Problem 4. (10 points) [Sample Problem A] Suppose that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal bases for $\mathbb{R}^2$. Define the 2×2 matrix

$$A = 2\mathbf{u}_1\mathbf{u}_1^T - \mathbf{u}_2\mathbf{u}_2^T.$$

(a) Find an orthogonal diagonalization of A.

(b) Find a formula for A^{-1} (if it exists) in terms of the given information.

(c) Find a singular value decomposition (SVD) for A.

Problem 4. (10 points) [Sample B] Consider the 3×3 matrix $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$.

(a) Diagonalize the matrix A.

$A = SDS^{-1}$ where $S = \boxed{}$ and $D = \boxed{}$.

(b) Compute and simplify e^{At} where t is a real number. Simplification should eliminate any explicit appearance of the complex number $i = \sqrt{-1}$. Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ may prove helpful.

$e^{At} = \boxed{}$

(c) Find the singular value decomposition (SVD) of A.

$A = U\Sigma V^T$ where $U = \boxed{}$ $\Sigma = \boxed{}$ $V = \boxed{}$

Problem 4. (10 points) [Sample Problem C] Suppose $A = QR$ where the orthogonal matrix Q and the upper trapezoidal matrix R are given by

$$Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \qquad \text{and} \qquad R = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) Explaining your solution process, find $\operatorname{rref}(A)$.

(b) Find an orthonormal basis for the left null space of A.

(c) Find the SVD of A.

Problem 4. (10 points) [Sample Problem D] (a) Suppose that A is a square matrix with characteristic polynomial

$$p_A(\lambda) = (1 - \lambda)^n.$$

Furthermore, suppose that $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a set of linearly independent eigenvectors of A. Compute and simplify A.

(b) Suppose that R is not a diagonal matrix but is a diagonalizable matrix which is a square root of I ($R^2 = I$). What is the spectrum (set of eigenvalues) of R?

Problem 5. (10 points) [Sample Problem A] Suppose that $\{\mathbf{u}_1, \mathbf{u}_2\}$ and $\{\mathbf{v}_1, \mathbf{v}_2\}$ are two orthonormal bases for $\mathbb{R}^2$. Define the 2×2 matrix

$$A = \mathbf{u}_1 \mathbf{u}_1^T - 2\mathbf{u}_2 \mathbf{u}_2^T.$$

(a) Find the singular value decomposition (SVD) for A.

(b) Find a formula for A^{-1} (if it exists) in terms of the given information.

(c) Find the singular value decomposition (SVD) of A^{-1}.

Problem 5. (10 points) [Sample Problem B] Suppose θ is real and $s \geq t \geq 0$. Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

(a) Find a full singular value decomposition (SVD) for A.

(b) Find an orthogonal diagonalization of $A^T A$.

(c) Find an orthogonal diagonalization of AA^T.

Problem 5. (10 points) [Sample Problem C]

(a) Find the orthogonal diagonalization of $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$.

(b) Find the orthogonal diagonalization of $B = \begin{bmatrix} 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & -3 \\ 3 & 4 & 0 & 0 \\ 4 & -3 & 0 & 0 \end{bmatrix}$.

Hint: If $\mathbf{v}$ is an eigenvector of A, then $\begin{bmatrix} \mathbf{v} \\ \mathbf{v} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{v} \\ -\mathbf{v} \end{bmatrix}$ are eigenvectors of B.

(c) Find the SVD of $B = \begin{bmatrix} 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & -3 \\ 3 & 4 & 0 & 0 \\ 4 & -3 & 0 & 0 \end{bmatrix}$.

Problem 5. (10 points) [Sample Problem C]

Suppose $A = \mathbf{u}_1\mathbf{v}_1^T$ where $\mathbf{u}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$ with $a^2 + b^2 = 1$ and $\mathbf{v}_1 = \begin{bmatrix} c \\ d \end{bmatrix}$ with $c^2 + d^2 = 1$.

(a) Find vectors $\mathbf{u}_2$ and $\mathbf{v}_2$ such that $\{\mathbf{u}_1, \mathbf{u}_2\}$ and $\{\mathbf{v}_1, \mathbf{v}_2\}$ are orthonormal bases for $\mathbb{R}^2$.

(b) Find the SVD of A.

Extra Credit. (5 points) [Sample Problem A] Suppose

$$\begin{aligned} \ddot{y} + \dot{y} &= 0 \\ y(0) &= 0 \\ \dot{y}(0) &= 1 \end{aligned}$$

Find and simplify

$$\lim_{t\to\infty} y(t).$$

Extra Credit. (5 points) [Sample Problem B] Showing your work, derive an explicit formula (the Binet formula) for the n^{th} Fibonacci number F_n where

$$F_{\ell+2} = F_{\ell+1} + F_\ell \qquad \text{for} \quad \ell = 0, 1, \ldots$$

and $F_0 = 0$ and $F_1 = 1$.

Extra Credit. (5 points) [Sample Problem C] Suppose $0 \leq a \leq 1$. Consider the stochastic matrix

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 1 & 0 \\ 0 & \frac{2}{3} & 0 & 0 & 0 & 1 \end{bmatrix}.$$

And define the iteration $\mathbf{p}_{\ell+1} = A\mathbf{p}_\ell$ where $\mathbf{p}_0 = \begin{bmatrix} a \\ 1-a \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Find $\lim_{\ell\to\infty} \mathbf{p}_\ell$.