26 Lecture 26 and 27: Singular Value Decomposition (SVD)

26.1 SVD: Definitions and Examples

Definition 26.1. *For each $m \times n$ matrix $\mathbf{A}$, the* **Singular Value Decomposition** *(abbreviation is SVD) is the factorization of the form*

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T,$$

where

- $\mathbf{U}$ *is an $m \times m$ orthogonal matrix.*
- $\mathbf{\Sigma}$ *is an $m \times n$ matrix of the form*

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \sigma_r & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix}, \tag{26.1}$$

  with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$, where r denotes the rank of matrix $\mathbf{A}$.

- $\mathbf{V}$ *is an $n \times n$ orthogonal matrix.*

The entries $\sigma_1, \sigma_2, \ldots, \sigma_r$ of matrix $\mathbf{\Sigma}$ are called **singular values** *of matrix $\mathbf{A}$.*

Example 26.1. *Matrix* $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \\ 2 & 1 \end{bmatrix}$ *has an SVD*

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T,$$

where

$$\mathbf{U} = \begin{bmatrix} -1/3 & -2/3 & -2/3 \\ -2/3 & 2/3 & -1/3 \\ -2/3 & -1/3 & 2/3 \end{bmatrix}, \quad \mathbf{\Sigma} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Property 26.1. *If $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$ is the SVD of an $m \times n$ matrix $\mathbf{A}$ then the Gram matrix of $\mathbf{A}$, $\mathbf{A}^T\mathbf{A}$, is written as*

$$\mathbf{A}^T\mathbf{A} = \mathbf{V}(\mathbf{\Sigma}^T\mathbf{\Sigma})\mathbf{V}^T = \mathbf{V}\mathbf{D}\mathbf{V}^T, \tag{26.2}$$

where the matrix

$$\mathbf{D} = \begin{bmatrix} \sigma_1^2 & & & & & & \\ & \sigma_2^2 & & & & & \\ & & \ddots & & & & \\ & & & \sigma_r^2 & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix}, \quad \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0 \tag{26.3}$$

is an $n \times n$ diagonal matrix.

We note that (26.2) is an orthogonal diagonalization of $\mathbf{A}^T\mathbf{A}$, where the diagonal matrix $\mathbf{D}$ is defined by selecting the the eigenvalues of the Gram matrix in descending order.

Property 26.2. *From the SVD of $\mathbf{A}$, $\mathbf{A} = \mathbf{U\Sigma V}^T$, we see that $\mathbf{AV} = \mathbf{U\Sigma}$. With matrix $\mathbf{\Sigma}$ defined in (26.1) and writing $\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_m \end{bmatrix}$, the multiplication $\mathbf{U\Sigma}$ gives*

$$\mathbf{U\Sigma} = \begin{bmatrix} \sigma_1\mathbf{u}_1 & \sigma_2\mathbf{u}_2 & \ldots & \sigma_r\mathbf{u}_r & \mathbf{0} & \ldots & \mathbf{0} \end{bmatrix}.$$

In addition,

$$\mathbf{AV} = \mathbf{A}\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \mathbf{Av}_1 & \mathbf{Av}_2 & \ldots & \mathbf{Av}_n \end{bmatrix}.$$

Thus, from equation $\mathbf{AV} = \mathbf{U\Sigma}$ and the calculations of $\mathbf{AV}$ and $\mathbf{U\Sigma}$ above, we are able to find r equations to calculate r first columns of matrix $\mathbf{U}$, which are

$$\mathbf{u}_j = \frac{1}{\sigma_j}\mathbf{Av}_j, \quad \textit{for all} \quad j = 1, 2, \ldots, r.$$

In addition, these r columns belong to the column space of $\mathbf{A}$ and form an orthonormal basis for $\mathrm{Col}(\mathbf{A})$. *The last $(m - r)$ columns of $\mathbf{U}$ are an orthonormal basis for* $\mathrm{Nul}(\mathbf{A}^T)$.

26.2 Constructing an SVD via a three-step process

Given an $m \times n$ matrix $\mathbf{A}$, we aim to find three matrix $\mathbf{U}, \mathbf{\Sigma}$, and $\mathbf{V}$ which satisfy the definition of the SVD such that

$$\mathbf{A} = \mathbf{U\Sigma V}^T.$$

In general, the construction of $\mathbf{U}, \mathbf{\Sigma}$, and $\mathbf{V}$ can be always done via three steps, described below:

Step 1: Find an orthogonal diagonalization for the Gram matrix $\mathbf{A}^T\mathbf{A}$ as $\mathbf{A}^T\mathbf{A} = \mathbf{PDP}^T$, such that the entries in the main diagonal of matrix $\mathbf{D}$ is in decreasing order., i.e., $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0$. Here, $r = rank(\mathbf{A})$.

Step 2: Set up two matrices $\mathbf{\Sigma}$ and $\mathbf{V}$. Define

- $\mathbf{\Sigma}$ is an $m \times n$ matrix of the form (26.1), where $\sigma_1 = \sqrt{\lambda_1}, \ldots, \sigma_r = \sqrt{\lambda_r}$.
- $\mathbf{V} = \mathbf{P}$ ($\mathbf{P}$ is obtained from Step 1).

Step 3: Constructing matrix $\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_m \end{bmatrix}$. The r first columns can be determined easily by

$$\mathbf{u}_j = \frac{1}{\sigma_j}\mathbf{A}\mathbf{v}_j, \quad \text{for all} \quad j = 1, 2, \dots, r.$$

If $r < m$, then the $(m-r)$ last columns of $\mathbf{U}$ can be constructed by: first finding an arbitrary basis for $\text{Nul}(\mathbf{A}^T)$ and then apply the Gram-Schmidt to find an orthonormal basis.

Example 26.2. *Find a singular value decomposition of the matrix*

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \\ 2 & 2 \end{bmatrix}$$

Solution: *We first compute $\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 9 & -1 \\ -1 & 9 \end{bmatrix}$. The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 10$ and $\lambda_2 = 8$ and the corresponding unit eigenvectors are $\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The singular values of $\mathbf{A}$ are*

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{10}, \;\; \sigma_2 = \sqrt{\lambda_2} = \sqrt{8} = 2\sqrt{2}.$$

Matrix $\mathbf{\Sigma}$ is the same size as $\mathbf{A}$, which is

$$\mathbf{\Sigma} = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 2\sqrt{2} \\ 0 & 0 \end{bmatrix}. \tag{26.4}$$

The matrix $\mathbf{V}$ is formed by two eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of the matrix $\mathbf{A}^T\mathbf{A}$, which is

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \tag{26.5}$$

$\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix}$ *is 3×3 matrix. Note that, rank$(\mathbf{A}) = 2$, so the 2 first columns of $\mathbf{A}$ are calculated easily.*

$$\mathbf{u}_1 = \frac{1}{\sigma_1}\mathbf{A}\mathbf{v}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & -1 \\ 2 & -2 \\ 2 & 2 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{2\sqrt{5}}\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

$$\mathbf{u}_2 = \frac{1}{\sigma_2}\mathbf{A}\mathbf{v}_2 = \frac{1}{2\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 2 & -2 \\ 2 & 2 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

To find the last column of $\mathbf{U}$*, we need to find a basis for* $\text{Nul}(\mathbf{A}^T)$*.*

$$\mathbf{A}^T = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

A basis for $\text{Nul}(\mathbf{A}^T)$*,* $\mathbf{w}_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$*. We see that, vector* $\mathbf{w}_3$ *is orthogonal to two vectors* $\mathbf{u}_1$ *and* $\mathbf{u}_2$ *but* $\mathbf{w}_3$ *is not a unique vector. We need to normalize* $\mathbf{w}_3$ *and choose* $\mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$*. Matrix* $\mathbf{U}$ *is*

$$\mathbf{U} = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \end{bmatrix}. \tag{26.6}$$

The SVD of $\mathbf{A}$ *is*

$$\mathbf{A} = \mathbf{U\Sigma V}^T,$$

where $\mathbf{U}, \mathbf{\Sigma}$*, and* $\mathbf{V}$ *are found in* (26.6)*,* (26.4)*, and* (26.5)*.*

Example 26.3. *Find a singular value decomposition of* $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$*.*

Solution: We first find an orthogonal diagonalization of $\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$. The eigenvalues of $\mathbf{A}^T\mathbf{A}$ are $\lambda_1 = 10$ and $\lambda_2 = 0$ with corresponding unit eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Let $\sigma_1 = \sqrt{18} = 3\sqrt{2}$ and $\sigma_2 = 0$. Since only σ_1 is nonzero, matrix $\mathbf{A}$ is rank one. We define matrix $\mathbf{\Sigma}$ as

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and matrix $\mathbf{V}$ as

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

To construct $\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix}$, we first construct the first column $\mathbf{u}_1$:

$$\mathbf{u}_1 = \frac{1}{3\sqrt{2}}\mathbf{A}\mathbf{v}_1 = \frac{1}{3\sqrt{2}}\begin{bmatrix} 2/\sqrt{2} \\ -4/\sqrt{2} \\ 4/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

The other columns of $\mathbf{U}$ are found by extending the set $\{\mathbf{u}_1\}$ to an orthonormal basis for $\mathbb{R}^3$, and they form an orthonormal basis for Nul$(\mathbf{A}^T)$. Thus, we need to find a basis for Nul$(\mathbf{A}^T)$, then apply the Gram-Schmidt process to find an orthonormal basis.

$$\mathbf{A}^T \sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

which is equivalent to the equation $x_1 - 2x_2 + 2x_3 = 0$. A basis for the solution set of this equation is

$$\mathbf{w}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

Obviously, $\mathbf{w}_1$ and $\mathbf{w}_2$ are each orthogonal to $\mathbf{u}_1$. Apply the Gram-Schmidt process (with normalizations) to $\{\mathbf{w}_1, \mathbf{w}_2\}$, and obtain

$$\mathbf{u}_2 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -2/\sqrt{45} \\ 4/\sqrt{45} \\ 5/\sqrt{45} \end{bmatrix}$$

Finally, set $\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix}$, take $\mathbf{\Sigma}$ and $\mathbf{V}^T$ from above, we have the SVD of $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ -2/3 & 1/\sqrt{5} & 4/\sqrt{45} \\ 2/3 & 0 & 5/\sqrt{45} \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

26.3 SVD of a real symmetric matrix

Assume that $\mathbf{A}$ is symmetric matrix, then $\mathbf{A}$ is square an orthogonally diagonalizable. We first find a orthogonal diagonalization of $\mathbf{A}$ of the form

$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^T,$$

where the absolute values of eigenvalues of $\mathbf{A}$ are in decreasing order when constructing matrix $\mathbf{D}$. That is $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$. We define matrix $\mathbf{\Sigma}$ as

$$\mathbf{\Sigma} = \begin{bmatrix} |\lambda_1| & 0 & \ldots & 0 \\ 0 & |\lambda_2| & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & |\lambda_n| \end{bmatrix}$$

We then constructing a diagonal matrix $\mathbf{W}$, where entries in the main diagonal are 1 or -1 such that

$$\mathbf{D} = \mathbf{W\Sigma}.$$

Obviously, W is an orthogonal matrix. By this constructions, the orthogonal diagonalization of $\mathbf{A}$ is rewritten as

$$\mathbf{A} = \mathbf{VW\Sigma V}^T = (\mathbf{VW})\mathbf{\Sigma V}^T. \tag{26.7}$$

Choose $\mathbf{U} = \mathbf{VW}$ then $\mathbf{U}$ is an orthogonal matrix due to $\mathbf{U}^T\mathbf{U} = (\mathbf{VW})^T(\mathbf{VW}) = \mathbf{W}^T\mathbf{V}^T\mathbf{VW} = \mathbf{I}$. Thus, equation (26.7) represents a SVD of $\mathbf{A}$.

Example 26.4. *Given that matrix* $\mathbf{A} = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$ *has an orthogonal diagonalization*

$$\mathbf{A} = \mathbf{PDP}^T,$$

where,

$$\mathbf{P} = \begin{bmatrix} -2/3 & 1/\sqrt{2} & -1/\sqrt{18} \\ -1/3 & 0 & 4/\sqrt{18} \\ 2/3 & 1/\sqrt{2} & 1/\sqrt{18} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}.$$

Find a SVD of $\mathbf{A}$.

Solution: We first redefine an orthogonal diagonalization corresponding to the decreased ordering of the absolute value of eigenvalues, which is given by

$$\mathbf{A} = \mathbf{V}\widetilde{\mathbf{D}}\mathbf{V}^T,$$

where

$$\widetilde{\mathbf{D}} = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad \text{and} \quad \mathbf{V} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \end{bmatrix}.$$

We next define

$$\mathbf{\Sigma} = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{W} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

then $\mathbf{D} = \mathbf{W}\mathbf{\Sigma}$. Finally, let $\mathbf{U} = \mathbf{V}\mathbf{W} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & 2/3 \\ 0 & 4/\sqrt{18} & 1/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & -2/3 \end{bmatrix}$, then $\mathbf{A}$ has the SVD

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T.$$

26.4 Applications of the Singular Value Decomposition

26.4.1 Importance of the first (largest) singular value:

$$\|\mathbf{A}\mathbf{x}\| \leq \sigma_1 \|\mathbf{x}\|$$

for all $\mathbf{x}$ such that the multiplication $\mathbf{A}\mathbf{x}$ is defined.

26.4.2 Reduced SVD (or truncated SVD) and Least-square solutions

Given an $m \times n$ matrix $\mathbf{A}$ of the rank r. From the SVD $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}$, we introduce a truncation $\mathbf{U}_r$, $\mathbf{\Sigma}_r$ and $\mathbf{V}_r$ for $\mathbf{U}$, $\mathbf{\Sigma}$ and $\mathbf{V}$ (respectively) as

$$\mathbf{U}_r = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_r \end{bmatrix}, \qquad \mathbf{\Sigma}_r = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & \sigma_r \end{bmatrix}, \qquad \mathbf{V}_r = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_r \end{bmatrix}.$$

Then, $\mathbf{A}$ is written as a multiplication

$$\mathbf{A} = \mathbf{U}_r\mathbf{\Sigma}_r\mathbf{V}_r^T.$$

The latter form is called the **reduced SVD** of $\mathbf{A}$.

Application: (use reduced SVD to solve a least-square problem.) Given the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$, we can apply the reduced SVD to find the least-square solution easily. The least-square solution is given as the form

$$\hat{\mathbf{x}} = \mathbf{V}_r\mathbf{\Sigma}_R^{-1}\mathbf{U}_r^T\mathbf{b}.$$

Indeed,

$$\mathbf{A}\hat{\mathbf{x}} = (\mathbf{U}_r\mathbf{\Sigma}_r\mathbf{V}_r^T)(\mathbf{V}_r\mathbf{\Sigma}_R^{-1}\mathbf{U}_r^T\mathbf{b}) = \mathbf{U}_r\mathbf{U}_r^T\mathbf{b}.$$

We see that, $\mathbf{U}_r\mathbf{U}_r^T$ is the orthogonal matrix onto Col($\mathbf{A}$). Thus, $\hat{b} = \mathbf{U}_r\mathbf{U}_r^T\mathbf{b}$ is the orthogonal projection of $\mathbf{b}$ onto the column space of $\mathbf{A}$, which implies, $\hat{\mathbf{x}}$ is the least-square solution.

26.4.3 SVD and bases for four fundamental subspaces

Given an $m \times n$ matrix $\mathbf{A}$ of rank r with an SVD

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T.$$

Then

- The set of r first columns of $\mathbf{U}$: $\{\mathbf{u}_1, \dots, \mathbf{u}_r\}$ is an orthomormal basis for Col($\mathbf{A}$).

- The set of $m-r$ last columns of $\mathbf{U}$: $\{\mathbf{u}_{r+1}, \ldots, \mathbf{u}_m\}$ is an orthomormal basis for $\text{Nul}(\mathbf{A}^T)$.
- Since $\mathbf{A}\mathbf{v}_j = \mathbf{0}$ for all $j = r+1 \ldots n$ and $\dim(\text{Nul}(\mathbf{A})) = n - r$, the last $n-r$ columns of $\mathbf{V}$, $\{v_{r+1}, \ldots, \mathbf{v}_n\}$ is an orthomormal basis for $\text{Nul}(\mathbf{A})$.
- The set of r first columns of $\mathbf{V}$: $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is an orthomormal basis for $\text{Row}(\mathbf{A})$.

Example 26.5. *The singular value decomposition of*

$$\mathbf{A} = \begin{bmatrix} 10 & 5 \\ 10 & 5 \\ 2 & 11 \\ 2 & 11 \end{bmatrix} \quad \textit{is} \quad \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T \ \textit{where}$$

$$\mathbf{U} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} 20 & 0 \\ 0 & 10 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{V} = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

(a) Find the orthogonal diagonalization of the Gram matrix $\mathbf{A}^T\mathbf{A}$.

(b) Find the orthogonal diagonalization of the Gram matrix of $\mathbf{A}^T$ which is $\mathbf{A}\mathbf{A}^T$

(c) Find an orthonormal basis for each of the four fundamental subspaces of $\mathbf{A}$.

Solution:

(a) From the SVD of $\mathbf{A}$ we have

$$\mathbf{A}^T\mathbf{A} = \mathbf{V}\boldsymbol{\Sigma}^T\mathbf{U}^T\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T = \mathbf{V}\boldsymbol{\Sigma}^T\boldsymbol{\Sigma}\mathbf{V}^T = \mathbf{V}\mathbf{D}\mathbf{V}^T \text{ where}$$

$$\mathbf{D} = \boldsymbol{\Sigma}^T\boldsymbol{\Sigma} = \begin{bmatrix} 20 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \end{bmatrix}\begin{bmatrix} 20 & 0 \\ 0 & 10 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 400 & 0 \\ 0 & 100 \end{bmatrix}$$

(b) The orthogonal diagonalization of the Gram matrix of $\mathbf{A}^T$ which is $\mathbf{A}\mathbf{A}^T$.

$$\mathbf{A}\mathbf{A}^T = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T\mathbf{V}\boldsymbol{\Sigma}^T\mathbf{U}^T = \mathbf{U}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T\mathbf{U}^T = \mathbf{U}\Lambda\mathbf{U}^T \text{ where}$$

$$\Lambda = \boldsymbol{\Sigma}\boldsymbol{\Sigma}^T = \begin{bmatrix} 20 & 0 \\ 0 & 10 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 20 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 400 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) Find orthonormal basis for each of the four fundamental subspaces of $\mathbf{A}$:

$$\text{An orthonormal basis for } \mathrm{Col}(\mathbf{A}) \text{ is: } \left\{ \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix} \right\}$$

$$\text{An orthonormal basis for } \mathrm{Nul}(\mathbf{A}^T) \text{ is: } \left\{ \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} \right\}$$

$$\text{An orthonormal basis for } \mathrm{Row}(\mathbf{A}) \text{ is: } \left\{ \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}, \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix} \right\}$$

$$\text{An orthonormal basis for } \mathrm{Nul}(\mathbf{A}) \text{ is: } \quad \emptyset$$

26.5 Suggested problems

Problem 26.1. *Find the SVD of matrix* $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \\ 2 & 1 \end{bmatrix}$.

Problem 26.2. *Suppose that* $\mathbf{A} = \mathbf{u}_1\mathbf{v}_1^T$*, where* $\mathbf{u}_1 = \begin{bmatrix} a \\ b \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} c \\ d \end{bmatrix}$ *such that* $\|\mathbf{u}_1\| = \|\mathbf{v}_1\| = 1$.

(a) Find two vectors $\mathbf{u}_2, \mathbf{v}_2 \in \mathbf{R}^2$ *such that* $\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}$ *and* $\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$ *are orthogonal matrices.*

(b) Find SVD of $\mathbf{A}$.