The Lane-Emden Equation: Derivation, Analysis, and Plotting for Stellar Polytropes

BY ARNAV WADALKAR
NIT ROURKELA, DEPARTMENT OF PHYSICS AND ASTRONOMY

CONTENT

ABSTRACT

White dwarfs are the remnants of low-mass stars in the Main Sequence (Mass less than 7–10.6 M⊙) of the Hertzsprung-Russell diagram, which exist with low luminosity and very high temperatures due to the degeneracy pressure of electrons.
The first White dwarf to be discovered was in the early 20th century and was a constituent of the famous binary system Sirius, Sirius B, a white dwarf. The luminosity and the spectral lines of Sirius B gave an estimate of its radius, which gave us a first estimate of its very high density, giving it drastically different characteristics than the Red Giants and the Massive stars. This difference can also be interpreted by investigating the Hertzsprung-Russell diagram. The white dwarf lies below the Main sequence due to its low luminosity and high temperature.
The existence of white Dwarfs is extremely fascinating, as it was one of the first discoveries which was described by the concepts of Quantum mechanics and Relativity simultaneously. The existence of a White dwarf also gave experimental results for the working of Pauli's exclusion principle.

In this article, we will be describing the structure of the white dwarf and its properties by going deep into the Polytropic model of equation of states and attempting to describe its behaviour mathematically by deriving the Lane-Emden equation and analysing its solutions. We will then use these equations to derive the Chandrasekhar limit for the white dwarf using Fermi-Dirac Statistics and Landau's approach in the latter article.

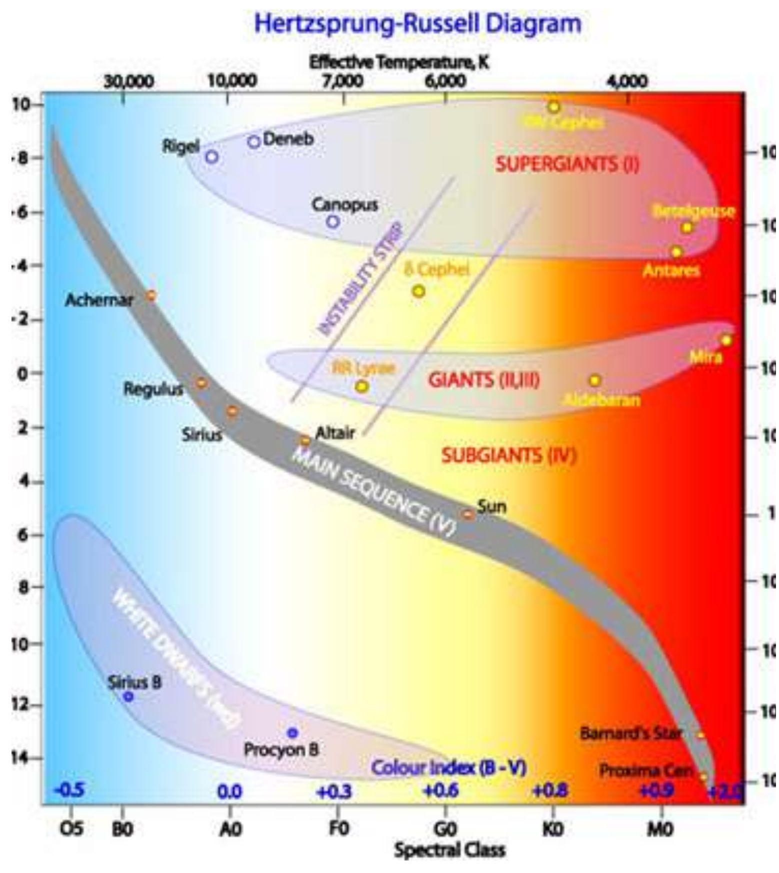

Fig. 1 Hertzsprung-Russell diagram.
From R. Hollow, CSIRO.

I. Hydrostatic Equilibrium of a Main Sequence Star

A main-sequence star maintains a pressure equilibrium against its gravitational pull by fusion. Fusion is a nuclear phenomenon occurring in the core of the star where two hydrogen nuclei combine to form a helium nucleus. The huge amount of energy released in this process is due to the difference in masses of the two hydrogen nuclei and the helium nucleus. The energy from the core is diffused outwards via photons. These photons transmit their momentum to the electrons they interact with, creating an outward pressure exerted throughout the star's layer called the radiation pressure.

The total energy flux from a blackbody surface is calculated by Stefan-Boltzmann's law, which also states that the amount of power radiated by a black body is directly proportional to the fourth power of the body's absolute temperature.

$$P = \sigma A T^4$$

where σ is called the Stefan-Boltzmann constant

$$\sigma = \left(\frac{2\pi^5 k^4}{15h^3c^2}\right)$$

Further, we get an expression for the radiation using Planck's law

$$P_{\text{rad}} = \frac{1}{3}aT^4, \quad \text{where} \quad a = \frac{4\sigma}{c}$$

But this does not constitute the entire pressure exerted outwards by a star.

The atoms create a net pressure on the surface of the star due to microscopic thermal motion.

$$P = P_{\text{gas}} + P_{\text{rad}}$$

Here, P_{gas} corresponds to the gas kinetic pressure, a macroscopic phenomenon occurring due to the microscopic random motion of the particles constituting the star(electrons, ions, etc). It was discovered through the Kinetic theory of gases by calculating for an ideal monatomic gas the pressure exerted by the random collisions due to atoms moving in all possible directions with all possible velocities imparting a net momentum to the walls of a container, generating a pressure called the gas kinetic pressure. Subsequently, we will be using its thermodynamical version for the gas kinetic pressure, which is

$$P = \frac{\rho k T}{\mu m_u} = n k_b T$$

where:

ρ: mass density of the star

k_b: Boltzmann constant

T: temperature in kelvin

μ: mean molecular weight

m_u: atomic mass unit

Now, we have the entire picture of the outward pressure generated by a star. The state where the outward pressure(radiation and gas pressure) and the inward gravitational pull are in mechanical equilibrium is called Hydrostatic equilibrium. The equation representing this equilibrium is

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

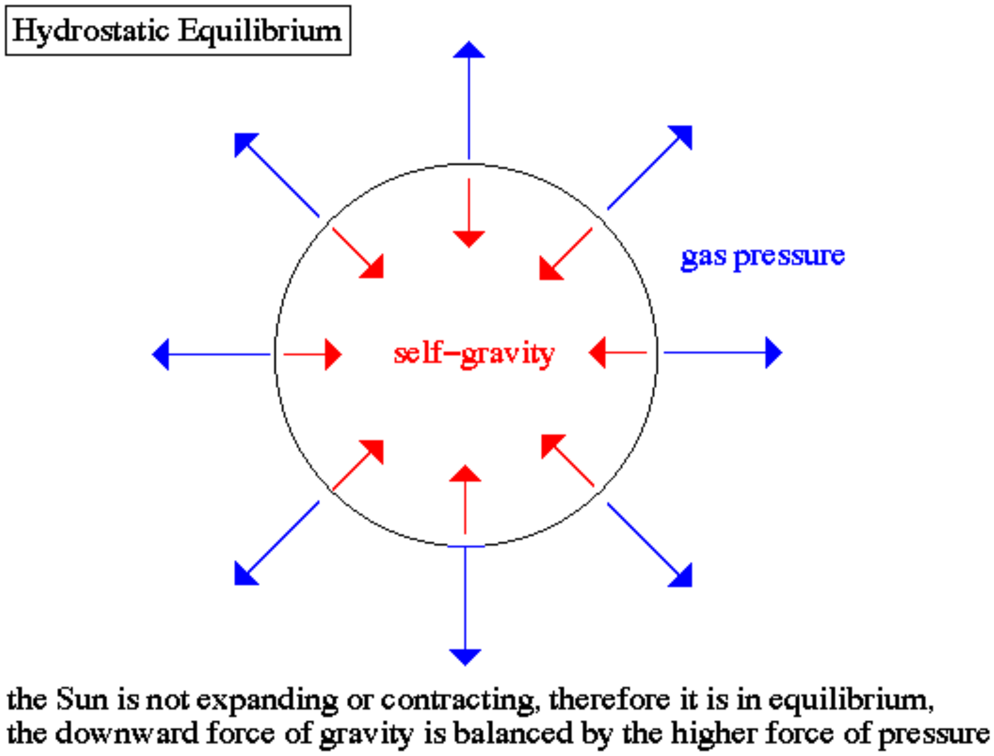

Fig.2 Here we have taken the Sun(a main sequence star) as an example to demonstrate Hydrostatic Equilibrium
From https://pages.uoregon.edu/jschombe/ast122/lectures/lec12.html

Substituting for total pressure, we construct the hydrostatic equilibrium equation

$$P = P_{\text{gas}} + P_{\text{rad}} = \frac{\rho kT}{\mu m_u} + \frac{1}{3}aT^4 \Rightarrow \frac{d}{dr}\left(\frac{\rho kT}{\mu m_u} + \frac{1}{3}aT^4\right) = -\frac{GM(r)\rho(r)}{r^2}$$

This equation represents the mechanical equilibrium of a star in the main sequence.

II. Physical Assumptions Made to Model a Self-Gravitating Gas Sphere

For further calculations, we will proceed to make a set of 4 assumptions regarding the mass distribution, nature and symmetry of stars.

1. The Star is static, i.e. it is neither contracting nor expanding

We have discussed in great detail the conditions and equations for the static nature of the star, specifically a main-sequence star. Hydrostatic Equilibrium will be assumed in all further calculations for a main-sequence star.

2. The star is spherically symmetric

The phenomenon breaking the spherical symmetry, such as rotation and magnetic field of the star, contributes to a negligible amount of energy compared to the gravitational binding energy in the majority of cases. Hence, we shall move forward with assuming spherical symmetry, which aids us in easing the calculations. Spherical symmetry states that the density, total pressure and other physical variables vary only with respect to the radial distance in a symmetric manner. The equation stating spherical symmetry for a body is:

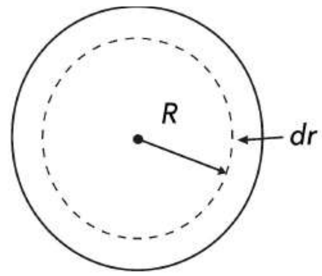

Fig.3 A spherical shell of radius R and Infenitismally small thickness dr

$$P = P(r) \quad \rho = \rho(r) \quad T = T(r)$$

A key consequence of spherical symmetry is the Mass continuum equation, which describes the change of mass with respect to the radial distance.

The Mass continuity equation:- $\dfrac{dM}{dr} = 4\pi r^2 \rho(r)$

$$dM = 4\pi r^2 \rho(r) dr$$

The mass of the infinitesimally thin spherical shell ‘cut out’ of the sphere is dM described by the above equation.

3. General Relativity is negligible

The curvature of the spacetime for a star is considered only if the compactness parameter of the star is of the order 0.1 or higher.
The compactness parameter of a star is the ratio of its gravitational potential and its rest mass. If the compactness parameter, also called the metric deviation for all the parts of the star, is much less than 1 (e.g. ~10^{-6} for the Sun, ~10^{-4} to 10^{-3} for white dwarfs), the spacetime curvature is too weak to consider, and Newton's equation suffices.

$$Compactness\ parameter = \frac{2GM(r)}{rc^2} \ll 1$$

The compactness parameter is a method of comparing the strength of the gravitational potential against its rest mass. General relativity corrections become crucial when the compactness parameter approaches unity. For entities such as Neutron stars and Black holes, one cannot move forward without considering General Relativity to study their phenomena.

4. Polytropic and Isothermal model of Gas spheres

A Polytropic Gas Sphere is a self-gravitating gaseous sphere in which Pressure and density are related through the polytropic equation of state.

$$P = K\rho^{1+\frac{1}{n}}$$

Here, n is called the polytropic index. Each Polytropic index describes a different system. We will talk about this in greater detail while deriving the Lane-Emden Equation.

If a gas is enclosed in a rigid spherical shell impenetrable to heat, i.e. an Isothermal gas sphere, it will transition into a state of gross thermal equilibrium through the process of conduction of heat.

$$P = K\rho$$

Under this equilibrium, the only temperature gradient allowed is one in accordance with the hydrostatic equilibrium, leading the pressure and density to be described by their respective polytropic index. We will be calculating the Equation of state for this case and deriving its Lane-Emden Equation

III. Milne's Theorem of Gravitational Equilibrium

Theorem 1: In a gaseous configuration in equilibrium in which the radiation pressure is negligible

$$\bar{T} > \frac{1}{6} \cdot \frac{\mu m_p}{k} \cdot \frac{GM}{R}$$

The mean temperature is defined as:

$$M\bar{T} = \int T\, dM(r)$$

We know from the ideal gas law:

$$P = \frac{k}{\mu m_p} \rho T$$

Rearranging, we get:

$$T = \frac{\mu m_p}{k} \cdot \frac{P}{\rho}$$

Substitute this expression for T into the definition of mean temperature:

$$M\bar{T} = \frac{\mu m_p}{k} \int \frac{P}{\rho}\, dM(r)$$

Using the relation, $dM = \rho\, dV$ we get:

$$M\bar{T} = \frac{\mu m_p}{k} \int P\, dV$$

From the virial theorem, we know:

$$\int P\, dV = -\frac{1}{3}\Omega$$

Hence,

$$M\bar{T} = -\frac{1}{3} \cdot \frac{\mu m_p}{k} \cdot \Omega$$

Now, for a self-gravitating system (like a star), the gravitational potential energy is approximately:

$$\Omega = -\frac{GM^2}{R}$$

Substitute into the equation:

$$\bar{T} = \frac{1}{M} \cdot \left(-\frac{1}{3} \cdot \frac{\mu m_p}{k} \cdot \Omega \right) = \frac{1}{3} \cdot \frac{\mu m_p}{k} \cdot \frac{GM}{R}$$

Finally, we get the inequality:

$$\bar{T} > \frac{1}{6} \cdot \frac{\mu m_p}{k} \cdot \frac{GM}{R}$$

Theorem 2: In any equilibrium configuration, the function $P + \frac{GM^2(r)}{8\pi r^4}$ *decreases outwards.*

We start with the equation of hydrostatic equilibrium:

$$\frac{dP}{dr} = -\frac{GM(r)}{4\pi r^4}\frac{dM(r)}{dr}$$

Now take the derivative of the function:

$$\frac{d}{dr}\left[P + \frac{GM^2(r)}{8\pi r^4}\right] = \frac{dP}{dr} + \frac{GM(r)}{4\pi r^4}\frac{dM(r)}{dr} - \frac{GM^2(r)}{2\pi r^5}$$

Substitute the expression for $\frac{dP}{dr}$:

$$\frac{d}{dr}\left[P + \frac{GM^2(r)}{8\pi r^4}\right] = -\frac{GM(r)}{4\pi r^4}\frac{dM(r)}{dr} + \frac{GM(r)}{4\pi r^4}\frac{dM(r)}{dr} - \frac{GM^2(r)}{2\pi r^5}$$

The first two terms cancel:

$$\frac{d}{dr}\left[P + \frac{GM^2(r)}{8\pi r^4}\right] = -\frac{GM^2(r)}{2\pi r^5} < 0$$

Thus, the function decreases outwards.

Corollary:
If P_c is the central pressure, then:

$$P_c \geq P + \frac{GM^2(r)}{8\pi r^4} > \frac{GM^2}{8\pi r^4}$$

Therefore,

$$P_c > \frac{GM^2}{8\pi r^4}$$

IV. The Lane-Emden Equation

1. Fundamental Equation of Equilibrium for Self-Gravitating Gas Sphere

From Hydrostatic Equilibrium, we extracted the mass continuum equation, which states the change of mass with respect to the sphere's radial distance.

$$dM = 4\pi r^2 \rho(r) dr$$

Take an infinitesimal cylinder and calculate the pressure on it. By the assumption of spherical symmetry, we can say:

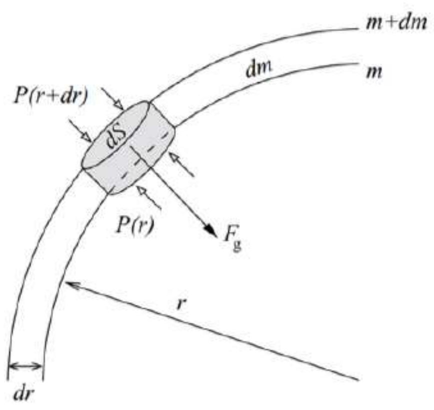

Fig.4 An infinitesimal cylinder under spherical symmetry
From Ravindu Kalhara - Hydrostatic Equilibrium Of Stars

$$P = P(r)$$

Therefore, the pressure difference created would be

$$dP = P(r + dr) - P(r)$$

The mechanical equilibrium is maintained as the pressure is counterbalanced by the inward gravitational pull of the star. Hence, we calculate the force exerted on the infinitesimal cylinder and equate it with its counteracting gravitational force.

$F = dP \,.\, (4\pi r^2)$ and $F = -F_g$

$$dP \cdot 4\pi r^2 = -\frac{GM(r)\, dm(r)}{r^2}$$

We know that $dm(r) = 4\pi r^2 \rho(r)\, dr$. Substituting the mass continuum equation, we get the final expression in hydrostatic equilibrium:

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

If we integrate the mass continuum equation from 0 to R, where R corresponds to the radius of the sphere, we will get the total mass of the gas sphere.

$$M(r) = \int_0^r 4\pi r'^2 \rho(r')\, dr' \Rightarrow M(r) = \frac{4\pi r^3}{3}$$

On substituting the mass in the expression of hydrostatic equilibrium

$$\frac{r^2}{\rho}\frac{dP}{dr} = \frac{-4\pi G\rho r^3}{3}$$

Solving further, we reach the fundamental equation of equilibrium:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -\,4\pi G\rho$$

To move further in the derivation of the Lane-Emden equation, we need to understand and apply the polytropic model of the equation of state.

2. Polytropic Equation of States

From thermodynamics, we know, for an adiabatic process,

$$P(\frac{1}{\rho})^{\gamma} = \text{const} \quad \Rightarrow \quad P = K\rho^{\gamma}$$

$$T = K\rho^{\gamma-1}$$

where K is a constant, and γ is known as the adiabatic index,$\gamma = \dfrac{C_p}{C_v}$

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V \; and \; C_P = \left(\frac{\partial H}{\partial T}\right)_P$$

Now, we introduce the polytropic index n, where n is related to γ in the following manner:

$$\gamma = 1 + \frac{1}{n}$$

Substituting γ in the form of polytropic index, we get the relation between pressure and density, which is called the Polytropic model of the Equation of State

$$P = K\rho^{1+\frac{1}{n}}$$

Assuming the ideal gas law equation, we can compute a relation between Temperature and Density. We substitute P to obtain a relation between temperature and density:

$$T = \frac{\mu m_H}{k_B}K\rho^{\frac{1}{n}}$$

Solving further, we get

$$\rho = \left(\frac{k_B T}{K\mu m_H}\right)^n$$

Hence, we have the Polytropic Equation of State:

$$P = K\rho(r)^{1+\frac{1}{n}} \text{ and } \rho(r) = \lambda\theta(r)^n$$

where:

$$\lambda = \left(\frac{k_B}{K\mu m_H}\right)^n$$

3. Deriving the Lane-Emden Equation for Self-Gravitating Gas Spheres

We have the fundamental equation of equilibrium:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

And the Polytropic EoS:

$$P = K\rho(r)^{1+\frac{1}{n}} \text{ and } \rho(r) = \lambda\theta(r)^n$$

where:
K = Constant
n = Polytropic index
θ = Normalised density
Hence, we have pressure in terms of θ:

$$P = K\lambda^{1+\frac{1}{n}}\theta(r)^{n+1}$$

Substituting for P and ρ in terms of θ in the fundamental equation of equilibrium, we get:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\lambda\theta(r)^n}\cdot\frac{d}{dr}\left(K\lambda^{\frac{n+1}{n}}\theta(r)^{n+1}\right)\right) = -4\pi G\lambda\theta(r)^n$$

Differentiating the inner bracket:

$$\frac{d}{dr}\left(K\lambda^{\frac{n+1}{n}}\theta(r)^{n+1}\right) = K\lambda^{\frac{n+1}{n}}(n+1)\theta(r)^n\frac{d\theta(r)}{dr}$$

Substituting the quantity, we get:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\lambda\theta^n}\cdot K\lambda^{\frac{n+1}{n}}(n+1)\theta^n\frac{d\theta}{dr}\right)=-4\pi G\lambda\theta^n$$

Now, we will make our final substitution to rescale the radial distance and simplify the equation:

$$r=a\xi$$

Where a is the scaling constant:

$$a=\left[\frac{(n+1)K\lambda^{\frac{1}{n}-1}}{4\pi G}\right]^{1/2}$$

On rescaling the radial distance, we obtain the famous Polytropic Lane-Emden Equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta(\xi)}{d\xi}\right)=-\theta(\xi)^n$$

4. Deriving the Lane-Emden Equation for Isothermal Gas Spheres

By the Ideal gas law, for an isothermal process:

$$P=K\rho$$

We recalculate, taking the condition for isothermal processes from:

$$\frac{dP}{dr}=-\frac{GM(r)\rho(r)}{r^2}$$

Hence, we get:

$$K\frac{1}{\rho}\frac{d\rho}{dr}=-\frac{GM(r)}{r^2}$$

We can write the LHS as:

$$\int Kd(ln(\rho))=\int_0^r-\frac{GM(r)}{r^2}dr$$

Hence, we get the solution ρ in exponential form

$$\rho(r)=\rho_c e^{-\theta}\quad \text{where}\quad \theta=\frac{\mu m_H}{k_BT}\left(\Phi(r)-\Phi(0)\right)$$

$\phi = Gravitational\ Potential\ Energy$

$\theta = Dimensionless\ \phi$ or $Polytropic\ Temperatue$

Once we have the revised solution for an Isothermal process, we follow the same steps by substituting the EoS in the fundamental equation of equilibrium

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

The new equations of state are:

$$P = \frac{k_b T}{\mu m_h}\rho$$

$$\rho(r) = \rho_c e^{-\theta} \quad \text{where} \quad \theta = \frac{\mu m_H}{k_B T}\left(\Phi(r) - \Phi(0)\right)$$

Therefore, the Lane-Emden Equation for an Isothermal Gas Sphere is:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = e^{-\theta}$$

This equation is considered when the system is in a Quasistatic Isothermal state, i.e. the process is so slow that the hydrostatic equilibrium is maintained at all times.

V. Analytic solution to the Polytropic Lane-Emden Equation for n = 0 and n = 1

Before solving the Lane-Emden Equation, we need to state its boundary conditions.

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta(\xi)}{d\xi}\right) = -\,\theta(\xi)^n$$

Like any second-order Differential Equation, the Polytropic Lane-Emden Equation has two boundary conditions which help us extract unique solutions from the general solutions of the differential equation.

1. $\theta(0) = 1$

This is a consequence of the normalisation of density.
θ It is the dimensionless normalised density we used in the EoS of the Polytropic Model, and it was used in the normalisation of ρ .

$$\rho = \rho_c \theta^n$$

At the Centre of the star $\xi = 0$, ρ is maximum. Hence, we have our solution $\rho = \rho_c$, meaning that at the centre $\theta(0) = 1$.

2. $\left.\frac{d\theta}{d\xi}\right|_{\xi=0} = 0$

This is a consequence of spherical symmetry. At the centre of the star, the density is equal in all directions; therefore, its derivative with respect to radial distance should be zero, meaning there can be no preferred direction at the centre.
Physically, this avoids a singularity and provides a symmetric profile around the centre of the star

Now, we are fully equipped to solve the Lane-Emden Equation at different Polytropic indices.

$i)\ n = 0$

We will take θ as a power series expansion of ξ near $\xi = 0$

$$\theta(\xi) = \sum_{k=0}^{\infty} a_k \xi^k$$

Applying the boundary conditions, we obtain that $a_0 = 1\ and\ a_1 = -1$

$$\theta(\xi) = 1 - \xi + a_2\xi^2 + a_3\xi^3 + a_4\xi^4 + \cdots$$

Since the system is spherically symmetric, at the centre, there should be no preference for any direction. For this to be true, we need to eliminate the odd power terms, leaving us with only the even powers.

$$\theta(\xi) = 1 + a_2\xi^2 + a_4\xi^4 + a_6\xi^6 + \cdots$$

Applying this power series to the Lane-Emden Equation, we get:

$$\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = \frac{d}{d\xi}(2a_2\xi^3 + 4a_4\xi^5 + \ldots.)$$

$$\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = 6a_2\xi^2 + 20a_4\xi^4 + \ldots.$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = 6a_2 + 20a_4\xi^2 + \ldots.$$

Now, we solve the RHS. We binomially expand the series with power n raised to it for small ξ

$$\theta^n = 1 + na_2\xi^2 + \ldots.$$

$$-\theta^n = -1 - na_2\xi^2 - \ldots.$$

Since ξ is small(in general cases $\xi < < 1$), we have the privilege to ignore higher powers of it.

Equating LHS and RHS, we get:

$$6a_2 + 20a_4\xi^2 = -1 - na_2\xi^2$$

Equating the coefficients with equal powers, we get:

$$a_2 = \frac{-1}{6} \; and \; a_4 = \frac{n}{120}$$

Therefore, the solution for small ξ is

$$\theta^n = 1 - \frac{\xi}{6} + \frac{n\xi^2}{120}\ldots.$$

We can solve for the coefficients of higher powers of ξ using the method of recursion

$$\theta(\xi) = 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 - \frac{n(8n-5)}{15120}\xi^6 + \cdots$$

Inputting $n = 0$, we get:

$$\theta(\xi) = 1 - \frac{1}{6}\xi^2$$

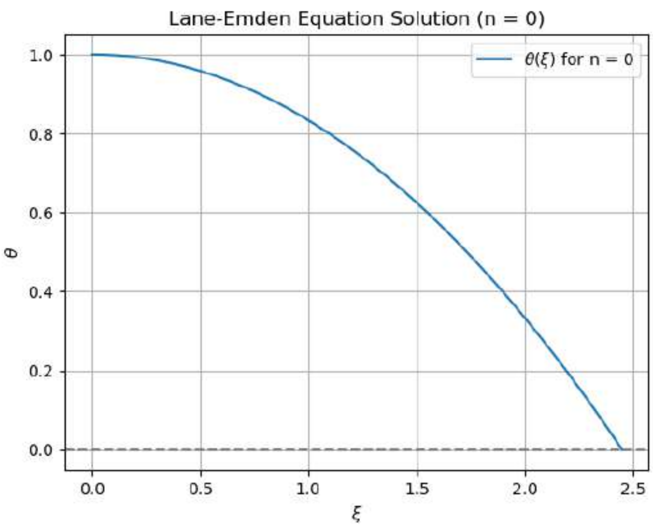

Fig. 5 Plotting the Lane-Emden Equation for n = 0 using matplotlib in Python

For $n = 1$, we obtain a very familiar Taylor series:

$$\theta(\xi) = 1 - \frac{1}{6}\xi^2 + \frac{1}{120}\xi^4 - \frac{1}{5040}\xi^6 + \cdots = \frac{sin(\xi)}{\xi}$$

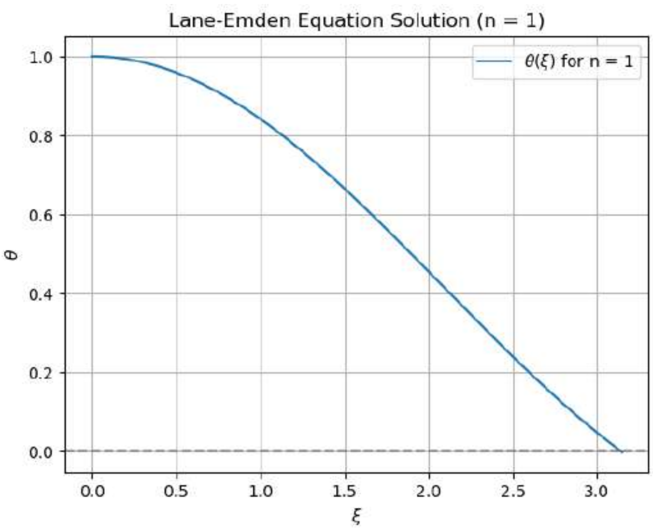

Fig. 6 Plotting the Lane-Emden Equation for n = 1 using matplotlib in Python

VI. Numerical solution to the Polytropic Lane-Emden Equation for n = 1.5, n = 3 and n = 5 using RK4 in Python

To calculate the solutions, we will be using the 4th-order Runge-Kutta Integrator(RK4) in Python.
The working of the 4th-order Runge-Kutta Integrator is based on minimising the local error to an order of $\mathcal{O}(\Delta\ t^5)$.
Suppose we have a vector field $\dot{x} = f(x, t)$ which varies with both position and time, such as a non-laminar flow. What the 4th-order Runge-Kutta Integrator does is update the vector 4 times in one loop

$$f_1 = f(x_k,\ t_k)$$

$$f_2 = f(x_k + \frac{\Delta t}{2} f_1,\ t_k + \frac{\Delta t}{2})$$

$$f_3 = f(x_k + \frac{\Delta t}{2} f_2,\ t_k + \frac{\Delta t}{2})$$

$$f_4 = f(x_k + (\Delta t) f_3,\ t_k + \Delta t)$$

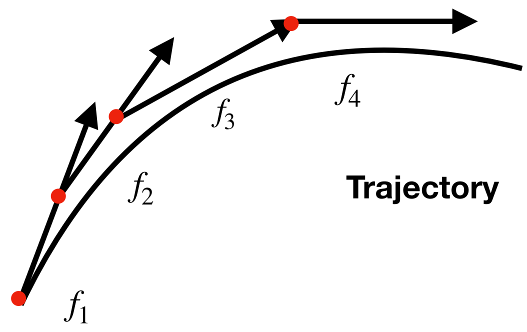

Fig.7 Depiction of the full and half forward Euler steps for a curved trajectory

First, the loop inputs the value of f_1, then it takes a half-forward Euler step in the direction of f_1. Proceeding to do the same, the RK4 estimates the derivative at the half point using the updated vector, i.e. f_2 and takes a half-forward Euler step in the direction of f_2 giving us f_3. Now, the RK4 again estimates the derivative at the end point of f_3 and takes a full forward Euler step, and gives us the final position, which f_4 is pointing at.

We now take a weighted average of these derivative estimates to compute the final update in such a manner that it minimises the local error. We can calculate the most efficient combination which minimises the local error to the order of $\mathcal{O}(\Delta\ t^5)$ and the global error to the order of $\mathcal{O}(\Delta\ t^4)$ for a non-chaotic system.

$$x_{k+1} = x_k + \frac{\Delta t}{6}(f_1 + 2f_2 + 2f_3 + f_4)$$

Now that we have understood the algorithm we will be using to code to obtain a numerical solution, let's get started with the logic of the code

$i)\ n = 5$

The Polytropic Lane-Emden Equation is a coupled second-order equation and can be expressed in terms of two first-order equations (a property of every second-order differential equation)

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta(\xi)}{d\xi}\right) = -\,\theta(\xi)^n$$

This can be expressed as two separate first-order differential equations by the following substitution: $y = \theta(\xi)$ and $z = \dfrac{d\theta}{d\xi}$

Now, we solve the Lane-Emden Equation using these substitutions

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 z\right) = -\,y^n \Rightarrow \frac{1}{\xi^2}(2\xi z + \xi^2\frac{dz}{d\xi}) = -\,y^n$$

$$\frac{dz}{d\xi} = \frac{-2z}{\xi} - y^n$$

Hence, we get the following pair of first-order differential equations:

$$z = \frac{dy}{d\xi}, \text{ and } \frac{dz}{d\xi} = \frac{-2z}{\xi} - y^n$$

For n = 5, we get:

$$z = \frac{dy}{d\xi} \text{ and } \frac{dz}{d\xi} = \frac{-2z}{\xi} - y^5$$

According to the boundary conditions $\theta \rightarrow 1$, when $\xi \rightarrow 0$
But here we find that this condition causes a singularity in the second equation. Hence, we work our way around it by taking ξ as not zero but a very small number $\epsilon \approx 10^{-5}$

Using the Taylor expansion near $\xi = 0$, we get

$$y \approx 1 - \frac{1}{6}\xi^2$$

$$z \approx -\frac{1}{3}\xi$$

Now, for the algorithm of RK4, we define the following:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$z_{n+1} = z_n + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

We will write the expressions for the respective k's and l's.

Here f_1 corresponds to $\frac{dy}{d\xi}$ and f_2 corresponds to $\frac{dz}{d\xi}$. We are moving a time step of h in the direction of the derivative.

$$k_1 = f_1(\xi_n, y_n, z_n) = hz$$

$$l_1 = hf_2(\xi_n, y_n, z_n) = h(-\frac{2}{\xi_n}z_n - y_n^5)$$

$$k_2 = hf_1(\xi_n + \frac{h}{2}, y_n + \frac{k_1}{2}, z_n + \frac{l_1}{2})$$

$$l_2 = hf_2(\xi_n + \frac{h}{2}, y_n + \frac{k_1}{2}, z_n + \frac{l_1}{2})$$

$$k_3 = hf_1(\xi_n + \frac{h}{2}, y_n + \frac{k_2}{2}, z_n + \frac{l_2}{2})$$

$$l_3 = hf_2(\xi_n + \frac{h}{2}, y_n + \frac{k_2}{2}, z_n + \frac{l_2}{2})$$

$$k_4 = hf_1(\xi_n + h, y_n + k_3, z_n + l_3)$$

$$l_4 = hf_2(\xi_n + h, y_n + k_3, z_n + l_3)$$

Once the necessary equations have been put down, we shall define the parameters required to define a system of polytropic index n = 5

$h = 0.01$ (Time step)

$\epsilon = 10^{-5}$

$y = 1 - \frac{1}{6}\xi^2$

$z = -\frac{1}{3}\xi$

y represents the normalised density, which is a non-negative quantity. Therefore, we will integrate till the surface, i.e. until $y \leq 0$

We can also solve the Lane-Emden equation analytically for n=5, which gives us the mathematical expression:

$$\theta(\xi) = \left(1 + \frac{\xi^2}{3}\right)^{-1/2}$$

We can see that when $\xi \to 0$, the function tends towards a singularity.

Solving and plotting for n = 5, we get

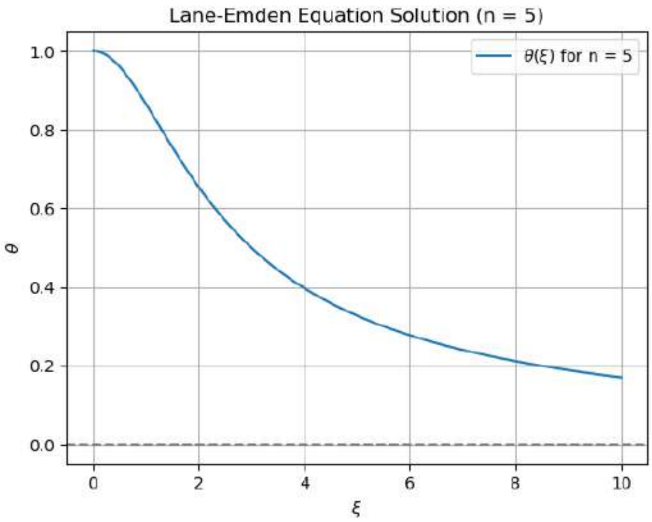

Fig. 8 Plotting the Lane-Emden Equation for n = 5 using matplotlib in Python

Now, we solve for n = 1.5 and n = 3, having their equations given below. Using the RK4 integrator in Python, we plot for both the polytropic indices.

$$\frac{dz}{d\xi} = \frac{-2z}{\xi} - y^{\frac{3}{2}} \text{ and } \frac{dz}{d\xi} = \frac{-2z}{\xi} - y^{3}$$

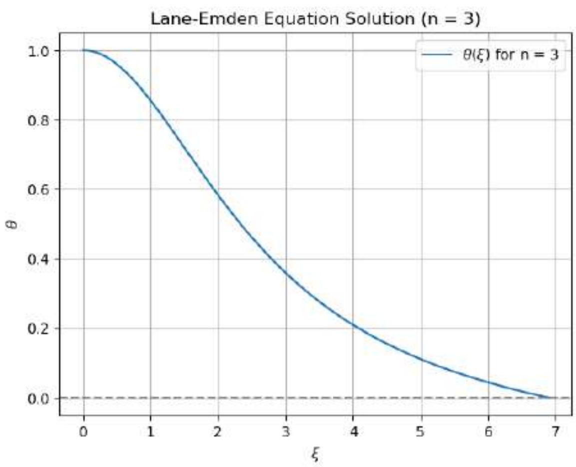

Fig. 9 Plotting the Lane-Emden Equation for n = 3 using matplotlib in Python

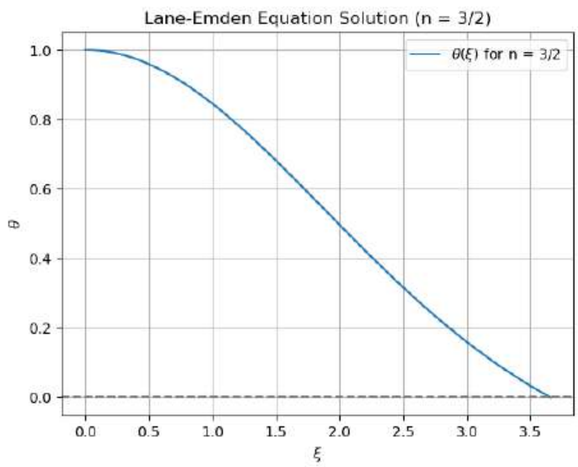

Fig. 10 Plotting the Lane-Emden Equation for n = 1.5 using matplotlib in Python

VII. Physical Profile and Interpretation of the Polytropic Index

Polytropic Index(n)	Physical Interpretation	Main Property	Systems explained by the EoS
$n = 0$	Incompressible Fluid	Constant Density $M \propto R^3$	Idealized model
$n = 1$	Moderate Compressibility	$P \propto \rho^2$	Isothermal Core Stars
$n = 5$	Limiting case	Finite Mass, Infinite Radius	Isothermal Spheres
$n = 3$	Relativistic degenerate electrons	Radiation Pressure dominates	High-Mass White Dwarfs, Very Massive Main Seqeunce Stars
$n = \frac{3}{2}$	Non - Relativistic degenerate electrons	$M \propto R^{-3}$	Low-Mass White Dwarfs, Brown Dwarfs

Before analysing and understanding the physical meaning behind each polytropic model, we have to derive the mass-radius relation, which will act as a very strong tool to intuitively understand the behaviour and structure of the stars through equations and plots.

Solving for equations of mass and radius will give us a better intuitive picture of the star and its basic characteristics.

The mass of the star is calculated by substituting the polytropic EoS equations in the mass continuum equation.

$$M(r) = \int_0^r 4\pi r'^2 \rho(r')\, dr'$$

Now we substitute for radius and density as

$$R = a\xi_1 \text{ and } \rho = \rho_c \theta^n$$

Here, R correspond to the stellar radius. On substituting, we get:

$$M(\xi_1) = a\int_0^{a\xi_1} 4\pi (a\xi)^2 \rho_c \theta^n d\xi$$

$$M(\xi_1) = 4\pi\rho_c a^3 \int_0^{a\xi_1} \xi^2 \theta^n d\xi$$

From the Polytropic Lane-Emden Equation, we know:

$$\frac{d}{d\xi}(\xi^2 \frac{d\theta(\xi)}{d\xi}) = -\xi^2 \theta^n$$

Using this to simplify the mass continuum equation, we get:

$$M(\xi_1) = -4\pi\rho_c a^3 \int_0^{a\xi_1} \frac{d}{d\xi}(\xi^2 \frac{d\theta(\xi)}{d\xi})d\xi$$

$$M(\xi_1) = 4\pi\rho_c a^3 \left| \xi_1^2 \frac{d\theta}{d\xi} \right|_{\xi=\xi_1} \Bigg|$$

This is the dimensionless mass-radius relation. Though useful, it won't give us a realistic or intuitive insight into the Star's Structure and Behaviour.

To get an insight into the physical structure and behaviour of the star, we will convert this equation of dimensionless mass into mass in solar masses by resubstituting and inputting values.

Inputting the value of a, we get:

$$R = a\ \xi_1 = \xi_1 \left[\frac{(n+1)K}{4\pi G}\ \rho_c^{\frac{1-n}{n}} \right]^{1/2}$$

$$M(\xi) = 4\pi \left(\frac{(n+1)K}{4\pi G} \right)^{3/2} \rho_c^{\frac{3-n}{2n}} \left| \xi_1^2 \frac{d\theta}{d\xi} \right|_{\xi=\xi_1} \Bigg|$$

Now, let's start to analyse how each polytropic model behaves

1) $n = 0$

Let’s understand the mathematical equations for this polytropic index since they have a lot of information in them.

$$\rho = \rho_c$$

By this equation, we understand that the density throughout the sphere is constant, making it incompressible. The solution of the Polytropic Lane-Emden Equation for n = 0 is:

$$\theta(\xi) = 1 - \frac{1}{6}\xi^2$$

When the dimensionless density becomes zero, it corresponds to the surface of the star. Hence, the value of ξ will correspond to the radial distance.

$$\xi_1 = \sqrt{6}$$

This model doesn't have any real physical implications since it has constant density throughout, but it is a useful mathematical model.

2) $n = 1$

For a polytropic index being 1, we have the following EoS

$\rho = \rho_c\theta$ and $P = K\rho^2$

This corresponds to mild compression and a more realistic model compared to n=0, since the Pressure changes quadratically with respect to ρ

We can analyse its compressibility by observing how Pressure changed with density:

$$\frac{dP}{d\rho} = 2K\rho$$

This quantity is also proportional to the speed of sound, which is finite in this case and infinite for n=0

We also have the analytical solution for this polytropic index:

$$\theta(\xi) = \frac{sin(\xi)}{\xi}$$

Here, $\xi_1 = \pi$ which corresponds to the dimensionless radial distance since $\theta(\xi_1) = 0$

To understand the Mass distribution, we will use the help of the expression we derived above

$$M(\xi_1) = 4\pi\rho_c a^3 \left| \xi_1^2 \frac{d\theta}{d\xi} \bigg|_{\xi=\xi_1} \right|$$

We know the solution for n=1 is $\theta(\xi) = \dfrac{sin(\xi)}{\xi}$, substituting the solution, we get the following expression of mass in terms of dimensionless radial distance:

$$M = 4\pi^2 a^3 \rho_c$$

Similarly, for density, we have the following expression

$$\rho = \rho_c \frac{sin(\xi)}{\xi}$$

We will now plot the Mass and Density against the dimensionless radial distance to visually understand the distribution throughout the structure.

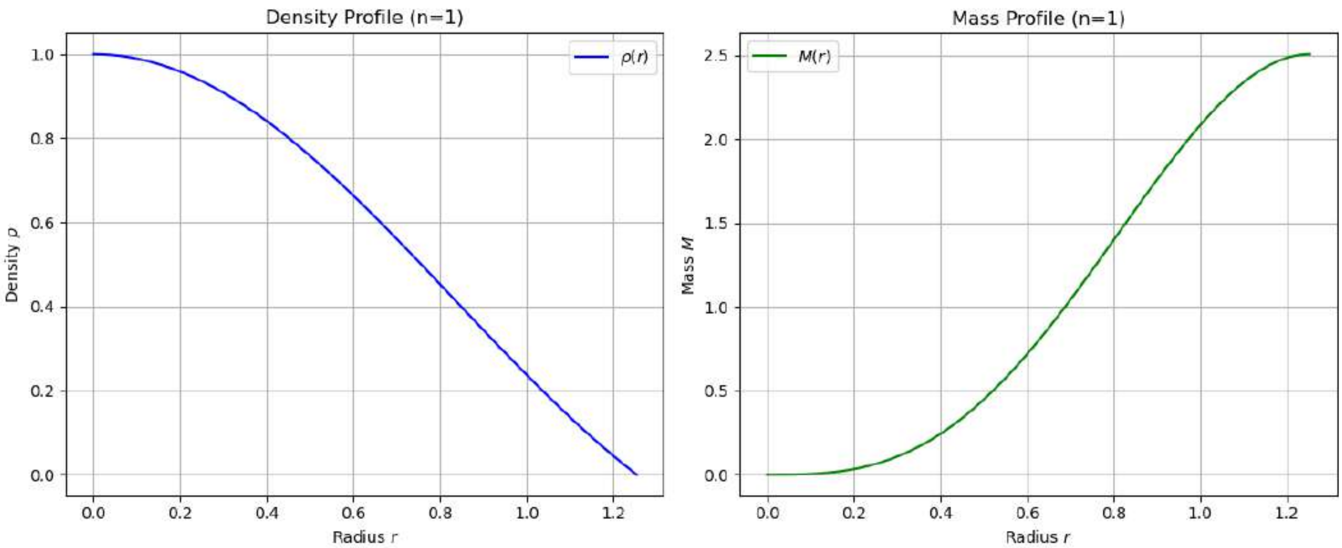

Fig. 11 Plotting the Mass and Density Profile for n=1 Polytropic Model

For a Real Star, the density curve is far steeper, but under some specific assumptions, this model can be used to describe cores of low-mass stars.

3) $n = 5$

For the polytropic index $n = 5$, we have an analytic solution. We will interrogate the solution and the EoS to extract information about this model

$$P = K\rho^{\frac{6}{5}} \text{ and } \rho = \rho_c \theta^5$$

$$\theta(\xi) = \left(1 + \frac{\xi^2}{3}\right)^{-1/2}$$

We observe that as $\theta \to 0$,

$$\xi \to \infty$$

This corresponds to the sphere having an infinite radius for there to be zero density. A very unrealistic model to describe a star and the limiting case for polytropic states.
Even though the radius is infinite, the mass is constant if we apply the solution in the mass-radius relation.
The reason for mass being constant, even though the model has an infinite radius, can be understood if we approximate the solution for large ξ
Thus, the approximated solution for large ξ is:

$$\theta(\xi) = \frac{\sqrt{3}}{\xi}$$

Therefore, $\rho \propto \xi^{-5}$ which corresponds to density steeply dropping as ξ increases

Using this in the following equation:

$$M(\xi_1) = 4\pi\rho_c a^3 \left| \xi_1^2 \frac{d\theta}{d\xi} \right|_{\xi=\xi_1} \Bigg|$$

We get the value of Mass at infinite radius as a constant, which is

$$M = 4\pi\rho_c a^3 m(\infty) = 4\pi\rho_c a^3\sqrt{3}$$

Where m corresponds to dimensionless mass.

4) $n = 1.5$ and $n = 3$

These are very important polytropic indices as they describe the non-relativistic and relativistic degenerate star which works on the principle of Electron Degeneracy pressure occurring due to the Quantum effect called Pauli's Exclusion principle, which forbids any two identical fermions to be in the same quantum state. This puts a constraint on the compactness of the star and forces the electrons to maintain different quantum states. The consequence of this phenomenon is interpreted as a macroscopic phenomenon, which is the outward pressure created due to the degeneracy of the electrons, called the Electron Degeneracy pressure. This Degenerate state is the final state of a White Dwarf with low luminosity and very high pressure. But this phenomenon has its limits. For massive stars which have a very high amount of gravitational inward pull, collapse even in the presence of degeneracy pressure and evolve either into Neutron Stars or Black holes. This famous threshold is called the 'Chandrasekhar limit', which is approximately 1.4 to

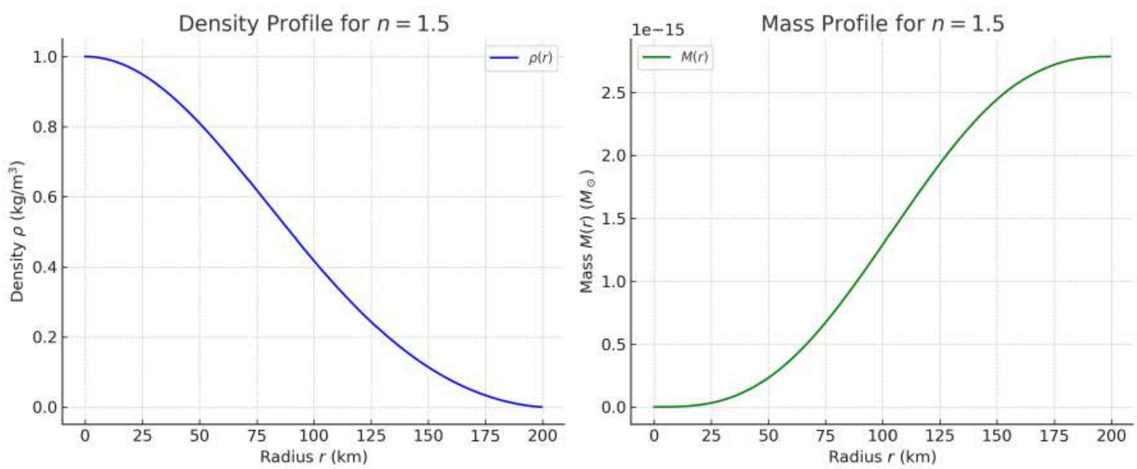

Fig. 12 Plotting the Mass and Density Profile for n=1.5 Polytropic Model

1.5 Solar Masses. Any Star with a mass higher than the Chandrasekhar limit is bound to collapse into a black hole or a neutron star. While plotting the Mass profile for the non-relativistic case. It isn't depicted in the plot since the non-relativistic case is only applicable for lower mass white dwarfs. As the mass increases, it is essential to adopt the relativistic case as relativistic effects become significant. Hence, it is crucial to consider a relativistic formulation for the electron degeneracy pressure.

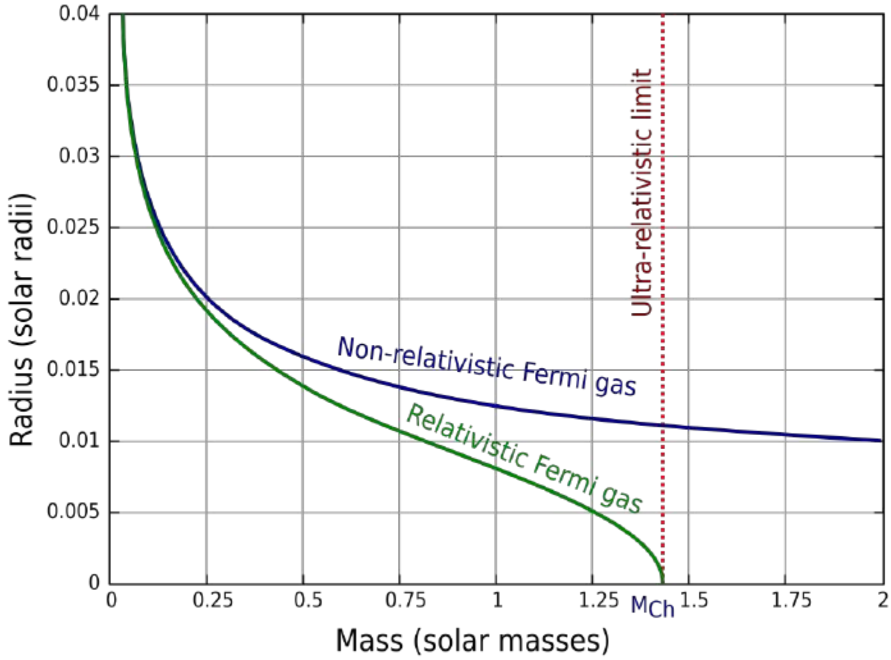

Fig. 13 Comparison between the Mass profiles of the Relativistic and Non-Relativistic cases From Wikipedia

Now, we take the relativistic case and plot the same profiles. It is depicted from the mass profile that there exists a limit on the mass, the Chandrasekhar limit.

Conclusion

In this article, we have studied and analysed the Lane-Emden equation and its solution from both a physical and a mathematical perspective. This derivation does not include the corrections from General Relativity and Coulomb Interactions. Through this idealised model, we were still able to learn the behaviour of white dwarfs to a great extent. This topic is a great example of the unification of various fields, such as quantum mechanics, where we understand the workings of microscopic phenomena and fields such as thermodynamics and astrophysics, where we study macroscopic phenomena. This unification provides a very strong theoretical framework

In the next article, using this polytropic equation and statistical mechanics, we will derive and understand the Chandrasekhar limit.
The Limit is calculated by approximating the Fermi-Dirac statistics at 0 Kelvin. We will understand the consequences and the math behind the Chandrasekhar limit, and also plot relevant profiles.

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