9.1.5

EE24BTECH11007 - Arnav Makarand Yadnopavit

Question: Solve the differential equation $\frac{d^2y}{dx^2} = \cos 3x + \sin 3x$ with initial conditions $y(0) = 0$ and $y'(0) = \frac{1}{3}$.
Solution:
Theoretical Solution:
Integrating with respect to x on both sides

$$\int \frac{d^2y}{dx^2} dx = \int \cos 3x + \sin 3x dx \tag{1}$$

$$\frac{dy}{dx} = \frac{\sin 3x}{3} - \frac{\cos 3x}{3} + c_1 \tag{2}$$

Using initial condition $y'(0) = -\frac{1}{3}$

$$c_1 = 0 \tag{3}$$

$$\implies \frac{dy}{dx} = \frac{\sin 3x}{3} - \frac{\cos 3x}{3} \tag{4}$$

$$\tag{5}$$

Again integrate on both sides with respect to x

$$\int \frac{dy}{dx} dx = \int -\frac{\sin 3x}{3} + \frac{\cos 3x}{3} dx \tag{6}$$

$$y = -\frac{\cos 3x}{9} - \frac{\sin 3x}{9} + c_2 \tag{7}$$

$$\tag{8}$$

Using initial condition $y(0) = \frac{-1}{9}$

$$\implies c_2 = \frac{1}{9} \tag{9}$$

$$\therefore y = -\frac{\cos 3x}{9} - \frac{\sin 3x}{9} + \frac{1}{9} \tag{10}$$

The theoretical solution is $f(x) = -\frac{\cos 3x}{9} - \frac{\sin 3x}{9} + \frac{1}{9}$

Computational Solution:
Using trapezoidal rule to get difference equation

$$x_0 = 0 \tag{11}$$

$$y_0 = 0 \tag{12}$$

$$h = 0.001 \tag{13}$$

$$x_{n+1} = x_n + h \tag{14}$$

$$y_{n+1} = y_n + \frac{h}{2}\left(y'_{n+1} + y'_n\right) \tag{15}$$

$$\implies y_{n+1} = y_n + \frac{h}{2}\left(\left(-\frac{\sin 3x_n}{3} + \frac{\cos 3x_n}{3}\right) + \left(-\frac{\sin 3x_{n-1}}{3} + \frac{\cos 3x_{n-1}}{3}\right)\right) \tag{16}$$

Another approach :
Consider (4). Let the Laplace transform of RHS be $X(s)$. Then,

$$g(t) = \frac{\sin 3t}{3} - \frac{\cos 3t}{3} \tag{17}$$

$$\frac{dy}{dt} = g(t) \tag{18}$$

Applying Laplace transform on both the sides of (18) , we have

$$sY(s) = X(s) \tag{19}$$

The transfer function, $H(s)$ can then be defined as

$$H(s) = \frac{Y(s)}{X(s)} \tag{20}$$

$$H(s) = \frac{1}{s} \tag{21}$$

Applying **Bi-linear transform** on both sides of (21), i.e., converting s-domain into z-domain, we have

$$s = \frac{2}{h}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \tag{22}$$

$$H(z) = \frac{h}{2}\left(\frac{1+z^{-1}}{1-z^{-1}}\right) \tag{23}$$

$$Y(z) = \frac{h}{2}\left(\frac{1+z^{-1}}{1-z^{-1}}\right)X(z) \tag{24}$$

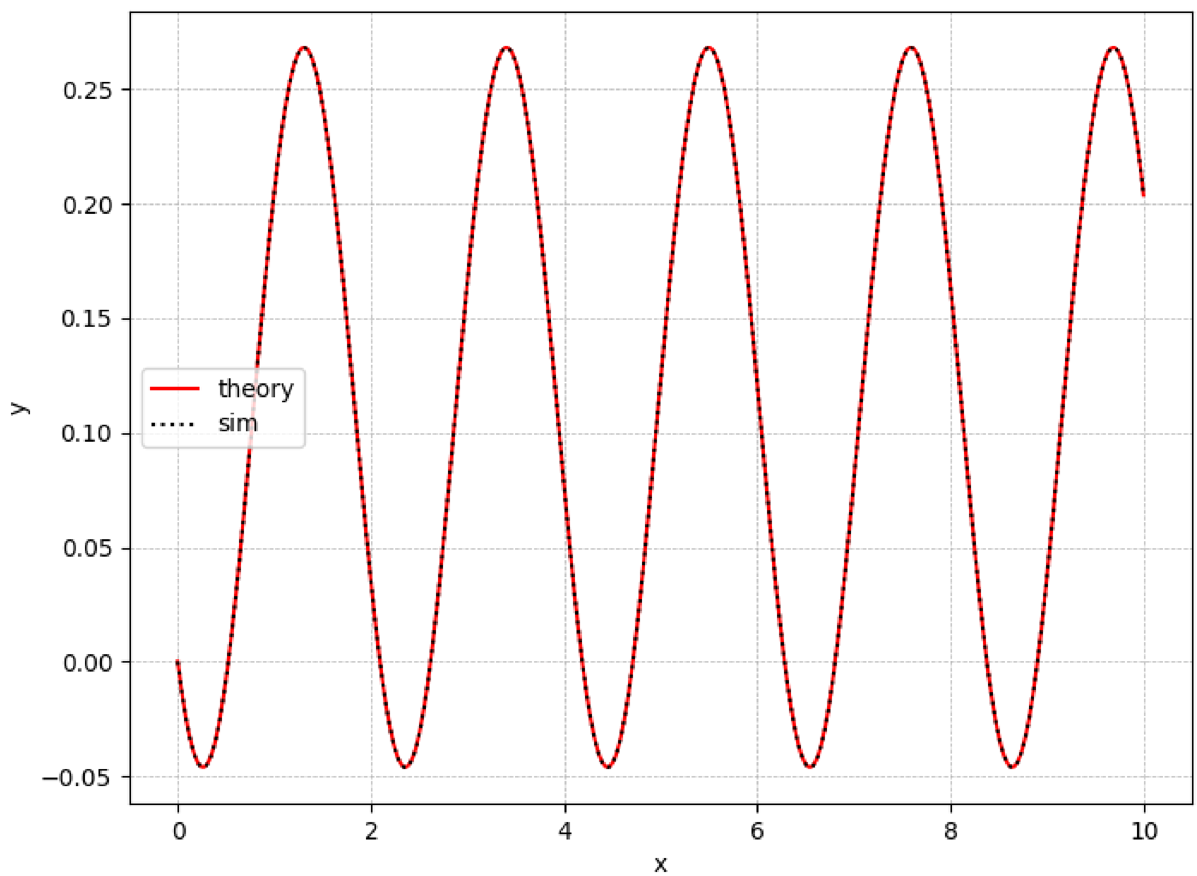
$$\left(1-z^{-1}\right)Y(z) = \frac{h}{2}\left(1+z^{-1}\right)X(z) \tag{25}$$

Taking **Inverse z-transform** on both the sides of (25) , we have

$$y_n - y_{n-1} = \frac{h}{2}\left(g(x_n) + g(x_{n-1})\right) \tag{26}$$

$$y_n = y_{n-1} + \frac{h}{2}\left(g(x_n) + g(x_{n-1})\right) \tag{27}$$

$$\implies y_{n+1} = y_n + \frac{h}{2}\left(\left(-\frac{\sin 3x_n}{3} + \frac{\cos 3x_n}{3}\right) + \left(-\frac{\sin 3x_{n-1}}{3} + \frac{\cos 3x_{n-1}}{3}\right)\right) \tag{28}$$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