12 Lecture 13: Characteristic Polynomials, Eigenvalues and Diagonalization of 2x2 Matrices

12.1 Eigenvalues and Eigenvectors

Definition 12.1. *An* **eigenvector** *for an $n \times n$ matrix $\mathbf{A}$ is a nonzero vector $\mathbf{v} \in \mathbb{R}^n$ (or $\mathbf{x} \in \mathbb{C}^n$) such that there exist a constant λ satisfying*

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}.$$

The constant λ above is called the **eigenvalue** *associated with the eigenvector $\mathbf{v}$.*

Equivalently, a constant λ is an **eigenvalue** of an $n \times n$ matrix $\mathbf{A}$ if the equation $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ has a *nontrivial solution.* The solution to this equation is an **eigenvector** associated with the eigenvalue λ.

Example 12.1. *Let* $\mathbf{A} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, *and* $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, *then*

$$\mathbf{A}\mathbf{u} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 50 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix},$$

thus $\mathbf{u}$ is an eigenvector of $\mathbf{A}$ corresponding to an eigenvalue -4.

$$\mathbf{A}\mathbf{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda \begin{bmatrix} 3 \\ -2 \end{bmatrix},$$

for all λ. Thus $\mathbf{v}$ is not an eigenvector of $\mathbf{A}$.

Remark: equation $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ is equivalent to $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$. Thus, if λ is an eigenvalue of an $n \times n$ matrix $\mathbf{A}$, then equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ has a nontrivial solution, that is, the matrix $(\mathbf{A} - \lambda\mathbf{I})$ is not invertible (or $(\mathbf{A} - \lambda\mathbf{I})$ is a singular matrix). As a result, $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.

- $\det(\mathbf{A} - \lambda\mathbf{I})$ is a polynomial of the variable λ. It is called the **characteristic polynomial** of the matrix $\mathbf{A}$.
- Equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ is called the **characteristic equation** of the matrix $\mathbf{A}$. Solving this equation, we obtain all eigenvalues of $\mathbf{A}$.

The set of all the eigenvalues of a matrix $\mathbf{A}$ is called the **spectrum** of the matrix $\mathbf{A}$ and is denoted $\sigma(\mathbf{A})$.

12.2 Characteristic Polynomials

The characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$, denoted by $p_{\mathbf{A}}$, and given by

$$p_{\mathbf{A}}(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}).$$

$p_{\mathbf{A}}$ is a polynomial of degree n and has n roots (counting complex roots and repeated roots). Roots of the characteristic polynomial are the eigenvalues of the matrix, and are solution of the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0. \tag{12.1}$$

Finding the eigenvalues of matrix $\mathbf{A}$ means solving the characteristic equation (12.1). Below are some simple cases:

a) If $\mathbf{A}$ is a 2×2 matrix, $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\mathbf{A} - \lambda \mathbf{I} = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$. The characteristic polynomial of $\mathbf{A}$:

$$p_{\mathbf{A}}(\lambda) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + (ad-bc).$$

Note that

- $(ad - bc)$ is the determinant of $\mathbf{A}$.
- $a + d$, the sum of entries in the main diagonal, is called the **trace** of $\mathbf{A}$ and denoted by $\mathrm{tr}(\mathbf{A})$.

Thus, the characteristic polynomial is expressed as

$$p_{\mathbf{A}}(\lambda) = \lambda^2 - tr(\mathbf{A})\,\lambda + \det(\mathbf{A}),$$

which is a quadratic polynomial and has two roots (eigenvalues):

$$\lambda_{1,2} = \frac{\mathrm{tr}(\mathbf{A}) \pm \sqrt{(\mathrm{tr}(\mathbf{A}))^2 - 4\det(\mathbf{A})}}{2}$$

Example 12.2. *Find the eigenvalues of the matrix* $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

Solution: The eigenvalue of matrix $\mathbf{A}$ is the roots of the characteristic polynomial

$$p_{\mathbf{A}}(\lambda) = \begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} = \lambda^2 + 4\lambda - 21 = (\lambda + 7)(\lambda - 3).$$

If $p_{\mathbf{A}}(\lambda) = 0$ then $\lambda = -7$ or $\lambda = 3$. Thus, the eigenvalues of $\mathbf{A}$ are 3 and -7 (or, $\sigma(\mathbf{A}) = \{3, -7\}$).

Example 12.3. *Find the eigenvalues of the matrix* $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ -3 & 6 \end{bmatrix}$.

Solution: The eigenvalue of matrix $\mathbf{A}$ is the roots of the characteristic polynomial

$$p_{\mathbf{A}}(\lambda) = \begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 21.$$

Equation $p_{\mathbf{A}}(\lambda) = 0$ has two roots $\lambda_{1,2} = 4 \pm \sqrt{-5} = 4 \pm \sqrt{5}i$. Thus, the eigenvalues of $\mathbf{A}$ are $4 + \sqrt{5}i$ and $4 - \sqrt{5}i$.

a) If $\mathbf{A}$ is a 3×3 matrix, $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then $\mathbf{A} - \lambda \mathbf{I} = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix}$.

The characteristic polynomial of $\mathbf{A}$ is

$$p_{\mathbf{A}}(\lambda) = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}$$
$$= -\lambda^3 + \mathrm{tr}(\mathbf{A})\lambda^2 - \left(\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \right) \lambda + \det(\mathbf{A})$$

Example 12.4. *Find the characteristic polynomial of the matrix*

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}.$$

Solution: The characteristic polynomial of $\mathbf{A}$:

$$p_{\mathbf{A}}(\lambda) = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 3-\lambda & 1 \\ 3 & 1 & 2-\lambda. \end{vmatrix}$$

We have,

$$\det(\mathbf{A}) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & -5 & -7 \end{vmatrix} = 1 \begin{vmatrix} -1 & -5 \\ -5 & -7 \end{vmatrix} = (-1)(-7) - (-5)(-5) = -18$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1, \quad \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} = -7, \quad \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 5.$$

Thus,

$$p_{\mathbf{A}}(\lambda) = -\lambda^3 + (1+3+2)\lambda^2 - \big(-1 + (-7) + 5\big)\lambda + (-18) = -\lambda^3 + 6\lambda^2 + 3\lambda - 18$$

Example 12.5. *Find the spectrum of the matrix* $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$.

Solution: From example 12.4, the characteristic polynomial of $\mathbf{A}$ is

$$p_{\mathbf{A}}(\lambda) = -\lambda^3 + 6\lambda^2 + 3\lambda - 18 = -(\lambda - 6)(\lambda^2 - 3).$$

Eigenvalues λ is such that $(\lambda - 6)(\lambda^2 - 3) = 0$, which are $\lambda = 6$, $\lambda = \pm\sqrt{3}$. The spectrum is

$$\sigma(\mathbf{A}) = \{6, \sqrt{3}, -\sqrt{3}\}.$$