Problem 1.1.

(a) (20 points) Clearly documenting your work, compute the reduced row-echelon form of the matrix

$$A = \begin{array}{c} \bullet \\ 2 \\ 3 \end{array} \begin{bmatrix} 2 & 4 & 2 & 4 \\ 4 & 8 & 5 & 10 \\ 6 & 12 & 7 & 14 \end{bmatrix} \sim \begin{array}{c} \\ \bullet \\ 1 \end{array} \begin{bmatrix} 2 & 4 & 2 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{array}{c} \times 1/2 \\ \\ \\ \end{array} \begin{bmatrix} 2 & 4 & 2 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim$$

(b) (1 points) Which columns of A have pivots? $\boxed{1,\ 3}$ (list column numbers)

$$\begin{array}{c} 1 \\ \bullet \\ \\ \end{array} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \boxed{\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

(c) (9 points) If possible, find the solution of the linear system below in parametric vector form.

$$\begin{cases} 2x_1 + 4x_2 + 2x_3 = 4 \\ 4x_1 + 8x_2 + 5x_3 = 10 \\ 6x_1 + 12x_2 + 7x_3 = 14 \end{cases}$$

Since the augmented matrix for this system is the matrix given in part (a), the rref matrix computed there is the rref for our system. Hence, we directly write the parameteric vector form of the solution set:

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}$$

Alternatively, we can use the system with the rref as the augmented matrix

$$\begin{cases} x_1 + 2x_2 = 0 \\ x_3 = 2 \\ 0 = 0 \end{cases}$$

to write the parametric form

$$\begin{cases} x_1 = 0 - 2x_2 \\ x_3 = 2 \end{cases}$$

which we then convert to the parametric vector form.

Problem 1.2. (8 points) Carefully simplify each valid expression. For any invalid expression, explain why it is invalid.

(a) $$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$

(b) $$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}$$

$$\begin{bmatrix} 4 & 5 \\ 10 & 11 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$$

(c) $$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}^T \qquad \boxed{\text{is undefined}}$$

$2\times1 \quad 2\times2 \qquad 1\times2 \quad 1\times5$ columns of first factors fail to match rows of second factors

(d) $$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix} = \boxed{\begin{bmatrix} 3 & 6 & 9 & 12 & 15 \\ 2 & 4 & 6 & 8 & 10 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}}$$

Problem 1.3. (12 points) For each matrix below and on the next page, compute the matrix inverse, if it exists. If it fails to exist explain why.

$$A_1 = \begin{bmatrix} 9 & 8 \\ 10 & 9 \end{bmatrix}$$

$$A_1^{-1} = \frac{1}{9 \cdot 9 - 8 \cdot 10} \begin{bmatrix} 9 & -8 \\ -10 & 9 \end{bmatrix} = \boxed{\begin{bmatrix} 9 & -8 \\ -10 & 9 \end{bmatrix}}$$

$$A_2 = \begin{bmatrix} 4 & 8 \\ 2 & 4 \end{bmatrix} \qquad \boxed{\text{no inverse}} \qquad \text{since} \quad 4{\cdot}4 - 2{\cdot}8 = 0$$

$$A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\begin{array}{c} \bullet \\ 1 \\ 1 \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{bmatrix} \sim \begin{array}{c} \\ \bullet \\ 1 \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{bmatrix} \sim \begin{array}{c} 1 \\ 1 \\ \bullet \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix} \sim$$

$$\begin{array}{c} 1 \\ \bullet \\ \\ \end{array} \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix} \qquad A_3^{-1} = \boxed{\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}}$$

check

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The augmentation of A_4 by I,

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & -3 & -4 & -5 & -6 & -7 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

is easily brought to rref by multiplying the the ith row by i and subtracting it from the first row for $i = 2, 3, \ldots 7$. Hence,

$$\boxed{A_4^{-1} = \begin{bmatrix} 1 & -2 & -3 & -4 & -5 & -6 & -7 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}$$

Problem 2. (10 points) Evaluate the truth of each statement below. If the statement is true write T in the box preceding the statement. Otherwise, write F.

(a) $\boxed{T}$ A linear system with more than one solution must have a free variable.

(b) $\boxed{T}$ All matrices have a reduced row-echelon form.

(c) $\boxed{T}$ If the matrix A is invertible, then $\mathrm{rref}(A) = I$.

(d) $\boxed{T}$ Some pairs of 2×2 matrices, A and B, commute (i.e., $AB = BA$).

(e) $\boxed{T}$ The average of two solutions of a linear system is also a solution.

Problem 3. (10 points) Suppose that the MATLAB function `scalebyleading` is defined by

```
function A=scalebyleading(A)
    [m n]=size(A);
    for r=1:m
        for c=1:n
            if A(r,c) ~= 0
                A(r,:)=A(r,:)/A(r,c);
                break;
            end
        end
    end
end
```

Assuming `scalebyleading` is in the current MATLAB search path, what is the output obtained by entering the following MATLAB commands:

```
>> A = [ 0 0 2 3 4; 0 0 0 0 0; 1 2 3 4 5; 0 0 2 0 0; 0 1 2 3 2; 0 0 0 0 0];
>> scalebyleading(A)
```

$$\texttt{ans} = \boxed{\begin{array}{ccccc} 0 & 0 & 1 & 3/2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}}$$

The function `scalebyleading` identifies the leading element of every row (if any) and scales the row so that the value of the leading element becomes 1.

Problem 4. (15 points) Suppose a is a real number (which may or may not be 0).

(a) Find all vectors $\mathbf{x}$ that satisfy the following matrix equation.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} [0 \quad 1 \quad a]\,\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Notice that for both sides of the matrix equation to have the same dimensions $\mathbf{x} \in \mathbb{R}^3$.

$$\begin{bmatrix} 0 & 1 & a \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 0 & 1 & a & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow$$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -a \\ 1 \end{bmatrix}}$$

(b) Find all vectors $\mathbf{y}$ that satisfy the following matrix equation.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} [0 \quad 1 \quad a]\,\mathbf{y} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Since $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is invertible $(4 - 6 \neq 0)$ multiplying both sides of the matrix equation (on the left) by the inverse gives the same matrix equation as in part (a) (with a new name for the unknowns). Hence, the solution for (b) is

$$\boxed{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + y_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y_3 \begin{bmatrix} 0 \\ -a \\ 1 \end{bmatrix}}$$

(c) Find all vectors $\mathbf{z}$ that satisfy the following matrix equation.

$$\begin{bmatrix} a & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{z} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

The linear system proclaims explicitly $z_2 = z_3 = z_4 = z_5 = z_6 = z_7 = z_8 = 1$. Therefore, we have $az_1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 0$. Hence, if $a \neq 0$ then $z_1 = -35/a$, and if $a = 0$ there are no solutions. Thus,

$$\boxed{\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ z_8 \end{bmatrix} = \begin{bmatrix} -35/a \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{if } a \neq 0 \qquad \text{and} \qquad \emptyset \quad \text{if } a = 0.}$$

It is also possible to solve this problem (for $a \neq 0$) using the inverse matrix:

$$\begin{bmatrix} a & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{a} & -\frac{2}{a} & -\frac{3}{a} & -\frac{4}{a} & -\frac{5}{a} & -\frac{6}{a} & -\frac{7}{a} & -\frac{8}{a} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 5. (15 points) Suppose a linear system with a 2×3 augmented matrix

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \end{bmatrix}$$

has as its solution set the line passing through the two points

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

(a) How many free variables does the linear system have? Explain.

$\boxed{1}$, a line is generated by 1 parameter:

$$\mathbf{x} = \alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 \qquad -\infty < \alpha < \infty.$$

(b) How many pivot columns does H have? Explain.

$\boxed{1}$, column 3 has no pivot since solutions exist; of columns 1 and 2 exactly one is free (i.e., has no pivot) implying the other must have a pivot.

(c) Find rref(H).

Method I: The two point formula gives an equation for the given line:

$$y - 2 \;=\; \frac{2-1}{1-3}(x-1) \qquad \Rightarrow \qquad \frac{1}{2}x \;+\; y \;=\; \frac{5}{2}.$$

Joining this with the trivial equation $0 = 0$ gives a system of two equations and two unknowns with the given solution set and the augmented matrix

$$\begin{array}{c} \times\, 2 \\ \\ \end{array} \begin{bmatrix} \frac{1}{2} & 1 & \frac{5}{2} \\ 0 & 0 & 0 \end{bmatrix} \sim \boxed{\begin{bmatrix} 1 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix}}$$

Method II: We use the affine combinations of the two given points to find the "equation" of the line (without using the two point formula).

$$\begin{bmatrix} x \\ y \end{bmatrix} \;=\; \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} \;+\; (1-\alpha)\begin{bmatrix} 3 \\ 1 \end{bmatrix} \;=\; \begin{bmatrix} 3 \\ 1 \end{bmatrix} \;+\; \alpha \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} x - 3 & = -2\alpha \\ y - 1 & = \alpha \end{cases}$$

Dividing the first equation by the second (to rediscover the two point formula)

$$\frac{x-3}{y-1} = -2 \quad \Rightarrow \quad x - 3 = -2(y-1) \quad \Rightarrow \quad x + 2y = 5.$$

Joining this equation with the trivial equation gives a linear system having the required solution set:

$$\begin{cases} x \;+\; 2y \;=\; 5 \\ \qquad\quad 0 \;=\; 0 \end{cases}$$

The augmented matrix for this system is already in reduced row-echelon form:

$$\boxed{\begin{bmatrix} 1 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix}}$$

Extra Credit. (5 points) Suppose $x_0, x_1, x_2, \ldots$ is a sequence of real numbers satisfying

$$
\begin{aligned}
x_{n+2} &= \frac{1}{2}(x_{n+1} + x_n) \qquad n = 0, 1, 2, \ldots, \\
x_0 &= 0, \\
x_1 &= 1.
\end{aligned}
$$

Find and simplify an algebraic formula for x_n.

Guess for special solutions: $x_n = \lambda^n$. Hence, $\lambda^{n+2} = \frac{1}{2}\lambda^{n+1} + \frac{1}{2}\lambda^n$ which implies the characteristic equation:

$$\lambda^2 - \frac{1}{2}\lambda - \frac{1}{2} = 0$$

Hence,

$$\lambda = \frac{\frac{1}{2} \pm \sqrt{\left(-\frac{1}{2}\right)^2 - 4\left(-\frac{1}{2}\right)}}{2} = \frac{\frac{1}{2} \pm \frac{3}{2}}{2} \;=\; 1,\; -\frac{1}{2}$$

We seek the solution as a linear combination of the two special solutions:

$$x_n = \alpha(1)^n + \beta(-\frac{1}{2})^n.$$

To be valid for $n = 0$ and $n = 1$ requires

$$0 = x_0 = \alpha + \beta \qquad \text{and} \qquad 1 = x_1 = \alpha - \frac{1}{2}\beta.$$

Solving these two linear equations gives $\alpha = \frac{2}{3}$ and $\beta = -\frac{2}{3}$. Hence,

$$\boxed{x_n \;=\; \frac{2}{3} - \frac{2}{3}\left(-\frac{1}{2}\right)^n}$$

.

Extra Extra Credit. (5 points) Showing your work, find the smallest positive integers x_1, x_2, x_3 and x_4 that balances the chemical equation for the combustion of octane:

$$(x_1)\ \mathrm{C_8H_{18}}\ +\ (x_2)\ \mathrm{O_2}\ \to\ (x_3)\ \mathrm{CO_2}\ +\ (x_4)\ \mathrm{H_2O}.$$

$$\boxed{\quad 2 \quad}\ \mathrm{C_8H_{18}}\ +\ \boxed{\quad 25 \quad}\ \mathrm{O_2}\ \to\ \boxed{\quad 16 \quad}\ \mathrm{CO_2}\ +\ \boxed{\quad 18 \quad}\ \mathrm{H_2O}.$$

$$\begin{array}{c} \mathrm{C} \\ \mathrm{H} \\ \mathrm{O} \end{array} \qquad x_1 \begin{bmatrix} 8 \\ 18 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{array}{c} \bullet \\ {\scriptstyle 9/4} \\ \end{array} \begin{bmatrix} 8 & 0 & -1 & 0 \\ 18 & 0 & 0 & -2 \\ 0 & 2 & -2 & -1 \end{bmatrix} \sim \begin{array}{c} \\ \downarrow \\ \uparrow \end{array} \begin{bmatrix} 8 & 0 & -1 & 0 \\ 0 & 0 & 9/4 & -2 \\ 0 & 2 & -2 & -1 \end{bmatrix} \sim \begin{array}{c} {\scriptstyle \times\, 18} \\ {\scriptstyle \times\, 1/2} \\ {\scriptstyle \times\, 4/9} \end{array} \begin{bmatrix} 8 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 \\ 0 & 0 & 9/4 & -2 \end{bmatrix} \sim$$

$$\begin{array}{c} {\scriptstyle -1/8} \\ {\scriptstyle -1} \\ \end{array} \begin{bmatrix} 1 & 0 & -1/8 & 0 \\ 0 & 1 & -1 & -1/2 \\ 0 & 0 & 1 & -8/9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -25/18 \\ 0 & 0 & 1 & -8/9 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} \frac{1}{9} \\ \frac{25}{18} \\ \frac{8}{9} \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 25 \\ 16 \\ 18 \end{bmatrix} \qquad (x_4 = 18)$$