10.4.1.2.3: Solving Rohan's Age Problem

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Question

Rohan's mother is 26 years older than him. The product of their ages (in years) 3 years from now will be 360. Find Rohan's present age.

Theoretical Solution

Let Rohan's age, $R = x$. Then Mother's age, $M = x + 26$. After 3 years, we have:

$$(M+3)(R+3) = 360,$$
$$(x+29)(x+3) = 360,$$
$$x^2 + 32x - 273 = 0.$$

Solving this quadratic equation:

$$x = 7 \quad \text{or} \quad x = -39.$$

Since age cannot be negative, Rohan's present age is:

$$\boxed{7}.$$

Computational Solution: Matrix-Based Method

The quadratic equation is:

$$x^2 + 32x - 273 = 0.$$

Construct the companion matrix:

$$A = \begin{bmatrix} 0 & 1 \\ -273 & -32 \end{bmatrix}.$$

The eigenvalues of A are the roots of the polynomial.

QR Decomposition

QR decomposition factors a given matrix A into:

$$A = QR,$$

where Q is an orthogonal matrix $Q^T Q = I$, and R is an upper triangular matrix.

Householder Transformations

Householder transformations are used to zero out elements below the diagonal of a matrix column. Given a vector v, the Householder matrix is:

$$H = I - 2\frac{vv^T}{v^T v}.$$

The process:

- Initialize Q as the identity matrix.
- Let $\vec{x}$ be the first column of A, and $\alpha = \|\vec{x}\|$.
- Compute:

$$\vec{u} = \vec{x} - \alpha \vec{e}_1, \quad \vec{v} = \frac{\vec{u}}{\|\vec{u}\|}, \quad Q = I - 2\vec{v}\vec{v}^T$$

- Result: $Q_1 A$ transforms A to:

$$Q_1 A = \begin{bmatrix} \alpha_1 & * & \dots & * \\ 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{bmatrix}$$

Householder Transformations

This process is repeated for A' (obtained from $Q_1 A$ by deleting the first row and column), resulting in Q_2'. Expand Q_2' to Q_k:

$$Q_k = \begin{bmatrix} I_{k-1} & 0 \\ 0 & Q_k' \end{bmatrix}$$

After $n-1$ iterations:

$$R = Q_{n-1} \dots Q_2 Q_1 A, \quad Q^T = Q_{n-1} \dots Q_2 Q_1, \quad Q = Q_1 Q_2 \dots Q_{n-1}.$$

QR Algorithm for Eigenvalues

The QR algorithm iteratively applies QR decomposition to a shifted matrix $A - \mu I$ and reconstructs it as:

$$A = RQ + \mu I,$$

converging to an upper triangular form with eigenvalues on the diagonal.

- μ is calculated as:

$$\mu = a_m - \frac{\delta}{|\delta|} \frac{b_{m-1}^2}{|\delta| + \sqrt{\delta^2 + b_{m-1}^2}},$$

  where B is the lower right 2×2 matrix of A, $B = \begin{bmatrix} a_{m-1} & b_{m-1} \\ b_{m-1} & a_m \end{bmatrix}$, and $\delta = \dfrac{a_{m-1} - a_m}{2}$.
- If $\delta = 0$, $\mu = a_m - b_{m-1}$.

Complex Eigenvalues

If a matrix has complex eigenvalues, a Hessenberg matrix (2×2 block) appears along the diagonal of the triangularized matrix A:

$$A = \begin{bmatrix} \lambda_1 & \dots & \dots & \dots \\ 0 & a & b & \dots \\ \vdots & c & d & \dots \\ 0 & \dots & \dots & \lambda_n \end{bmatrix}$$

Eigenvalues are computed as:

$$\lambda_2 = \frac{a + d + \sqrt{(a+d)^2 - 4(ad - bc)}}{2}, \quad \lambda_3 = \frac{a + d - \sqrt{(a+d)^2 - 4(ad -}}{2}$$

Companion Matrix Example:

$$A = \begin{bmatrix} 0 & 1 \\ 273 & -32 \end{bmatrix}$$

Eigenvalues:

$$x_1 = 7.000000 + 0.000000i, \quad x_2 = -39.000000 + 0.000000i$$

Computational Solution: Newton-Raphson Method

Start with an initial guess x_0, and iterate using:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{3.1}$$

where:

$$f(x) = x^2 + 32x - 273, \tag{3.2}$$

$$f'(x) = 2x + 32. \tag{3.3}$$

The update equation becomes:

$$x_{n+1} = x_n - \frac{x_n^2 + 32x_n - 273}{2x_n + 32}. \tag{3.4}$$

Graphical Representation

The quadratic equation: $y = x^2 + 32x - 273$ intersects the x-axis at $x = 7$ and $x = -39$. The solution is illustrated below: