1.5 Review Exercises

1. What is a propositional function?
2. What is a domain of discourse?
3. What is a universally quantified statement?
4. What is a counterexample?
5. What is an existentially quantified statement?
6. State the generalized De Morgan's laws for logic.
7. Explain how to prove that a universally quantified statement is true.
8. Explain how to prove that an existentially quantified statement is true.
9. Explain how to prove that a universally quantified statement is false.
10. Explain how to prove that an existentially quantified statement is false.
11. State the universal instantiation rule of inference.
12. State the universal generalization rule of inference.
13. State the existential instantiation rule of inference.
14. State the existential generalization rule of inference.

1.5 Exercises

In Exercises 1–6, tell whether the statement is a propositional function. For each statement that is a propositional function, give a domain of discourse.

1. $(2n + 1)^2$ is an odd integer.
2. Choose an integer between 1 and 10.
3. Let x be a real number.
4. The movie won the Academy Award as the best picture of 1955.
5. $1 + 3 = 4$.
6. There exists x such that $x < y$ (x, y real numbers).

Let $P(n)$ be the propositional function "n divides 77." Write each proposition in Exercises 7–15 in words and tell whether it is true or false. The domain of discourse is $\mathbf{Z}^+$.

7. $P(11)$
8. $P(1)$
9. $P(3)$
10. $\forall n\, P(n)$
11. $\exists n\, P(n)$
12. $\forall n \neg P(n)$
13. $\exists n \neg P(n)$
14. $\neg(\forall n\, P(n))$
15. $\neg(\exists n\, P(n))$

Let $P(x)$ be the propositional function "$x \geq x^2$." Tell whether each proposition in Exercises 16–24 is true or false. The domain of discourse is $\mathbf{R}$.

16. $P(1)$
17. $P(2)$
18. $P(1/2)$
19. $\forall x\, P(x)$
20. $\exists x\, P(x)$
21. $\neg(\forall x\, P(x))$
22. $\neg(\exists x\, P(x))$
23. $\forall x \neg P(x)$
24. $\exists x \neg P(x)$

Suppose that the domain of discourse of the propositional function P is $\{1, 2, 3, 4\}$. Rewrite each propositional function in Exercises 25–31 using only negation, disjunction, and conjunction.

25. $\forall x\, P(x)$
26. $\forall x \neg P(x)$
27. $\neg(\forall x\, P(x))$
28. $\exists x\, P(x)$
29. $\exists x \neg P(x)$
30. $\neg(\exists x\, P(x))$
31. $\forall x((x \neq 1) \to P(x))$

Let $P(x)$ denote the statement "x is taking a math course." The domain of discourse is the set of all students. Write each proposition in Exercises 32–37 in words.

32. $\forall x\, P(x)$
33. $\exists x\, P(x)$
34. $\forall x \neg P(x)$
35. $\exists x \neg P(x)$
36. $\neg(\forall x\, P(x))$
37. $\neg(\exists x\, P(x))$
38. Write the negation of each proposition in Exercises 32–37 symbolically and in words.

Let $P(x)$ denote the statement "x is a professional athlete," and let $Q(x)$ denote the statement "x plays soccer." The domain of discourse is the set of all people. Write each proposition in Exercises 39–46 in words. Determine the truth value of each statement.

39. $\forall x\, (P(x) \to Q(x))$
40. $\exists x\, (P(x) \to Q(x))$
41. $\forall x\, (Q(x) \to P(x))$
42. $\exists x\, (Q(x) \to P(x))$
43. $\forall x\, (P(x) \vee Q(x))$
44. $\exists x\, (P(x) \vee Q(x))$
45. $\forall x\, (P(x) \wedge Q(x))$
46. $\exists x\, (P(x) \wedge Q(x))$
47. Write the negation of each proposition in Exercises 39–46 symbolically and in words.

Let $P(x)$ denote the statement "x is an accountant," and let $Q(x)$ denote the statement "x owns a Porsche." Write each statement in Exercises 48–51 symbolically.

48. All accountants own Porsches.
49. Some accountant owns a Porsche.
50. All owners of Porsches are accountants.
51. Someone who owns a Porsche is an accountant.
52. Write the negation of each proposition in Exercises 48–51 symbolically and in words.

Determine the truth value of each statement in Exercises 53–58. The domain of discourse is $\mathbf{R}$. Justify your answers.

53. $\forall x(x^2 > x)$
54. $\exists x(x^2 > x)$

55. $\forall x(x > 1 \to x^2 > x)$

56. $\exists x(x > 1 \to x^2 > x)$

57. $\forall x(x > 1 \to x/(x^2 + 1) < 1/3)$

58. $\exists x(x > 1 \to x/(x^2 + 1) < 1/3)$

59. Write the negation of each proposition in Exercises 53–58 symbolically and in words.

60. Could the pseudocode of Example 1.5.7 be written as follows?

```
for i = 1 to n
  if (¬P(d_i))
    return false
  else
    return true
```

What is the literal meaning of each statement in Exercises 61–71? What is the intended meaning? Clarify each statement by rephrasing it and writing it symbolically.

61. From *Dear Abby:* All men do not cheat on their wives.

62. From the *San Antonio Express-News:* All old things don't covet twenty-somethings.

63. All 74 hospitals did not report every month.

64. Economist Robert J. Samuelson: Every environmental problem is not a tragedy.

65. Comment from a Door County alderman: This is still Door County and we all don't have a degree.

66. Headline over a Martha Stewart column: All lampshades can't be cleaned.

67. Headline in the *New York Times:* A World Where All Is Not Sweetness and Light.

68. Headline over a story about subsidized housing: Everyone can't afford home.

69. George W. Bush: I understand everybody in this country doesn't agree with the decisions I've made.

70. From *Newsweek:* Formal investigations are a sound practice in the right circumstances, but every circumstance is not right.

71. Joe Girardi (manager of the New York Yankees): Every move is not going to work out.

72. (a) Use a truth table to prove that if p and q are propositions, at least one of $p \to q$ or $q \to p$ is true.

(b) Let $I(x)$ be the propositional function "x is an integer" and let $P(x)$ be the propositional function "x is a positive number." The domain of discourse is $\mathbf{R}$. Determine whether or not the following proof that all integers are positive or all positive real numbers are integers is correct.

By part (a),

$$\forall x\,((I(x) \to P(x)) \vee (P(x) \to I(x)))$$

is true. In words: For all x, if x is an integer, then x is positive; or if x is positive, then x is an integer. Therefore, all integers are positive or all positive real numbers are integers.

73. Prove Theorem 1.5.14, part (b).

74. Analyze the following comments by film critic Roger Ebert: No good movie is too long. No bad movie is short enough. *Love Actually* is good, but it is too long.

75. Which rule of inference is used in the following argument? Every rational number is of the form p/q, where p and q are integers. Therefore, 9.345 is of the form p/q.

In Exercises 76–78, give an argument using rules of inference to show that the conclusion follows from the hypotheses.

76. Hypotheses: Everyone in the class has a graphing calculator. Everyone who has a graphing calculator understands the trigonometric functions. Conclusion: Ralphie, who is in the class, understands the trigonometric functions.

77. Hypotheses: Ken, a member of the Titans, can hit the ball a long way. Everyone who can hit the ball a long way can make a lot of money. Conclusion: Some member of the Titans can make a lot of money.

78. Hypotheses: Everyone in the discrete mathematics class loves proofs. Someone in the discrete mathematics class has never taken calculus. Conclusion: Someone who loves proofs has never taken calculus.

79. Show that universal generalization (see Table 1.5.1) is valid.

80. Show that existential instantiation (see Table 1.5.1) is valid.

81. Show that existential generalization (see Table 1.5.1) is valid.

1.6 Nested Quantifiers

Consider writing the statement

The sum of any two positive real numbers is positive,

symbolically. We first note that since two numbers are involved, we will need two variables, say x and y. The assertion can be restated as: If $x > 0$ and $y > 0$, then $x + y > 0$. The given statement says that the sum of *any* two positive real numbers is positive, so we need two universal quantifiers. If we let $P(x, y)$ denote the expression $(x > 0) \wedge (y > 0) \to (x + y > 0)$, the given statement can be written symbolically as

$$\forall x \forall y\, P(x, y).$$

In words, for every x and for every y, if $x > 0$ and $y > 0$, then $x + y > 0$. The domain of discourse of the two-variable propositional function P is $\mathbf{R} \times \mathbf{R}$, which means that each variable x and y must belong to the set of real numbers. Multiple quantifiers such as $\forall x \forall y$ are said to be **nested quantifiers.** In this section we explore nested quantifiers in detail.

Example 1.6.1 Restate $\forall m \exists n(m < n)$ in words. The domain of discourse is the set $\mathbf{Z} \times \mathbf{Z}$.

SOLUTION We may first rephrase this statement as: For every m, there exists n such that $m < n$. Less formally, this means that if you take any integer m whatsoever, there is an integer n greater than m. Another restatement is then: There is no greatest integer. ◀

Example 1.6.2 Write the assertion

Everybody loves somebody,

symbolically, letting $L(x, y)$ be the statement "x loves y."

SOLUTION "Everybody" requires universal quantification and "somebody" requires existential quantification. Thus, the given statement may be written symbolically as

$$\forall x \exists y\, L(x, y).$$

In words, for every person x, there exists a person y such that x loves y.

Notice that

$$\exists x \forall y\, L(x, y)$$

is *not* a correct interpretation of the original statement. This latter statement is: There exists a person x such that for all y, x loves y. Less formally, someone loves everyone. The order of quantifiers is important; changing the order can change the meaning. ◀

By definition, the statement $\forall x \forall y\, P(x, y)$, with domain of discourse $X \times Y$, is true if, for *every* $x \in X$ and for *every* $y \in Y$, $P(x, y)$ is true. The statement $\forall x \forall y\, P(x, y)$ is false if there is *at least one* $x \in X$ and *at least one* $y \in Y$ such that $P(x, y)$ is false.

Example 1.6.3 Consider the statement

$$\forall x \forall y((x > 0) \wedge (y > 0) \rightarrow (x + y > 0)).$$

The domain of discourse is $\mathbf{R} \times \mathbf{R}$. This statement is true because, for every real number x and for every real number y, the conditional proposition

$$(x > 0) \wedge (y > 0) \rightarrow (x + y > 0)$$

is true. In words, for every real number x and for every real number y, if x and y are positive, their sum is positive. ◀

Example 1.6.4 Consider the statement

$$\forall x \forall y((x > 0) \wedge (y < 0) \rightarrow (x + y \neq 0)).$$

The domain of discourse is $\mathbf{R} \times \mathbf{R}$. This statement is false because if $x = 1$ and $y = -1$, the conditional proposition

$$(x > 0) \wedge (y < 0) \rightarrow (x + y \neq 0)$$

is false. We say that the pair $x = 1$ and $y = -1$ is a counterexample. ◀

Example 1.6.5 Suppose that P is a propositional function with domain of discourse $\{d_1, \ldots, d_n\} \times \{d_1, \ldots, d_n\}$. The following pseudocode determines whether $\forall x \forall y\, P(x, y)$ is true or false:

```
for i = 1 to n
   for j = 1 to n
      if (¬P(d_i, d_j))
         return false
return true
```

The for loops examine members of the domain of discourse. If they find a pair d_i, d_j for which $P(d_i, d_j)$ is false, the condition $\neg P(d_i, d_j)$ in the if statement is true; so the code returns false [to indicate that $\forall x \forall y\, P(x, y)$ is false] and terminates. In this case, the pair d_i, d_j is a counterexample. If $P(d_i, d_j)$ is true for every pair d_i, d_j, the condition $\neg P(d_i, d_j)$ in the if statement is always false. In this case, the for loops run to completion, after which the code returns true [to indicate that $\forall x \forall y\, P(x, y)$ is true] and terminates. ◀

By definition, the statement $\forall x \exists y\, P(x, y)$, with domain of discourse $X \times Y$, is true if, for *every* $x \in X$, there is *at least one* $y \in Y$ for which $P(x, y)$ is true. The statement $\forall x \exists y\, P(x, y)$ is false if there is *at least one* $x \in X$ such that $P(x, y)$ is false for *every* $y \in Y$.

Example 1.6.6 Consider the statement

$$\forall x \exists y (x + y = 0).$$

The domain of discourse is $\mathbf{R} \times \mathbf{R}$. This statement is true because, for every real number x, there is at least one y (namely $y = -x$) for which $x + y = 0$ is true. In words, for every real number x, there is a number that when added to x makes the sum zero. ◀

Example 1.6.7 Consider the statement

$$\forall x \exists y (x > y).$$

The domain of discourse is $\mathbf{Z}^+ \times \mathbf{Z}^+$. This statement is false because there is at least one x, namely $x = 1$, such that $x > y$ is false for every positive integer y. ◀

Example 1.6.8 Suppose that P is a propositional function with domain of discourse $\{d_1, \ldots, d_n\} \times \{d_1, \ldots, d_n\}$. The following pseudocode determines whether $\forall x \exists y\, P(x, y)$ is true or false:

```
for i = 1 to n
   if (¬ exists_dj(i))
      return false
return true
exists_dj(i) {
   for j = 1 to n
      if (P(d_i, d_j))
         return true
   return false
}
```

If for each d_i, there exists d_j such that $P(d_i, d_j)$ is true, then for each i, $P(d_i, d_j)$ is true for some j. Thus, *exists_dj*(i) returns true for every i. Since $\neg$ *exists_dj*(i) is always false, the first for loop eventually terminates and true is returned to indicate that $\forall x \exists y\, P(x, y)$ is true.

If for some d_i, $P(d_i, d_j)$ is false for every j, then, for this i, $P(d_i, d_j)$ is false for every j. In this case, the for loop in *exists_dj*(i) runs to termination and false is returned. Since $\neg$ *exists_dj*(i) is true, false is returned to indicate that $\forall x \exists y\, P(x, y)$ is false. ◀

By definition, the statement $\exists x \forall y\, P(x, y)$, with domain of discourse $X \times Y$, is true if there is *at least one* $x \in X$ such that $P(x, y)$ is true for *every* $y \in Y$. The statement $\exists x \forall y\, P(x, y)$ is false if, for *every* $x \in X$, there is *at least one* $y \in Y$ such that $P(x, y)$ is false.

Example 1.6.9 Consider the statement $\exists x \forall y (x \le y)$. The domain of discourse is $\mathbf{Z}^+ \times \mathbf{Z}^+$. This statement is true because there is at least one positive integer x (namely $x = 1$) for which $x \le y$ is true for every positive integer y. In words, there is a smallest positive integer (namely 1). ◀

Example 1.6.10 Consider the statement $\exists x \forall y (x \ge y)$. The domain of discourse is $\mathbf{Z}^+ \times \mathbf{Z}^+$. This statement is false because, for every positive integer x, there is at least one positive integer y, namely $y = x + 1$, such that $x \ge y$ is false. In words, there is no greatest positive integer. ◀

By definition, the statement $\exists x \exists y\, P(x, y)$, with domain of discourse $X \times Y$, is true if there is *at least one* $x \in X$ and *at least one* $y \in Y$ such that $P(x, y)$ is true. The statement $\exists x \exists y\, P(x, y)$ is false if, for *every* $x \in X$ and for *every* $y \in Y$, $P(x, y)$ is false.

Example 1.6.11 Consider the statement

$$\exists x \exists y ((x > 1) \wedge (y > 1) \wedge (xy = 6)).$$

The domain of discourse is $\mathbf{Z}^+ \times \mathbf{Z}^+$. This statement is true because there is at least one integer $x > 1$ (namely $x = 2$) and at least one integer $y > 1$ (namely $y = 3$) such that $xy = 6$. In words, 6 is composite (i.e., not prime). ◀

Example 1.6.12 Consider the statement

$$\exists x \exists y ((x > 1) \wedge (y > 1) \wedge (xy = 7)).$$

The domain of discourse is $\mathbf{Z}^+ \times \mathbf{Z}^+$. This statement is false because for every positive integer x and for every positive integer y,

$$(x > 1) \wedge (y > 1) \wedge (xy = 7)$$

is false. In words, 7 is prime. ◀

The generalized De Morgan's laws for logic (Theorem 1.5.14) can be used to negate a proposition containing nested quantifiers.

Example 1.6.13 Using the generalized De Morgan's laws for logic, we find that the negation of $\forall x \exists y\, P(x, y)$ is

$$\neg(\forall x \exists y\, P(x, y)) \equiv \exists x \neg(\exists y\, P(x, y)) \equiv \exists x \forall y\, \neg P(x, y).$$

Notice how in the negation, $\forall$ and $\exists$ are interchanged. ◀

Example 1.6.14 Write the negation of $\exists x \forall y (xy < 1)$, where the domain of discourse is $\mathbf{R} \times \mathbf{R}$. Determine the truth value of the given statement and its negation.

SOLUTION Using the generalized De Morgan's laws for logic, we find that the negation is

$$\neg(\exists x \forall y(xy < 1)) \equiv \forall x \neg(\forall y(xy < 1)) \equiv \forall x \exists y \neg(xy < 1) \equiv \forall x \exists y(xy \geq 1).$$

The given statement $\exists x \forall y(xy < 1)$ is true because there is at least one x (namely $x = 0$) such that $xy < 1$ for every y. Since the given statement is true, its negation is false. ◀

We conclude with a logic game, which presents an alternative way to determine whether a quantified propositional function is true or false. André Berthiaume contributed this example.

Example 1.6.15

The Logic Game Given a quantified propositional function such as $\forall x \exists y\, P(x, y)$, you and your opponent, whom we call Farley, play a logic game. Your goal is to try to make $P(x, y)$ true, and Farley's goal is to try to make $P(x, y)$ false. The game begins with the first (left) quantifier. If the quantifier is $\forall$, Farley chooses a value for that variable; if the quantifier is $\exists$, you choose a value for that variable. The game continues with the second quantifier. After values are chosen for all the variables, if $P(x, y)$ is true, you win; if $P(x, y)$ is false, Farley wins. We will show that if you can always win regardless of how Farley chooses values for the variables, the quantified statement is true, but if Farley can choose values for the variables so that you cannot win, the quantified statement is false.

Consider the statement

$$\forall x \exists y(x + y = 0). \tag{1.6.1}$$

The domain of discourse is $\mathbf{R} \times \mathbf{R}$. Since the first quantifier is $\forall$, Farley goes first and chooses a value for x. Since the second quantifier is $\exists$, you go second. Regardless of what value Farley chose, you can choose $y = -x$, which makes the statement $x + y = 0$ true. You can always win the game, so the statement (1.6.1) is true.

Next, consider the statement

$$\exists x \forall y(x + y = 0). \tag{1.6.2}$$

Again, the domain of discourse is $\mathbf{R} \times \mathbf{R}$. Since the first quantifier is $\exists$, you go first and choose a value for x. Since the second quantifier is $\forall$, Farley goes second. Regardless of what value you chose, Farley can always choose a value for y, which makes the statement $x + y = 0$ false. (If you choose $x = 0$, Farley can choose $y = 1$. If you choose $x \neq 0$, Farley can choose $y = 0$.) Farley can always win the game, so the statement (1.6.2) is false.

We discuss why the game correctly determines the truth value of a quantified propositional function. Consider $\forall x \forall y\, P(x, y)$. If Farley can always win the game, this means that Farley can find values for x and y that make $P(x, y)$ false. In this case, the propositional function is false; the values Farley found provide a counterexample. If Farley cannot win the game, no counterexample exists; in this case, the propositional function is true.

Consider $\forall x \exists y\, P(x, y)$. Farley goes first and chooses a value for x. You choose second. If, no matter what value Farley chose, you can choose a value for y that makes $P(x, y)$ true, you can always win the game and the propositional function is true. However, if Farley can choose a value for x so that every value you choose for y makes $P(x, y)$ false, then you will always lose the game and the propositional function is false.

An analysis of the other cases also shows that if you can always win the game, the propositional function is true; but if Farley can always win the game, the propositional function is false.

The logic game extends to propositional functions of more than two variables. The rules are the same and, again, if you can always win the game, the propositional function is true; but if Farley can always win the game, the propositional function is false. ◀

1.6 Problem-Solving Tips

- To prove that $\forall x \forall y\, P(x, y)$ is true, where the domain of discourse is $X \times Y$, you must show that $P(x, y)$ is true for all values of $x \in X$ and $y \in Y$. One technique is to argue that $P(x, y)$ is true using the symbols x and y to stand for *arbitrary* elements in X and Y.
- To prove that $\forall x \forall y\, P(x, y)$ is false, where the domain of discourse is $X \times Y$, find one value of $x \in X$ and one value of $y \in Y$ (*two* values suffice—one for x and one for y) that make $P(x, y)$ false.
- To prove that $\forall x \exists y\, P(x, y)$ is true, where the domain of discourse is $X \times Y$, you must show that for all $x \in X$, there is at least one $y \in Y$ such that $P(x, y)$ is true. One technique is to let x stand for an arbitrary element in X and then find a value for $y \in Y$ (*one* value suffices!) that makes $P(x, y)$ true.
- To prove that $\forall x \exists y\, P(x, y)$ is false, where the domain of discourse is $X \times Y$, you must show that for at least one $x \in X$, $P(x, y)$ is false for every $y \in Y$. One technique is to find a value of $x \in X$ (again *one* value suffices!) that has the property that $P(x, y)$ is false for every $y \in Y$. Having chosen a value for x, let y stand for an arbitrary element of Y and show that $P(x, y)$ is always false.
- To prove that $\exists x \forall y\, P(x, y)$ is true, where the domain of discourse is $X \times Y$, you must show that for at least one $x \in X$, $P(x, y)$ is true for every $y \in Y$. One technique is to find a value of $x \in X$ (again *one* value suffices!) that has the property that $P(x, y)$ is true for every $y \in Y$. Having chosen a value for x, let y stand for an arbitrary element of Y and show that $P(x, y)$ is always true.
- To prove that $\exists x \forall y\, P(x, y)$ is false, where the domain of discourse is $X \times Y$, you must show that for all $x \in X$, there is at least one $y \in Y$ such that $P(x, y)$ is false. One technique is to let x stand for an arbitrary element in X and then find a value for $y \in Y$ (*one* value suffices!) that makes $P(x, y)$ false.
- To prove that $\exists x \exists y\, P(x, y)$ is true, where the domain of discourse is $X \times Y$, find one value of $x \in X$ and one value of $y \in Y$ (*two* values suffice—one for x and one for y) that make $P(x, y)$ true.
- To prove that $\exists x \exists y\, P(x, y)$ is false, where the domain of discourse is $X \times Y$, you must show that $P(x, y)$ is false for all values of $x \in X$ and $y \in Y$. One technique is to argue that $P(x, y)$ is false using the symbols x and y to stand for *arbitrary* elements in X and Y.
- To negate an expression with nested quantifiers, use the generalized De Morgan's laws for logic. Loosely speaking, $\forall$ and $\exists$ are interchanged. Don't forget that the negation of $p \to q$ is equivalent to $p \wedge \neg q$.

1.6 Review Exercises

1. What is the interpretation of $\forall x \forall y P(x, y)$? When is this quantified expression true? When is it false?
2. What is the interpretation of $\forall x \exists y P(x, y)$? When is this quantified expression true? When is it false?
3. What is the interpretation of $\exists x \forall y P(x, y)$? When is this quantified expression true? When is it false?
4. What is the interpretation of $\exists x \exists y P(x, y)$? When is this quantified expression true? When is it false?
5. Give an example to show that, in general, $\forall x \exists y P(x, y)$ and $\exists x \forall y P(x, y)$ have different meanings.
6. Write the negation of $\forall x \forall y P(x, y)$ using the generalized De Morgan's laws for logic.

7. Write the negation of $\forall x \exists y P(x, y)$ using the generalized De Morgan's laws for logic.

8. Write the negation of $\exists x \forall y P(x, y)$ using the generalized De Morgan's laws for logic.

9. Write the negation of $\exists x \exists y P(x, y)$ using the generalized De Morgan's laws for logic.

10. Explain the rules for playing the logic game. How can the logic game be used to determine the truth value of a quantified expression?

1.6 Exercises

In Exercises 1–33, the set D_1 consists of three students: Garth, who is 5 feet 11 inches tall; Erin, who is 5 feet 6 inches tall; and Marty, who is 6 feet tall. The set D_2 consists of four students: Dale, who is 6 feet tall; Garth, who is 5 feet 11 inches tall; Erin, who is 5 feet 6 inches tall; and Marty, who is 6 feet tall. The set D_3 consists of one student: Dale, who is 6 feet tall. The set D_4 consists of three students: Pat, Sandy, and Gale, each of whom is 5 feet 11 inches tall.

In Exercises 1–21, $T_1(x, y)$ is the propositional function "x is taller than y." Write each proposition in Exercises 1–4 in words.

1. $\forall x \forall y\, T_1(x, y)$
2. $\forall x \exists y\, T_1(x, y)$
3. $\exists x \forall y\, T_1(x, y)$
4. $\exists x \exists y\, T_1(x, y)$
5. Write the negation of each proposition in Exercises 1–4 in words and symbolically.

In Exercises 6–21, tell whether each proposition in Exercises 1–4 is true or false if the domain of discourse is $D_i \times D_j$ for the given values of i and j.

6. $i = 1,\ j = 1$
7. $i = 1,\ j = 2$
8. $i = 1,\ j = 3$
9. $i = 1,\ j = 4$
10. $i = 2,\ j = 1$
11. $i = 2,\ j = 2$
12. $i = 2,\ j = 3$
13. $i = 2,\ j = 4$
14. $i = 3,\ j = 1$
15. $i = 3,\ j = 2$
16. $i = 3,\ j = 3$
17. $i = 3,\ j = 4$
18. $i = 4,\ j = 1$
19. $i = 4,\ j = 2$
20. $i = 4,\ j = 3$
21. $i = 4,\ j = 4$

In Exercises 22–27, $T_2(x, y)$ is the propositional function "x is taller than or the same height as y." Write each proposition in Exercises 22–25 in words.

22. $\forall x \forall y\, T_2(x, y)$
23. $\forall x \exists y\, T_2(x, y)$
24. $\exists x \forall y\, T_2(x, y)$
25. $\exists x \exists y\, T_2(x, y)$
26. Write the negation of each proposition in Exercises 22–25 in words and symbolically.
27. Tell whether each proposition in Exercises 22–25 is true or false if the domain of discourse is $D_i \times D_j$ for each pair of values i, j given in Exercises 6–21. The sets $D_1, \ldots, D_4$ are defined before Exercise 1.

In Exercises 28–33, $T_3(x, y)$ is the propositional function "if x and y are distinct persons, then x is taller than y." Write each proposition in Exercises 28–31 in words.

28. $\forall x \forall y\, T_3(x, y)$
29. $\forall x \exists y\, T_3(x, y)$
30. $\exists x \forall y\, T_3(x, y)$
31. $\exists x \exists y\, T_3(x, y)$
32. Write the negation of each proposition in Exercises 28–31 in words and symbolically.
33. Tell whether each proposition in Exercises 28–31 is true or false if the domain of discourse is $D_i \times D_j$ for each pair of values i, j given in Exercises 6–21. The sets $D_1, \ldots, D_4$ are defined before Exercise 1.

Let $L(x, y)$ be the propositional function "x loves y." The domain of discourse is the Cartesian product of the set of all living people with itself (i.e., both x and y take on values in the set of all living people). Write each proposition in Exercises 34–37 symbolically. Which do you think are true?

34. Someone loves everybody.
35. Everybody loves everybody.
36. Somebody loves somebody.
37. Everybody loves somebody.
38. Write the negation of each proposition in Exercises 34–37 in words and symbolically.

Let $A(x, y)$ be the propositional function "x attended y's office hours" and let $E(x)$ be the propositional function "x is enrolled in a discrete math class." Let S be the set of students and let T denote the set of teachers—all at Hudson University. The domain of discourse of A is $S \times T$ and the domain of discourse of E is S. Write each proposition in Exercises 39–42 symbolically.

39. Brit attended someone's office hours.
40. No one attended Professor Sandwich's office hours.
41. Every discrete math student attended someone's office hours.
42. All teachers had at least one student attend their office hours.

Let $P(x, y)$ be the propositional function $x \geq y$. The domain of discourse is $\mathbf{Z}^+ \times \mathbf{Z}^+$. Tell whether each proposition in Exercises 43–46 is true or false.

43. $\forall x \forall y\, P(x, y)$
44. $\forall x \exists y\, P(x, y)$
45. $\exists x \forall y\, P(x, y)$
46. $\exists x \exists y\, P(x, y)$
47. Write the negation of each proposition in Exercises 43–46.

Determine the truth value of each statement in Exercises 48–65. The domain of discourse is $\mathbf{R} \times \mathbf{R}$. Justify your answers.

48. $\forall x \forall y (x^2 < y + 1)$
49. $\forall x \exists y (x^2 < y + 1)$
50. $\exists x \forall y (x^2 < y + 1)$
51. $\exists x \exists y (x^2 < y + 1)$
52. $\exists y \forall x (x^2 < y + 1)$
53. $\forall y \exists x (x^2 < y + 1)$
54. $\forall x \forall y (x^2 + y^2 = 9)$
55. $\forall x \exists y (x^2 + y^2 = 9)$

56. $\exists x \forall y (x^2 + y^2 = 9)$

57. $\exists x \exists y (x^2 + y^2 = 9)$

58. $\forall x \forall y (x^2 + y^2 \geq 0)$

59. $\forall x \exists y (x^2 + y^2 \geq 0)$

60. $\exists x \forall y (x^2 + y^2 \geq 0)$

61. $\exists x \exists y (x^2 + y^2 \geq 0)$

62. $\forall x \forall y ((x < y) \to (x^2 < y^2))$

63. $\forall x \exists y ((x < y) \to (x^2 < y^2))$

64. $\exists x \forall y ((x < y) \to (x^2 < y^2))$

65. $\exists x \exists y ((x < y) \to (x^2 < y^2))$

66. Write the negation of each proposition in Exercises 48–65.

67. Suppose that P is a propositional function with domain of discourse $\{d_1, \dots, d_n\} \times \{d_1, \dots, d_n\}$. Write pseudocode that determines whether

$$\exists x \forall y\, P(x, y)$$

is true or false.

68. Suppose that P is a propositional function with domain of discourse $\{d_1, \dots, d_n\} \times \{d_1, \dots, d_n\}$. Write pseudocode that determines whether

$$\exists x \exists y\, P(x, y)$$

is true or false.

69. Explain how the logic game (Example 1.6.15) determines whether each proposition in Exercises 48–65 is true or false.

70. Use the logic game (Example 1.6.15) to determine whether the proposition

$$\forall x \forall y \exists z ((z > x) \wedge (z < y))$$

is true or false. The domain of discourse is $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$.

71. Use the logic game (Example 1.6.15) to determine whether the proposition

$$\forall x \forall y \exists z ((z < x) \wedge (z < y))$$

is true or false. The domain of discourse is $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$.

72. Use the logic game (Example 1.6.15) to determine whether the proposition

$$\forall x \forall y \exists z ((x < y) \to ((z > x) \wedge (z < y)))$$

is true or false. The domain of discourse is $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$.

73. Use the logic game (Example 1.6.15) to determine whether the proposition

$$\forall x \forall y \exists z ((x < y) \to ((z > x) \wedge (z < y)))$$

is true or false. The domain of discourse is $\mathbf{R} \times \mathbf{R} \times \mathbf{R}$.

Assume that $\forall x \forall y\, P(x, y)$ is true and that the domain of discourse is nonempty. Which of Exercises 74–76 must also be true? Prove your answer.

74. $\forall x \exists y\, P(x, y)$

75. $\exists x \forall y\, P(x, y)$

76. $\exists x \exists y\, P(x, y)$

Assume that $\exists x \forall y\, P(x, y)$ is true and that the domain of discourse is nonempty. Which of Exercises 77–79 must also be true? Prove your answer.

77. $\forall x \forall y\, P(x, y)$

78. $\forall x \exists y\, P(x, y)$

79. $\exists x \exists y\, P(x, y)$

Assume that $\exists x \exists y\, P(x, y)$ is true and that the domain of discourse is nonempty. Which of Exercises 80–82 must also be true? Prove your answer.

80. $\forall x \forall y\, P(x, y)$

81. $\forall x \exists y\, P(x, y)$

82. $\exists x \forall y\, P(x, y)$

Assume that $\forall x \forall y\, P(x, y)$ is false and that the domain of discourse is nonempty. Which of Exercises 83–85 must also be false? Prove your answer.

83. $\forall x \exists y\, P(x, y)$

84. $\exists x \forall y\, P(x, y)$

85. $\exists x \exists y\, P(x, y)$

Assume that $\forall x \exists y\, P(x, y)$ is false and that the domain of discourse is nonempty. Which of Exercises 86–88 must also be false? Prove your answer.

86. $\forall x \forall y\, P(x, y)$

87. $\exists x \forall y\, P(x, y)$

88. $\exists x \exists y\, P(x, y)$

Assume that $\exists x \forall y\, P(x, y)$ is false and that the domain of discourse is nonempty. Which of Exercises 89–91 must also be false? Prove your answer.

89. $\forall x \forall y\, P(x, y)$

90. $\forall x \exists y\, P(x, y)$

91. $\exists x \exists y\, P(x, y)$

Assume that $\exists x \exists y\, P(x, y)$ is false and that the domain of discourse is nonempty. Which of Exercises 92–94 must also be false? Prove your answer.

92. $\forall x \forall y\, P(x, y)$

93. $\forall x \exists y\, P(x, y)$

94. $\exists x \forall y\, P(x, y)$

Which of Exercises 95–98 is logically equivalent to $\neg(\forall x \exists y\, P(x, y))$? Explain.

95. $\exists x \neg(\forall y\, P(x, y))$

96. $\forall x \neg(\exists y\, P(x, y))$

97. $\exists x \forall y\, \neg P(x, y)$

98. $\exists x \exists y\, \neg P(x, y)$

99. [Requires calculus] The definition of

$$\lim_{x \to a} f(x) = L$$

is: For every $\varepsilon > 0$, there exists $\delta > 0$ such that for all x if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$. Write this definition symbolically using $\forall$ and $\exists$.

100. [Requires calculus] Write the negation of the definition of limit (see Exercise 99) in words and symbolically using $\forall$ and $\exists$ but not $\neg$.

⋆**101.** [Requires calculus] Write the definition of "$\lim_{x \to a} f(x)$ does not exist" (see Exercise 99) in words and symbolically using $\forall$ and $\exists$ but not $\neg$.

102. Consider the headline: Every school may not be right for every child. What is the literal meaning? What is the intended meaning? Clarify the headline by rephrasing it and writing it symbolically.