20 Lecture 20: The Gram-Schmidt Orthonormalization Process and the QR Factorization

20.1 The Gram-Schmidt Orthonormalization Process

Definitions 20.1. *For a basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ of a subspace V, we define* $\mathbf{A} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_p \end{bmatrix}$.

- *S is an* **orthogonal basis** *if and only if $\mathbf{A}^T\mathbf{A}$ is a diagonal matrix.*
- *S is an* **orthonormal basis** *if and only if $\mathbf{A}^T\mathbf{A}$ is the identity matrix.*

Let $V = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ be a subspace in $\mathbf{R}^m$, where the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is a basis for V. The Gram-Schmidt orthonormalization process (also known as the Gram-Schmidt process) is to find an orthogonal (or orthonormal) basis for V from the knowledge of the basis S. The main idea is for each $\ell = 1, 2, \dots, p$, find an **orthogonal set** $U = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_p\}$ such that

$$\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_\ell\} = \text{Span}\{\mathbf{w}_1, \dots, \mathbf{w}_\ell\}$$

The Gram-Schmidt process Algorithm to find an orthogonal basis

- Step 1: Let $\mathbf{w}_1 := \mathbf{v}_1$
- Step 2: For each $\ell = 1, 2, \dots, p$

$$\mathbf{w}_\ell = \mathbf{v}_\ell - \frac{\mathbf{w}_1 \cdot \mathbf{v}_\ell}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 - \frac{\mathbf{w}_2 \cdot \mathbf{v}_\ell}{\mathbf{w}_2 \cdot \mathbf{w}_2}\mathbf{w}_2 - \dots - \frac{\mathbf{w}_{\ell-1} \cdot \mathbf{v}_\ell}{\mathbf{w}_{\ell-1} \cdot \mathbf{w}_{\ell-1}}\mathbf{w}_{\ell-1}$$

To find an **orthonormal basis**, we first apply the Gram-Schmidt process to find an orthogonal basis then normalize each vector in the basis (i.e., divide each vector to its length).

Example 20.1. *Apply the Gram-Schmidt process to find an orthonormal basis of a subspace V whose a basis is*

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

Solution: Denote

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

We first apply the Gram-Schmidt process to find an orthogonal basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ for V. Let

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{w}_1 \cdot \mathbf{v}_2}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \frac{1(1)+1(2)+1(1)}{1^2+1^2+1^2}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\mathbf{w}_1 \cdot \mathbf{v}_3}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 - \frac{\mathbf{w}_2 \cdot \mathbf{v}_3}{\mathbf{w}_2 \cdot \mathbf{w}_2}\mathbf{w}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1(-1)+1(0)+1(1)}{1^2+1^2+1^2}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\frac{1}{3}\Big((-1)(-1)+2(0)+(-1)(1)\Big)}{(\frac{1}{3})^2\Big((-1)^2+2^2+(-1)^2\Big)}\frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

To find an orthonormal basis, we divide each vector in the orthogonal basis by its length. Let

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

The orthonormal basis is $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

20.2 QR Factorization

If $\mathbf{A}$ is an $m \times n$ matrix with **linearly independent columns**, then $\mathbf{A}$ can be factored as $\mathbf{A} = \mathbf{QR}$, where $\mathbf{Q}$ is an $m \times n$ matrix whose columns form an orthonormal basis for Col$\mathbf{A}$ and $\mathbf{R}$ is an $n \times n$ upper triangular invertible matrix with positive entries on its diagonal.

Method to find Q and R

- Apply the Gram-Schmidt process to columns of $\mathbf{A}$ to find an orthonormal basis $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ for Col$\mathbf{A}$. The matrix $\mathbf{Q}$ is formed by the orthonormal basis:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \ldots & \mathbf{w}_n \end{bmatrix}$$

- $\mathbf{R} = \mathbf{Q}^T\mathbf{A}$

Example 20.2. *Find a QR-factorization of the matrix*

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution: *The columns of* $\mathbf{A}$ *are the vectors* $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ *in Example 20.1. An orthonormal basis for* $\mathrm{Col}\mathbf{A} = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ *was found in that example:*

$$\mathbf{u}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\2\\-1\end{bmatrix}, \quad \mathbf{u}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix},$$

as the columns of matrix $\mathbf{Q}$

$$\mathbf{Q} = \begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0\\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}}\end{bmatrix}$$

By construction, the matrix $\mathbf{A}$ *in the QR-factorization is*

$$\mathbf{R} = \mathbf{Q}^T\mathbf{A} = \begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}}\\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1 & 1 & -1\\ 1 & 2 & 0\\ 1 & 1 & 1\end{bmatrix} = \begin{bmatrix}\sqrt{3} & 2\sqrt{3} & 0\\ 0 & \sqrt{6} & 0\\ 0 & 0 & \sqrt{2}\end{bmatrix}.$$

Example 20.3. *Find a QR factorization of* $\mathbf{A} = \begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$.

Solution: Denote $\mathbf{v}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$.We first apply the Gram-Schmidt to find an orthogonal basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ for the column space of $\mathbf{A}$. Let

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}.$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{w}_1\cdot\mathbf{v}_2}{\mathbf{w}_1\cdot\mathbf{w}_1}\mathbf{w}_1 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix} - \frac{3}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \frac{1}{4}\begin{bmatrix}-3\\1\\1\\1\end{bmatrix}$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\mathbf{w}_1 \cdot \mathbf{v}_3}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 - \frac{\mathbf{w}_2 \cdot \mathbf{v}_3}{\mathbf{w}_2 \cdot \mathbf{w}_2}\mathbf{w}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\frac{1}{4}(2)(\frac{1}{4})}{\left(\frac{1}{4}\right)^2(12)}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

We then normalize the three vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ to obtain columns of matrix $\mathbf{Q}$

$$\mathbf{Q} = \begin{bmatrix} \frac{1}{2} & \frac{-3}{\sqrt{12}} & 0 \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

The matrix $\mathbf{R}$ in the fQR-actorization $\mathbf{A} = \mathbf{QR}$ is

$$\mathbf{R} = \mathbf{Q}^T\mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{-3}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} \\ 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & \frac{3}{2} & 1 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{3}} \\ 0 & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}.$$

20.3 Extension to dependent sets of vectors

20.3.1 Gram-Schmidt process for a subspace which spanned by a dependent set

To find an orthogonal basis for a subspace $V = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, where the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a dependent set, we apply the Gram-Schmidt to the the set and ignore all zero vectors produced by the process. The orthogonal basis is the set of all non-zeros vectors produced by the process.

Example 20.4. *Let*

$$V = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix} \right\}$$

be a subspace. Find an orthonormal basis for V.

Solution:

Denote $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}$. We first apply the Gram-Schmidt process to find an orthogonal basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$ for V. From Example 20.3, we have found

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{w}_2 = \frac{1}{4}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{w}_3 = \frac{1}{3}\begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}.$$

Apply the Gram-Schmidt to find $\mathbf{w}_4$, we obtain

$$\mathbf{w}_4 \quad = \mathbf{v}_4 - \tfrac{\mathbf{w}_1\cdot\mathbf{v}_4}{\mathbf{w}_1\cdot\mathbf{w}_1}\mathbf{w}_1 - \tfrac{\mathbf{w}_2\cdot\mathbf{v}_4}{\mathbf{w}_2\cdot\mathbf{w}_2}\mathbf{w}_2 - \tfrac{\mathbf{w}_3\cdot\mathbf{v}_4}{\mathbf{w}_3\cdot\mathbf{w}_3}\mathbf{w}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix} - \tfrac{9}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \tfrac{5}{12}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \tfrac{2}{6}\begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

which implies $\mathbf{w}_4 \in \text{Span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. Thus, $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is an orthogonal basis for V. By normalizing three vectors in this orthogonal basis, we have an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where

$$\mathbf{u}_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} \frac{-3}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

20.3.2 QR-factorization for the matrix with dependent columns

Suppose that an $m \times n$ matrix $\mathbf{A}$ with $m \geq n$ has the rank $r < n$. That is the columns of $\mathbf{A}$ form a linear dependent set. Then, there are two way to find the QR-factorization for $\mathbf{A}$:

Way 1: Apply the Gram-Schmidt to find an orthonormal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_r\}$ for Col$\mathbf{A}$ then define $\mathbf{Q} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \dots & \mathbf{u}_p \end{bmatrix}$ and $\mathbf{R} = \mathbf{Q}^T\mathbf{A}$.

Example 20.5. *Find a QR-factorization for the matrix*

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 3 \end{bmatrix}$$

Solution: Use the calculation in Example 20.4, we have an orthonormal basis for Col$\mathbf{A}$ is

$$\mathbf{u}_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} \frac{-3}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}.$$

Thus,

$$\mathbf{Q} = \mathbf{Q} = \begin{bmatrix} \frac{1}{2} & \frac{-3}{\sqrt{12}} & 0 \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} \end{bmatrix}.$$

Matrix $\mathbf{R}$ is

$$\mathbf{R} = \mathbf{Q}^T\mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{-3}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} \\ 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & \frac{3}{2} & 1 & \frac{9}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{3}} & \frac{5}{\sqrt{12}} \\ 0 & 0 & \frac{2}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}.$$

Way 2: After using the Gram-Schmidt process to find an orthonormal basis for Col$\mathbf{A}$ (including r vectors), we use addition vectors in the standard basis for $\mathbf{R}^m$ to complete a set of n vectors in order to build matrix $\mathbf{Q}$ with the same size of $\mathbf{A}$. The matrix $\mathbf{R}$ is still $\mathbf{R} = \mathbf{Q}^T\mathbf{A}$.

Example 20.6. *Find a QR-factorization for the matrix*

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 3 \end{bmatrix},$$

where $\mathbf{Q}$ *has the same size as* $\mathbf{A}$.

Solution: Use the calculation in Example 20.4, we have an orthonormal basis for Col$\mathbf{A}$ is

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}.$$

Let us apply the Gram-Schmidt to $\mathbf{e}_1$ and define

$$\mathbf{w}_4 = \mathbf{e}_1 - \frac{\mathbf{w}_1 \cdot \mathbf{e}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 - \frac{\mathbf{w}_2 \cdot \mathbf{e}_1}{\mathbf{w}_2 \cdot \mathbf{w}_2}\mathbf{w}_2 - \frac{\mathbf{w}_3 \cdot \mathbf{e}_1}{\mathbf{w}_3 \cdot \mathbf{w}_3}\mathbf{w}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{-3}{12}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{0}{6}\begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, $\mathbf{e}_1 \in \text{Span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ and we have to ignore it. Let us try apply the Gram-Schmidt to $\mathbf{e}_2$ and redefine $\mathbf{w}_5$:

$$\mathbf{w}_4 = \mathbf{e}_2 - \frac{\mathbf{w}_1 \cdot \mathbf{e}_2}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 - \frac{\mathbf{w}_2 \cdot \mathbf{e}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\mathbf{w}_2 - \frac{\mathbf{w}_3 \cdot \mathbf{e}_2}{\mathbf{w}_3 \cdot \mathbf{w}_3}\mathbf{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{12}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{-2}{6}\begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

which implies $\mathbf{e}_2$ is not satisfied. We need to apply the Gram-Schmidt to $\mathbf{e}_3$ and redefine $\mathbf{w}_5$:

$$\mathbf{w}_4 = \mathbf{e}_3 - \frac{\mathbf{w}_1 \cdot \mathbf{e}_3}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 - \frac{\mathbf{w}_2 \cdot \mathbf{e}_3}{\mathbf{w}_2 \cdot \mathbf{w}_2}\mathbf{w}_2 - \frac{\mathbf{w}_3 \cdot \mathbf{e}_3}{\mathbf{w}_3 \cdot \mathbf{w}_3}\mathbf{w}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{12}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 6 \\ -6 \end{bmatrix}.$$

Since $\mathbf{w}_4$ is not a zero vector, the set of four vectors $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$ is an orthogonal set, and the associated orthonormal set $\left\{\frac{\mathbf{w}_1}{\|\mathbf{w}_1\|}, \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|}, \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|}, \frac{\mathbf{w}_4}{\|\mathbf{w}_4\|}\right\}$. We define matrix $\mathbf{Q}$:

$$\mathbf{Q} = \mathbf{Q} = \begin{bmatrix} \frac{1}{2} & \frac{-3}{\sqrt{12}} & 0 & 0 \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \end{bmatrix}.$$

Matrix $\mathbf{R}$ is

$$\mathbf{R} = \mathbf{Q}^T\mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{-3}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} \\ 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & \frac{3}{2} & 1 & \frac{9}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{3}} & \frac{5}{\sqrt{12}} \\ 0 & 0 & \frac{2}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The Cholesky factorization Suppose that $\mathbf{G}$ is the Gram matrix of a nonsingular $n \times n$matrix $\mathbf{A}$. From the QR factorization of $\mathbf{A}$, it follows that

$$\mathbf{G} = \mathbf{A}^T\mathbf{A} = (\mathbf{QR})^T(\mathbf{QR}) = \mathbf{R}^T(\mathbf{Q}^T\mathbf{Q})\mathbf{R} = \mathbf{R}^T\mathbf{R}$$

That is, is the Gram matrix of an upper triangular matrix $\mathbf{R}^T\mathbf{R}$. This factorization is called the Cholesky factorization of $\mathbf{G}$.