CS 241 - Arnav Kucheriya - Homework 8

1. Convert 89 to base 6 and base 7.

- Divide 89 by 6:

$89 \div 6 = 14 remainder 5$

$14 \div 6 = 2 remainder 2$

$2 \div 6 = 0 remainder 2$

Therefore:

$$89_{10} = 225_6$$

To convert 89 to base 7:

- Divide 89 by 7:

$89 \div 7 = 12 remainder 5$

$12 \div 7 = 1 remainder 5$

$1 \div 7 = 0 remainder 1$

Therefore,

$89_{10} = 155_7$

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2. Consider the string "2110" in base 3, base 4, and base 5.

Solution

To convert "2110" from base 3:

$$2110_3 = 2 \cdot 3^3 + 1 \cdot 3^2 + 1 \cdot 3^1 + 0 \cdot 3^0 = 66_{10}$$

To convert "2110" from base 4:

$$2110_4 = 2 \cdot 4^3 + 1 \cdot 4^2 + 1 \cdot 4^1 + 0 \cdot 4^0 = 148_{10}$$

To convert "2110" from base 5:

$$2110_5 = 2 \cdot 5^3 + 1 \cdot 5^2 + 1 \cdot 5^1 + 0 \cdot 5^0 = 280_{10}$$

Section 5.1 - Exercise 3

Problem

Define quotient.

Solution

The **quotient** of two integers n and d, where $deq0$, is the integer q such that:

$$n = dq + r, Where, 0 \leq r < |d|$$

Here, q is the quotient, and r is the remainder.

Section 5.1 - Exercise 11

Problem

Find the prime factorization of $11!$.

Solution

$$11! = 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$
$$11! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$$

Section 5.1 - Exercise 23

Problem

Simplify the expression:

$$3^2 \cdot 7^3 \cdot 11, \quad 2^3 \cdot 5 \cdot 7$$

Solution

- First expression:

$$3^2 \cdot 7^3 \cdot 11 = 9 \cdot 343 \cdot 11 = 33957$$

- Second expression:

$$2^3 \cdot 5 \cdot 7 = 8 \cdot 5 \cdot 7 = 280$$

Section 5.1 - Exercise 25

Problem

Find the least common multiple (LCM) of the given integers.

Solution

The least common multiple of two integers is given by:

$lcm(a, b) = \frac{|a \cdot b|}{gcd(a,b)}$

Section 5.2 - Exercise 12

Problem

Explain how to compute $a^n \mod z$ using repeated squaring.

Solution

To compute $a^n \mod z$ efficiently, use the **repeated squaring** algorithm:

1. Initialize:

$$result = 1, \quad x = a \mod z$$

2. While $n > 0$:

- If $n \mod 2 == 1$, set:

$$result = (result \cdot x) \mod z$$

- Update:

$$x = (x \cdot x) \mod z, \quad n = \lfloor n/2 floor$$

3. Return:

$$result$$

Section 5.2 - Exercise 18

Problem

Convert the decimal number 61 to binary.

Solution

- Follow the steps:

$$61 \div 2 = 30 remainder 1$$

$$30 \div 2 = 15 remainder 0$$

$$15 \div 2 = 7 remainder 1$$

$$7 \div 2 = 3 remainder 1$$

$$3 \div 2 = 1 remainder 1$$

$$1 \div 2 = 0 remainder 1$$

Therefore:

$$61_{10} = 111101_2$$

Section 5.2 - Exercise 19

Problem

Prove that the number of pr\cdot is infinite.

Solution

Proof by contradiction:

- Assume that the number of pr\cdot is finite, say $p_1, p_2, \ldots, p_n$.
- Consider the number:

$$N = p_1 p_2 \ldots p_n + 1$$

- N is not divisible by any prime in the list, so it is either prime or divisible by some prime not in the list.
- This contradiction implies that there are infinitely many pr\cdot.

Section 5.3 - Exercise 3

Problem

Use the Euclidean algorithm to find the greatest common divisor of 220 and 1400.

Solution

$$1400 \div 220 = 6 remainder 80$$

$$220 \div 80 = 2 remainder 60$$

$$80 \div 60 = 1 remainder 20$$

$$60 \div 20 = 3 remainder 0$$

Therefore:

$$gcd(220, 1400) = 20$$

Section 5.3 - Exercise 7

Problem

Use the Euclidean algorithm to find the greatest common divisor of 27 and 27.

Solution

$$27 \div 27 = 1 remainder 0$$

Therefore:

$$gcd(27, 27) = 27$$

Geometric Sum Solutions

Problem 1

Find an expression for:

$\sum_{k=0}^{n} 2^k$

Solution

Using the geometric sum formula:

$$\sum k = 0n2k = 2n + 1 - 12 - 1 = 2n + 1 - 1 \sum_{k=0}^{n} 2^k = \frac{2^{n+1}-1}{2-1} = 2^{n+1} - 1$$

In binary, a number of the form $2n + 1 - 12^{n+1} - 1$ is written as a string of $n + 1n + 1$ ones. For example:

- For $n = 3n = 3$, we have:

$$23 + 1 - 1 = 1510 = 111122^{3+1} - 1 = 15_{10} = 1111_2$$

Thus, the number is written in binary as a sequence of $n + 1n + 1$ ones.

Problem 2

Find an expression for:

$\sum_{k=0}^{n} 10^k$

Solution

Using the geometric sum formula:

$$\sum k = 0n10k = 10n + 1 - 110 - 1 = 10n + 1 - 19 \sum_{k=0}^{n} 10^k = \frac{10^{n+1}-1}{10-1} = \frac{10^{n+1}-1}{9}$$

For example:

- If $n = 2n = 2$:

$$102 + 1 - 19 = 9999 = 111 \frac{10^{2+1}-1}{9} = \frac{999}{9} = 111$$

Problem 3

Find an expression for:

$\sum_{k=0}^{n} 3^k$

How would this number be written in base 3?

Solution

Using the geometric sum formula:

$$\sum k = 0n3k = 3n + 1 - 13 - 1 = 3n + 1 - 12 \sum_{k=0}^{n} 3^k = \frac{3^{n+1}-1}{3-1} = \frac{3^{n+1}-1}{2}$$

In base 3, a number of the form $3n + 1 - 13^{n+1} - 1$ is written as a string of $n + 1n + 1$ ones. For example:

- For $n = 3n = 3$, we have:

$$33 + 1 - 1 = 8010 = 111133^{3+1} - 1 = 80_{10} = 1111_3$$

Problem 4

Solution

The value of $111b111_b$ is:

$$1{\times}b2 + 1{\times}b1 + 1{\times}b0 = b2 + b + 11 \times b^2 + 1 \times b^1 + 1 \times b^0 = b^2 + b + 1$$

For a string of kk ones in base bb, the value is:

$$\sum_{i=0}^{k-1} b^i = \frac{b^k-1}{b-1}$$

For example:

- If $b = 2b = 2andk = 3k = 3$:

$$23 - 12 - 1 = 7\frac{2^3-1}{2-1} = 7$$

Problem 5

Solution

The value of 3334 in decimal is:

$$3{\times}43 + 3{\times}42 + 3{\times}41 + 4{\times}403 \times 4^3 + 3 \times 4^2 + 3 \times 4^1 + 4 \times 4^0 = 3{\times}64 + 3{\times}16 + 3{\times}4 + 4 = 192 + 48$$

The value of $dddbdddb$, where $d = b - 1d = b - 1$, is:

$$(b-1)(b3 + b2 + b + 1)(b-1)(b^3 + b^2 + b + 1)$$

Simplify using the geometric sum formula:

$(b-1)\times b4-1b-1=b4-1(b-1)\times\frac{b^4-1}{b-1}=b^4-1$

For example, if $b=5b=5$:

$4\times(53+52+5+1)=4\times156=6244\times(5^3+5^2+5+1)=4\times156=624$
