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Connectives

1. Explanation of Theorem 1.1.22 (d), (h), (k)

- **(d) Distributive Law:** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C) A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  - This follows because taking the intersection of AA with the union $B \cup C B \cup C$ means we consider elements in AA that also belong to either BB or CC. This is equivalent to taking the union of the two intersections.
- **(h) De Morgan's Law:** $\neg(A \cap B) = \neg A \cup \neg B \neg(A \cap B) = \neg A \cup \neg B$
  - The left-hand side represents the complement of the intersection, meaning elements that are missing from at least one of AA or BB. This is equivalent to taking the union of the complements.
- **(k) Involution Law:** $\neg(\neg A) = A \neg(\neg A) = A$
  - Applying negation twice restores the original set.

2. Conversion of Set Identities to Propositional Logic

- $A \cap B \to p \wedge q A \cap B \to p \wedge q$
- $A \cup B \to p \vee q A \cup B \to p \vee q$
- $\neg A \to \neg p \neg A \to \neg p$
- $U \to T U \to T$
- $\emptyset \to F \emptyset \to F$

Some converted expressions:

- $A \cup A = A \Rightarrow p \vee p = p A \cup A = A \Rightarrow p \vee p = p$
- $A \cap A = A \Rightarrow p \wedge p = p A \cap A = A \Rightarrow p \wedge p = p$
- $\neg(\neg A) = A \Rightarrow \neg(\neg p) = p \neg(\neg A) = A \Rightarrow \neg(\neg p) = p$

3. Applying De Morgan's Laws to Three Variables

$\neg(p1 \vee p2 \vee p3) = \neg((p1 \vee p2) \vee p3) \neg(p_1 \vee p_2 \vee p_3) = \neg((p_1 \vee p_2) \vee p_3)$ Applying De Morgan's Theorem: $= (\neg p1 \wedge \neg p2) \wedge \neg p3 = (\neg p_1 \wedge \neg p_2) \wedge \neg p_3$

Final result: $\neg p1 \wedge \neg p2 \wedge \neg p3 \neg p_1 \wedge \neg p_2 \wedge \neg p_3$

Similarly, for sets:
$\neg(A \cap B \cap C) = \neg A \cup \neg B \cup \neg C \neg(A \cap B \cap C) = \neg A \cup \neg B \cup \neg C \neg(A \cup B \cup C) = \neg A \cap \neg B \cap \neg C \neg(A \cup$

4. Writing Equivalent Expressions Using $\neg, \wedge$

- $\neg(p \vee q) = \neg p \wedge \neg q \neg(p \vee q) = \neg p \wedge \neg q$
- $\neg(\neg p \vee \neg q) = p \wedge q \neg(\neg p \vee \neg q) = p \wedge q$
- $\neg(p \vee \neg q) = \neg p \wedge q \neg(p \vee \neg q) = \neg p \wedge q$
- $p \vee q = \neg(\neg p \wedge \neg q) p \vee q = \neg(\neg p \wedge \neg q)$
- $p \vee \neg q = \neg(\neg p \wedge q) p \vee \neg q = \neg(\neg p \wedge q)$
- $\neg p \vee \neg q = \neg(p \wedge q) \neg p \vee \neg q = \neg(p \wedge q)$

5. Expressing Logical Connectives Using $\neg, \wedge, \vee$

- $p \rightarrow r = \neg p \vee r p \rightarrow r = \neg p \vee r$
- $\neg(p \rightarrow r) = p \wedge \neg r \neg(p \rightarrow r) = p \wedge \neg r$
- $p \oplus r = (p \wedge \neg r) \vee (\neg p \wedge r) p \oplus r = (p \wedge \neg r) \vee (\neg p \wedge r)$
- $p \oplus (r \wedge s) = (p \wedge \neg(r \wedge s)) \vee (\neg p \wedge (r \wedge s)) p \oplus (r \wedge s) = (p \wedge \neg(r \wedge s)) \vee (\neg p \wedge (r \wedge s))$
- $p \leftrightarrow r = (p \wedge r) \vee (\neg p \wedge \neg r) p \leftrightarrow r = (p \wedge r) \vee (\neg p \wedge \neg r)$

6. Proving Functional Completeness Using NOR $(\downarrow)$

- $\neg p = p \downarrow p \neg p = p \downarrow p$
- $p \vee q = (p \downarrow p) \downarrow (q \downarrow q) p \vee q = (p \downarrow p) \downarrow (q \downarrow q)$
- $p \wedge q = (p \downarrow q) \downarrow (p \downarrow q) p \wedge q = (p \downarrow q) \downarrow (p \downarrow q)$

7. Proving Functional Completeness of $\neg, \vee$

Using De Morgan’s Laws:

- $\neg p = \neg p \neg p = \neg p$
- $p \vee q = p \vee q p \vee q = p \vee q$
- $p \wedge q = \neg(\neg p \vee \neg q) p \wedge q = \neg(\neg p \vee \neg q)$

8. Proving Functional Completeness of $\neg, \rightarrow$

Using transformations:

- $\neg p = \neg p \neg p = \neg p$
- $p \vee q = \neg(p \rightarrow \neg q) p \vee q = \neg(p \rightarrow \neg q)$
- $p \wedge q = \neg(p \rightarrow \neg q) \vee \neg(q \rightarrow \neg p) p \wedge q = \neg(p \rightarrow \neg q) \vee \neg(q \rightarrow \neg p)$

9. Functional Completeness of $\neg, \diamond$

Define $p \diamond q \equiv \neg(p \to q) p \diamond q \equiv \neg(p \to q)$:

- $\neg p = p \diamond p \neg p = p \diamond p$
- $p \vee q = \neg(\neg p \diamond \neg q) p \vee q = \neg(\neg p \diamond \neg q)$
- $p \wedge q = \neg((p \diamond \neg q) \diamond (q \diamond \neg p)) p \wedge q = \neg((p \diamond \neg q) \diamond (q \diamond \neg p))$

Normal Forms

10. Expressing R3, R4, and Q3 in $\neg, \vee, \wedge$

- (a) $R3 = \neg p \wedge q \wedge \neg r R_3 = \neg p \wedge q \wedge \neg r$
- (b) $R4 = \neg p \wedge q \wedge \neg r R_4 = \neg p \wedge q \wedge \neg r$
- (c) $R4 = \neg p \wedge \neg q \wedge \neg r R_4 = \neg p \wedge \neg q \wedge \neg r$
- (d) $Q3 = p \vee \neg q \vee r Q_3 = p \vee \neg q \vee r$
- (e) $Q4 = p \vee \neg q \vee \neg r Q_4 = p \vee \neg q \vee \neg r$
- (f) $Q8 = p \vee q \vee r Q_8 = p \vee q \vee r$

11. Finding the DNF of Logical Expressions

- $p \oplus q = (p \wedge \neg q) \vee (\neg p \wedge q) p \oplus q = (p \wedge \neg q) \vee (\neg p \wedge q)$
- $p \leftrightarrow q = (p \wedge q) \vee (\neg p \wedge \neg q) p \leftrightarrow q = (p \wedge q) \vee (\neg p \wedge \neg q)$
- $\neg(p \to q) = p \wedge \neg q \neg(p \to q) = p \wedge \neg q$

12. Truth Table R_1

p	q	r	R1	R2	R3
T	T	T	T	T	F
T	T	F	F	T	T
T	F	T	T	F	T
T	F	F	F	F	F
F	T	T	T	T	T
F	T	F	F	F	T
F	F	T	F	F	F
F	F	F	F	T	F

13. Finding DNF and CNF of R_1

- **DNF**:
  $R_1 = (p \land q \land r) \lor (p \land \neg q \land r) \lor (\neg p \land q \land r)$
- **CNF**:
  $R_1 = (p \lor q \lor r) \land (p \lor \neg q \lor r) \land (\neg p \lor q \lor r)$

14. Completing R_2 and Finding its CNF

Fill in the values for R_2:

- **CNF**:
  $R_2 = (p \lor q \lor r) \land (p \lor q \lor \neg r) \land (\neg p \lor q \lor r) \land (\neg p \lor \neg q \lor \neg r)$

15. Completing R_3 and Finding its DNF

- **DNF**:
  $R_3 = (p \land q \land r) \lor (p \land q \land \neg r) \lor (\neg p \land q \land r) \lor (\neg p \land \neg q \land \neg r)$

Circuit Boolean Expressions

Circuit	Boolean Expression
D1	$\neg((x_1 \land x_2) \lor (x_2 \land x_3))$
D2	$(x_2 \land x_3) \lor (x_1 \oplus x_2)$
D3	$\neg x_1 \land \neg x_2 \land \neg x_3$
D4	$\neg(x_3 \lor (x_1 \oplus x_2))$
D5	$x_1 \land x_2$
D6	$x_1 \land (x_1 \land x_2)$
D7	$x_1 \land x_2$
D8	$x_1 \oplus x_2$
D9	$x_1 \oplus x_2$
D10	$(x_1 \oplus x_2) \lor (x_1 \oplus x_2)$
D11	$(x_1 \land x_2) \lor (\neg x_1 \oplus x_2)$
D12	$(\neg x_1 \oplus x_2) \lor (x_1 \land x_2)$