CS 241 - Homework 9 - Arnav Kucheriya

Homework 9: Functions, Sequences, Strings

---

1. Definitions

Please refer to definitions 3.1.1, 3.1.22, 3.1.29, 3.1.35, 3.1.47 from the textbook.

---

2. Functions f, g, h, k

Let $A = \{1, 2, 3, 4, 5\}$

- $f = (1, 2), (2, 3), (3, 4), (4, 5), (5, 2) f = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 2)\}$
- $g = (1, 5), (2, 3), (3, 2), (4, 4), (5, 1) g = \{(1, 5), (2, 3), (3, 2), (4, 4), (5, 1)\}$
- $h = (1, 3), (2, 4), (3, 5), (4, 1), (5, 2) h = \{(1, 3), (2, 4), (3, 5), (4, 1), (5, 2)\}$
- $k = (1, 3), (2, 5), (3, 4), (4, 5), (1, 1) k = \{(1, 3), (2, 5), (3, 4), (4, 5), (1, 1)\}$

(a) Determine if f, g, h, k are functions and whether they are injective, surjective, bijective

- **f**: Is a function. Not injective (5 and 1 both map to 2), not surjective (no element maps to 1).
- **g**: Function. Bijective. Each input maps to a unique and complete output. Inverse: $g - 1 = (5, 1), (3, 2), (2, 3), (4, 4), (1, 5) g^{-1} = \{(5, 1), (3, 2), (2, 3), (4, 4), (1, 5)\}$
- **h**: Function. Bijective. Inverse: $h - 1 = (3, 1), (4, 2), (5, 3), (1, 4), (2, 5) h^{-1} = \{(3, 1), (4, 2), (5, 3), (1, 4), (2, 5)\}$
- **k**: Not a function. Two mappings for input 1: $(1, 3) and (1, 1)$.

(b) Composition of functions

Let's denote $\circ$ as function composition:

- $g \circ h = (1, 2), (2, 1), (3, 4), (4, 5), (5, 3) g \circ h = \{(1, 2), (2, 1), (3, 4), (4, 5), (5, 3)\}$
- $h \circ g = (1, 2), (2, 5), (3, 4), (4, 1), (5, 3) h \circ g = \{(1, 2), (2, 5), (3, 4), (4, 1), (5, 3)\}$
- $g2 = g \circ g = (1, 1), (2, 2), (3, 3), (4, 4), (5, 5) g^2 = g \circ g = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$
- $g3 = g \circ g \circ g = g g^3 = g \circ g \circ g = g$

- $h2 = (1,5),(2,1),(3,2),(4,3),(5,4)h^2 = \{(1,5),(2,1),(3,2),(4,3),(5,4)\}$
- $f2 = (1,3),(2,4),(3,5),(4,2),(5,3)f^2 = \{(1,3),(2,4),(3,5),(4,2),(5,3)\}$
- $h5 = (1,1),(2,2),(3,3),(4,4),(5,5)h^5 = \{(1,1),(2,2),(3,3),(4,4),(5,5)\}$

---

3. Compositions with bijection b

$Let X = 1,2,3,4,5, Y = a,b,c,d,e X = \{1,2,3,4,5\}, Y = \{a,b,c,d,e\}. Let b : X \to Y b : X \to Y$ be a bijection.

$b - 1 \circ b =$ identity on $X b^{-1} \circ b = \text{identity on } X$

$b \circ b - 1 =$ identity on $Y b \circ b^{-1} = \text{identity on } Y$

$Domains : b - 1 \circ b : X \to X b^{-1} \circ b : X \to X, b \circ b - 1 : Y \to Y b \circ b^{-1} : Y \to Y$

---

4. Floor and Ceiling Functions

$g : R \to R, g(x) = \lfloor x \rfloor g : \mathbb{R} \to \mathbb{R}, g(x) = \lfloor x \rfloor$: Not surjective since not all real values are outputs $(e.g., 2.5)$.

$h : R \to Z, h(x) = \lfloor x \rfloor h : \mathbb{R} \to \mathbb{Z}, h(x) = \lfloor x \rfloor$: Surjective because every integer is reachable.

h is not injective: $h(2.1) = h(2.9) = 2h(2.1) = h(2.9) = 2$

---

5. Section Problems (3.1 & 3.2)

Section 3.1 – Functions

3.1.14 – Define onto function

A function $f : X \to Y$ is **onto** (surjective) if for every $y \in Y$, there exists an $x \in X$ such that $f(x) = y$
Example: $f(x) = x^3$ is onto $\mathbb{R} \to \mathbb{R}$.

3.1.23 – $f(m,n) = m - n$, $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

- One-to-one: No. $f(1,0) = 1 = f(2,1)$
- Onto: Yes. For any $z \in \mathbb{Z}$, $f(z,0) = z$

Conclusion: Not one-to-one, but onto.

3.1.24 – $f(m,n) = m$

- One-to-one: No. $f(1,0) = f(1,1) = 1$
- Onto: Yes.

Conclusion: Onto but not injective.

3.1.29 – Prove $f(m,n) = 2^m 3^n$ is injective but not onto

- Suppose $f(m,n) = f(p,q) \Rightarrow 2^m 3^n = 2^p 3^q$
  - By unique prime factorization: $m = p$, $n = q$
  - Therefore, injective.
- Not all integers are of the form $2^m 3^n$, e.g., 5 is not.

Conclusion: One-to-one but not onto.

3.1.44 – Inverse of $f(x) = 3x$

Inverse: $f^{-1}(y) = \frac{y}{3}$

3.1.47 – Inverse of $f(x) = 4x^3 - 5$

Let $y = 4x^3 - 5 \Rightarrow x^3 = \frac{y+5}{4} \Rightarrow x = \sqrt[3]{\frac{y+5}{4}}$

Inverse: $f^{-1}(y) = \sqrt[3]{\frac{y+5}{4}}$

3.1.59 – Decompose $f(x) = \frac{1}{(\cos(6x))^3}$

Let:

- $g(x) = \cos(6x)$
- $h(x) = g(x)^3$
- $f(x) = \frac{1}{h(x)}$

3.1.62 – $f = (a, b), (b, a), (c, b)$

- $f \circ f = (a, a), (b, b), (c, a)$
- $f^3 = f \circ f \circ f = (a, b), (b, a), (c, b)$

Pattern repeats every 2 applications.

3.1.96–100 – Bit string encoding from subsets

Let $X = a, b, c$, define $S : P(X) \to 0, 1^3$ by:

- $S(a, c) = 101$
- $S(\emptyset) = 000$
- $S(X) = 111$
- S is injective because different subsets yield unique bit strings
- S is surjective because every 3-bit string maps to a subset

Conclusion: S is a bijection

Section 3.2 – Sequences and Strings

3.2.10 – Define string

A **string** is a finite sequence of symbols from a given alphabet.

3.2.12 – What is X^* for a finite set X?

X^* is the set of all finite-length strings (including the null string λ) over X.

3.2.18 – $t_n = 2n - 1$; is it increasing?

Yes, $t_{n+1} - t_n = 2 > 0$, so it is strictly increasing.

3.2.21 – Is $t_n = 2n - 1$ nonincreasing?

No, it is increasing, not nonincreasing.

3.2.108 – Recursive sequence $u_1 = 3$, $u_n = 3 + u_{n-1}$

Solution: $u_n = 3n$ by induction.

3.2.109–112 – $s_n = 2n - 1$

- First 7 terms: 1, 3, 5, 7, 9, 11, 13
- Subsequence (odd indices): 1, 5, 9, 13, ...
- Formula for k^{th} term of the subsequence: $S_{2k-1} = 4k - 3$

3.2.121–124 – $r_n = 3 \cdot 2^n - 4 \cdot 5^n$

- $r_0 = 3 - 4 = -1$
- $r_1 = 6 - 20 = -14$
- $r_2 = 12 - 100 = -88$
- $r_3 = 24 - 500 = -476$

3.2.125–128 – Recurrence relation

Given $r_n = 3 \cdot 2^n - 4 \cdot 5^n$, show:

$rn = 7rn - 1 - 10rn - 2r_n = 7r_{n-1} - 10r_{n-2}rn = 7rn - 1 - 10rn - 2$

3.2.142 (b), (e), (k) – String operations

Let:

- $\alpha = \texttt{baab}$
- $\beta = \texttt{caaba}$
- $\gamma = \texttt{bbab}$

Answers:

- (b) $\beta\alpha = \texttt{caababaab}$
- (e) $|\alpha\beta| = 9$
- (k) $\alpha\beta\gamma = \texttt{baabcaababbab}$

3.2.156 – $f(\alpha) = \alpha\alpha$

- Not one-to-one: $f(a) = aa = f(aa)$
- Not onto: Can't generate odd-length strings like aba

6. Summations

(a)

$\sum k = 1nk = n(n+1)2 \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

(b)

$\sum k = 1nB = nB \sum_{k=1}^{n} B = nB$

(c)

$\sum k = 1n(A \cdot Xk) = A \cdot \sum k = 1nXk \sum_{k=1}^{n} (A \cdot X_k) = A \cdot \sum_{k=1}^{n} X_k$

(d)

$\sum k = 1n(A \cdot k + B) = A \cdot n(n+1)2 + nB \sum_{k=1}^{n} (A \cdot k + B) = A \cdot \frac{n(n+1)}{2} + nB$

(e)

- $\sum k = 1n(2n-1) = n(2n-1) \sum_{k=1}^{n}(2n-1) = n(2n-1)$
- $\sum k = 1n(4n+1) = n(4n+1) \sum_{k=1}^{n}(4n+1) = n(4n+1)$

(f)

$\sum k = 120(4n+1) = 20(4\cdot 20+1) = 20\cdot 81 = 1620 \sum_{k=1}^{20}(4n+1) = 20(4\cdot 20+1) = 20\cdot 81 = 1620$
$\sum k = 1120(4n+1) = \sum k = 120 - \sum k = 110 = 1620 - 10\cdot 41 = 1210 \sum_{k=11}^{20}(4n+1) = \sum_{k=1}^{20} - \sum_{k=}^{10}$

---

7. Triangular Numbers

(a)

$\triangle 1 = 1, \triangle 2 = 3, \triangle 3 = 6, \triangle 4 = 10, \triangle 5 = 15 \triangle_1 = 1, \triangle_2 = 3, \triangle_3 = 6, \triangle_4 = 10, \triangle_5 = 15$

(b/c)

$\sum i = 1n \triangle i = n(n+1)(n+2)6 \sum_{i=1}^{n} \triangle_i = \frac{n(n+1)(n+2)}{6}$

(d)

$\triangle n - 1 + \triangle n = (n-1)n2 + n(n+1)2 = n2 \triangle_{n-1} + \triangle_n = \frac{(n-1)n}{2} + \frac{n(n+1)}{2} = n^2$

---

8. Geometric Sum

(a)

∑k=0na·rk=a·rn+1−1r−1for r≠1\sum_{k=0}^n a \cdot r^k = a \cdot \frac{r^{n+1} - 1}{r - 1} \quad \tex

(b)

Adding 1 causes a left-shift and reset to 0.

(c)

$\sum k = 0n2\cdot 3k = 2\cdot 3n+1-13-1 = 2\cdot 3n+1-12 = 3n+1-1 \sum_{k=0}^{n} 2\cdot 3^k = 2\cdot \frac{3^{n+1}-1}{3-1} = 2\cdot \frac{3^{n+1}-1}{2}$

(d)

$\sum k = 0n3 \cdot 4k = 3 \cdot 4n + 1 - 13 = 4n + 1 - 1 \sum_{k=0}^{n} 3 \cdot 4^k = 3 \cdot \frac{4^{n+1}-1}{3} = 4^{n+1} - 1$

(e)

$Letd = b - 1d = b - 1, then : \sum k = 0nd \cdot bk = d \cdot bn + 1 - 1b - 1 = bn + 1 - 1 \sum_{k=0}^{n} d \cdot b^k = d \cdot \frac{b^{n+1}-1}{b-1}$
