \\ 1 \end{bmatrix} = 0, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 0, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 0$$

These results are consequence of the relationships: $\text{row}(A)^\perp = \text{nul}(A)$ and $\text{col}(A)^\perp = \text{nul}(A^T)$. Hence, the calculations are not particularly useful except as check on the bases.