9.1.8

EE24BTECH11007 - Arnav Makarand Yadnopavit

Question: Solve the differential equation $y' + y = e^x$ with initial condition $y(0) = 1/2$.
Solution:
Theoretical Solution:

$$\frac{dy}{dx} + y = e^x \tag{1}$$

$$\frac{dy}{dx}e^x + ye^x = e^{2x} \tag{2}$$

$$d\left(ye^x\right) = e^{2x}dx \tag{3}$$

$$\int \frac{d\left(ye^x\right)}{dx} = \int e^{2x}dx \tag{4}$$

$$ye^x = \frac{e^{2x}}{2} + c \tag{5}$$

$$y = \frac{e^x}{2} + ce^{-x} \tag{6}$$

Substituting values from initial conditions

$$y(0) = 0 \tag{7}$$

$$\implies c = 0 \tag{8}$$

$$\therefore y = \frac{e^x}{2} \tag{9}$$

The theoretical solution is $f(x) = \frac{e^x}{2}$

Solution Using Laplace Transform:
Laplace Transform

$$\mathcal{L}\left(f(x)\right) = F(s) = \int_0^\infty f(x)e^{-sx}dx \tag{10}$$

Laplace Transform is a linear transformation, since integration is a linear operation.
Laplace transform of some functions:

$$f(x) = 0 \implies F(s) = 0 \tag{11}$$

$$f(x) = 1 \implies F(s) = \frac{1}{s} \text{ for } Re(s) > 0 \tag{12}$$

$$f(x) = x^n \implies F(s) = \frac{n!}{s^{n+1}} \text{ for } Re(s) > 0 \tag{13}$$

$$f(x) = e^{ax} \implies F(s) = \frac{1}{s-a} \text{ for } Re(s) > a \tag{14}$$

$$f(x) = \sin ax \implies F(s) = \frac{a}{s^2+a^2} \text{ for } Re(s) > 0 \tag{15}$$

$$f(x) = \cos ax \implies F(s) = \frac{s}{s^2+a^2} \text{ for } Re(s) > 0 \tag{16}$$

Some other useful results include :

$$\mathcal{L}\left(f'(x)\right) = sF(s) - f(0^-) \tag{17}$$

$$\mathcal{L}\left(f''(x)\right) = s^2F(s) - sf(0^-) - f'(0^-) \tag{18}$$

Applying Laplace transform to (1) on both sides:

$$\mathcal{L}\left(\frac{dy}{dx} + y\right) = \mathcal{L}\left(e^x\right) \tag{19}$$

$$sF(s) - f(0^-) + F(s) = \frac{1}{s-1} \tag{20}$$

$$F(s) = \frac{1}{s^2-1} + \frac{f(0^-)}{s+1} \tag{21}$$

Using initial condition, $y(0^-) = 1/2$, we have

$$F(s) = \frac{s+1}{2(s^2-1)} \tag{22}$$

$$F(s) = \frac{1}{2(s-1)} \tag{23}$$

$$f(x) = \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) \tag{24}$$

Using (14) we can deduce solution to be the following

$$f(x) = \frac{e^x}{2} \tag{25}$$

Computational Solution:
We can plot a curve with individual closely spaced points such that

$$x_0 = 0 \tag{26}$$

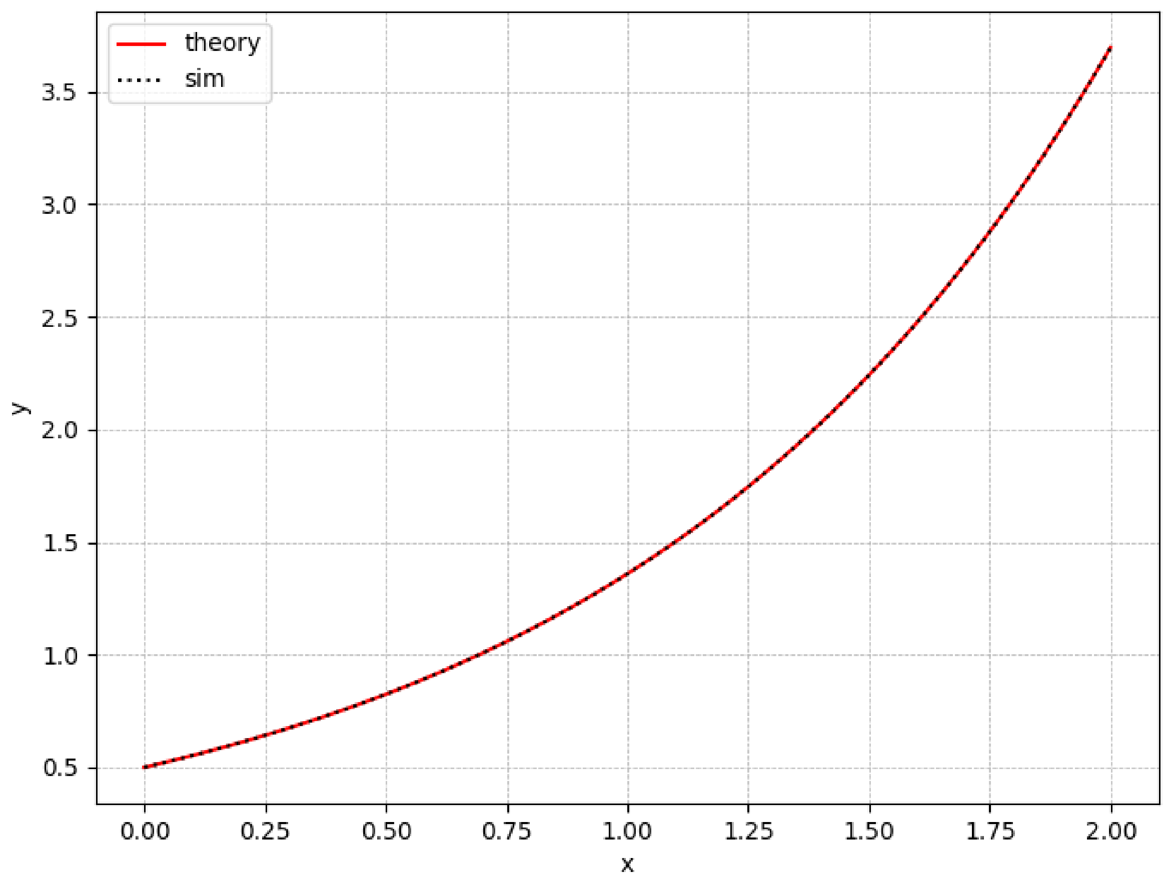
$$y_0 = \frac{1}{2} \tag{27}$$

$$h = 0.001 \tag{28}$$

$$x_{n+1} = x_n + h \tag{29}$$

$$y_{n+1} = y_n + hy' \tag{30}$$

$$\implies y_{n+1} = y_n + h\left(e^x - y\right) \tag{31}$$


theory
sim
3.5
3.0
2.5
2.0
1.5
1.0
0.5
y
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
x