Constraining the Hubble Constant via Type Ia Supernovae: A Study through the FLRW Metric and Friedmann Equations

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CONTENT

ABSTRACT

In this article, we will construct the Hubble Diagram using the Pantheon+SH0ES dataset of Type Ia Supernovae and derive the Friedmann Equation through Newtonian approximations to study the expansion of the universe and determine the cosmological parameters H_0 and Ω_m. By plotting the Hubble Diagram, we aim to understand how the universe's expansion evolves with time and compare our locally fitted values with early-universe predictions such as those from Planck 2018, which worked on the CMB.

We study the FLWR metric and the Friedmann equation through which we develop the framework required for the cosmological model called ΛCDM.

We model the luminosity distance using the standard ΛCDM framework, assuming a flat universe with negligible radiation. We then compute the theoretical distance modulus and fit it to the observed data using a least-squares method with the help of scipy curve_fit. Through this process, we obtain the fitted values, H_0 = 72.97±0.26 km/s/Mpc and Ω_m=0.351±0.019

These are consistent with local measurements but differ from the Planck 2018 estimate, which is derived from CMB data at high redshifts and based on early-universe physics. This suggests the Hubble tension may originate from the difference in temporal scales: Planck uses early-universe information, while our analysis captures late-universe local expansion. To understand these differences, we will segregate the redshift data into three ranges and understand why the Hubble constant is different for each range.

Residuals and the covariance matrix are analysed to quantify the uncertainties and correlations between the parameters. The diagonal elements reflect individual parameter uncertainty, while off-diagonal elements encode how tightly coupled these parameters are. We also calculate the age of the universe and define the young and old universe

I. FLRW Metric and Hubble's Law

From the Special Theory of Relativity, we are familiar with the Minkowski space. It contains 10 symmetries (4 translational, 3 rotational and 3 Lorentz boosts). Lorentz boost symmetry corresponds to symmetry under the Lorentz transformation.
The metric for Minkowski space is:

$$ds^2 = -c^2dt^2 + dx^2$$

For the FLWR Metric, we have fewer symmetries, 6 symmetries(3 transistional + 3 rotational), no boost symmetry since it does not stay the same under Lorentz transformation, and also no time symmetry.

1.1 CURVATURE METRICS

To understand the FLWR Metric, we first need to derive the structure for a curved space. In our case, we will take a space with positive curvature. We know the metric for flat Euclidean space is:

$$ds^2 = dx^2 + dy^2 + dz^2$$

The Euclidean space transformed using spherical coordinates is:

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$$

$$ds^2 = dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

Now, we consider a homogeneous and isotropic space, as our metric should conform to the cosmological principle. We construct a three-dimensional sphere embedded in a four-dimensional Euclidean space.

$$R^2 = x^2 + y^2 + z^2 + \omega^2$$

Hence, we can say

$$R^2 = \omega^2 + r^2$$

$$\omega^2 = R^2 - r^2 \Rightarrow d\omega = -\frac{r}{\omega}dr \Rightarrow dw = -\frac{r}{\sqrt{R^2 - r^2}}\,dr$$

On further simplification, we have the metric for Positive curvature as:

$$ds^2 = \frac{R^2}{R^2 - r^2}\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

Substituting $r = Rsin(\chi)$ our final reduced form is:

$$ds^2 = R^2 \left[d\chi^2 + \sin^2\chi \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)\right]$$

Where each coordinate is represented by 3 angles

$$x = R\sin\chi\sin\theta\cos\phi, \; y = R\sin\chi\sin\theta\sin\phi, \; z = R\sin\chi\cos\theta, \; w = R\cos\chi$$

One key characteristic to note for this metric is that it has finite volume, whereas the Flat Euclidean space(R^4) or the negative curvature space(H^3) have infinite volume. To further generalise the metric, we add a variable integer k, which takes specific values to transform into its respective metric

$$ds^2 = \frac{dr^2}{1 - kr^2/R^2} + r^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \text{ where } k = \begin{cases} +1 & \text{Spherical (closed)} \\ 0 & \text{Euclidean (flat)} \\ -1 & \text{Hyperbolic (open)} \end{cases}$$

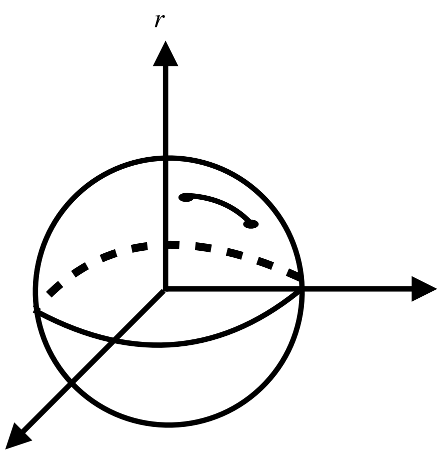
1.2 FLWR METRIC

Now that we have the appropriate curvatures and their structures, we input the general metric in the Minkowski metric, replacing the Flat Euclidean dx^2

$$ds^2 = -c^2dt^2 + a^2(t)[\frac{dr^2}{1 - kr^2/R^2} + r^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right)]$$

Important thing to note here is that if the space is expanding or contracting, the radius of the sphere, i.e. the curvature itself, does not change; rather, the expansion and contraction phenomena are accounted for by the scaling factor $a(t)$. We can also collapse the three-dimensional spherical term to one to get the familiar structure of the FLWR metric, which is:

$$ds^2 = -c^2dt^2 + a^2(t)[\frac{dr^2}{1 - kr^2/R^2} + r^2d^2\Omega]$$



The spatial coordinates are called comoving coordinates. Suppose you have two points on the surface; for intuitive understanding, we will assume the sphere to be a balloon. Hence, the expansion is just filling up the balloon. On expansion, i.e., $a(t)$ increases by a factor of 2, the physical distance between the two dots correspondingly increases by a factor of 2.
Now, suppose you are observing two particles at a distance of 1.01 Mpc and 1.02 Mpc. After expansion, they will be at 2.02 Mpc and 2.04 Mpc.

Hence, we see that the further the particle is, the faster it goes away. This is called the Hubble law. It roughly says that the galaxies that are n times further away would recede n times faster. We can verify isotropy of the universe by the Hubble law since it is rotationally invariant. We can also verify Homogeneity, which states that no matter where you are in space, it will behave identically. If you were at any point on the surface of the balloon, you would see it as expanding in any direction. Also, the scaling factor is only dependent on time, which ensures the expansion or contraction of the universe is the same everywhere at a given time.

1.3 HUBBLE PARAMETER AND HUBBLE'S LAW

Under rescaling of coordinates, i.e. $a \to \lambda a$, $r \to \lambda r$, and $R \to \lambda R$, the metric remains unchanged.

Hence, we take $a(t_0) = a_0$ as unity where t_0 is the present time.

Now, we choose a function $a(t)$ such that its derivative is positive, which states that the system is expanding. The comoving coordinates trace a trajectory $x(t)$, and the physical distance corresponds to:

$$x_{phys}(t) = a(t)\ x(t)$$

$$v_{phys}(t) = \frac{dx_{phys}}{dt} = \dot{a}x + a\frac{dx}{dt} = Hx_{phys} + v_{pec}$$

The second term is known as peculiar velocity corresponding the the inherent motion of the galaxy with respect to the cosmological frame, which can also be said as the motion arising due to local gravitational interactions and not due to the expansion of the universe. The first term, which is the Hubble Parameter, is hence defined as

$$H(t) = \frac{\dot{a(t)}}{a(t)}$$

For a great range of time, we can assume the Hubble parameter to be approximately constant for a time t_0

$$H(t) \approx H_0$$

The present-day value of the Hubble parameter can be approximated to $70\ kms^{-1}Mpc^{-1}$. Therefore, a galaxy 0.1 Mpc away would be seen as retreating at a speed of $7\ kms^{-1}Mpc^{-1}$

Assuming that the peculiar velocity is negligible compared to the rate of expansion of the universe, we find a linear relationship between physical velocity and physical distance. This relation is referred to as Hubble's law.

$$v_{phys} = H_0\ x_{phys}$$

II. Cosmological Distances

If we input the speed of light in the FLWR metric, we find out that $ds^2 = 0$. This is consistent with the Minkowski space, where massless particles have an invariant interval between two events always equal to zero. On further calculations, we have

$$c\,dt = \pm \frac{a(t)\,dr}{\sqrt{1-\frac{kr^2}{R^2}}}$$

We will now send 2 signals at a time t_1 and $t_1 + \delta t$ sitting at a comoving coordinate r_1 with respect to the origin with time t_0. We have the following two equations:-

$$\int_{t_1}^{t_0} \frac{c\,dt}{a(t)} = \int_0^{r_1} \frac{dr}{\sqrt{1-\frac{kr^2}{R^2}}} \quad \text{and} \quad \int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{c\,dt}{a(t)} = \int_0^{r_1} \frac{dr}{\sqrt{1-\frac{kr^2}{R^2}}}$$

Equating the LHS since the RHS is equal, we get:

$$\int_{t_0}^{t_0+\delta t_0} \frac{c\,dt}{a(t)} = \int_{t_1}^{t_1+\delta t_1} \frac{c\,dt}{a(t)} \Rightarrow \frac{\delta t_0}{\delta t_1} = \frac{a(t_0)}{a(t_1)}$$

We know $\delta t_n = \frac{\lambda_n}{c}$. Therefore, substituting further, we get:

$$\frac{\lambda_0}{\lambda_1} = \frac{a(t_0)}{a(t_1)}$$

The light itself hasn't stretched; the surface it is on has expanded, which can be mistaken for the wavelength being stretched. This phenomenon is called the cosmological redshift due to the expansion of the universe.

We introduce z, where z is the redshift parameter

$$1 + z = \frac{1}{a(t_1)} \ \ where\ z = \frac{\lambda_0 - \lambda_1}{\lambda_1}$$

2.1 LUMINOSITY DISTANCE

The luminosity distance is a way to measure how far an object is from the laboratory. It is measured by how dim we receive the light compared to its intrinsic brightness. We know the Luminosity of the object is measured by the product of energy flux, which is energy

received per unit area per unit time and surface area of radius D, where the energy spreads out, which is very large due to the expansion of the universe.
The energy we observe and the energy emitted are not equal. The Observed energy is reduced due to cosmological redshift, which essentially means the photon has lost energy, and the loss can be calculated by the redshift parameter z

$$E_{obs} = \frac{E_{emit}}{1+z}$$

In earlier calculations, we also found the relation between time differences in the initial and final states after expansion, which states:

$$\Delta t_{obs} = (1+z)\Delta t_{emit}$$

This can also be understood as frequency decreasing due to time dilation caused by the expansion of the spacetime fabric.
Therefore, Luminosity, which is proportional to Energy per unit time, reduces by a factor of $(1+z)^2$. Calculating the proper surface area considering the comoving radius:

$$A = 4\pi(a_0 r)^2$$

We move forward assuming $a_0 = 1$. Hence, the observed flux will be a ratio of the reduced luminosity (observed luminosity) and the proper surface area

$$F = \frac{L}{4\pi r^2 (1+z)^2}$$

Compared to the standard definition, we obtain the value of luminosity distance

$$d_L = (1+z)\,.\,r(z)$$

Where $r(z)$ is the comoving distance for a flat universe. Inputting the expression for the comoving distance, we get:

$$d_L(z) = (1+z)\int_0^z \frac{dz'}{H(z')}$$

Converting this equation from natural units where c =1 and time/distance is measured in inverse Hubble units, to physical units, we obtain the final expression

$$d_L(z) = (1+z)\cdot\frac{c}{H_0}\int_0^z \frac{dz'}{E(z')}$$

Where $E(z)$ is the normalised Hubble parameter

$$E(z) = \frac{H(z)}{H_0}$$

2.2 STANDARD CANDLES AND DISTANCE MODULUS

A general issue comes in when we try to measure distances, which is whether the object is far away or just small. Correspondingly, for luminosity distance, the issue translates to is the object is far away or simply intrinsically dim? To tackle this issue, astronomers developed ‘references’ that one can compare to the observed brightness of these references to understand the distance of the object. These references are called standard candles. One of the most reliable standard candles is the type 1a supernovae. Through its data, we will analyse the Hubble parameter, redshift and the expansion of the Universe. Type 1a supernovae occur when a white dwarf pulls too much matter from its companion star, with which it is in orbit, reaching the Chandrasekhar mass limit, at which point the star collapses, leading to a thermonuclear explosion and often shoots out light brighter than all the stars of the galaxy combined during its peak luminosity.

With the understanding of standard candles and the luminosity distance, we will now develop a function to relate the luminosity distance to the actual measured distance. The distance modulus is the difference between the observed luminosity to the intrinsic luminosity. Say m is the observed luminosity and M is the intrinsic luminosity, we have the scaling system as follows:

$$\mu = m - M = -2.5 \log_{10}\left(\frac{F}{F_{10\text{pc}}}\right)$$

The RHS can further be simplified by integrating the inverse square law for distances, stating the energy flux(F) is inversely proportional to the Luminosity distance. Also, converting pc to Mpc, we get the familiar result:

$$\mu = 5 \log_{10}\left(\frac{d_L}{\text{Mpc}}\right) + 25$$

This formula is used to relate the luminosity distance to the observed brightness of the standard candles, such as Type 1a supernovae.

Until now, we have stated two important quantities for the analysis of the Type 1a Supernovae data and plotting and fitting the Hubble diagram, which are:

$$\mu = 5 \log_{10}\left(\frac{d_L}{\text{Mpc}}\right) + 25 \text{ and } d_L(z) = (1+z) \cdot \frac{c}{H_0} \int_0^z \frac{dz'}{E(z')}$$

We measure the observed luminosity and obtain the distance modulus by subtracting the intrinsic luminosity we have due to standardisation with the help of standard candles. Once we obtain the value for d_L from the distance modulus and compare it with the formula derived from the FLWR metric, we find the value of the redshift parameter z. The famous plot of the Redshift parameter and the distance modulus is called the Hubble Diagram, which depicts a linear relationship between the two quantities.
To further our understanding, we will learn about the fitting parameters and the Friedmann Equations.

III. Friedmann Equations and Thermodynamics of the Expansion

The Friedmann equation describes the expansion of the universe as a function of time and is an approximation of Einstein's field equation, which contains the information about the curvature and behaviour of the spacetime fabric. To reach from Einstein's field equation to the Friedmann equation explaining the expansion of the universe, we make two critical assumptions:-

1. The matter behaves as a perfect fluid whose information is contained in the stress-energy tensor.
2. Symmetry of the Cosmological Principle is followed, i.e. we will work with the FLWR metric.

3.1 PERFECT FLUIDS

A perfect fluid in our context has different characteristics compared to the ideal fluid in classical mechanics. Both have a non-viscous nature and dont conduct heat, but the perfect fluid is compressible, while the ideal fluid is not. Another key similarity is that the pressure for both systems is isotropic. The homogeneous and isotropic perfect fluid requires only two quantities to characterise it, namely the energy density(mass-energy per unit volume) $\rho(t)$ and the pressure $P(t)$.

We will consider two cases, a relativistic fluid and a non-relativistic fluid and differentiate them by considering Einstein's energy-mass relation for moving particles

$$E = m^2c^4 \ + \ p^2c^2$$

For the non-relativistic case(dust), $pc \ll mc^2$ that means that the energy is mass-dominated.
For the relativistic case(radiation) where the speed is comparable to the speed of light, the energy is momentum-dominated. Hence, we can say

$$E \approx pc$$

We define $n(p)$ as the number density, which is a distribution for different values of momentum.

$$\frac{N}{V} = \int_0^\infty dp\, n(p)$$

Pressure in our context can be defined as the flux of momentum across a unit area of surface. For a particle in a box, the pressure on a surface of our choice, say a surface on the xy plane, from the kinetic theory of gases

$$P = \int_0^\infty dp(v_z p_z)n(p)$$

From the equipartition theorem, the pressure exerted in each direction is the same. Hence, we get:

$$P = \frac{1}{3}\int_0^\infty dp(vp)n(p)$$

Approximating the total energy as $E \approx pc$ and substituting further

$$P = \frac{1}{3}\int_0^\infty dp(E)n(p) \Rightarrow P = \frac{N\langle E\rangle}{3V}$$

Where the energy density is $\rho = \dfrac{N\langle E\rangle}{V}$. Therefore, we get a simple relation between energy density and pressure, which corresponds to the equation of state for a relativistic, homogeneous, isotropic perfect fluid.

$$P = \frac{1}{3}\rho$$

These types of gases are referred to as radiation, and their general equation of state is:

$$P = \omega\rho$$

Whereas for dust, i.e. non-relativistic gases $\omega = 0$ and $\omega = 1/3$ for radiation, i.e. relativistic gas.

3.2 FRIEDMANN EQUATION

In this article, we will derive the Friedmann equation through Newtonian physics. Ideally, it is formulated through Einstein's field equation using the FLRW metric and the stress-energy tensor. Alas, we move forward with the Newtonian approximation:

The total energy of a particle in a gravitational field is:

$$E = \frac{1}{2}m\dot{r}^2 - \frac{GMm}{r}$$

We can also express the mass M enclosed within radius r linearly in terms of its density, since it's a homogeneous medium:

$$E = \frac{1}{2}m\dot{r}^2 - \frac{Gm}{r}\cdot\frac{4}{3}\pi r^3\rho \Rightarrow E = \frac{1}{2}m\dot{r}^2 - \frac{4\pi G}{3}mr^2\rho$$

We will now use the earlier substitutions and include the comoving coordinates

$$x_{phys}(t) = a(t)\ x(t)$$

In our context, it'll look like:

$$r = a(t)x(t) \Rightarrow \dot{r} = \dot{a}x \Rightarrow \dot{r} = \dot{a}\frac{r}{a}$$

On further substitution, we get:

$$E = \frac{1}{2}mr^2\left[\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G}{3}\rho\right]$$

We will now shamelessly define the curvature parameter k discussed earlier in a manner derived from General relativity:

$$E = -\frac{1}{2}mr^2\frac{kc^2}{a^2}$$

Solving both equations simultaneously, we have arrived at the expression which defines the expansion of the universe as a function of time, the Friedmann Equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

3.3 THERMODYNAMIC INTERPRETATION OF EXPANSION

The first law of thermodynamics for adiabatic processes, i.e. no exchange of energy with the surroundings, hence equating the term $-TdS$ to zero, giving us:

$$E = -PdV$$

The time-dependent version of the first law goes as follows:

$$\frac{dE}{dt} = -P\frac{dV}{dt}$$

Now, for a small region of fluid, the relation between the physical and comoving volume is:

$$V(t) = a^3(t)V_0 \Rightarrow \frac{dV}{dt} = 3a^2\dot{a}V_0$$

For a volume in space, the total energy in the volume will be its volume multiplied by its energy density. Hence, we get a new expression of Energy:

$$E = \rho a^3 V_0 \Rightarrow \frac{dE}{dt} = \left(\dot{\rho}\ a^3 + 3\rho\ a^2\dot{a}\right)V_0$$

Now that we have an expression of both the LHS and the RHS for the time-dependent first law in terms of the scaling factor to account for the expansion of the universe, we give the first law a new structure:

$$\dot{\rho} + 3H(\rho + P) = 0$$

This form of the first law is also called the Continuity Equation in Cosmology.

For dust, $\omega = 0$ which corresponds to $\Rightarrow \dot{\rho} + 3H(\rho) = 0$. On further calculation, we get the relation:

$$\rho = \frac{C}{a^3}$$

This works since the density is decreasing as volume is increasing due to the expansion of the universe. Hence, the dilution of the mass accounts for the conservation of energy.

For radiation, $\omega = 1/3$, doing similar calculations, we obtain the relationship between energy density and scaling factor as:

$$\rho = \frac{C}{a^4}$$

The extra a^{-1} which leads to energy not being conserved is due to the cosmological redshift, which decreases the frequency of a photon over time and hence the energy.

For vacuum energy (Dark Energy), $\omega = -1$. This is the actual fascinating world we live in, where energy is not conserved, but the energy density is constant even though the universe is expanding. Here, the total energy increases; one might think this disobeys Noether's theorem, but our system is not time independent, i.e. it is not symmetric in time.

If we simultaneously solve the time-differentiated Friedmann Equation and the Continuity equation, we obtain the second Friedmann Equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$$

From the second Friedmann Equation, we observe a very crucial phenomenon, that the universe is not only expanding, but it is also accelerating. When $\omega = -1$, we observe that $\ddot{a} > 0$, mathematically proving that our universe is expanding at an accelerating rate.

3.4 ΛCDM MODEL

The ΛCDM Model (Lambda Cold Dark Matter) is the standard model of cosmology based on several important assumptions, some we have discussed and some we have not. We will mention some of the assumptions in the scope of our article below:

$1)$ *Cosmological Principle and FLWR metric*

The cosmological principle states that the universe is homogeneous and isotropic. We describe this universe using the FLWR metric in spherical coordinates with curvature $k = 0$ and a time-dependent scaling factor $a(t)$.

$2)$ *General Relativity governs gravity*

We are considering all objects with a compressibility factor either less than or equal to 1. Hence, using the field equations of general relativity is the optimal and formal route to choose.

$3)$ *Thermal History and Matter Domination*

Radiation-dominated era$(\omega = 1/3)$: until $z \sim 3400$

$$a(t) \sim t^{1/2}$$

Matter-dominated era$(\omega = 0)$: CMB era, i.e. structure formation occurred here

$$a(t) \sim t^{2/3}$$

Λ-dominated era$(\omega = -1)$: acceleration begins $z \sim 0.7$

$$a(t) \sim e^{Ht}$$

IV. Measuring Cosmological Parameters Using Type Ia Supernovae

In this section, we will work with data from the Pantheon+SH0ES dataset of Type 1a supernovae to measure the Hubble constant. We will initially plot the log scale curve for lower z and then fit the cosmological model to the Type 1a supernovae data.

4.1 PLOTTING OF THE HUBBLE DIAGRAM

Ignoring all other data in the dataset, we will extract the relevant data, which includes the redshift parameter for their specific distance modulus and its corresponding uncertainty. We will represent the error as a vertical bar for each datapoint through the function errorbar.
For lower z, we see a linear log relationship between the distance modulus and the redshift

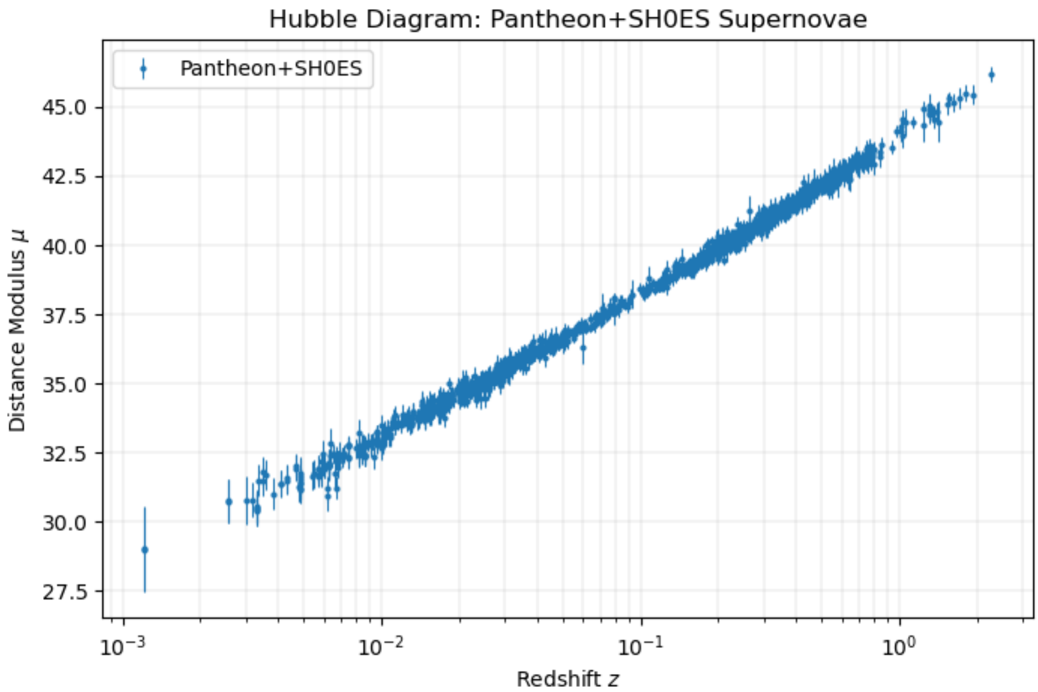


FIG 1. THE FAMOUS HUBBLE DIAGRAM PLOTS THE DISTANCE MODULUS ON THE Y AXIS AND THE REDSHIFT PARAMETER ON A LOG SCALE ON THE X AXIS, PROVIDING US A LINEAR RELATIONSHIP FOR LOWER Z

Now, we will define the cosmological model, i.e. the ΛCDM model. We will use three equations:

The Dimensionless Hubble parameter:- $E(z) = \sqrt{\Omega_m(1+z)^3 + (1-\Omega_m)}$

Distance Modulus:- $\mu(z) = 5\log_{10}(d_L/\text{Mpc}) + 25$

Luminosity Distance:- $d_L(z) = (1+z) \cdot \frac{c}{H_0} \int_0^z \frac{dz'}{E(z')}$

We have talked about the Luminosity distance and the distance modulus and their corresponding significance before, but this new structure of the dimensionless Hubble parameter was not discussed.

4.2 DIMENSIONLESS HUBBLE PARAMETER

The dimensionless Hubble parameter, or in other terms, the normalised Hubble parameter, describes the normalised rate at which the universe is expanding. The normalisation helps rescale any redshift to the present-day value of H_0.

$$E(z) = \frac{H(z)}{H_0}$$

To derive the form used in the ΛCDM model, we will need the help of the Friedmann Equation with the cosmological constant

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}$$

The energy density is segregated into 3 different forms, each of which we have discussed earlier, and how they change with the scaling factor for the expansion

$$\rho = \rho_m + \rho_r + \rho_\Lambda$$

Where the energy densities correspond to matter-dominated, radiation-dominated, and Dark energy-dominated universes. Their relation with the scaling factor is as follows:

$$\rho_m = \frac{C}{a^3}$$

$$\rho_r = \frac{C}{a^4}$$

$$\rho_\Lambda = C$$

Now we consider a flat space, i.e. k = 0 and substitute the specific conditions into the Friedmann equation:

$$H^2 = \frac{8\pi G}{3}\left[\rho_{m0}(1+z)^3 + \rho_{r0}(1+z)^4\right] + \frac{\Lambda c^2}{3}$$

Assigning appropriate density parameters, we finally arrive at our result:

$$E(z)^2 = \Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_\Lambda$$

Neglecting radiation at mid to higher values of the redshift parameter and using the substitution

$$\Omega_m + \Omega_\Lambda = 1 \quad \Rightarrow \quad \Omega_\Lambda = 1 - \Omega_m$$

We obtain the Dimensionless Hubble parameter for low redshifts:

$$E(z) = \sqrt{\Omega_m(1+z)^3 + (1-\Omega_m)}$$

4.3 COVARIANCE MATRIX

The covariance matrix is a square matrix that contains the information about the spread of the uncertainty(variance of the variable) in each dimension and the relationship between each dimension's uncertainty with the other(covariance).

$$\mathbf{C} = \begin{bmatrix} \mathrm{Var}(H_0) & \mathrm{Cov}(H_0, \Omega_m) \\ \mathrm{Cov}(H_0, \Omega_m) & \mathrm{Var}(\Omega_m) \end{bmatrix}$$

The diagonal elements of the matrix correspond to the variation of the Parameters we are considering, i.e. the independent uncertainty each variable possesses. The off-diagonal elements contain the information about how the uncertainty of one variable affects the other, i.e. their interrelationships. If covariance is positive, when one variable increases, the other increases. On the other hand, if covariance is negative, when one variable increases, the other decreases.
For non-linear square fitting, the covariance matrix is approximated by the Jacobian matrix

$$C = \sigma^2 \left(J^T J\right)^{-1}$$

where the Jacobian is defined as he partial derivative of the distance modulus with the parameters in consideration

$$J_{ij} = \frac{d\mu(z)}{\partial p_j} \text{ where } p_j \in \{H_0, \Omega_m\}$$

Correlation is normalised covariance, which gives us a better understanding of the interdependence since relative measurements are often misleading.

$$\rho(H_0,\, \Omega_m) = \frac{\mathrm{Cov}(H_0,\, \Omega_m)}{\sigma_{H_0}\sigma_{\Omega_m}}, \quad \rho \in [-1,1]$$

4.4 FITTING THE ΛCDM MODEL TO THE TYPE 1A SUPERNOVAE DATA

We will now use the ΛCDM model and fit it to our observational data from the Pantheon+SH0ES dataset to obtain the values for H_0 and Ω_m best fitting the Type 1a Supernovae data.
Fitting is a statistical method by which one finds the value of a parameter by comparing the theoretical predictions with observable data. It explores all the possibilities of combinations of the parameters in the neighbourhood of the theoretical prediction to calculate and try to match with the observed data, minimising the discrepancy.
In this analysis, we use the curve_fit function from the SciPy library to obtain the value for H_0 and Ω_m with their respective error.
Our initial theoretical guess is $(H_0, \Omega_m) = (70\ km/s/Mpc,\ 0.3)$. Using the curve_fit function for the observed values and the uncertainty represented by the Covariance matrix, we obtain the value of the scaling factor and the density parameter for the ΛCDM model as:

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Fitted H0 = 72.97 ± 0.26 km/s/Mpc
Fitted Omega_m = 0.351 ± 0.019
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Suppose we kept Ω_m constant as 0.3, which is the theoretical value from the equation of state, where $\omega = -1$, i.e. a fixed matter density is maintained. Fitting the curve to the observable data for finding the value of the Hubble constant H_0, we obtain:

`Fitted H0 = 73.23 ± 0.14 km/s/Mpc when` $\Omega_m = 0.3$

4.5 UNCERTAINTIES IN SUPERNOVAE COSMOLOGY

The uncertainty in the modulus, which essentially arises from the measurement inaccuracy of the luminosity distance, or inaccuracy in the intrinsic luminosity of a standard candle.

1) *Measurement of Luminosity Distance*
An error in the measurement of luminosity distance through photometry is a very common source of residuals(a statistical representation of error).

2) *Intrinsic luminosity of Type 1a Supernovae*
Type 1a Supernovae are not perfect standard candles. We can only move forward with a set of assumptions, leading them to be accepted as Standard Candles. When the assumptions meet their limit, the intrinsic value or the absolute magnitude will show uncertainty.

3) *High Redshift*
The formulation we have done was under the assumption that we will work with a lower range of redshift parameters. Once we reach a range of higher redshifts. Errors will pile up due to the non-linear nature of the curve for higher redshifts. We will compare the higher redshift and lower redshift samples quite shortly.

4.6 RESIDUAL ANALYSIS

A Residual is the vertical distance between a data point and the regression line. They correspond to the uncertainty in the measurement
It is crucial to do residual analysis since it gives us an idea of how well our cosmological model fits the observable data.

$$Residual = \mu_{obs} - \mu_{model}$$

A good model fit should show residual scatter randomly around zero without any significant structure, since that nullifies as much uncertainty as possible.

We can see that the mean residual is near zero. Since our equations were approximated for lower redshifts, the residual will increase as we move forward in the redshift parameter values. After plotting, we observe that there is no systematic trend depicting wrong or missing physics. Hence, we can confidently say that our best-fitted values of the ΛCDM model are in the acceptable range.

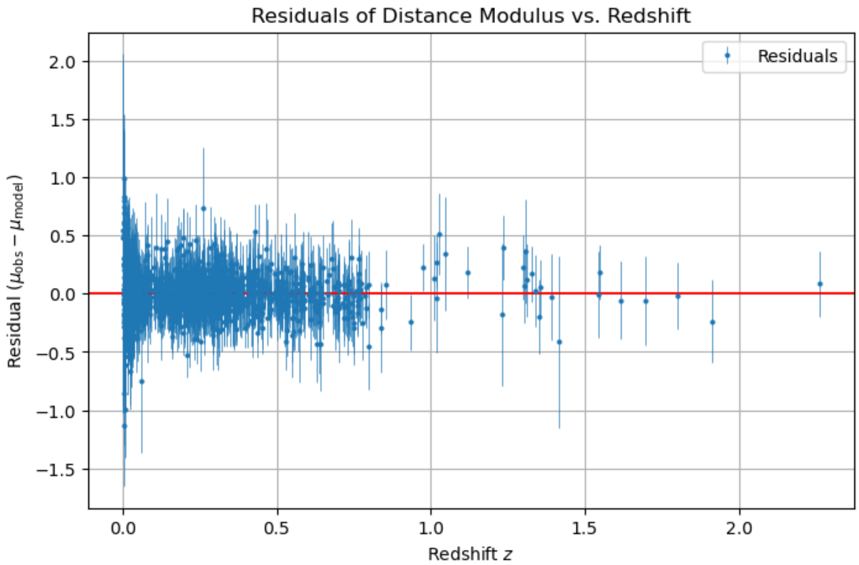

FIG 3. PLOTTING THE RESIDUALS AGAINST REDSHIFT TO IDENTIFY POSSIBLE SYMMETRIES OR PATTERNS LEADING TO NON ZERO MEAN

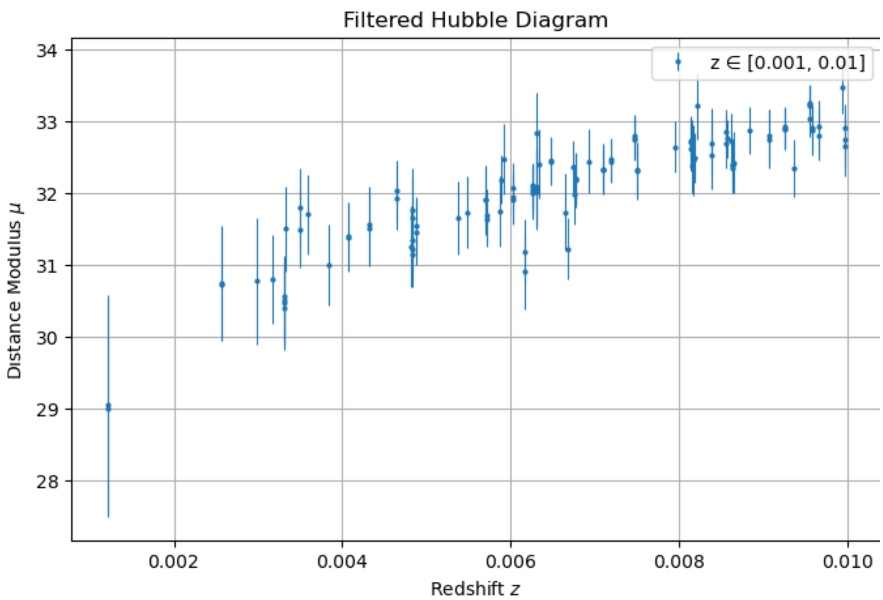

FIG 2. ZOOMED INTO THE HUBBLE DIAGRAM TO DEPICT THE RESIDUALS

V. Results and Interpretation

This is the last section of the article, where we will make our analytical conclusions and relevant comparisons.

5.1 BEST-FITTED HUBBLE PARAMETER AND COMPARISON WITH PLANCK 18 RESULTS

The value we get for the Hubble constant by fitting the ΛCDM cosmological model to the observational data of the Standard candle Type 1a supernovae is

```
Fitted H0 = 72.97 ± 0.26 km/s/Mpc
Fitted Omega_m = 0.351 ± 0.019
```

If we fix $\Omega_m = 0.3$, corresponding to a dark matter energy-dominated universe, which is our universe. We will again fit the model, but this time only the Hubble constant with an already assumed density parameter, we get:

Fitted H0 = 73.23 ± 0.14 km/s/Mpc when $\Omega_m = 0.3$

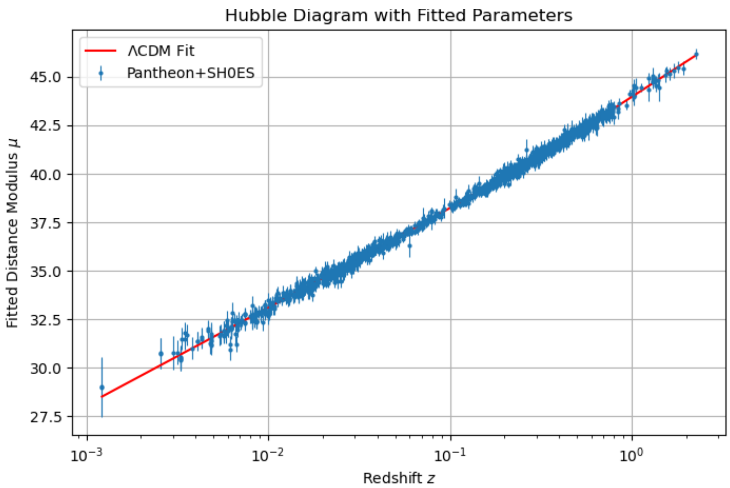

FIG 4. THE FITTED HUBBLE DIAGRAM ON A LOG SCALE

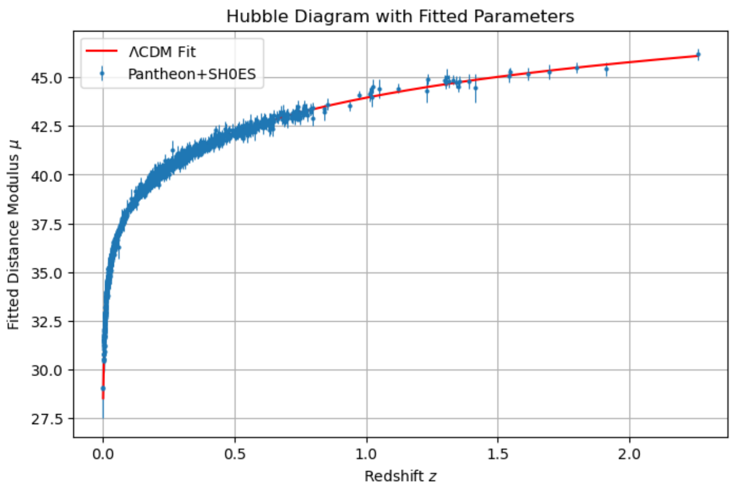

FIG 5. THE FITTED HUBBLE DIAGRAM ON A LINEAR SCALE

We will now input these values to obtain a modelled distance module through which we will plot a revised Hubble diagram, or so to speak, a Hubble Diagram with fitted parameters. The results for the higher redshifts are consistent with our prediction, which states that at higher z, the residuals will deviate from the ΛCDM fit.

Now, to compare the results of the Planck 2018, we will try to achieve this via plotting for the parameters obtained by the Planck 2018 dataset:

PARAMETERS	MY RESULTS	PLANCK 2018
H_0	72.97 ± 0.26 km/s/Mpc	67.4±0.5 km/s/Mpc
Ω_m	0.351±0.019	0.315±0.007

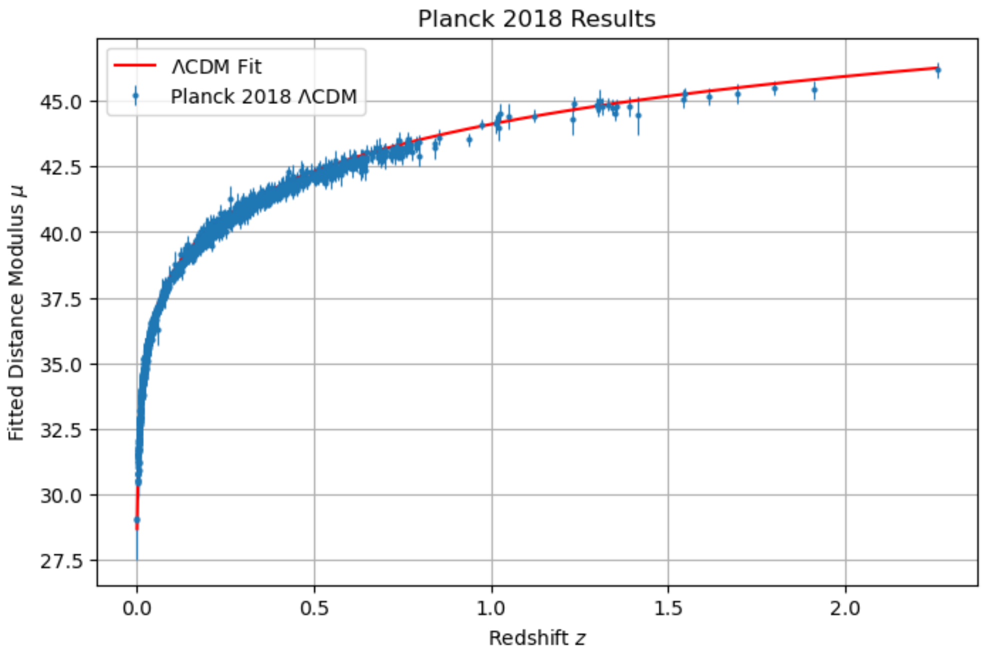

FIG 6. THE HUBBLE DIAGRAM MEASURED VIA THE PLANCK 18 DATASET

5.2 COMPARING THE VALUES OF THE HUBBLE CONSTANT FOR DIFFERENT RANGES OF Z

We will separate the data into three sections, low, mid, and high redshift and see the fitted value of H_0 for each. Low z ranges from 0 to 0.05, mid z ranges from 0.05 to 0.3, and high z ranges from 0.3 to infinity.

We apply these ranges and use the same function we coded to fit the previous data of z, and we obtain:

```
Low-z fit (z < 0.1):        H0 = 66.74 ± 3.12 km/s/Mpc
Mid-z fit (0.1 ≤ z < 1.0):  H0 = 73.37 ± 0.20 km/s/Mpc
High-z fit (z ≥ 1.0):       H0 = 73.98 ± 0.32 km/s/Mpc
```

Redshift is a tag on light which gives us the information on how far or how long the light has travelled. When we discuss high redshift values, which are significantly greater than one, we are referring to light that has travelled through space for a long time, leading to the conclusion that it originates from a younger and denser universe.
For redshift values less than one, the light was emitted more recently, from a closer object.
A redshift of around 1 will bring us halfway back in cosmic time, whereas a redshift value of 1100 corresponds to the CMB era.

From our previous sections, we know

$$\lambda_{obs} = a(t_0)\lambda_{emit}$$

$$z = \frac{\lambda_{obs} - \lambda_{emit}}{\lambda_{emit}} = \frac{a(t_0)}{a(t_{emit})} - 1$$

From the above formula, we see that redshift and the time-dependent scaling factor are inversely proportional. Hence, the higher the redshift of a photon, the further back in time it is, or it belongs to $1/a(t)$ times smaller universe. Redshift is therefore a cosmic clock.
The Huubbl law at low z can be approximated to

$$v \approx cz = H_0 d \quad \Rightarrow \quad z \propto d$$

This is linear at lower redshift, but non-linear as redshift increases. This phenomenon reflects the accelerated expansion of the universe due to dark matter.

For a flat ΛCDM universe:

$$H(z)^2 = H_0^2 \left[\Omega_m(1+z)^3 + \Omega_\Lambda\right]$$

Say we take a limit $z \to 0$, the matter term decreases while the dark matter term stays constant. This means that $\ddot{a} > 0$ when $z \to 0$.

For the low z range, our best-fitted value is highly in agreement with the Planck 18 observation. This is because the Planck 18 measures the value of H_0 at z corresponding to 1100, i.e. the early universe and then uses the ΛCDM model to calculate the value for $z \to 0$. This method is model-dependent as it assumes that the model works in the entire range of z.
By the direct fit method, the best-fitted values for lower z are in accordance with Planck 18's measurement. The benefit of this method is that it is relatively less dependent on the ΛCDM Model, but it generates large errors since it uses a very short range of z.
Our method of fitting using standard candles gives a deviation from the result of lower z because of the phenomenon called the Hubble tension.

5.3 THE HUBBLE TENSION

The Hubble tension arises from the large discrepancies between the two methods of measurement of the Hubble constant, which sets the absolute scale of the universe.
The Planck 18 satellite measurement and the SH0ES dataset measurement have a discrepancy of 5.6 km/s/Mpc. The giant difference in the value of the Hubble Constant between the two datasets is due to the difference in the method of measurement. The Planck satellite determines the Hubble constant via Cosmic microwave radiation(CMB), where the model is used to extrapolate the value of H_0 over a long time frame.
On the other hand, SH0ES data uses the Luminosity of standard candles to calculate the distance modulus. This type of measurement is called a local measurement, which does not involve the evolution of a system with time and is measured in a relatively short time frame. In our context, we refer to the late universe.
The Hubble tension could potentially have various theoretical implications, such as the breakdown of our current cosmological model or time-varying dark energy. The issue is not resolved yet.

A possible solution to the Hubble Tension is the relaxation of the ΛCDM Model, meaning to allow more freedom and loosen the strict assumptions. Some of the strict assumptions of the ΛCDM Model are:

- A constant dark energy
- A flat universe
- Negligible Early dark energy(radiation and matter dominated universe)

The corresponding relaxed model will have such assumptions:

- Dark energy varies with time
- Presence of Early dark energy

5.4 THE AGE OF THE UNIVERSE

From our earlier calculations, we have:

$$a(t) = \frac{1}{1+z} \text{ and } H(t) = \frac{\dot{a(t)}}{a(t)}$$

Now, we begin our derivation

$$t_0 = \int_0^{t_0} dt$$

$$t_0 = \int_{a=0}^{a=1} \frac{da}{\dot{a}} = \int_0^1 \frac{da}{aH(a)}$$

After doing a change of variable $a(t) = \dfrac{1}{1+z}$, we get the following expression

$$t_0 = \int_0^{\infty} \frac{1}{(1+z)H(z)}\, dz$$

We will now solve this integral through Python using the built-in quad function of scipy. Taking the best-fitted values of H_0 and Ω_m obtained from the Pantheon+SH0ES dataset, which are:

```
Fitted H0 = 72.97 ± 0.26 km/s/Mpc
Fitted Omega_m = 0.351 ± 0.019
```

We get the estimated value of the age of the universe as 12.37 billion years.

If we consider Planck 18's values of H_0 and Ω_m, we will get a closer answer to the actual age of the universe (13.8 billion years), which is:-

```
H0 = 67.4 ± 0.5 km/s/Mpc
Omega_m = 0.315 ± 0.007
```

We get a better estimate of the age of the universe as 13.39 billion years.

The Pantheon+SH0ES dataset has a larger Hubble constant (H_0) value compared to the Planck 18 datasets' Hubble constant. This larger H_0 corresponds to a faster-growing universe. Hence, reaching the present state faster. This universe appears younger. Whereas for the Planck 18 dataset's Hubble constant, the universe is older as the expansion rate is relatively slower.
Hence, as a conclusion, the SH0ES database implies a younger universe, and the Planck 18 database implies an older universe. This again is a consequence of the Hubble tension.

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