Homework 9 - Functions, Sequences, Strings

CS241

Finding the inverse function

If f is a bijection $A \to B$, it has an inverse, $f^{-1} : B \to A$, defined by:

$$f^{-1} = \{(b, a) | (a, b) \in f\}$$

. f^{-1} is also a bijection, and

$$\forall_{a \in A} . f^{-1} \circ f(a) = a$$

$$\forall_{b \in B} . f \circ f^{-1}(b) = b$$

Let $y = f(x)$ a bijection. To find f^{-1} we need to find x in terms of y. Example:
Let $f : [-5, \infty) \to \mathbb{R}^{+} \cup 0, \quad f(x) = \sqrt{x+5}$.

Then

$$y = \sqrt{x+5}$$

we would like to find x in terms of y:

$$y^2 = x + 5$$

$$y^2 - 5 = x$$

Therefore, if $y = \sqrt{x+5}$, it means $x = y^2 - 5$:
The inverse of $f(x) = \sqrt{x+5}$ is $f^{-1}(y) = y^2 - 5$,
f^{-1} is a bijection $\mathbb{R}^{+} \cup \{0\} \to [-5, \infty)$.

Functions, bijections

1. Read definitions 3.1.1, 3.1.22, 3.1.22, 3.1.29, 3.1.35, 3.1.47

2. $Let A = \{1, 2, 3, 4, 5\},\ f, g, h, k \subseteq A \times A$

$f = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 2)\},$

$g = \{(1, 5), (2, 3), (3, 2), (4, 4), (5, 1)\},$

$h = \{(1, 3), (2, 4), (3, 5), (4, 1), (5, 2)\},$

$k = \{(1, 3), (2, 5), (3, 4), (4, 5), (1, 1)\}$

   (a) For each of f, g, h, k; determine if it is a function. If it is a function, determine if it is injective, surjective, bijective.
   If it is bijective, find the inverse function. Explain you answers.

   (b) Write in set notation: $g \circ h, \quad h \circ g, \quad g^2, \quad g^3, \quad h^2, \quad f^2, \quad h^5$

3. Let $X = \{1, 2, 3, 4, 5\}$, $Y = \{a, b, c, d, e\}$, $b : X \to Y$ is a bijection.
   Consider the compositions $b^{-1} \circ b, \quad b \circ b^{-1}$.
   Find the domain, co domain, and write in set notation.

4. Read about the floor and ceiling functions in example 3.1.17.

   - Let $g : \mathbb{R} \to \mathbb{R}\ , \quad g(x) = \lfloor x \rfloor$. Prove that g is not surjective.
   - Let $h : \mathbb{R} \to \mathbb{Z}\ , \quad h(x) = \lfloor x \rfloor$. Prove that h is surjective.
   - Prove that h is not injective.

5. Section 3.1. 14, 23, 24, 29, 44, 47, 59, 62, 96-100

Sequences, Strings

Section 3.2. 10, 12, 14, 16, 18, 19 - 22, 108, 109-112, 121-128 (example 3.2.7), 142(b), 142(e), 142(k), 156

Summation

6. Let A, B be constants, and let $\{X\}_{k=1}^{n}$ be a sequence.

   (a) Find an expression for the sum $\sum_{k=1}^{n} k$
   (more about this sum: https://www.youtube.com/watch?v=KRUuTPTcs2E and https://www.youtube.com/shorts/yLDf6Hw4NsY)

(b) Prove $\sum_{k=1}^{n} B = n \cdot B$

(c) Prove $\sum_{k=1}^{n} (A \cdot X_k) = A \cdot \sum_{k=1}^{n} X_k$

(d) Find a formula for the sum: $\sum_{k=1}^{n} (A \cdot k + B)$.

(e) Use the formula to find expressions for: $\sum_{k=1}^{n} (2n-1)$ and $\sum_{k=1}^{n} (4n+1)$

(f) Calculate: $\sum_{k=1}^{20} (4n+1)$, $\sum_{k=11}^{20} (4n+1)$

The triangular numbers

7. Denote $\triangle_n$ is the nth triangular number; $\triangle_n = \sum_{k=1}^{n} k$.

(a) Find the first 5 triangular numbers. $(\triangle_1, \triangle_2, \triangle_3, \triangle_4, \triangle_5)$.

(b) Prove exercise 2 in section 2.4

(c) Use the previous exercise to prove:

$$\sum_{i=1}^{m} \triangle_i = \frac{n(n+1)(n+2)}{6}$$

A visualization of the claim: https://www.youtube.com/watch?v=N0ETyJ5K6j0

(d) Prove that for all $n \in \mathbb{N}, n \geq 2$: $\triangle_{n-1} + \triangle_n = n^2$. Show an algebraic and a geometrical explanation

The geometric sum

8. Read example 2.4.4.

(a) Prove $\sum_{k=0}^{n} a \cdot (r^k) = a \cdot \sum_{k=0}^{n} (r^k)$.

(b) Evaluate 22_3, 222_3, 2222_3, 33_4, 333_4. Add 1 to each of this numbers (in the given base) Can you see a pattern?

(c) Find an expression for $\sum_{k=0}^{n} 2 \cdot (3^k)$. How would this number be written in base 3?

(d) Find an expression for $\sum_{k=0}^{n} 3 \cdot (4^k)$. How would this number be written in base 4?

(e) Based on the previous parts, use the geometric sum to evaluate the number $dd...dd_b$ where b is a base (≥ 2) and d is the maximal digit (so $d = b - 1$)