# Rodrigues' Formula, and Detailed Examples

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## 0. Functional-Analytic Background (brief)

Let $w(x) > 0$ on an interval $(a, b)$. Define the weighted Hilbert space

$$L_w^2(a,b) = \{f : (a,b) \to \mathbb{R} \mid \|f\|^2 = \langle f, f \rangle = \int_a^b |f(x)|^2 w(x)\, dx < \infty\}.$$

Inner product:

$$\langle f, g \rangle = \int_a^b f(x) g(x) w(x)\, dx.$$

A sequence of polynomials $\{p_n\}$ with $\deg p_n = n$ is orthogonal w.r.t. $w$ if

$$\langle p_n, p_m \rangle = 0 \quad (n \neq m).$$

Orthogonal polynomials often arise as eigenfunctions of a self-adjoint Sturm–Liouville operator and therefore serve as natural bases for expansions of physical wavefunctions.

## 1. Rodrigues' Formula — motivation and statement

Rodrigues' formula gives an explicit differential operator representation of classical orthogonal polynomials. A common pattern: the weight $w$ satisfies a Pearson-type identity

$$\frac{d}{dx}\big[\sigma(x) w(x)\big] = \tau(x) w(x),$$

with $\sigma(x)$ (usually polynomial degree $\leq 2$) and $\tau(x)$ (degree $\leq 1$). For the classical families there exists a constant $A_n$ such that

$$\boxed{p_n(x) = \frac{1}{A_n\, w(x)} \frac{d^n}{dx^n}\big[\, \sigma(x)^n w(x)\, \big]}$$

produces a polynomial of degree $n$ which satisfies a second-order differential equation of Sturm–Liouville type and is orthogonal w.r.t. $w$. Below we demonstrate this concretely for Legendre, Hermite and (generalized) Laguerre polynomials.

## 3. Worked Example: Legendre polynomials

### Data

Weight $w(x) = 1$ on $[-1, 1]$. Take $\sigma(x) = 1 - x^2$. Rodrigues formula (classical form):

$$\boxed{P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\big[(x^2 - 1)^n\big].}$$

## (i) Polynomial degree and leading term

$(x^2-1)^n$ has degree $2n$. Differentiating $n$ times gives degree $n$. The leading term: from $(x^2)^n = x^{2n}$,

$$\frac{d^n}{dx^n}x^{2n} = \frac{(2n)!}{n!}x^n,$$

so prefactor $1/(2^n n!)$ yields leading coefficient $\dfrac{(2n)!}{2^n(n!)^2}$ (conventional normalization).

## (ii) Verify the Legendre ODE

Legendre ODE:

$$\frac{d}{dx}\big[(1-x^2)P_n'(x)\big] + n(n+1)P_n(x) = 0.$$

To verify, substitute $P_n = \dfrac{1}{2^n n!}Q^{(n)}$ with $Q(x) = (x^2-1)^n$. We compute $(1-x^2)P_n'$ as

$$(1-x^2)P_n'(x) = \frac{1}{2^n n!}(1-x^2)Q^{(n+1)}(x).$$

Differentiate once more:

$$\frac{d}{dx}\big[(1-x^2)P_n'\big] = \frac{1}{2^n n!}\big[(1-x^2)Q^{(n+2)} - 2xQ^{(n+1)}\big].$$

One then uses repeated Leibniz expansions of derivatives of $Q = (x^2-1)^n$. Because $Q$ contains factor $(x^2-1)^n$, many lower-order terms vanish or cancel, and the leftover simplifies to $-n(n+1)P_n(x)$. (This algebra is mechanical but lengthy; for teaching you may expand for small $n$ to see the cancellations explicitly.)

## (iii) Orthogonality (integration by parts)

Let $m < n$. Consider

$$\int_{-1}^{1} P_n(x)\,P_m(x)\,dx = \frac{1}{2^n n!}\int_{-1}^{1} P_m(x)\,Q^{(n)}(x)\,dx.$$

Integrate by parts $n$ times, moving derivatives onto $P_m$. Boundary terms vanish because $Q = (x^2-1)^n$ and its first $n-1$ derivatives are zero at $x = \pm 1$ (factor of $(x \pm 1)^n$ kills them). After $n$ integrations we obtain an integral involving $P_m^{(n)}(x)$, but $\deg P_m = n < m$ so $P_m^{(n)} \equiv 0$. Therefore the integral is zero: orthogonality proven.

## (iv) Explicit small $n$ check

Compute directly:

$$P_0 = \frac{1}{1}\frac{d^0}{dx^0}(x^2-1)^0 = 1,$$

$$P_1 = \frac{1}{2\cdot 1!}\frac{d}{dx}(x^2-1) = x,$$

$$P_2 = \frac{1}{4\cdot 2!}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{2}(3x^2-1).$$

These match the standard Legendre polynomials.

# 4. Worked Example: Hermite polynomials

## Data

Weight $w(x) = e^{-x^2}$ on $(-\infty, \infty)$. The Rodrigues formula (physicist's Hermite) is

$$\boxed{H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\big(e^{-x^2}\big).}$$

This family satisfies

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0,$$

and orthogonality

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x)H_m(x)\,dx = \sqrt{\pi}\,2^n n!\,\delta_{nm}.$$

## (i) Polynomial degree and small $n$

Let $R(x) = e^{-x^2}$. Then $H_n(x) = (-1)^n e^{x^2} R^{(n)}(x)$. Differentiating $R$ $n$ times produces $e^{-x^2}$ times a polynomial of degree $n$; multiplication by $e^{x^2}$ cancels the exponential leaving a polynomial of degree $n$.

Check:

$$H_0 = 1, \quad H_1 = -e^{x^2}\frac{d}{dx}e^{-x^2} = 2x, \quad H_2 = e^{x^2}\frac{d^2}{dx^2}e^{-x^2} = 4x^2 - 2.$$

## (ii) Verify the Hermite ODE (sketch)

Using $R' = -2xR$ and induction on derivatives, one shows

$$e^{x^2}\frac{d}{dx}\Big(e^{-x^2}\frac{d}{dx}H_n\Big) = -2nH_n,$$

which expands to the ODE above. For a direct check, you can substitute the small-$n$ polynomials above into the ODE.

## (iii) Orthogonality via integration by parts

For $m < n$,

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x)H_m(x)\,dx = (-1)^n \int_{-\infty}^{\infty} H_m(x)\,\big(R^{(n)}(x)\big)\,dx,$$

with $R = e^{-x^2}$. Integrate by parts $n$ times moving derivatives onto $H_m$. The boundary terms at $\pm\infty$ vanish because $R$ decays faster than any polynomial; after $n$ steps one gets an integral with $H_m^{(n)}$, which is zero if $m < n$. Thus orthogonality holds.

## (iv) Physics connection

Hermite polynomials appear in the quantum harmonic oscillator wavefunctions:

$$\psi_n(x) = \mathcal{N}_n e^{-x^2/2} H_n(x),$$

where the Gaussian factor is part of the weight and $H_n$ supplies the polynomial part of the $n$-th excited eigenstate.

# 5. Worked Example: Generalised Laguerre polynomials

## Data

Weight $w(x) = x^\alpha e^{-x}$ on $[0, \infty)$ with $\alpha > -1$. Rodrigues formula:

$$\boxed{L_n^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha} e^x \frac{d^n}{dx^n}\big(e^{-x} x^{n+\alpha}\big).}$$

They satisfy

$$xy'' + (\alpha + 1 - x)y' + ny = 0,$$

and orthogonality

$$\int_0^\infty x^\alpha e^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x)\, dx = \frac{\Gamma(n + \alpha + 1)}{n!}\, \delta_{nm}.$$

## (i) Polynomial degree and small $n$

Inside the $n$-th derivative there is a factor $x^{n+\alpha}$; differentiating $n$ times produces a leading term proportional to $x^\alpha$. Multiplying by $x^{-\alpha} e^x$ cancels the power and exponential leaving a polynomial of degree $n$.

Check for $\alpha$ generic:

$$L_0^{(\alpha)}(x) = 1, \qquad L_1^{(\alpha)}(x) = -x + (\alpha + 1).$$

## (ii) Verify the Laguerre ODE (sketch)

Set $S(x) = e^{-x} x^{n+\alpha}$, so

$$L_n^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha} e^x S^{(n)}(x).$$

Using product-rule expansions and the Pearson identity for the weight, algebraic cancellation yields the Laguerre ODE with eigenvalue $n$. As with Legendre, the full expansion is mechanical; checking small $n$ explicitly is instructive.

## (iii) Orthogonality

For $m < n$ consider

$$\int_0^\infty x^\alpha e^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x)\, dx.$$

Substitute the Rodrigues form of $L_n^{(\alpha)}$ and integrate by parts $n$ times, moving derivatives onto the other polynomial times the weight. Boundary terms at $x \to \infty$ vanish due to $e^{-x}$; at $x = 0$ the behavior is controlled by $x^\alpha$ with $\alpha > -1$, ensuring boundary terms vanish as well. After $n$ integrations the integrand contains derivatives of order $n$ of $L_m^{(\alpha)}$ which vanish if $m < n$. Hence orthogonality holds.

## (iv) Quantum application

Laguerre polynomials (with $\alpha = 2\ell + 1$ or related values) appear in radial parts of hydrogen-like wavefunctions, where the radial equation reduces to a Laguerre-type ODE after separation of variables.

# 6. Practical remarks and exercises

## Practical remarks

- The Rodrigues representation is extremely useful for deriving explicit expressions, recurrence relations, and normalization constants.
- The three-term recurrence (Favard) can also be derived from orthogonality and gives a tridiagonal (Jacobi) representation of the multiplication operator $M_x$ in the polynomial basis.
- Generating functions are complementary: once you have Rodrigues (or recurrence) you can often derive the generating function and vice versa.

## Exercises (suggested)

1. Using Gram–Schmidt on $\{1, x, x^2, x^3\}$ with weight $w(x) = 1$ on $[-1, 1]$, compute $P_0, \ldots, P_3$ explicitly (show integrals).
2. Verify that Hermite polynomials obtained from Rodrigues satisfy the orthogonality constant $\sqrt{\pi}2^n n!$ by evaluating the Gaussian integrals for small $n$.
3. Using the Rodrigues form for Laguerre with $\alpha = 0$, compute $L_0, L_1, L_2$ and verify the orthogonality numerically (a short Python/Mathematica check).
4. Derive the three-term recurrence for Legendre polynomials from the differential equation and show it matches the standard $(n + 1)P_{n+1} = (2n + 1)xP_n - nP_{n-1}$.