The Chandrasekhar Limit: Derivation, Plotting and Analysis of the Core Collapse

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CONTENT

ABSTRACT

In the previous Article, we investigated the internal structure and workings of a main-sequence star and how we mathematically depict its gravitational equilibrium state. In this study, we derived and thoroughly studied both the numerical and analytical solutions of the Lane-Emden equations, a dimensionless form of the hydrostatic equilibrium equation and the implications of different Polytropic Indices.

In this Article, we will focus on the astrophysical significance of cases n = 3/2 and n = 3, which correspond to non-relativistic and Highly Relativistic degenerate electron gas, respectively. This will ultimately lead to the derivation and implications of the Chandrasekhar Limit.

Polytropic Index(n)	Physical Interpretation	Main Property	Systems explained by the EoS
$n = 0$	Incompressible Fluid	Constant Density $M \propto R^3$	Idealized model
$n = 1$	Moderate Compressibility	$P \propto \rho^2$	Isothermal Core Stars
$n = 5$	Limiting case	Finite Mass, Infinite Radius	Isothermal Spheres
$n = 3$	Relativistic degenerate electrons	Radiation Pressure dominates	High-Mass White Dwarfs, Very Massive Main Seqeunce Stars
$n = \frac{3}{2}$	Non - Relativistic degenerate electrons	$M \propto R^{-3}$	Low-Mass White Dwarfs, Brown Dwarfs

I. Fermi-Dirac Statistics and Pressure Integral

The phenomenon underlying the Chandrasekhar limit is Electron Degeneracy Pressure, whose integral we will derive in the following pages. Electron Degeneracy Pressure is a quantum phenomenon occurring due to Pauli's Exclusion Principle, which states that no two fermions can have the same quantum state simultaneously. It also states that electrons cannot all occupy the lowest energy states. This quantum restriction forces the electrons to occupy higher energy states, creating an outward pressure against the gravitational collapse of the white dwarf. This pressure is the Electron Degeneracy Pressure.

In the derivation, we will make two assumptions regarding the Fermi-Dirac statistics

1. Absolute Zero Limit, that is, the system has a temperature of $T = 0k$
2. Non-interacting fermion gas model for electrons in the white dwarf

At zero kelvin, the electrons occupy all energy levels up to a maximum energy known as the Fermi energy level. This structure of electrons, where each is in its unique quantum state, is known as the Fermi sea. Even at zero Kelvin, there is momentum distribution in the Fermi sea due to Pauli's Exclusion principle

1.1 NON-RELATIVISTIC DEGENERATE GAS MODEL

For a non-relativistic electron of mass m_e moving in a three-dimensional box (infinite potential well), we know that the wavelength is quantised with respect to the length of the box. Using the discreteness of wavelength and the following momentum relation:

$$p = \frac{hk}{2\pi}$$

where k is the angular wavenumber, we get the following expression of Energy

$$E_{n_x,n_y,n_z} = \frac{h^2}{8m_eL^2}\left(n_x^2 + n_y^2 + n_z^2\right) = \frac{h^2}{8m_eV^{2/3}}\left(n_x^2 + n_y^2 + n_z^2\right)$$

We will now use an abstract idea, consider a three-dimensional space with axes n_x, n_y, n_z. Energy then will be a sphere in the ' number space ', and its volume will be

$$V = \frac{4}{3}\pi n^3$$

But for our physical implications, the volume represents the total number of states. We will only take the positive values of n since our wavelength is a positive quantity. Hence, only $1/8$th portion of the volume is to be considered. Therefore,

$$N(E) = V = \frac{1}{6}\pi n^3$$

Substituting the value of n in terms of energy from our previous expression, we get the following expression for the number of states with Energy less than or equal to E (this condition arises since we are considering the volume, not the surface area of the sphere in the number space) in a three-dimensional cubic box

$$N(E) = \frac{\pi}{6}\left(\frac{8m_e L^2}{h^2}\right)^{3/2} E^{3/2}$$

The factor of consideration is not the total number of states, but rather the density of states, which is given by the expression

$$g(E) = \frac{dN(E)}{dE} = \frac{\pi V}{4}\left(\frac{8m_e}{h^2}\right)^{3/2} E^{1/2}$$

Quantites	**Physical Interpretation**	**Number Space Interpretation**
$N(E)$	Total number of states with energy less than or equal to E	$1/8$th Volume of the Sphere of radius n^3
$g(E)$	$g(E) = \frac{dN(E)}{dE}$	Density of states i.e. density of the sphere
$d\epsilon$	Energy Interval ranging from ϵ to $\epsilon + d\epsilon$	Shell of radius ϵ and thickness $d\epsilon$
dN	$dN = g(\epsilon)f(\epsilon)d\epsilon$	Number of occupied states in a n energy interval of E to $E + dE$

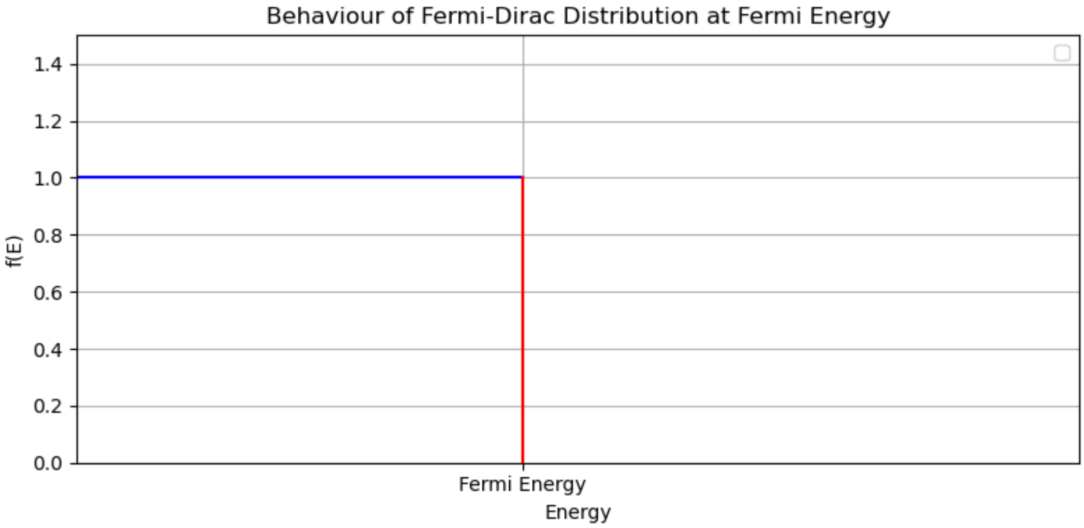
The Fermi-Dirac distribution provides us with a distribution stating the probability of finding an electron in a particular energy interval $d\epsilon$

$$f(\epsilon) = \frac{1}{e^{\frac{\epsilon - \mu}{k_B T}} + 1}$$

where

$\mu =$ Change in energy when a particle enters or leaves the system, while keeping the entropy and volume constant.
This quantity is introduced in the grand canonical ensemble, where two systems, one being a large heat reservoir, can exchange both particles and energy with the large reservoir

Introducing this function is crucial becuase we do not need the number of possible states. Rather, we need the number of occupied states in an energy interval.
At the absolute temperature limit $T \rightarrow 0$, the Fermi-Dirac distribution converts into a step function,



Caption

$$f(E) = \begin{cases} 1, & \epsilon < E_F \\ 0, & \epsilon > E_F \end{cases}$$

Using this property, we will formulate the final expression of the number of occupied states in an energy interval.

$$dN = \begin{cases} 4\sqrt{8}\,\pi V \left(\frac{m_e}{h^2}\right)^{3/2} \sqrt{E}\,dE, & E < E_F \\ 0, & E > E_F \end{cases}$$

$$N = \int_0^{\epsilon_F} dN = 4\sqrt{8}\,\pi V \left(\frac{m_e}{h^2}\right)^{3/2} \cdot \frac{2}{3}\epsilon_F^{3/2}$$

To calculate the average energy, we have the formula

$$U = \int_0^{\epsilon_F} \epsilon\, dN$$

$$U = 4\sqrt{8}\,\pi V \left(\frac{m_e}{h^2}\right)^{3/2} \cdot \frac{2}{5}\epsilon_F^{5/2}$$

From the following expressions of Average Energy and Average number of Particles for the absolute kelvin limit, we find the following final relation

$$\frac{U}{N} = \frac{3}{5} E_F$$

From the Kinetic Theory of gases, we know the relation between the average energy and pressure exerted by the particles inside a three-dimensional box.

$$P = \frac{2}{5}\frac{U}{V} = \frac{3}{5} N E_F$$

The number of states with an angular wavevector less than k_f

$$N = 2\frac{V}{(2\pi)^3}\frac{4\pi}{3}k_F^3$$

$$N = \frac{V}{\pi^2}\frac{k_F^3}{3} \Rightarrow k_F^3 = 3\pi^2\frac{N}{V} = 3\pi^2 n \Rightarrow k_F = (3\pi^2 n)^{1/3}$$

$$E_F = \frac{\hbar^2 k_F^2}{2m_e} = \frac{\hbar^2}{2m_e}\left(3\pi^2 n\right)^{2/3}$$

For further thoroughness, we will derive the pressure from its differential form

$$P = -\left(\frac{\partial U}{\partial V}\right)_N = -\frac{dU}{dn}\cdot\frac{dn}{dV}$$

where n is the number density

Calculating the following derivatives, we arrive at an expression of pressure in terms of number density:

$$P = \frac{3}{2}\cdot\frac{h^2}{20m_e}\left(\frac{3}{\pi}\right)^{\frac{2}{3}} n^{5/3}$$

Inputting this into the pressure equation, we finally obtain a relation between the number density and pressure of the gas

$$P = \frac{\hbar^2}{5m_e}\left(3\pi^2\right)^{2/3} n^{5/3}$$

From the Polytropic equation of states, we identify that the polytropic index for non-relativistic degenerate gas is $n = 3/2$

1.2 RELATIVISTIC DEGENERATE GAS MODEL

For the ultra-relativistic case, we have the following relation between energy and momentum:

$$E = pc$$

Following the same steps, we first take an abstract momentum space to define the density of states in the momentum space:

$$g(p)dp = \frac{V}{h^3}(4\pi p^2 dp)$$

Applying the same absolute zero kelvin limit, our distribution is converted into the following:

$$f(p) = \begin{cases} 1, & p < p_F \\ 0, & p > p_F \end{cases}$$

$$Momentum\ \ Density(\mathscr{P}) = \int_0^{p_F} p\,.f(p)\,.\,\nu(p)\,.\,g(p)\,.\,dp$$

where $\nu = dE/dp$

Inputting all the values, we get the following expression

$$\mathscr{P} = \int_0^{p_f} \frac{V}{h^3}\frac{4\pi}{3}c{p_f}^3 dp$$

Further, we calculate the value of p_f via the number density

$$n_e = \frac{2}{h^3}\int_0^{p_f} dp^3 \rightarrow p_F = \left(\frac{3n_e h^3}{8\pi}\right)^{\frac{1}{3}}$$

Substituting for p_f into $\mathscr{P}$, we get:

$$\Rightarrow \mathscr{P} = \frac{2\pi c}{h^3}\left(\frac{3n_e h^3}{8\pi}\right)^{\frac{4}{3}}$$

Now, we relate the Pressure to $\mathscr{P}$ and energy flux(ξ) through radiation theory

$$P = \frac{1}{3}\xi \sim \frac{1}{3c}\mathscr{P}$$

Therefore, we can say the relation between Pressure and number density is

$$P \propto n_e^{4/3} \rightarrow P \propto \rho^{4/3}$$

From the Polytropic equation of states, we conclude that the polytropic index for an ultrarelativistic electron degenerate gas is $n = 3$

In conclusion, we have thoroughly derived the polytropic indices for both the non-relativistic(n=3/2) and the ultra-relativistic case(n=3) for an electron-degenerate white dwarf

II. *Investigating the Chandrasekhar Limit*

We have the Mass Relation via the Lane-Emden Equation and the mass continuum equation (refer to the previous article for complete derivation)

$$M(\xi) = 4\pi \left(\frac{(n+1)K}{4\pi G}\right)^{3/2} \rho_c^{\frac{3-n}{2n}} \left| \xi_1^2 \frac{d\theta}{d\xi}\bigg|_{\xi=\xi_1} \right|$$

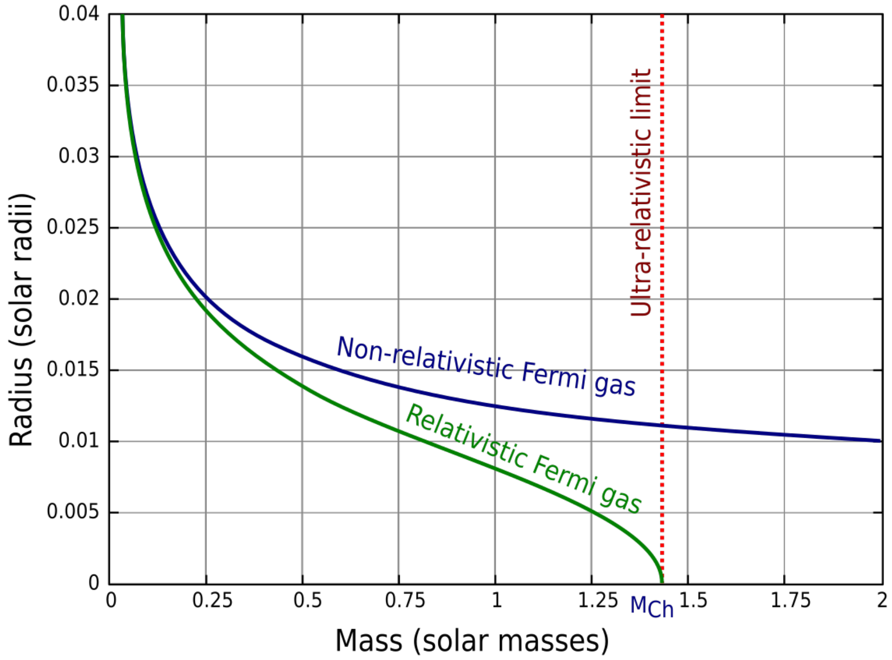


Fig. 2 Comparison between the Mass profiles of the Relativistic and Non-Relativistic cases From Wikipedia

2.1 MASS RELATION FOR THE NON-RELATIVISTIC DEGENERATE GAS MODEL

There is no analytical solution for the Lane-Emden Equation for a polytropic index of 1.5. Hence, we will use some numerical results and move forward with the calculations

$$M = 4\pi \left(\frac{5K}{8\pi G}\right)^{3/2} \rho_c^{1/2} \cdot \left| \xi_1^2 \frac{d\theta}{d\xi}\bigg|_{\xi=\xi_1} \right|$$

From the Standard Results:

$$\xi_1 = 3.65375 \text{ and } -\left(\xi^2 \frac{d\theta}{d\xi}\right)_{\xi=\xi_1} = 2.71406$$

Hence,

$$M(\xi_1) = 4\pi \left(\frac{5K}{8\pi G}\right)^{3/2} \rho_c^{1/2} \cdot 2.71406$$

Inputting the value of Non-Relativistic Polytropic constant K, we get the Chandrasekhar limit for low-mass White Dwarfs:

$$M \approx 9.86 \times 10^{29}\,\text{kg} \approx 0.496\, M_\odot \quad \text{for} \quad \rho_c = 10^9\,\text{kg/m}^3$$

where $M_\odot$ corresponds to solar mass

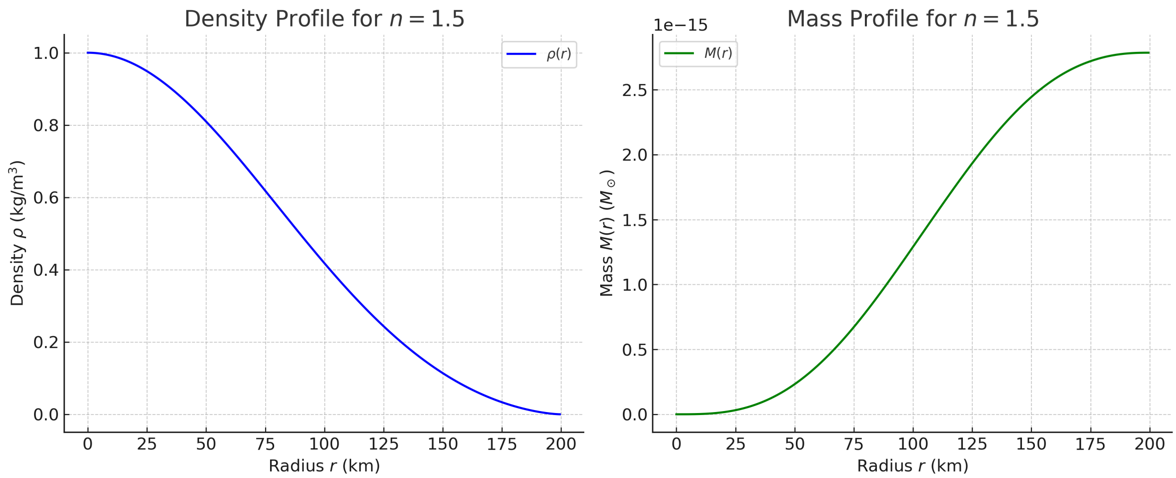


Fig. 3 Plotting the Mass and Density Profile for n=1.5 Polytropic Model

2.2 MASS RELATION FOR THE RELATIVISTIC DEGENERATE GAS MODEL

In the non-relativistic case, the mass does not have a limit. As the Mass of the White Dwarf keeps on increasing, the electrons start showing relativistic behaviour and hence the correct mathematical polytropic model changes to a polytropic index of n=3. Similar to the previous case, since there isn't an analytical solution, we will use the results from the standard values table and solve for mass numerically

$$M = 4\pi \left(\frac{K}{\pi G}\right)^{3/2} \left| \xi_1^2 \frac{d\theta}{d\xi}\bigg|_{\xi=\xi_1} \right|$$

An Important observation here is that the mass does not depend on central density.

Inputting $\xi_1 = 6.89685$ and $-\left(\xi^2 \frac{d\theta}{d\xi}\right)_{\xi=\xi_1} = 2.01824$, we get our final expression

$$M = 4\pi \left(\frac{K}{\pi G}\right)^{3/2} \cdot 2.01824$$

Inputting the value of the Relativistic Polytropic constant, we arrive at the Chandrasekhar Mass limit for a relativistic degenerate electron gas model:

$$M_{\text{Ch}} \approx 1.44\, M_\odot$$

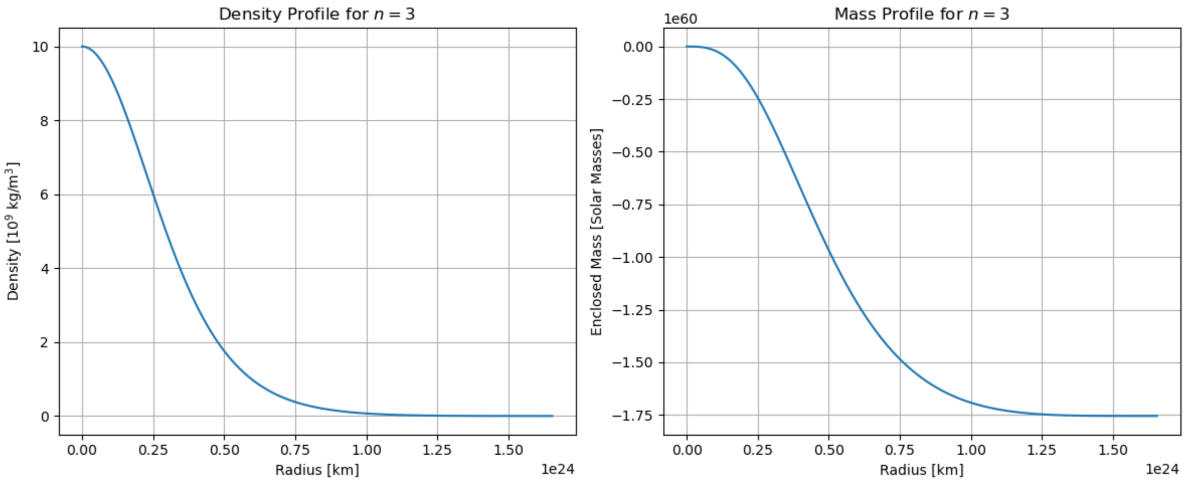


Fig. 4 Plotting the Mass and Density Profile for n=3 Polytropic Model, the code will be mentioned below

The Chandrasekhar limit is a great example of how microscoping Quantum Mechanical phenomena are showing significant effects on the cosmological scale. Experimentally, we have found no massive White Dwarfs exceeding M_{Ch} which gives strong relevance to the limit.

2.3 TYPE 1A SUPERNOVAE EXPLOSION

One of the most important astrophysical effects of the Chandrasekhar limit is the Type Ia supernovae explosion triggered by the Chandrasekhar limit. Before learning about the involvement of the limit, we will understand more about the type Ia supernovae explosion. A general issue comes in when we try to measure distances, which is whether the object is far away or just small. Correspondingly, for luminosity distance, the issue translates to 'is the object far away or simply intrinsically dim?' To tackle this issue, astronomers developed 'references' that one can compare to the observed brightness of these references to understand the distance of the object. These references are called 'standard candles'. One of the most reliable standard candles is the type 1a supernovae. Through its data, we analyse the Hubble parameter, redshift and the expansion of the Universe. Type 1a supernovae occur when a white dwarf pulls too much matter from its companion star, with which it is in orbit, reaching the Chandrasekhar mass limit, at which point the star collapses, leading to a thermonuclear explosion and often shoots out light brighter than all the stars of the galaxy combined during its peak luminosity. To understand how to calculate the Hubble parameter, redshift and the expansion of the Universe from these 'Standard Candles', you can refer to my paper mentioned below.

The Chandrasekhar limit is the triggering point for these White Dwarfs. Type Ia SN light curves and spectra provide indirect evidence of the Chandrasekhar-triggered explosion. Until recently, all SN Ia were nearly M_{Ch}, but new analyses show some $super - M_{Ch}$ explosions. In our calculations, we only give our attention towards the gravitational balance, but other physical phenomena, such as rotations and magnetic fields, can increase the limit. There are observations of a massive White Dwarf of $M \sim 2.6M_{\odot}$, these white dwarfs are often called 'Super-Chandrasekhar' White Dwarfs. Nonetheless, our value is correct for non-rotating, non-magnetic White Dwarfs.

2.3 PYTHON CODE FOR NUMERICALLY SOLVING THE CHANDRASEKHAR LIMIT

The code to solve the Lane-Emden equations using solve_ivp and plotting the Chandrasekhar limit is given below:

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp, cumtrapz
#solve_ivp is used to solve initial value problems of ODE in
the SciPy library
#cumtrapz is used for definite integrals in the SciPy library

#Constants

G = 6.67430e-11             # Gravitational constant [m^3 kg^-1 s^-2]
h = 6.62607015e-34          # Planck's constant
[J·s] c = 2.99792458e8      # Speed of light [m/
s] m_e = 9.10938356e-31     # Electron mass [kg]
m_u = 1.66053906660e-27     # Atomic mass unit [kg]
M_sun = 1.98847e30          # Solar mass [kg]

# Parameters
mu_e = 2                    # Mean molecular weight per electron
rho_c = 1e10                # Central density
n = 3                       # Polytropic index

# Relativistic polytropic constant K
K = (1/8) * (3/np.pi)**(1/3) * (h * c) / ((mu_e * m_u)**(4/3) *
m_e**(4/3))
# Here it is important not to take the non-relativistic value of K,
as we are dealing with relativistic electrons.

def lane_emden_rhs(xi, y, n):
    theta, phi = y
    if xi == 0:
        return [phi, 0]
    return [phi, -2/xi * phi - theta**n]

# xi corresponds to the dimensionless radius (r/R) and theta is the
dimensionless density (rho/rho_c).
# y is a vector having two components: theta and phi, where phi is the
```

```
derivative of theta with respect to xi.
# n is our polytropic index for the relativistic electron gas model.

# Solve the Lane-Emden equation
xi_max = 20
sol = solve_ivp(
    fun=lambda xi, y: lane_emden_rhs(xi, y, n),
    t_span=(1e-10, xi_max), #Starting from just above zero avoids
complications of the case xi=0.
    y0=[1.0, 0.0], # States the Initial condition
    dense_output=True,
    # The following values are for the solver's accuracy.
    max_step=0
    .01,
    rtol=1e-8,

    atol=1e-10
)

# Extract and truncate the solution at the first zero of theta
xi = sol.t    # array of xi values
theta = sol.y[0] # array of theta values
phi = sol.y[1] # array of phi values (derivative of theta)
i_zero = np.where(theta <= 0)[0][0] # to find the index in the
array where, for the first time, theta reaches the value of
zero, i.e. the surface of the star.
# slicing the arrays to only include values up to the surface of
the star
xi  = xi[:i_zero+1]
theta =
theta[:i_zero+1] phi
= phi[:i_zero+1]

# Scaling factor and physical radius/mass
xi_1 = xi[-1]
dtheta_dxi =
phi[-1]
a = np.sqrt((n + 1) * K / (4 * np.pi * G) * rho_c**((1 - n) /
n)) R = a * xi_1              # Radius in meters
M = 4 * np.pi * a**3 * rho_c * (-xi_1**2 * dtheta_dxi)  # Mass in kg

print(f"Chandrasekhar mass (computed): {M / M_sun:.3f} M_sun")
print(f"White dwarf radius: {R / 1e3:.2f} km")

# Mass and density profiles
r = a * xi
rho = rho_c * theta**n
dm_dxi = -4 * np.pi * a**3 * rho_c * xi**2 *
theta**n m = cumtrapz(dm_dxi, xi, initial=0)
m_solar = m / M_sun
r_km = r / 1e3
```

```
Chandrasekhar mass (computed):
1754926812632832025308697335730067877135268753374314655383552.000
M_sun
White dwarf radius: 1654583660933735516209152.00 km

/var/folders/x2/mg4hfy393fngj494k59zy5z80000gn/T/
```

```
ipykernel_12494/2861509941.py:8: DeprecationWarning:
'scipy.integrate.cumtrapz' is deprecated in favour of
'scipy.integrate.cumulative_trapezoid' and will be removed in SciPy
1.14.0
  m = cumtrapz(dm_dxi, xi, initial=0)
```

```
# Plotting
plt.figure(figsize=(12, 5))
plt.subplot(1, 2, 1)
plt.plot(r_km, rho / 1e9)
plt.xlabel("Radius [km]”)
plt.ylabel(r"Density [$10^9$ kg/m$^3$]")
plt.title("Density Profile for $n = 3$")
plt.grid(True)
plt.subplot(1, 2, 2)
plt.plot(r_km, m_solar)
plt.xlabel("Radius [km]”)
plt.ylabel("Enclosed Mass [Solar Masses]")
plt.title("Mass Profile for $n = 3$")
plt.grid(True)
plt.tight_layout()
plt.show()
```

III. Microphysical processes involved in destabilising the White Dwarf Core

Once the white dwarf exceeds the Chandrasekhar limit, it converts into a Neutron star. We will investigate the microphysical phenomena involved in the conversion, the involvement of conservation laws and an overview of the mass limit for a neutron star to convert into a black hole.

Electrons in the standard model are categorised under Fermions, i.e. particles have half-integer spins. In the family of fermions are Leptons and Quarks, which are elementary particles. The particles formed via a combination of quarks are called Hadrons, which contain Baryons(a combination of 3 quarks) and Mesons(a pair of quark and anti-quark). When the mass of a massive White Dwarf crosses the Chandrasekhar limit $M_{\text{Ch}} \approx 1.44\, M_{\odot}$, the electrons in the Fermi sea attain higher momentum, becoming more and more relativistic. For the relativistic degenerate electron gas model, the relation between pressure and density is 'weak' $P \propto \rho^{4/3}$, i.e. the pressure doesn't increase as fast as the density does. This leads to the gravitational pull overwhelming the outward electron degeneracy pressure, and from then on, the electron degeneracy pressure is no longer sufficient for gravitational stability of the White Dwarf. Certain microphysical processes begin in the core of the White Dwarf, softening the adiabatic index $\Gamma = 1 + 1/n$. We will discuss the key microphysical processes occurring due to the exceeding of the limit.

3.1 ELECTRON CAPTURE

Electrons in the standard model are categorised as Fermions, i.e. particles have half-integer spins. Fermions are divided into two families, namely Leptons and Quarks, which are elementary particles. The particles formed via a combination of quarks are called Hadrons, which contain Baryons(a combination of 3 quarks) and Mesons(a pair of quark and anti-quark), whereas leptons contain three generations of elementary particles. In this discussion, we will focus only on the electron family containing electrons and electron-neutrinos. Every lepton and antilepton has an assigned 'lepton number' of 1 and -1. Lepton number is a bookkeeping tool to ensure of conservation of angular momentum by keeping track of the number of leptons in a reaction. The conservation of lepton number states that the system should have equal lepton numbers before and after the reaction occurs. Any reaction breaking this conservation law can be said to be forbidden.

Due to the increasing momentum of the electrons in the Fermi sea, the Fermi energy of the degenerate electron gases exceeds the energy threshold required for the conversion of a proton to a neutron in the core of the star. This allows inverse beta decay to occur.

$$E_f = p_f c$$

Due to this increase in energy, electrons are forced to occupy higher energy states and making the inverse beta decay process energy favourable.

$$e^- + p \rightarrow n + \nu_e$$

The condition for electron capture is contained in the energy threshold; the Fermi energy should be higher than the mass difference between the neutron and the proton(1.293 MeV). The electron then attains enough energy to overcome the mass difference.

$$Condition \rightarrow E_f > 1.293\ MeV$$

This process converts electrons and protons into neutrons and electron-neutrinos, reducing the number density of electrons (n_e) , which leads to a decrease in the electron degeneracy pressure, and the collapse accelerates.

$$P \propto n_e^{\frac{1}{3}}$$

3.2 NEUTRINO LOSS AND NEUTRINO COOLING

A crucial property of neutrinos is that they only interact with weak forces; this is the reason why all neutrinos are left-handed, i.e. the particle's spin is antiparallel to its momentum. The left-handedness property of the neutrinos is directly related to a symmetry principle called 'Parity'(P), which states that all physical laws have mirror symmetry and the Standard Model's chiral structure. However, weak interactions violate this symmetry and hence neutrinos violate this symmetry as well, stating that parity is no longer a fundamental symmetry for all interactions. Tho it was further saved by introducing two further properties, charge conjugation and time reversal. While Parity(P) and charge conjugation(C) are violated by weak interactions, the combination of all three symmetries is assumed to be conserved under all interactions This conserved quantity is called the TCP theorem, though there haven't been a lot of experimental proof for time reversal of reactions since that it is very difficult to replicate exact initial conditions.

The electron-neutrino released in the inverse beta decay escapes into the nearby surroundings.
In Nuclear and Particle physics, each reaction has two main quantities, the decay rate(τ) and the cross section(σ). The cross-section is directly proportional to the probability of the particle's interaction with matter. The reason neutrinos interact with matter and escape but photons dont is explained by the astronomical difference in their cross section(10^{19}), this is becuase photons interact via electromagnetic forces whereas neutrinos interact via weak forces. Another crucial quantity required to justify neutrinos escape is the mean free path, i.e. the average distance a particle travels before interacting. This quantity is inversely proportional to the cross section:

$$\lambda \propto \frac{1}{\sigma}$$

The mean free path of the left-handed neutrino is much larger than the stellar radius of the white dwarf, allowing it to freely stream through.

$$\lambda \sim 10^{14}cm \text{ and } R \sim 10^{9}cm \implies \lambda >> R$$

This leads to neutrinos carrying away energy and lepton number away from the star with a great efficiency since their scattering probability is very less, allowing no loss of energy by interactions with matter. This loss of lepton number due to electron capture and electron-neutrino escape decreases the Y_e and accelerates the process of collapse. The Chandrasekhar limit is directly dependent on Y_e due to the electron degeneracy pressure. Therefore, a consequence of this decrease in Y_e is the decrease in the Chandrasekhar limit, which will then accelerate the collapse further. Another consequence of this removal of leptons from the core of the star is the formation of a neutron-rich state of the core. The efficient energy loss due to neutrinos escaping freely (important to note that they aren't diffusing, rather leaving as a free stream) creates an extremely efficient cooling mechanism, which has various astrophysical consequences. The cooling rate can be quantified by calculating the energy loss per unit mass per unit time due to neutrino emission.

The mechanism behind this cooling system contains four reactions:

1. Cycling beta and inverse beta decay reaction (Dominant in collapsing cores) $\propto T^6$
   Condition: High Temperature ($T \gtrsim 10^9$) and Moderate Density ($\rho \gtrsim 10^{10}$)

$$n \rightarrow p + e^- + \bar{\nu}_e \text{ and } p + e^- \rightarrow n + \nu_e$$

2. Plasmon decay $\propto T^9$
   Condition: High Temperature ($T \gtrsim 10^8$) and Moderate Density ($\rho \lesssim 10^{10}$)

$$\gamma^* \rightarrow \nu + \bar{\nu}$$

3. Pair annihilation $\propto T^9$
   Condition: High Temperature ($T \gtrsim 10^9$) and Density ranging from $\rho \sim 10^6 - 10^9$

$$e^- + e^+ \rightarrow \nu + \bar{\nu}$$

4. Photon-neutrino process
   Condition: High Temperature ($T \gtrsim 10^9$) and Density ranging from $\rho \sim 10^6 - 10^9$

$$\gamma + e^- \rightarrow e^- + \nu + \bar{\nu}$$

The cooling rate can be quantified by calculating the energy loss per unit mass per unit time due to all processes having neutrino emission.

$$\epsilon_\nu \approx 10^{20} \left[9 \times 10^4 \, Y_e \left(\frac{\rho}{10^9} \right) \left(\frac{T}{10^9} \right)^6 + 25 \left(\frac{T}{10^9} \right)^8 \left(\frac{\rho}{10^6} \right) + 1000 \left(\frac{T}{10^9} \right)^9 \right]$$

One of the major consequences is the suppression of thermal pressure buildup occurring due to compression. Naturally, as the core compresses more and more due to electron capture and neutrino loss, the thermal pressure should build up and become a stabilising factor against the gravitational collapse. However, in the core of White Dwarfs exceeding the Chandrasekhar limit, this is prevented by the extremely efficient cooling due to neutrino loss. This breaking of thermal pressure buildup via cooling also keeps the core from igniting prematurely. Also, this extreme extraction of energy reduces the evolutionary time scale. In Main-Sequence stars, the energy is lost via diffusion of photons; this is not the case here, the free-like flow of neutrinos is far more efficient in carrying energy, and hence the star loses energy rapidly. This rapid nature makes the collapse irreversible.

Another major consequence is the inhomogeneous nature of compression produced due to the rapidly collapsing environment in the core. There is no time for pressure waves to propagate and maintain hydrostatic equilibrium, breaking the homogeneity of compression. In a White Dwarf below the Chandrasekhar limit, the pressure waves propagate through the medium to distribute energy and pressure, maintaining a homogeneous equilibrium. Once the density of the core reaches a value where the mean free path is no longer greater than the radius of the star and neutrinos from then on remain trapped. This phenomenon is known as neutrino trapping, where neutrinos begin to diffuse rather than stream freely, drastically reducing the efficiency of cooling.

3.3 PYTHON CODE FOR SIMULATING THE CORE COLLAPSE OF A MASSIVE WHITE DWARF VIA THE ELECTRON CAPTURE MECHANISM

In this simulation, I've only included electron degeneracy pressure and electron capture, as these are the dominant effects responsible for triggering core collapse. Processes like photodisintegration and neutrino emission become relevant in the later stages and are omitted here, since the focus is on modelling the onset of collapse and the point at which degeneracy pressure fails.

```python
import numpy as np
import matplotlib.pyplot as plt

#constants

h_bar = 1.0545718e-34  # Planck's constant (Joule second)
m_e = 9.10938356e-31  # Electron mass (kg)
c = 2.998e8  # Speed of light (m/s)
MeV = 1.60218e-13  # Conversion factor

threshold = 1.293*MeV  # Threshold energy for inverse beta decay

# defining formulae
def fermi_energy(n_e):

    p_f = h_bar * (3 * np.pi**2 * n_e)**(1/3)
    E_f= ( p_f**2 * c**2 + m_e**2 * c**4)**(1/2) - m_e * c**2
    return E_f

# initial parameters of the white dwarf

rho = 1e9  # Density in kg/m^3
Y_e = 0.5  # Electron fraction
m_p = 1.6726219e-27  # Proton mass (kg)

def electron_density(rho, Y_e):
    return Y_e * rho / m_p

# empty arrays to store information and evolution
densities = []
fermi_energies = []
collapse_speeds = []
```

```python
states = []

collapse_started = False

for step in range(1000):
    n_e = electron_density(rho, Y_e)
    E_f = fermi_energy(n_e)

    densities.append(rho)
    fermi_energies.append(E_f / MeV)  # Convert to MeV

    if not collapse_started:
        if E_f >= threshold:
            collapse_started = True
            print(f"Collapse begins at density ≈ {rho:.2e} kg/m³")
            states.append('Electron Capture Begins')
        else:
            states.append('Stable')
            rho *= 1.01 # Increase density by 1% each step
            collapse_speeds.append(0.01)
            continue

    if rho > 5e17:
        states.append('Black Hole Forms')
        break
    elif rho > 1e17:
        states.append('Neutron Star Forms')
    else:
        states.append('Collapsing')
        rho *= 1.05
        collapse_speeds.append(0.05)

plt.figure(figsize=(10,6))
plt.plot(densities, fermi_energies, label='Fermi Energy (MeV)')
plt.axhline(1.293, color='r', linestyle='--', label='Threshold for
Inverse Beta Decay')
plt.xlabel('Core Density (kg/m³)')
plt.ylabel('Fermi Energy (MeV)')
plt.title('Fermi Energy vs Density in White Dwarf Core')
plt.grid(True, which='both', linestyle='--')
plt.legend()
plt.xscale('log')
plt.tight_layout()
plt.show()
```

```
Collapse begins at a density ≈ 7.66e+10 kg/m³
```

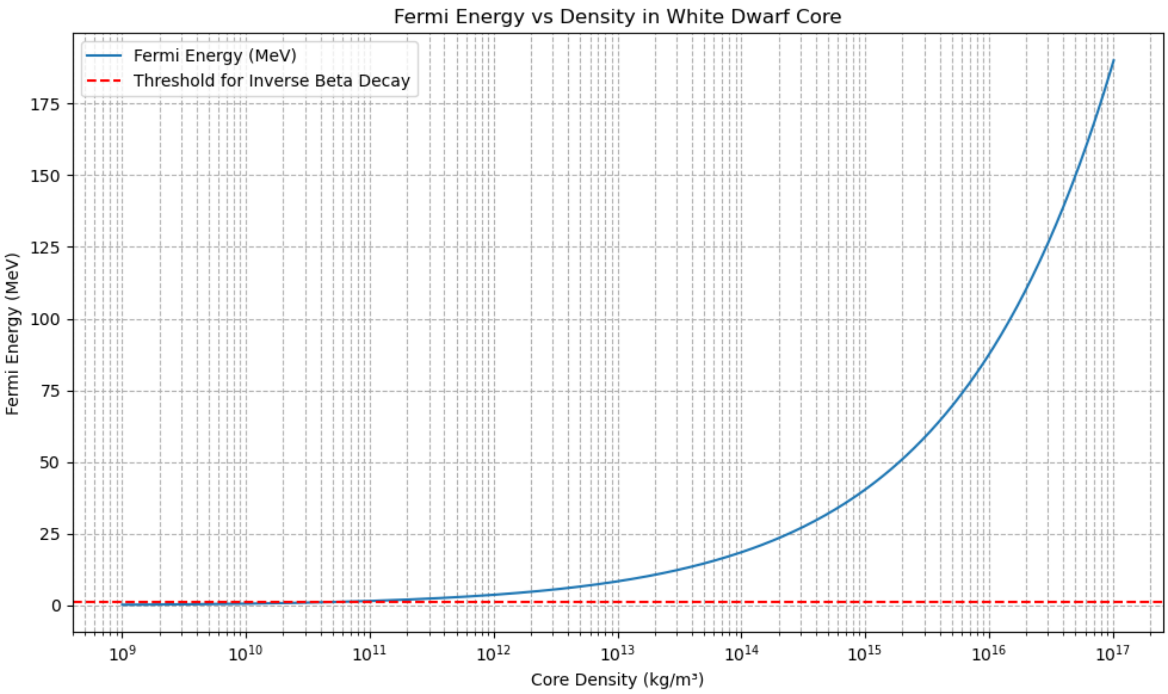


Fig 5. Simulating the collapse of the White Dwarf core via electron capture mechanism

IV. Conclusion

This article was a direct continuation of the Lane-Emden article, where I explored the derivation, solutions and plotting of the Lane-Emden equation. In this article, we derive the Chandrasekhar limit with a first-principle derivation via Fermi-Dirac statistics. Later, we discussed the microphysical processes involved which contribute to accelerating the collapse of the core of the star. We will now discuss some observational data which supports our limit value and its consequences. Further, we will discuss what really does happen.

We discussed the Type 1a supernovae in detail and how the Chandrasekhar limit triggers these cosmological events to occur. The uniformity in SN 1a Light curves points towards an indirect validation towards the limit. The Sloan Digital Sky Survey (SDSS DR7) analysed over 27,000 white dwarfs using spectroscopic techniques and found a very quick drop in the number of white dwarfs as the mass approached $1.4M_{\odot}$, consistent with our derived limit.

Once a critical density is reached $\rho_c \sim 2.8\ X\ 10^{17}$, these types of stars are classified as Neutron stars. The Neutron Star is immensely cold due to its efficient cooling system by the freely streaming neutrinos, until the star's core reaches a stage called 'neutrino trapping' where neutrinos no longer freely stream, but rather they diffuse. Once Neutronization is complete, the Star is stable by the mechanics of Neutron Degeneracy pressure. The physical meaning is equivalent to electron degeneracy pressure since Pauli's exclusion principle is to be obeyed by all fermions. In this step, the core of the Neutron Star is extremely neutron-rich, and the pressure balance is restored. Alas, this balance isn't permanent, and there comes another limit after which the gravitational pressure overwhelms even the Neutron Degeneracy pressure. This leads to the final stage of star evolution, possibly a Black Hole. The new critical mass limit derived through General Relativity is analogous to the Chandrasekhar limit and is called the Tolman–Oppenheimer–Volkoff (TOV) limit $2.0 \sim 2.3M_{\odot}$

From a quantum mechanical integral over momenta to the collapse of stars and the ignition of supernovae, the Chandrasekhar limit is one of the most elegant connections of gravity and quantum effects working simultaneously and also one of the clearest demonstrations of how microscopic events govern cosmological structures and the fate of the Universe.

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