24 Lecture 24: Linear system of Differential Equations

Consider an initial value problem described by a **linear system of differential equation**

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t), \tag{24.1}$$

and an **initial condition at** $t = 0$**,** $\mathbf{x}(0) = \mathbf{x}_0$, where $\mathbf{x} = \mathbf{x}(t)$ is a vector-valued function of a scalar variable t and $\mathbf{A}$ is a real square matrix. This lecture discuss the application of the diagonalization to solve the initial value problem above.

24.1 Solutions to initial value problems

The solution to the initial value problem

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0$$

is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0$$

The calculation of $e^{\mathbf{A}t}$**:** Firstly,

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{(\mathbf{A}t)^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{A}^k.$$

Assume that $\mathbf{A}$ is diagonalizable, $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$. Then,

$$\sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{A}^k = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{S}\mathbf{D}^k\mathbf{S}^{-1} = \mathbf{S}\left(\sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{D}^k\right)\mathbf{S}^{-1} = \mathbf{S}e^{\mathbf{D}t}\mathbf{S}^{-1}.$$

Furthermore,

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{D}^k = \sum_{k=0}^{\infty} \frac{t^k}{k!}\begin{bmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_n^k \end{bmatrix} &= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{t^k}{k!}\lambda_1^k & 0 & \dots & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{t^k}{k!}\lambda_2^k & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \sum_{k=0}^{\infty} \frac{t^k}{k!}\lambda_n^k \end{bmatrix} \\
&= \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix}.
\end{aligned}$$

Thus,

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 = \mathbf{S}e^{\mathbf{D}t}\mathbf{S}^{-1}\mathbf{x}_0 = \mathbf{S}\begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix}\mathbf{S}^{-1}\mathbf{x}_0.$$

Example 24.1. *Suppose a particle is moving in a planar force field and its position vector x satisfies $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ and $\mathbf{x}(0) = \mathbf{x}_0$, where*

$$\mathbf{A} = \begin{bmatrix} 4 & -5 \\ -2 & 1 \end{bmatrix}, \qquad \mathbf{x}_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Solve this initial value problem for $t \geq 0$.

Solution: Eigenvalues of $\mathbf{A}$ are $\lambda_1 = 6$ and $\lambda_2 = -1$ with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (respectively). The solution to the initial value problem is

$$\begin{aligned}
\mathbf{x}(t) &= \begin{bmatrix} -5 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} e^{6t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} -5 & 1 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\
&= -\frac{1}{7} \begin{bmatrix} -5 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} e^{6t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & -5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = -\frac{1}{7} \begin{bmatrix} -5e^{6t} & e^{-t} \\ 2e^{6t} & e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ -9 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 5e^{6t} + 9e^{-t} \\ -2e^{6t} + 9e^{-t} \end{bmatrix}.
\end{aligned}$$

Example 24.2. *Find the solution of the initial value problem $\mathbf{x}' = \mathbf{A}\mathbf{x}$ and $\mathbf{x}(0) = \mathbf{x}_0$, where*

$$\mathbf{A} = \begin{bmatrix} -2 & -2.5 \\ 10 & -2 \end{bmatrix}, \qquad \mathbf{x}_0 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Solution: Matrix $\mathbf{A}$ has complex eigenvalues $\lambda_1 = -2+5i$ and $\lambda_2 = -2-5i$ and the corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} i \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -i \\ 2 \end{bmatrix}$. The solution to the initial value problem is

$$\begin{aligned}
\mathbf{x}(t) &= \begin{bmatrix} i & -i \\ 2 & 2 \end{bmatrix} \begin{bmatrix} e^{(-2+5i)t} & 0 \\ 0 & e^{(-2-5i)t} \end{bmatrix} \begin{bmatrix} i & -i \\ 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\
&= \frac{1}{4i} \begin{bmatrix} i & -i \\ 2 & 2 \end{bmatrix} \begin{bmatrix} e^{(-2+5i)t} & 0 \\ 0 & e^{(-2-5i)t} \end{bmatrix} \begin{bmatrix} 2 & i \\ -2 & i \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{e^{-2t}}{4} \begin{bmatrix} i & -i \\ 2 & 2 \end{bmatrix} \begin{bmatrix} e^{5it} & 0 \\ 0 & e^{-5it} \end{bmatrix} \begin{bmatrix} -2i & 1 \\ 2i & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\
&= \frac{e^{-2t}}{4} \begin{bmatrix} 2(e^{5it} + e^{-5it}) & i(e^{5it} - e^{-5it}) \\ -4i(e^{5it} - e^{-5it}) & 2(e^{5it} + e^{-5it}) \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{e^{-2t}}{2} \begin{bmatrix} 4\cos 5t & -2\sin 5t \\ 8\sin 5t & 4\cos 5t \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= e^{-2t} \begin{bmatrix} 2\cos 5t - \sin 5t \\ 2\cos 5t + 4\sin 5t \end{bmatrix}
\end{aligned}$$

Remark 24.1. *Euler's formula:*

$$e^{i\varphi} = \cos\varphi + i\sin\varphi, \quad \textit{for all} \quad \varphi \in \mathbf{R}.$$

24.2 Application to damped spring-mass oscillation

Motion of a mass on a vibrating with forced-free is described by a second-order differential equation

$$my''(t) + cy'(t) + ky(t) = 0,$$

where the mass m, the damping constant c, and the spring constant k are non-negative. This system can be reformulate as a linear system of differential equations discussed above. To do so, we denote $v(t) = y'(t)$ the velocity of the the mass, then $y''(t) = v'(t)$, which is the acceleration. The second-order differential equation is reformulated as

$$\begin{cases} v'(t) = -\frac{c}{m}v(t) - \frac{k}{m}y(t) \\ y'(t) = v(t), \end{cases} \quad \text{or} \quad \begin{bmatrix} u' \\ y' \end{bmatrix} = \begin{bmatrix} -\frac{c}{m} & -\frac{k}{m} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}$$

We introduce vector $\mathbf{x}$ by $\mathbf{x}(t) = \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}$, then $\mathbf{x}'(t) = \begin{bmatrix} u' \\ y' \end{bmatrix}$ and the system becomes

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t), \quad \text{where} \quad \mathbf{A} = \begin{bmatrix} -\frac{c}{m} & -\frac{k}{m} \\ 1 & 0 \end{bmatrix}.$$

This system is a particular case of the general form introduced in (24.1). Furthermore, eigenvalues of matrix $\mathbf{A} = \begin{bmatrix} -\frac{c}{m} & -\frac{k}{m} \\ 1 & 0 \end{bmatrix}$ are:

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}.$$

Three statements of the spring-mass oscillation:

(a) If $c^2 - 4mk > 0$, both eigenvalues are real numbers. This oscillation is called **overdamped**, and the general solution is of the form

$$y(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t},$$

where A and B are constants. If the initial condition $\mathbf{x}(0)$ is given, values of A and B are determined uniquely.

(b) If $c^2 - 4mk < 0$, both eigenvalues are complex numbers. This oscillation is called **underdamped**. We express the eigenvalue λ_1 as $\lambda_1 = \alpha + i\beta$, then the general solution is of the form

$$\mathbf{x}(t) = e^{\alpha t}(A\cos\beta t + B\sin\beta t),$$

where A and B are constants. If the initial condition $\mathbf{x}(0)$ is given, values of A and B are determined uniquely.

(c) If $c^2 - 4mk = 0$, matrix $\mathbf{A}$ has repeated eigenvalues $\lambda_1 = \lambda_2 = -\frac{c}{2m}$. This oscillation is called **critical damped**. The general solution is

$$y(t) = e^{-\frac{c}{2m}t}(A + Bt),$$

where A and B are constants. If the initial condition $\mathbf{x}(0)$ is given, values of A and B are determined uniquely.

See examples on Canvas - common course.

24.3 Suggested problems

Problem 24.1. *Let* $\mathbf{u} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} & 0 \end{bmatrix}^T$ *and* $\mathbf{A} = \mathbf{u}\mathbf{u}^T$. *Assume that* $\mathbf{x}(t)$ *is the solution to the initial value problem:*

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t), \quad \mathbf{x}(0) = \begin{bmatrix} 2 & 2 & 0 \end{bmatrix}^T$$

(a) Find $\mathbf{x}(t)$

(b) Calculate $\lim_{t\to\infty} \mathbf{x}(t)$.