# 7 Lecture 7: Matrix Inverse

**Definition 7.1.** *A matrix of the size $n \times n$ (i.e., the number of rows equals the number of columns) is called a* **square matrix***. The square matrix*

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

*is called the* **identity** *matrix. In some context, the identity matrix of size $n \times n$ (or simply of "size $n$") is denoted by $\mathbf{I}_n$.*

**Note that:**

- For any square matrix $\mathbf{A}$, we have $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$.
- For any $m \times n$ matrix $\mathbf{A}$, we have $\mathbf{I}_m\mathbf{A} = \mathbf{A}\mathbf{I}_n = \mathbf{A}$.

## 7.1 The inverse matrix

A square matrix $\mathbf{A}$ is said to be **invertible** if there exists a square matrix $\mathbf{B}$ (of the same size) such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I} \tag{7.1}$$

In this case, $\mathbf{B}$ is an **inverse** of $\mathbf{A}$. The inverse of $\mathbf{A}$ is denoted by $\mathbf{A}^{-1}$.

**Example 7.1.** *If* $\mathbf{A} = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ *and* $\mathbf{B} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$*, then*

$$\mathbf{AB} = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \textit{and}$$

$$\mathbf{BA} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

*Thus,* $\mathbf{B} = \mathbf{A}^{-1}$.

The inverse of $\mathbf{A}$ is unique. Indeed, if $\mathbf{C}$ is another inverse of $\mathbf{A}$, we will show that $\mathbf{C} = \mathbf{B}$. By the relation (7.1), $\mathbf{AC} = \mathbf{CA} = \mathbf{I}$. Therefore,

$$\mathbf{C} = \mathbf{CI} = \mathbf{C}(\mathbf{AB}) = (\mathbf{CA})\mathbf{B} = \mathbf{IB} = \mathbf{B}.$$

A matrix that is *not invertible* is sometimes called a **singular matrix**, and an invertible matrix is called a **nonsingular matrix**.

## 7.2 Calculation of inverse matrices

### 7.2.1 For a $2 \times 2$ matrix

Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ If $ad - bc \neq 0$ then $\mathbf{A}$ is invertible and the inverse of $\mathbf{A}$ is

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The term $ad - bc$ is known as the **determinant** of $\mathbf{A}$. The determinant of a matrix $\mathbf{A}$ is written as: $det(\mathbf{A})$

**Example 7.2.** *Find the inverse of* $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$.

**Solution:** Since $det(\mathbf{A}) = 3(6) - 4(5) = -2 \neq 0$, $A$ is invertible, and

$$\mathbf{A}^{-1} = \frac{1}{-2} \begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 6/(-2) & -4/(-2) \\ -5/(-2) & 3/(-2) \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$$

### 7.2.2 For any $n \times n$ matrix:

Using Gauss-Jordan Algorithm: Row reduce the augmented matrix $\begin{bmatrix} \mathbf{A} & \mathbf{I} \end{bmatrix}$. If $\mathbf{A}$ is row equivalent to $\mathbf{I}$, then $\begin{bmatrix} \mathbf{A} & \mathbf{I} \end{bmatrix}$ is row equivalent to $\begin{bmatrix} \mathbf{I} & \mathbf{A}^{-1} \end{bmatrix}$. Otherwise, $\mathbf{A}$ does not have an inverse.

**Example 7.3.** *Find the inverse of the matrix* $\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, *if it exists.*

**Solution:** We first row reduce the matrix $\begin{bmatrix} \mathbf{A} & \mathbf{I} \end{bmatrix}$:

$$\begin{bmatrix} \mathbf{A} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}.$$

Since $\mathbf{A} \sim \mathbf{I}$, that $\mathbf{A}$ is invertible, and

$$\mathbf{A}^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$$

**Example 7.4.** *Find the inverse of the matrix* $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$, *if it exists.*

**Solution:** We first row reduce the matrix $\begin{bmatrix} \mathbf{A} & \mathbf{I} \end{bmatrix}$:

$$\begin{bmatrix} \mathbf{A} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 2 & -2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 & -3 & 1 \\ 0 & 1 & 2 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}.$$

Since matrix $\mathbf{A}$ cannot be row reduced to the identity matrix, $\mathbf{A}$ is not invertible.

## 7.3 Properties

(1) If $\mathbf{A}$ is an invertible matrix, then $\mathbf{A}^{-1}$ is invertible and

$$\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$$

(2) If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ invertible matrices, then so is $\mathbf{AB}$, and the inverse of $\mathbf{AB}$ is the product of the inverses of $\mathbf{A}$ and $\mathbf{B}$ in the reverse order. That is,

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

(3) If $\mathbf{A}$ is an invertible matrix, then so is $\mathbf{A}^T$, and the inverse of $\mathbf{A}^T$ is the transpose of $\mathbf{A}^{-1}$. That is,

$$\left(\mathbf{A}^T\right)^{-1} = \left(\mathbf{A}^{-1}\right)^T$$

The set of all nonsingular matrices is known as the *general linear group of degree $n$ and denoted by* $\mathbf{GL}_n$.

## 7.4 Application to linear systems with unique solutions

*If $\mathbf{A} \in \mathbf{GL}_n$, the linear system $\mathbf{Ax} = \mathbf{b}$ has a unique solution for any $\mathbf{b} \in \mathbb{R}^n$, and the solution is*

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}.$$

**Example 7.5.** *Find the solution of the linear system*

$$\begin{cases} & +x_2 & +2x_3 = 2 \\ x_1 & & +3x_3 = -4 \\ 4x_1 & -3x_2 & +8x_3 = 0 \end{cases}$$

**Solution:** *The linear system is written as*

$$\mathbf{A}\mathbf{x} = \mathbf{b},$$

*where* $\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$ *and* $\mathbf{b} = \begin{bmatrix} 2 \\ -4 \\ 0 \end{bmatrix}$. *We see from Example 7.3 that the coefficient matrix of the linear system,* $\mathbf{A}$, *is invertible. Thus, the system has a unique solution, which is*

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} -33 \\ -20 \\ 11 \end{bmatrix}.$$

# 8 Lecture 8: Matrix Factorization with Elementary Matrices

## 8.1 Upper triangular, lower triangular, and diagonal matrices

*(i) An* $n \times n$ *matrix* $\mathbf{A} = [a_{ij}]$ *is* **upper triangular** *if and only if* $a_{ij} = 0$ *when* $i > j$*. Examples,*

$$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 4 & 5 & 6 \\ 0 & 0 & 6 & 7 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

*In particular, an upper triangle matrix whose entries in the main diagonal are 1 is called a* **upper unitriangular***.*

*(ii) An* $n \times n$ *matrix* $\mathbf{A} = [a_{ij}]$ *is* **lower triangular** *if and only if* $a_{ij} = 0$ *when* $i < j$*. Examples,*

$$\begin{bmatrix} 1 & 0 \\ 1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 7 & 0 & 6 & 0 \\ 1 & 1 & 6 & 8 \end{bmatrix}$$

*(iii) An* $n \times n$ *matrix* $\mathbf{A} = [a_{ij}]$ *is* **diagonal** *if and only if* $a_{ij} = 0$ *when* $i \neq j$*. Examples,*

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

## 8.2 Upper trapezoidal and lower trapezoidal matrices

*(a) An* $m \times n$ *matrix* $\mathbf{A} = [a_{ij}]$ *is* **upper trapezoidal** *if and only if* $a_{ij} = 0$ *when* $i > j$*. Examples,*

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 4 & 5 & 0 & 6 \\ 0 & 0 & 6 & 2 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

*(b) An* $m \times n$ *matrix* $\mathbf{A} = [a_{ij}]$ *is* **lower trapezoidal** *if it is the transpose of an upper trapezoidal matrix.*

### 8.3 Elementary matrices

## Elementary Matrices

*An* **elementary matrix** *is obtained by applying a* **single** *elementary row operation on an identity matrix. The following example shows the three kinds of elementary matrices.*

**Example 8.1.**

$$\mathbf{E}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, \text{ applying the replacement - Replace row 3 by row 3 + (-4) times row 1)}$$

$$\mathbf{E}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ applying the interchange - interchange row 1 and row 2}$$

$$\mathbf{E}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \text{appying the scaling - scale the last row by 5.}$$

**Note** *Elementary matrices are invertible (row operations are reversible). The inverse of an elementary matrix is an elementary matrix of the same kind.*

**Example 8.2.** *With three matrices* $\mathbf{E}_1, \mathbf{E}_2$ *and* $\mathbf{E}_3$ *in Example 8.1,*

$$\mathbf{E}_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}, \text{ applying the replacement - Replace row 3 by row 3 + 4 times row 1)}$$

$$\mathbf{E}_2^{-1} = \mathbf{E}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ applying the interchange - interchange row 1 and row 2}$$

$$\mathbf{E}_{3^{-1}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}, \text{appying the scaling - scale the last row by } \tfrac{1}{5}.$$

*Let* $\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ *be a* $3 \times 3$ *matrix in a general form and* $\mathbf{E}_1, \mathbf{E}_2$ *and* $\mathbf{E}_3$ *be three matrices in Example 8.1, we see that*

$$\mathbf{E}_1\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{bmatrix}.$$

*The matrix on the right-hand side is also obtained by replacing row 3 of matrix* $\mathbf{A}$ *with row 3 + (-4) times row 1. This is the same operation to produce the elementary matrix* $\mathbf{E}_1$ *from the identity matrix. Similarly,*

$$\mathbf{E}_2\mathbf{A} = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}$$

*equals the matrix obtained by interchange row 1 and row 2 of matrix* $\mathbf{A}$*, and*

$$\mathbf{E}_3\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix}$$

$\mathbf{E}_3\mathbf{A}$ *equals the matrix obtained by multiplying the last row of matrix* $\mathbf{A}$ *by 5. These illustrations show that a single row operation applied to a matrix equals this matrix multiplied by an elementary matrix on the left side (or left-multiplication).*

## 8.4 Gauss-Jordan through products of elementary matrices and LU factorization

*Suppose that to row reduce a matrix* $\mathbf{A}$ *to its proto-row reduce echelon form, denoted by* $\text{pref}(\mathbf{A})$*, we need to apply* $\ell$ *times elementary row operation. Each elementary row operation* $j^{th}$ *associated with an elementary matrix* $\mathbf{E}_j$*, then from the discussion above we have*

$$\text{pref}(\mathbf{A}) = \mathbf{E}_\ell\mathbf{E}_{\ell-1}\dots\mathbf{E}_2\mathbf{E}_1\mathbf{A}$$

*Each elementary matrix is invertible. Thus, the multiplication is invertible. Therefore, matrix* $\mathbf{A}$ *can be written as a multiplication as follows*

$$\mathbf{A} = (\mathbf{E}_\ell\mathbf{E}_{\ell-1}\dots\mathbf{E}_2\mathbf{E}_1)^{-1}\text{pref}(\mathbf{A})$$

*When no row exchanges are required in the forward phase,* $\mathbf{L} := (\mathbf{E}_\ell\mathbf{E}_{\ell-1}\dots\mathbf{E}_2\mathbf{E}_1)^{-1}$ *is lower unitriangular and* $\mathbf{U} := \text{pref}(\mathbf{A})$*, the proto-row-echelon form of* $\mathbf{A}$*, is upper trapezoidal. Thus, we can write as the product of a lower unitriangular matrix and an upper trapezoidal matrix.*

$$\mathbf{A} = \mathbf{L}\mathbf{U}$$

*This result is known as the LU factorization of* $\mathbf{A}$*.*

**Remark 8.1.** *Some important observations:*

1. *The LU factorization does not exist if row exchange is needed in the forward phase of the Gauss-Jordan algorithm.*

2. *The LU factorization may not be unique; for some matrices, there is more than one way to factor the matrix as the product of a lower unitriangular matrix and the proto-row-echelon form of the matrix.*

### 8.5 LU factorization and Algorithms

1. *Reduce* **A** *to an echelon form* **U** *by a sequence of* **row replacement operations**, *if possible.*
2. *Place entries in* **L** *such that the same sequence of row operations reduces* **L** *to* **I**.

*The following example illustrates how to make a matrix* **L** *when the LU factorization exists:*

**Example 8.3.** *Find an LU factorization of*

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

**Solution:** *Matrix* **A** *has four rows, so if exists then* **L** *is an* $4 \times 4$ *matrix. We find a matrix* **L** *of the form*

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix}$$

*The first column of* **L** *is the first column of* **A** *divided by the top pivot entry:*

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & * & 1 & 0 \\ -3 & * & * & 1 \end{bmatrix}$$

*Row reducing matrix* **A** *into its proto row echelon form without using the scaling operation gives*

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix} \sim \mathbf{A}_1 = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix}$$

$$\sim \mathbf{A}_2 = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix} \sim \mathrm{pref}(\mathbf{A}) = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = \mathbf{U}.$$

*To determine two entries under the pivot position in the second column of* **L**, *we use the second column of* $\mathbf{A}_1$. *Similarly to the way building the first column, divide the second column of* $\mathbf{A}_1$ *by the* *pivot, the two entries below the normalized pivot gives the two corresponding entries in the second column of* $\mathbf{L}$*, i.e., after this step, matrix* $\mathbf{L}$ *becomes*

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & * & 1 \end{bmatrix}.$$

*Doing similarly, the last missing entry in the matrix* $\mathbf{L}$ *is obtain from the fourth column of* $\mathbf{A}_2$*, and equal the last entry in the column divided by the pivot (column 3 of matrix* $\mathbf{A}_2$ *is ignored since it is not a pivot column). Thus, matrix* $\mathbf{L}$ *is*

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix}.$$

### 8.6 CR Factorization

*If* $\mathbf{A}$ *is an* $m \times n$ *matrix with the rank* $r$*, then* $\mathbf{A}$ *has the factorization of* $\mathbf{A} = \mathbf{CR}$*, where*

- $\mathbf{C}$ *is the* $m \times r$ *matrix consisting of the pivot columns of* $\mathbf{A}$ *and*
- $\mathbf{R}$ *the compact reduced row-echelon matrix of* $\mathbf{A}$ *(i.e., the matrix consisting of the nonzero rows of* $rref(\mathbf{A})$*).*

**Example 8.4.** *Find the CR-factorization of the matrix*

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 2 & 1 & 3 \\ 1 & 0 & 1 & 2 & 1 & 3 \\ 1 & 0 & 1 & 2 & 1 & 3 \end{bmatrix}$$

*Apply the Gauss-Jordan Algorithm to row reduce matrix* $\mathbf{A}$

$$\mathbf{A} \sim \begin{bmatrix} 1 & 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

*Thus,* $rank(\mathbf{A}) = 1$ *and the pivot column of* $\mathbf{A}$ *is* $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$*. The* $CR$ *factorization of* $\mathbf{A}$ *is*

$$\mathbf{A} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 2 & 1 & 3 \end{bmatrix}$$

**Example 8.5.** *Find an CR factorization of*

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 9 \end{bmatrix}$$

*We first apply the Gauss-Jordan algorithm to find the reduced-row-echelon form of* $\mathbf{A}$, *i.e.,* $rref(\mathbf{A})$*:*

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 9 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 11 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1/2 & 5/2 & -1 \\ 0 & 1 & 1/3 & 2/3 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1/2 & 0 & -7/2 \\ 0 & 1 & 1/3 & 0 & -5/3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7/6 & 0 & -1/6 \\ 0 & 1 & 1/3 & 0 & -5/3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

*Matrix* $\mathbf{A}$ *has three pivot columns 1, 2, 4, which are* $\begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -5 \\ -5 \end{bmatrix}, \begin{bmatrix} 5 \\ -8 \\ 1 \end{bmatrix}$. *Thus, the CR-factorization is*

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 5 \\ -4 & -5 & -8 \\ 2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -7/6 & 0 & -1/6 \\ 0 & 1 & 1/3 & 0 & -5/3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$