10.3.2.5: LU Factorization

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Question

Half the perimeter of a rectangular garden, whose length is 4 m more than its width, is 36 m. Find the dimensions of the garden.

Matrix Representation

Let the length and width of the garden be x and y, respectively.

$$x + y = 36,$$
$$x - y = 4.$$

We represent this system in matrix form:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 36 \\ 4 \end{bmatrix}, \quad x = \begin{bmatrix} x \\ y \end{bmatrix}.$$

LU Factorization Using Update Equations

- Given a matrix **A** of size $n \times n$, LU decomposition is performed row by row and column by column.
- The update equations are as follows:

Step-by-Step Procedure

1. **Initialization:**
   - Start by initializing $\mathbf{L}$ as the identity matrix $\mathbf{L} = \mathbf{I}$ and $\mathbf{U}$ as a copy of $\mathbf{A}$.
2. **Iterative Update:**
   - For each pivot $k = 1, 2, \ldots, n$:
     1. Compute the entries of $\mathbf{U}$ using the first update equation.
     2. Compute the entries of $\mathbf{L}$ using the second update equation.
3. **Result:**
   - After completing the iterations, the matrix $\mathbf{A}$ is decomposed into $\mathbf{L} \cdot \mathbf{U}$, where $\mathbf{L}$ is a lower triangular matrix with ones on the diagonal, and $\mathbf{U}$ is an upper triangular matrix.

Update for $U_{k,j}$ (Entries of U)

For each column $j \geq k$, the entries of U in the k-th row are updated as:

$$U_{k,j} = A_{k,j} - \sum_{m=1}^{k-1} L_{k,m} \cdot U_{m,j}, \quad \text{for } j \geq k.$$

This equation computes the elements of the upper triangular matrix **U** by eliminating the lower triangular portion of the matrix.

Update for $L_{i,k}$ (Entries of L)

For each row $i > k$, the entries of L in the k-th column are updated as:

$$L_{i,k} = \frac{1}{U_{k,k}} \left(A_{i,k} - \sum_{m=1}^{k-1} L_{i,m} \cdot U_{m,k} \right), \quad \text{for } i > k.$$

This equation computes the elements of the lower triangular matrix $\mathbf{L}$, where each entry in the column is determined by the values in the rows above it.

LU Decomposition Result

Using code, we compute:

$$L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}.$$

Forward Substitution: Solve $Ly = b$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 36 \\ 4 \end{bmatrix}.$$

- From the first row: $y_1 = 36$.
- From the second row: $y_1 + y_2 = 4 \implies y_2 = -32$.

Thus:

$$y = \begin{bmatrix} 36 \\ -32 \end{bmatrix}.$$

Back Substitution: Solve $Ux = y$

$$\begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 36 \\ -32 \end{bmatrix}.$$

- From the second row: $-2y = -32 \implies y = 16$.
- Substitute $y = 16$ into the first row: $x + y = 36 \implies x = 20$.

Thus:

$$x = 20, \quad y = 16.$$

Graphical Representation