9.1.5: Solving a Differential Equation

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Question

Solve the differential equation:

$$\frac{d^2y}{dx^2} = \cos 3x + \sin 3x \tag{2.1}$$

with initial conditions:

$$y(0) = 0, \quad y'(0) = \frac{1}{3}. \tag{2.2}$$

Theoretical Solution

Integrate with respect to x:

$$\int \frac{d^2y}{dx^2}dx = \int \cos 3x + \sin 3x \, dx, \tag{3.1}$$

$$\frac{dy}{dx} = \frac{\sin 3x}{3} - \frac{\cos 3x}{3} + c_1. \tag{3.2}$$

Using $y'(0) = \frac{1}{3}$:

$$c_1 = 0. \tag{3.3}$$

Integrate again:

$$y = -\frac{\cos 3x}{9} - \frac{\sin 3x}{9} + c_2. \tag{3.4}$$

Using $y(0) = 0$:

$$c_2 = \frac{1}{9}. \tag{3.5}$$

$$\therefore y = -\frac{\cos 3x}{9} - \frac{\sin 3x}{9} + \frac{1}{9}. \tag{3.6}$$

Computational Solution

$$y'(x) = \frac{\sin 3x}{3} - \frac{\cos 3x}{3}. \tag{3.7}$$

Let the Laplace transform of RHS be $X(s)$. Then,

$$g(t) = \frac{\sin 3t}{3} - \frac{\cos 3t}{3}. \tag{3.8}$$

$$\frac{dy}{dt} = g(t). \tag{3.9}$$

Applying Laplace transform on both sides, we have

$$sY(s) = X(s) \tag{3.10}$$

The transfer function, $H(s)$ can then be defined as

$$H(s) = \frac{Y(s)}{X(s)} \tag{3.11}$$

$$H(s) = \frac{1}{s} \tag{3.12}$$

Computational Solution

Applying **Bi-linear transform** on both sides of (3.12), i.e., converting s-domain into z-domain, we have:

$$s = \frac{2}{h}\frac{1 - z^{-1}}{1 + z^{-1}}, \tag{3.13}$$

$$ROC : |z| > 1 \tag{3.14}$$

$$H(z) = \frac{h}{2}\frac{1 + z^{-1}}{1 - z^{-1}}, \tag{3.15}$$

$$Y(z) = \frac{h}{2}\frac{1 + z^{-1}}{1 - z^{-1}}X(z), \tag{3.16}$$

$$(1 - z^{-1})Y(z) = \frac{h}{2}\left(1 + z^{-1}\right)X(z). \tag{3.17}$$

The resulting difference equation becomes:

$$y_n = y_{n-1} + \frac{h}{2}\left[g(x_n) + g(x_{n-1})\right]. \tag{3.18}$$

Graphical Representation