Problem 1.1.

(a) (20 points) Clearly documenting your work, compute the reduced row-echelon form of the matrix

$$A = \begin{bmatrix} 2 & 4 & 2 & 4 \\ 4 & 8 & 5 & 10 \\ 6 & 12 & 7 & 14 \end{bmatrix}$$

(b) (1 points) Which columns of A have pivots? $\boxed{}$ (list column numbers)

Problem 1.1 continues on the next page.

(c) (9 points) If possible, find the solution of the linear system below in parametric vector form.

$$\begin{cases} 2x_1 + 4x_2 + 2x_3 = 4 \\ 4x_1 + 8x_2 + 5x_3 = 10 \\ 6x_1 + 12x_2 + 7x_3 = 14 \end{cases}$$

Problem 1.2. (8 points) Carefully simplify each valid expression. For any invalid expression, explain why it is invalid.

(a) $$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(b) $$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix}$$

(c) $$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}^T$$

(d) $$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

Problem 1.3. (12 points) For each matrix below and on the next page, compute the matrix inverse, if it exists. If it fails to exist explain why.

$$A_1 = \begin{bmatrix} 9 & 8 \\ 10 & 9 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 4 & 8 \\ 2 & 4 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Problem 1.3 continues on the next page.

$$A_4 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 2. (10 points) Evaluate the truth of each statement below. If the statement is true write T in the box preceding the statement. Otherwise, write F.

(a) $\boxed{}$ A linear system with more than one solution must have a free variable.

(b) $\boxed{}$ All matrices have a reduced row-echelon form.

(c) $\boxed{}$ If the matrix A is invertible, then $\text{rref}(A) = I$.

(d) $\boxed{}$ Some pairs of 2×2 matrices, A and B, commute (i.e., $AB = BA$).

(e) $\boxed{}$ The average of two solutions of a linear system is also a solution.

Problem 3. (10 points) What is the result of the MATLAB commands below.

```
>> A = [1 2 3; 2 3 4];
>> rref(A)

ans =


     1     0    -1
     0     1     2

>> rref([2 2; 0 2]*A)


ans =
```

Problem 4. (15 points) Suppose a is a real number (which may or may not be 0).

(a) Find all vectors $\mathbf{x}$ that satisfy the following matrix equation.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & a \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(b) Find all vectors $\mathbf{y}$ that satisfy the following matrix equation.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & a \end{bmatrix} \mathbf{y} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Problem 4 continues on the next page.

(c) Find all vectors $\mathbf{z}$ that satisfy the following matrix equation.

$$\begin{bmatrix} a & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{z} \;=\; \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Problem 5. (15 points) Suppose a linear system with a 2×3 augmented matrix

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \end{bmatrix}$$

has as its solution set the line passing through the two points

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

(a) How many free variables does the linear system have? Explain.

(b) How many pivot columns does H have? Explain.

Problem 5 continues on the next page.

(c) Find rref(H).

Extra Credit. (5 points) Suppose $x_0, x_1, x_2, \ldots$ is a sequence of real numbers satisfying

$$\begin{aligned} x_{n+2} &= \frac{1}{2}(x_{n+1} + x_n) \qquad n = 0, 1, 2, \ldots, \\ x_0 &= 0, \\ x_1 &= 1. \end{aligned}$$

Find and simplify an algebraic formula for x_n.

Extra Extra Credit. (5 points) Showing your work, find the smallest positive integers x_1, x_2, x_3 and x_4 that balances the chemical equation for the combustion of octane:

$$(x_1)\ C_8H_{18}\ +\ (x_2)\ O_2\ \rightarrow\ (x_3)\ CO_2\ +\ (x_4)\ H_2O.$$

$$\boxed{}\ C_8H_{18}\ +\ \boxed{}\ O_2\ \rightarrow\ \boxed{}\ CO_2\ +\ \boxed{}\ H_2O.$$