2021-MA-'40-52'

EE24BTECH11007 - Arnav Makarand Yadnopavit

40) The equation $xy - z\log y + e^{xz} = 1$ can be solved in a neighborhood of the point $(0, 1, 1)$ as $y = f(x, z)$ for some continuously differentiable function f. Then

a) $\nabla f(0, 1) = (2, 0)$
b) $\nabla f(0, 1) = (0, 2)$
c) $\nabla f(0, 1) = (0, 1)$
d) $\nabla f(0, 1) = (1, 0)$

41) Consider the following topologies on the set $\mathbb{R}$ of all real numbers.
T_1 is the upper limit topology having all sets (a, b) as basis.
$T_2 = \{U \subset \mathbb{R} : \mathbb{R}\backslash U \text{is finite}\} \cup \{\phi\}$.
T_3 is the standard topology having all sets (a, b) as basis. Then

a) $T_2 \subset T_3 \subset T_1$
b) $T_1 \subset T_2 \subset T_3$
c) $T_3 \subset T_2 \subset T_1$
d) $T_2 \subset T_1 \subset T_3$

42) Let $\mathbb{R}$ denote the set of all real numbers. Consider the following topological spaces.
$X_1 = (\mathbb{R}, T_1)$, where T_1 is the upper limit topology having all sets (a, b) as basis.
$X_2 = (\mathbb{R}, T_2)$, where $T_2 = \{U \subset \mathbb{R} : \mathbb{R}\backslash U \text{is finite}\} \cup \{\phi\}$. Then
a) both X_1 and X_2 are connected
b) X_1 is connected and X_2 is NOT connected
c) X_1 is NOT connected and X_2 is connected
d) neither X_1 nor X_2 is connected

43) Let $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be an inner product on the vector space $\mathbb{R}$ over $\mathbb{R}$. Consider the following statements:
$P : |\langle u, v\rangle| \leq \frac{1}{2}\left(\langle u, u\rangle + \langle v, v\rangle\right)$ for all $u, v \in \mathbb{R}^n$.
$Q : If \langle u, v\rangle = \langle 2u, -v\rangle$ for all $v \in \mathbb{R}^n$, then $u = 0$. Then
a) both P and Q are TRUE
b) P is TRUE and Q is FALSE
c) P is FALSE and Q is TRUE
d) both P and Q are FALSE

44) Let G be a group of order 5^4 with center having 5^2 elements. Then the number of conjugacy classes in G is ___________.

45) Let F be a finite field and $F^\times$ be the group of all nonzero elements of F under multiplication. If $F^\times$ has a subgroup of order 17, then the smallest possible order of the field F is ___________

46) Let $R = \{z = x + iy \in \mathbb{C} : 0 < x < 1 \text{and} - 11\pi < y < 11\pi\}$ and Γ be the positively oriented boundary of R. Then the value of the integral

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{e^z dz}{e^z - 2}$$

is ___________.

47) Let $D = \{z \in \mathbb{C} : |z| < 2\pi\}$ and $f : D \to \mathbb{C}$ be the function defined by

$$f(z) = \begin{cases} \frac{3z^2}{(1-\cos z)} & \text{if } z \neq 0, \\ 6 & \text{if } z = 0. \end{cases}$$

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $z \in D$, then $6a_2 =$___________

48) The number of zeroes (counting multiplicity) of $P(z) = 3z^5 + 2iz^2 + 7iz + 1$ in the annular region $\{z \in \mathbb{C} : 1 < |z| < 7\}$ is __________.

49) Let A be a square matrix such that $\det(xI - A) = x^4(x-1)^2(x-2)^3$, where $\det(M)$ denotes the determinant of a square matrix M.
If $\operatorname{rank}\left(A^2\right) < \left(A^3\right) = \left(A^4\right)$, then the geometric multiplicity of the eigenvalue 0 of A is __________.

50) If $y = \sum_{k=0}^{\infty} a_k x^k$, $(a_0 \neq 0)$ is the power series solution of the differential equation $\frac{d^2y}{dx^2} - 24x^2 y = 0$, the $\frac{a_4}{a_0} =$ __________.

51) If $u(x,t) = Ae^{-t}\sin x$ solves the following initial boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0,$$
$$u(0,t) = u(\pi,t) = 0, \quad t > 0,$$
$$u(x,0) = \begin{cases} 60, 0 < x \leq \frac{\pi}{2}, \\ 40, \frac{\pi}{2} < x < \pi, \end{cases}$$

then $\pi A =$__________

52) Let $V = \left\{p : p(x) = a_0 + a_1 x + a_2 x^2, a_0, a_1, a_2 \in \mathbb{R}\right\}$ be the vector space of all polynomials of degree at most 2 over the real field $\mathbb{R}$. Let $T : V \to V$ be the linear operator given by

$$T(p) = (p(0) - p(1)) + (p(0) + p(1))x + p(0)x^2.$$

Then the sum of the eigenvalues of T is __________.