Problem 0.1. (2 points) Find the reduced row-echelon form of the matrix

$$A = \begin{array}{c} \bullet \\ 4 \\ 7 \end{array} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 2 \\ 7 & 8 & 9 & 3 \end{bmatrix} \sim \begin{array}{c} \\ \bullet \\ 2 \end{array} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -2 \\ 0 & -6 & -12 & -4 \end{bmatrix} \sim \begin{array}{c} \\ \times\ -1/3 \\ \\ \end{array} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim$$

$$\begin{array}{c} 2 \\ \bullet \\ \\ \end{array} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -1/3 \\ 0 & 1 & 2 & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{rref}(A) = \boxed{\begin{bmatrix} 1 & 0 & -1 & -1/3 \\ 0 & 1 & 2 & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

Problem 0.2. (2 points) Find the solution set of the linear system below in parametric vector form.

$$\left\{ \begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 1 \\ 4x & + & 5y & + & 6z & = & 2 \\ 7x & + & 8y & + & 9z & = & 3 \end{array} \right.$$

Since the augmented matrix for the system is the matrix in Problem 0.1, we have

$$\left\{ \begin{array}{rcrcrcr} x & & & - & z & = & -1/3 \\ & & y & + & 2z & = & 2/3 \end{array} \right. \quad \Rightarrow \quad \left\{ \begin{array}{rcrcr} x & = & -1/3 & + & z \\ y & = & 2/3 & - & 2z \end{array} \right. \quad \Rightarrow$$

$$\text{solution set} \quad \boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1/3 \\ 2/3 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}$$

Problem 0.3. (2 points) Simplify the following matrix expression:

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix} + \begin{bmatrix} e \\ f \\ g \end{bmatrix} \begin{bmatrix} e & f \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e \\ f \\ g \end{bmatrix} \begin{bmatrix} e & f \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = ca + ge$$

$$\text{simplified expression} \quad \boxed{ca + ge}$$

Problem 0.4. (2 points) Find and simplify the inverse of the matrix

$$A = \begin{array}{c} \bullet \\ a \\ \end{array}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ a & 1 & 0 & 0 & 1 & 0 \\ 0 & a & 1 & 0 & 0 & 1 \end{array}\right] \sim \begin{array}{c} \\ \bullet \\ a \end{array}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -a & 1 & 0 \\ 0 & a & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -a & 1 & 0 \\ 0 & 0 & 1 & a^2 & -a & 1 \end{array}\right]$$

$$A^{-1} = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ a^2 & -a & 1 \end{bmatrix}}$$

Problem 0.5. (2 points) Find the LU-factorization of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 2 \\ 7 & 8 & 9 & 3 \end{bmatrix}$$

From Problem 0.1

$$L = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix}} \qquad U = \boxed{\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

Problem 0.6. (2 points) Find the CR-factorization of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 2 \\ 7 & 8 & 9 & 3 \end{bmatrix}$$

From Problem 0.1

$$C = \boxed{\begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}} \qquad R = \boxed{\begin{bmatrix} 1 & 0 & -1 & -1/3 \\ 0 & 1 & 2 & 2/3 \end{bmatrix}}$$

Problem 0.7. (2 points) Showing/explaining your work, find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix} \qquad \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{vmatrix} = -1 \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = -1(1+1) = -2$$

$$\det A = \boxed{-2}$$

Problem 0.8. (2 points) If possible, diagonalize the matrix below.

$$A = \begin{bmatrix} 5 & -1 \\ 9 & -1 \end{bmatrix} \qquad p_A(\lambda) = \lambda^2 - (5-1)\lambda + (-5+9) = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2$$

$$\underline{\lambda_1 = \lambda_2 = 2}: \qquad A - \lambda_1 I = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}$$

Because there is only one independent eigenvector, the matrix A is $\boxed{\text{not diagonalizable.}}$

$A = SDS^{-1}$ where $S = \boxed{\qquad}$ and $D = \boxed{\qquad}$

Problem 0.9. (2 points) Showing your work, deterimine if the following set is linearly dependent or linearly independent.

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\} \qquad \begin{array}{c} \bullet \\ 2 \\ 3 \end{array} \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \sim \begin{array}{c} \\ \bullet \\ 2 \end{array} \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

No pivot in the third column.

Circle one: $\boxed{\text{Linearly dependent}}$ Linearly independent

---

Problem 0.10. (4 points) Find bases for the four fundamental subspaces of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 7 & 4 \end{bmatrix}$$

$$\begin{array}{c} \bullet \\ 3 \end{array} \left[\begin{array}{ccc|cc} 1 & 2 & 1 & 1 & 0 \\ 3 & 7 & 4 & 0 & 1 \end{array}\right] \sim \begin{array}{c} 2 \\ \bullet \end{array} \left[\begin{array}{ccc|cc} 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & -3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|cc} 1 & 0 & -1 & 7 & -2 \\ 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

basis for $\mathrm{Col}(A)$: $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \end{bmatrix} \right\}$

basis for $\mathrm{Row}(A)$: $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

basis for $\mathrm{Nul}(A)$: $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

basis for $\mathrm{Nul}(A^T)$: $\emptyset$

Problem 0.11. (4 points) Diagonalize, if possible.

$$A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 3 & -1 & -2 \end{bmatrix}$$

$$\begin{array}{c} {\scriptstyle 2} \\ \bullet \\ {\scriptstyle 3} \end{array}\begin{vmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 3 & -1 & -2 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{vmatrix} = -\begin{vmatrix} -1 & 1 \\ -1 & 1 \end{vmatrix} = 0$$

$$p_A(\lambda) = -\lambda^3 + (2+0-2)\lambda^2 - \left(\begin{vmatrix} 0 & -1 \\ -1 & -2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 3 & -2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} \right) \lambda = -\lambda^3 + \lambda \; = \; -\lambda(\lambda^2 - 1) = -\lambda(\lambda-1)(\lambda+1)$$

$$\underline{\lambda_1 = 0}: \qquad A - \lambda_1 I = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 3 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\underline{\lambda_2 = 1}: \qquad A - \lambda_2 I = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 3 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 3 & -1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\underline{\lambda_3 = -1}: \qquad A - \lambda_3 I = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 3 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 3 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -4 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}_3 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$A = SDS^{-1}$ where $S = \boxed{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}}$ and $D = \boxed{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}$

Problem 0.12. (2 points) Find the least squares solutions of $\begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$$\mathbf{x} = \boxed{\begin{bmatrix} 1/3 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

Multiplying on the left by the transpose of the coeffient matrix gives the normal equations:

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Applying Gauss-Jordan to the augmented matrix,

$$\begin{array}{c} \bullet \\ {\scriptstyle -1} \end{array} \begin{bmatrix} 3 & -3 & 1 \\ -3 & 3 & -1 \end{bmatrix} \sim \begin{array}{c} {\scriptstyle \times\, 1/3} \\ \end{array} \begin{bmatrix} 3 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1/3 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} 1/3 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Problem 1.1. (10 points) Find the QR-factorization of the matrix

$$A = \begin{bmatrix} 6 & -5 & 0 \\ 8 & 10 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$$\mathbf{v}_1 = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix} \quad ||\mathbf{v}_1|| = \sqrt{36+64} = 10 \quad \mathbf{w}_1 = \begin{bmatrix} 3/5 \\ 4/5 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_2 = \begin{bmatrix} -5 \\ 10 \\ 0 \end{bmatrix} - \frac{-30+80+0}{36+64+0} \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 10 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} -8 \\ 6 \\ 0 \end{bmatrix} \quad ||\mathbf{v}_2|| = \sqrt{64+36} = 10 \quad \mathbf{w}_2 = \begin{bmatrix} -4/5 \\ 3/5 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} - \frac{0}{100} \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix} - \frac{0}{100} \begin{bmatrix} -8 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} \quad ||\mathbf{v}_3|| = 7 \quad \mathbf{w}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad R = \begin{bmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & -5 & 0 \\ 8 & 10 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$$Q = \boxed{\begin{bmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \qquad R = \boxed{\begin{bmatrix} 10 & 5 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 7 \end{bmatrix}}$$

Problem 1.2. (10 points) Find an orthogonal diagonalization of the following matrix:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1+1 = 2$$

$$\begin{aligned} p_A(\lambda) &= -\lambda^3 + (0+0+0)\lambda^2 - \left(\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \right) \lambda + 2 \\ &= -\lambda^3 + 3\lambda + 2 \qquad\qquad \text{note: } p_A(-1) = 1 - 3 + 2 = 0 \\ &= -(\lambda+1)(\lambda^2 - 1\lambda - 2) \;=\; -(\lambda+1)(\lambda+1)(\lambda-2) \;=\; -(\lambda+1)^2(\lambda-2) \end{aligned}$$

$\underline{\lambda_1 = \lambda_2 = -1:}$

$$A - \lambda_1 I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \quad \tilde{\mathbf{v}}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \tilde{\mathbf{v}}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1+0+0}{1+1+0}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$\underline{\lambda_3 = 2:}$

$$A - \lambda_3 I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{u}_1 = \frac{1}{||\mathbf{v}_1||}\mathbf{v}_1 = \frac{1}{\sqrt{1+1+0}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{u}_2 = \frac{1}{||\mathbf{v}_2||}\mathbf{v}_2 = \frac{1}{\sqrt{1+1+4}}\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$$\mathbf{u}_3 = \frac{1}{||\mathbf{v}_3||}\mathbf{v}_3 = \frac{1}{\sqrt{1+1+1}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A = UDU^T \text{ where } U = \boxed{\begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \end{bmatrix}} \quad \text{and} \quad D = \boxed{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}$$

Problem 1.3. (9 points) Find the singular value decomposition (SVD) of the matrix:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & -2 \\ 2 & 0 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 2 & 2 \\ 4 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & -2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 20 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 20 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^T$$

Note: Although the Gram matrix is orthogonally diagonalized with minimal effort, the eigenvalues and eigenvectors must ordered so that the eigenvalues are monotonically decreasing.

Hence, $\sigma_1 = \sqrt{20} = 2\sqrt{5},\ \sigma_2 = \sqrt{9} = 3,\ \Sigma = \begin{bmatrix} 2\sqrt{5} & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix},\ V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = [\mathbf{v}1 \quad \mathbf{v}_2].$

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{2\sqrt{5}} \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2\sqrt{5}/5 \\ -\sqrt{5}/5 \\ 0 \end{bmatrix} \qquad \mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

The remaining columns of U (just one in this case) are found as an orthonormal basis for $\text{nul}(A^T)$:

$$A^T = \begin{array}{c} \bullet \\ 4 \end{array} \begin{bmatrix} 1 & 2 & 2 \\ 4 & -2 & 0 \end{bmatrix} \sim \begin{array}{c} \\ \times -1/10 \end{array} \begin{bmatrix} 1 & 2 & 2 \\ 0 & -10 & -8 \end{bmatrix} \sim \begin{array}{c} 2 \\ \bullet \end{array} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 4/5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2/5 \\ 0 & 1 & 4/5 \end{bmatrix} \quad \tilde{\mathbf{u}}_3 = \begin{bmatrix} -2/5 \\ -4/5 \\ 1 \end{bmatrix}$$

$$\mathbf{u}_3 = \frac{1}{||\mathbf{u}_3||} \mathbf{u}_3 = \frac{1}{\sqrt{4+16+25}} \begin{bmatrix} -2 \\ -4 \\ 5 \end{bmatrix} = \frac{1}{3\sqrt{5}} \begin{bmatrix} -2 \\ -4 \\ 5 \end{bmatrix}$$

$A = U\Sigma V^T$
where

$$U = \boxed{\begin{bmatrix} \frac{2\sqrt{5}}{5} & \frac{1}{3} & -\frac{2\sqrt{5}}{15} \\ -\frac{\sqrt{5}}{5} & \frac{2}{3} & -\frac{4\sqrt{5}}{15} \\ 0 & \frac{2}{3} & \frac{\sqrt{5}}{3} \end{bmatrix}} \qquad \Sigma = \boxed{\begin{bmatrix} 2\sqrt{5} & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}} \qquad V = \boxed{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$$

Problem 2. (10 points) Evaluate the truth of each statement below. If the statement is true write T in the box preceding the statement. Otherwise, write F.

(a) $\boxed{T}$ The set of pivot columns of a matrix is linearly independent.

(b) $\boxed{T}$ If $\det(A) = 1$, then A is invertible.

(c) $\boxed{T}$ If $1 \in \sigma(A)$, then $A - I$ is singular.

(d) $\boxed{T}$ For an $n \times n$ matrix A with n signular values, $||A\mathbf{x}|| \geq \sigma_n ||\mathbf{x}||$.

(e) $\boxed{T}$ The normal equations $A^T A\mathbf{x} = A^T\mathbf{b}$ always have at least one solution.

Problem 3. (10 points) The built-in MATLAB function `null` returns a matrix having columns that form an orthonormal basis for the null space of the input matrix. For example,

```
>> null([0 1 0; 0 2 0; 0 3 0])

ans =
          0    -1
          0     0
          1     0
```

Provide the output expected from MATLAB for the given commands in the answer boxes:

```
>> A = [1 1 1 0; 1 1 1 0 ; 1 1 1 0; 0 0 0 0];
>> rref(A)

ans =
      1   1    1   0
      0   0    0   0
      0   0    0   0
      0   0    0   0

>> N=null(A);
>> A*N
```

```
ans =
        0     0     0
        0     0     0
        0     0     0
        0     0     0
```

```
>> transpose(N)*N       % transpose outputs the transpose of the input matrix
```

```
ans =
        1     0     0
        0     1     0
        0     0     1
```

Since A is 4×4 with one pivot, N is 4×3; i.e., a basis for $\text{nul}(A)$ contains three vectors.

Since N has orthonormal columns $N^T N = I$.