CS241 - Arnav Kucheriya - Homework 4

Section 1.5:

Exercise 7, 10, 12, 14: "n Divides 77" - True or False

1. **P(11)**
   - True (11 divides 77)
2. **$\forall$n P(n)**
   - False (Not every number divides 77)
3. **$\forall$n ¬P(n)**
   - False (Some numbers do divide 77, such as 1, 7, 11, and 77)
4. **¬($\forall$n P(n))**
   - True (Since not all numbers divide 77, the negation of the universal quantifier is true)

Exercise 20, 22, 24: "$x^2 \geq x$" - True or False

5. **$\exists$x P(x)**
   - True (At least one value, such as $x = 2$, satisfies $x^2 \geq x$)
6. **¬($\exists$x P(x))**
   - False (Since at least one x satisfies $x^2 \geq x$, the negation of existential quantifier is false)
7. **$\exists$x ¬P(x)**
   - True (There exists an x, such as $x = 1/2$, where $x^2 < x$)

Exercise 25, 29: Rewrite Using Only Negation, Disjunction, and Conjunction

8. **$\forall$x P(x) → ¬$\exists$x ¬P(x)**
   - Equivalent to **¬$\exists$x ¬P(x)**
9. **$\exists$x ¬P(x)**
   - Equivalent to **¬$\forall$x P(x)**

Exercise 53, 55, 59: Truth Values

10. **$\forall x(x^2 > x)$**

- False
- Counterexample: $x = 1$ violates the statement since $1^2 = 1$ is not greater than 1.

11. **$\forall x(x > 1 \rightarrow x^2 > x)$**

- True
- For any $x > 1$, squaring x always results in a larger value.

12. **Negation of 53 and 55**

- Negation of **53**: **$\exists x(x^2 \leq x)$**
- Negation of **55**: **$\exists x(x > 1 \wedge x^2 \leq x)$**

Section 1.6 Exercises

Exercise 34-38: Negation in Words and Symbols

13. Someone loves everybody.
    Negation: No one loves everybody.
    Symbolic: $\neg\exists x\ \forall y\ L(x, y) \Leftrightarrow \forall x\ \exists y\ \neg L(x, y)$
14. Everybody loves everybody.
    Negation: There is someone who does not love everybody.
    Symbolic: $\neg\forall x\ \forall y\ L(x, y) \Leftrightarrow \exists x\ \exists y\ \neg L(x, y)$
15. Somebody loves somebody.
    Negation: Nobody loves anybody.
    Symbolic: $\neg\exists x\ \exists y\ L(x, y) \Leftrightarrow \forall x\ \forall y\ \neg L(x, y)$
16. Everybody loves somebody.
    Negation: There exists someone who loves nobody.
    Symbolic: $\neg\forall x\ \exists y\ L(x, y) \Leftrightarrow \exists x\ \forall y\ \neg L(x, y)$

Exercise 48-51: Truth Values

17. **$\forall x \forall y\ (x^2 < y + 1)$**

- **True**
- Justification: For any real number x, we can always find y such that x^2 is smaller than $y + 1$.

18. **$\forall x \exists y\ (x^2 < y + 1)$**

- **True**
- Justification: For each x, we can always find a y such that $x^2 < y + 1$ (for example, $y = x^2$).

19. **$\exists x \forall y \ (x^2 < y + 1)$**

- **False**
- Counterexample: If we assume $x = 0$, for every y, it must satisfy $0^2 < y + 1$, which holds, but choosing x very large does not guarantee this holds for all y.

20. **$\exists x \exists y \ (x^2 < y + 1)$**

- **True**
- Justification: We can find at least one pair (e.g., $x = 0$, $y = 1$) where this statement holds.

Exercise 54-57: Truth Values

21. **$\forall x \forall y \ (x^2 + y^2 = 9)$**

- **False**
- Counterexample: Choosing $x = 0$, $y = 0$ leads to $0^2 + 0^2 \neq 9$.

22. **$\exists x \exists y \ (x^2 + y^2 = 9)$**

- **True**
- Justification: (3,0) or (0,3) satisfies this equation.

23. **$\exists x \forall y \ (x^2 + y^2 = 9)$**

- **False**
- Counterexample: If $x = 0$, the equation $y^2 = 9$ must hold for all y, which is not true.

24. **$\exists x \exists y \ (x^2 + y^2 = 9)$**

- **True**
- Justification: Same reasoning as problem 55.

Exercise 62: Logical Proposition

25. **$\forall x \forall y \ ((x < y) \rightarrow (x^2 < y^2))$**

- **False**
- Counterexample: Consider $x = -2$ and $y = -1$. We have $-2 < -1$, but $(-2)^2 = 4$ is not less than $(-1)^2 = 1$.

Problem 3: Truth Values and Negations

(a) **$\forall x \forall y\ (x - y = 7)$ in** $\mathbb{Z}$

- **In words**: "For all integers x and y, x - y is always 7."
- **Truth value**: **False** (This is not always true for all x and y.)
- **Negation**: **$\exists x \exists y\ (x - y \neq 7)$**
  ("There exist some integers x and y such that x - y is not 7.")

(b) **$\exists x \exists y\ (x - y = 7)$ in** $\mathbb{Z}$

- **In words**: "There exist integers x and y such that x - y = 7."
- **Truth value**: **True** (For example, x = 10 and y = 3 satisfy this equation.)

(c) **$\forall x \exists y\ (x - y = 7)$ in** $\mathbb{Z}$

- **In words**: "For every integer x, there exists an integer y such that x - y = 7."
- **Truth value**: **True** (For every x, choosing y = x - 7 makes the equation hold.)

(d) **$\exists x \forall y\ (x - y = 7)$ in** $\mathbb{Z}$

- **In words**: "There exists an integer x such that for all y, x - y = 7."
- **Truth value**: **False** (If x were fixed, x - y = 7 must hold for all y, which is impossible.)
- **Negation**: **$\forall x \exists y\ (x - y \neq 7)$**
  ("For every integer x, there exists an integer y such that x - y is not 7.")

(e) **$\forall x \exists y\ (xy = 7)$ in** $\mathbb{Q}$

- **In words**: "For every rational number x, there exists a rational number y such that xy = 7."
- **Truth value**: **False** (For x = 0, there is no y such that $0 \cdot y = 7$.)
- **Negation**: **$\exists x \forall y\ (xy \neq 7)$**
  ("There exists some x such that for all y, xy is not 7.")

(f) **$\forall x \exists y\ ((x \neq 0) \rightarrow xy = 7)$ in** $\mathbb{Q}$

- **In words**: "For every rational number x, if $x \neq 0$, then there exists a rational number y such that xy = 7."
- **Truth value**: **True** (For any $x \neq 0$, choosing y = 7/x ensures xy = 7.)

(g) **$\forall x \exists y\ (xy = 7)$ in $\mathbb{Q}$ - {0}**

- **In words**: "For every nonzero rational number x, there exists a rational number y such that xy = 7."
- **Truth value**: **True** (For any $x \neq 0$, choosing $y = 7/x$ satisfies the equation.)

(h) **$\forall x \exists y$ (x > y) in** $\mathbb{Z}$

- **In words**: "For every integer x, there exists an integer y such that x > y."
- **Truth value**: **True** (For any integer x, choosing $y = x - 1$ satisfies $x > y$.)

(i) **$\exists y \forall x$ ($x \leq y$) in** $\mathbb{Z}$

- **In words**: "There exists an integer y such that for all integers x, $x \leq y$."
- **Truth value**: **False** (There is no maximum integer.)
- **Negation**: **$\forall y \exists x$ (x > y)**
  ("For every integer y, there exists an integer x such that x > y.")
